Exam dates: 20 Feb, 30 & 31 Mar,
13/14 Apr and then every week Tuesday afternoon

19.4 THEOREM

(1) Every good cut with minimum cost rate can be found by augmenting path algorithms for max flow in $O(n^3 \log n)$

(2) The zero flow is feasible at t_{max}

The current flow remain feasible when the capacities are changed

$\Rightarrow$ $H(t)$ is convex

(3) $H(t)$ can be calculated in $O(\underbrace{\#\text{ cuts calculation}}_{\geq \#\text{ breakpoints of } H(t)} \cdot n^3 \log n)$

can be exponential

Proof: (1) flow theory

(2) easy verification

(3) obvious (except for exponential example)

This algorithm based on

19.1 THEOREM: Let x be optimal for $t > C_{max}(\alpha)$ [= min. makespan]

Then there exists a good cut $[S,T]$ in D_{crit} with ass. ~~δ~~ $\delta > 0$ s.t.

for every $\rho \in]0, \delta]$, the change of processing times according to (19.3)

By ρ yields an optimal processing time vector y^ρ for $t - \rho$.

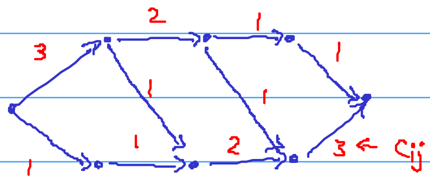
So

$$y^\rho_{ij} = \begin{cases} x_{ij} - \rho & (i,j) \text{ is a forward arc of } [S,T] \\ x_{ij} + \rho & (i,j) \text{ is a backward arc of } [S,T] \text{ with } x_{ij} < b_{ij} \\ x_{ij} & \text{otherwise,} \end{cases}$$

The total cost grows by the amount $\rho \cdot \text{cost rate}(S,T)$

Proof of Thm. 19.1: [Sketch on an example]

Expl.



every job has

$a_{ij} = 1$, $b_{ij} = 10$

Consider $t \in [t_{min}, t_{max}]$ and x optimal for t

Problem: an optimal proc. time vector for $t - \rho$ may be obtained by shortening several jobs in D_{crit} that are scattered all over the network

Must show: this can be done at the same cost on a good cut

Expl: $x = (10, \ldots, 10)$ is optimal for $t = 40 = t_{max}$ and every job is critical

Consider $\bar{\delta}$ defined as δ, but over all arcs of D_{crit}

Expl: $\bar{\delta}_1 = \infty$ (all arcs are critical)

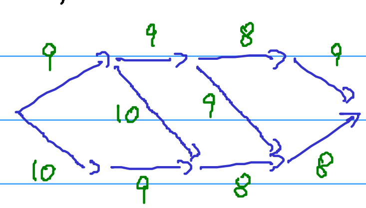
$\bar{\delta}_2 = 9$ (all arcs can be shortened by 9)

$\bar{\delta}_3 = \infty$ (no arc can be lengthened)

$\Rightarrow \bar{\delta} = 9$

Let $\rho_0 \in \,]0, \bar{\delta}]$ and let y be optimal for $t - \rho_0$

Expl. $\rho_0 = 5$ y

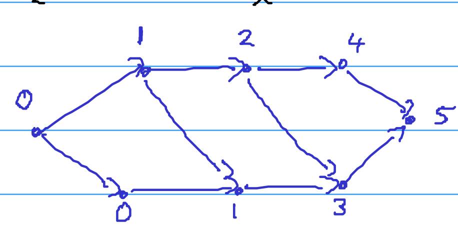


$t - \rho_0 = 35$

Define for each $i \in N$ the potential difference $\delta\pi_i = \pi_i(x) - \pi_i(y)$

Let $\Delta\pi_1 < \Delta\pi_2 < \ldots \Delta\pi_\ell$ be the different $\delta\pi_i$ values

