B Project scheduling with time lags and start dependent cost

time between jobs

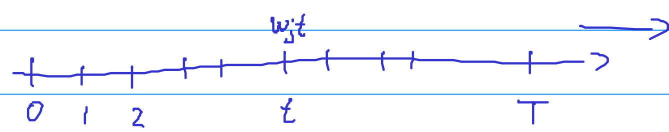
$S_i + d_{ij} \le S_j$

w_{jt} cost for starting j at t

$V = \{0, \dots, n\}$ det. processing times $p_j \in \mathbb{N} \cup \{0\}$

projectstart project end

$p_0 = p_n = 0$ time lags with $0, n$ can be arbitrary



can be positive or negative, or 0

$$\min \sum_j (\text{start time cost of } j) \quad \text{over all feasible schedules}$$

(assume $\exists$ feasible schedule)

Formulation as IP

$$x_{jt} = \begin{cases} 1 & \text{if } j \text{ is started at } t \\ 0 & \text{otherwise} \end{cases}$$

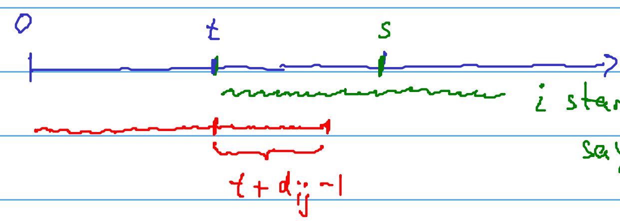
$$\min \sum_j \sum_t w_{jt} x_{jt} \qquad (1)$$

$$\text{s.t.} \quad \sum_t x_{jt} = 1 \quad \forall j \qquad (2)$$

every job is started exactly once

$$\sum_{s=t}^{T} x_{is} + \sum_{s=0}^{t+d_{ij}-1} x_{js} \le 1 \quad \forall (i,j) \in L \ \forall t \qquad (3)$$

every time lag $S_i + d_{ij} \le S_j$ is respected



i starts somewhere here, say at s

so if i starts at a and j at b

and $a + d_{ij} > b$, then (3) is violated for $t = a$

$$x_{jt} \in \{0, 1\} \qquad (4)$$

We can compute in advance

- earliest start $e(j) \geq 0 \quad \forall j$
- lastest start $\ell(j) \leq T - p_j \quad \forall j$

Because of (2), we lift all w_{jt} by a constant M such that

$\tilde{w}_{jt} := w_{jt} + M \geq 0$

This changes the objective only by the constant $(n+1)M$

$\Rightarrow$ will assume w.l.o.g. that $w_{jt} \geq 0$

Transformation into a min cut problem

$D = (N, A)$ is the graph for the min cut problem

Nodes : one for every possible start time of every job and $\ell(j)+1$

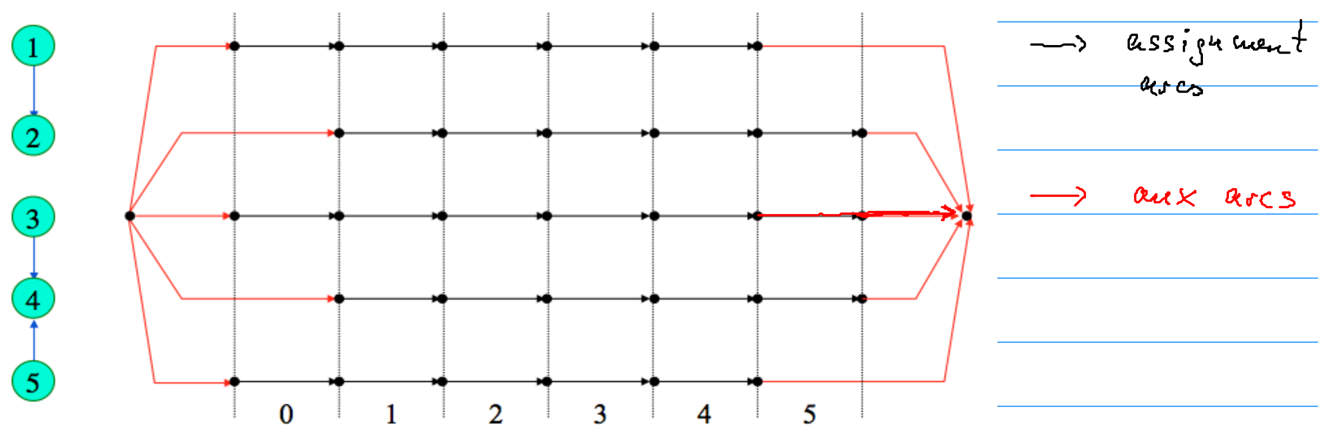
+ source a, sink b

$\Rightarrow N = \{ v_{jt} \mid j \in V, \ t = e(j), e(j)+1, \ldots \ell(j)+1 \} \cup \{a, b\}$

Arcs $A = \{$assignment arcs$\} \cup \{$temporal arcs$\} \cup \{$auxiliary arcs$\}$

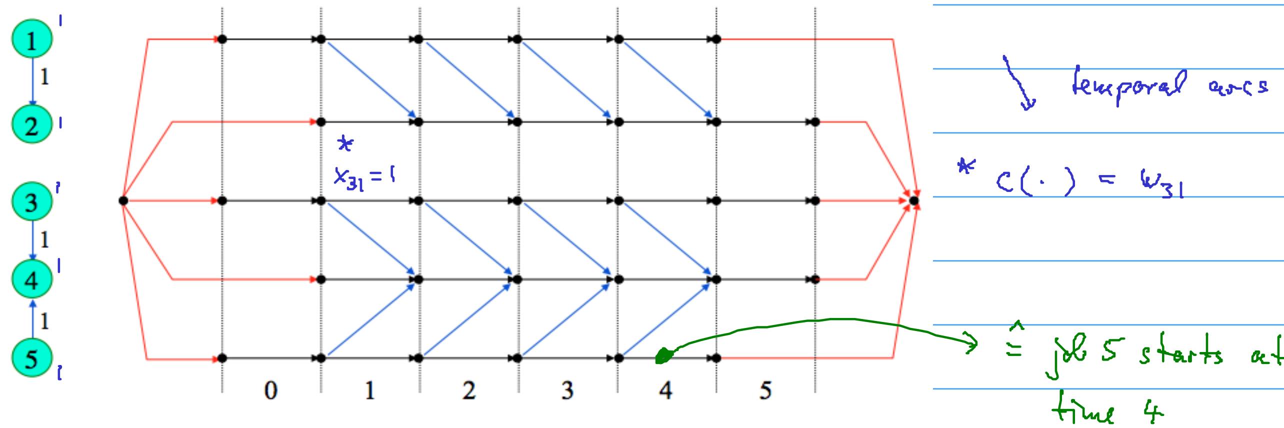
assignment arcs : $(v_{jt}, v_{j,t+1}) \quad \forall j \in V$

auxiliary " : $(a, v_{j,e(j)}) \ (v_{j,\ell(j)+1}, b) \quad \forall j \in V$



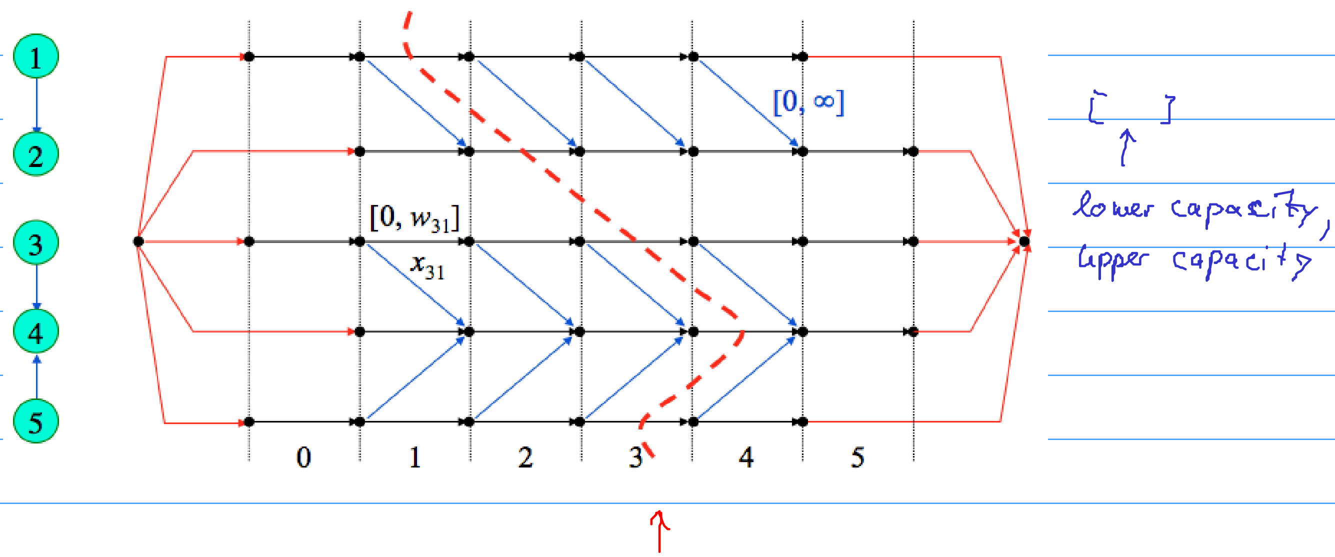
all $p_j = 1$, all $d_{ij} = 1$ $\quad T = 6$