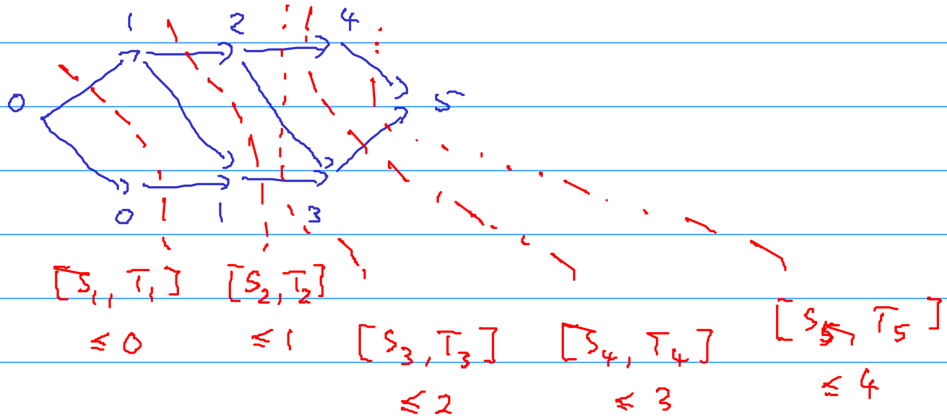
Expl.



$\Delta\pi_1 < \Delta\pi_2 < \Delta\pi_3 < \Delta\pi_4 < \Delta\pi_5 < \Delta\pi_\ell$
$\quad 0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$

(1) $[S_k, T_k]$ with $S_k = \{ i \in N \mid \delta\pi_i \leq \Delta\pi_k \}$ $\quad T_k = N \setminus S_k$
is a good cut $\forall k$

Expl.



(2) y is obtained from x as follows
for $k := 1$ to $\ell - 1$ do
(i) $\Delta_k := \Delta\pi_k - \Delta\pi_{k-1}$
(ii) $x_{ij} := x_{ij} - \Delta_k$ for all forward arcs in $[S_k, T_k]$
(iii) $x_{rs} := x_{rs} + \Delta_k$ for all backward arcs in $[S_k, T_k]$

Expl: $\Delta_k = 1 \ \forall k$, no backward arcs
$\Rightarrow$ subtract 1 on every forward arc of every cut

(3) Let $[S_r, T_r]$ be the cut with smallest cost rate among the $[S_k, T_k]$
Let z be the poc. time vector obtained by changing x on $[S_r, T_r]$ by ϱ_0. (possible since $\varrho_0 \leq \bar{\delta}$)
Then z is optimal for $t - \varrho_0$

Proof: change of total cost
for $x \to y$: $\sum_{k=1}^{\ell-1} \text{costrate}\,[S_k, T_k] \cdot \Delta_k$

$$\geq \sum_k \text{costrate}[S_r, T_r] \cdot \Delta_k$$

$$= \text{costrate}[S_r, T_r] \cdot \rho_0 \quad \hat{=} \text{ from } x \text{ to } z$$

(4) Decrease on cut $[S_r, T_r]$ is optimal for <u>all</u> $\rho_0 \in [0, \tilde{\delta}]$
(ρ_0 was fixed so far)

Let $\rho_1, \rho_2 \in [0, \tilde{\delta}]$ with best cuts $[\bar{S}_1, \bar{T}_1]$, $[\bar{S}_2, \bar{T}_2]$
according to (3)

assume w.l.o.g. $\rho_1 \leq \rho_2$

Show $\text{costrate}[\bar{S}_1, \bar{T}_1] = \text{costrate}[\bar{S}_2, \bar{T}_2]$

If "<" then the decrease on $[\bar{S}_1, \bar{T}_1]$ by $\min\{\rho_1, \rho_2\}$
gives a better solution for $t - \rho_1$ on the other cut, contradiction

(5) So far decrease by at most $\tilde{\delta}$

Now by at most δ (defined by the best cut $[S_r, T_r] =: [S, T]$

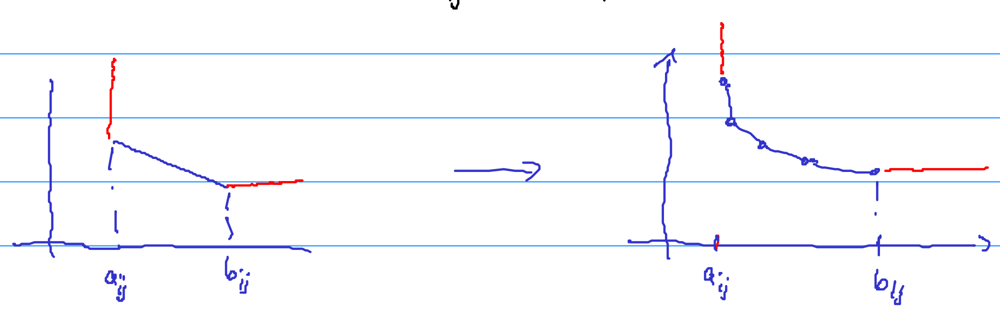
$\tilde{\delta} \leq \delta \overset{(4)}{\Rightarrow} [S, T]$ is optimal for $t - \tilde{\delta}$ and z