Temporal arcs $(v_{it}, v_{j,t+d_{ij}})$ $\forall$ time lags $(i,j) \in L$
$\forall t$ with $e(i)+1 \leq t \leq l(i)$
$e(j)+1 \leq t+d_{ij} \leq l(j)$



Idea: use assignment arcs to indicate the start of a job

arc capacities: assignment arcs: $c(v_{jt}, v_{j,t+1}) = w_{jt}$
other arcs $c(\cdot) := \infty$
all lower capacities are 0



such a cut will induce start times (maybe infeasible)

For our result, we are interested in special cuts, so called n-cuts.
An n-cut contains exactly one forward arc for every job

21.2 Lemma (n-cuts are the important ones)

Let $(X, \bar{X})$ be a minimum (a,b)-cut and $c(X,\bar{X}) < \infty$

Then $\exists$ an n-cut with the same value

It can be computed from $(X,\bar{X})$ in $O(n \cdot T)$ time

Proof

(1) $(X,\bar{X})$ contains at least one assignment arc for every job

obvious

(2) if $(X,\bar{X})$ contains more than one assignment arc for some job

the we can construct from it an n-cut with the same capacity

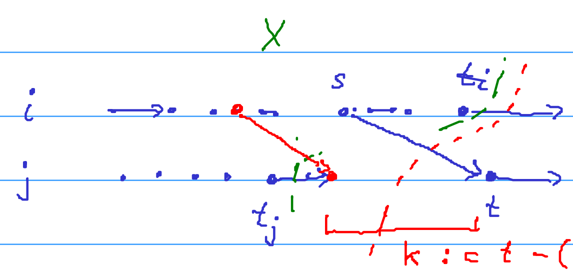
for such job let t_j = first index s.t. $v_{jt} \to v_{j,t+1}$ is forward arc in $(X,\bar{X})$

define $X^* := \{a\} \cup \bigcup_j \{v_{jt} \mid t \le t_j\}$ $\quad \bar{X}^* = N \setminus X^*$

(2.1) $(X^*, \bar{X}^*)$ does not contain a temporal arc as forward arc

suppose it does contain $v_{is} \to v_{jt}$ as forward arc

$\hookrightarrow \hat{=}$ time lag $d_{ij} = t - s$



$k := t - (t_j + 1) \ge 0$

t_j possible start time $\Rightarrow$ $\boxed{e(j) + 1 \le t - k}$ (i)

$\boxed{s - k \le \ell(i)}$ (ii)

$\uparrow$ since $s - k \le s \le t_i \le \ell(i)$

$\boxed{s-k \geq e(i)+1}$ (iii)

otherwise $s-k < e(i)+1$

$\Rightarrow \underline{t-k} = s-k+(t-s) < e(i)+(t-s)+1$

(i,j) time lag with $d_{ij} = t-s \Rightarrow e(i)+(t-s) \leq e(j)$ $\Big\} \Rightarrow$

$\Rightarrow t-k < e(j)+1$, contradiction to (i)

(iii) $\Rightarrow$ $v_{i,s-k}$ is a start node for i, $v_{i,s-k} \in X^*$ by def of X^*

(i) $\Rightarrow$ $v_{j,t-k} = v_{j,t_j+1} \in \bar{X}$

So the temporal arc $v_{i,s-k} \longrightarrow v_{j,t+1}$ is a forward arc of $(X,\bar{X})$

So $c(X,\bar{X}) = \infty$, a contradiction $\Rightarrow$ (2.1)

(2.2) $(X^*,\bar{X}^*)$ has the same capacity as $(X,\bar{X})$

clear since $(X^*,\bar{X}^*)$ contains fewer assignment arcs, no temporal arcs, and $w_{jt} \geq 0$

(3) $(X^*,\bar{X}^*)$ can be constructed in poly time from $(X,\bar{X})$

In $O(nT)$

clear from def of X^*

21.3 Thm

There is a one-to-one correspondance between n-cuts of D with finite capacity and feasible solutions of our scheduling problem defined by (1)-(4), namely

$$x_{jt} = \begin{cases} 1 & \text{if } (v_{jt}, v_{j,t+1}) \text{ is in the cut } (X,\bar{X}) \text{ of } D \\ 0 & \text{otherwise} \end{cases}$$

The capacity $c(X,\bar{X})$ is equal to the value $w(x)$ of the scheduling problem

21.4 Corollary

The scheduling problem with start-time dependent cost and time lags can be solved by computing a maximum a,b-flow and a corresponding a,b-cut in the digraph D