All cuts that can be decreased at time $t - \tilde{\delta}$ and z
can also be decreased at time t and x (by def of $\tilde{\delta}$)

$\Rightarrow [S, T]$ is still the best at $t - \tilde{\delta}$ and now can be
further decreased until δ $\square$

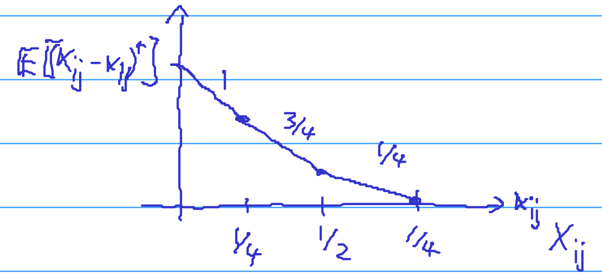
Exercise 19.1

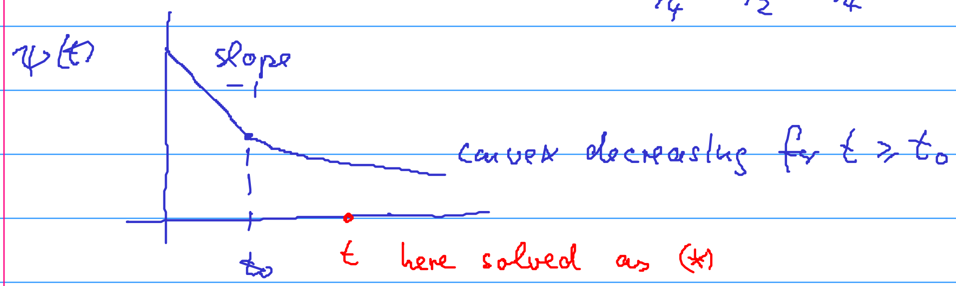
Generalize the computation of $H(t)$ by flow methods to the case
that all cost functions k_{ij} are piecewise linear and convex



Back to the computation of $\psi(t) = \min \sum \overbrace{E[(X_{ij} - x_{ij})^+]}^{k_{ij}}$ (*)

s.t. $C_{max}(x) \le t$





Theorem 18.2 b) $\exists$ random variable Z with $\psi(t) = E[(Z-t)^+]$

$\hookrightarrow$ first piece of $\psi(t)$ will have slope -1

So in the tcto computation with cost functions $k_{ij}(x_{ij}) = E[(X_{ij} - x_{ij})^+]$
stop when the cost rate of the current min cut is ≥ 1
and set the slope to the left of the current time to -1

§ 20 More on the complexity of project scheduling with resource constraints

have seen complexity results for machine scheduling problems
$\underbrace{\text{project scheduling}}_{\text{partial order}}$ with $\underbrace{\text{resource constraints}}_{\text{system of forbidden sets}}$ is much harder

$\llcorner$ jobs may require several resource types

will show hardness of approximation and reoptimization already
for C_{max} and deterministic processing times

Problem is called RCPSP $\hat{=}$ resource constrained project scheduling problem

20.1 Theorem (Folklore)

RCPSP is as hard to approximate as vertex coloring of graphs, i.e., unless P = NP, there is no approx algo with a performance guarantee of $n^{1-\varepsilon}$ for every $\varepsilon \in [0,1[$, $n = \#$ jobs

Proof: Transform instance of COLORING to instance of RCPSP

COLORING	$\Rightarrow$	instance of RCPSP
Given: graph G (undirected)		Given $V = V(G)$ with $x_j = 1 \ \forall j \in V$
number $k \in \mathbb{N}$		no precedence constraints
Question: $\exists$? vertex coloring		$\mathcal{F} = E(G)$, $k \in \mathbb{N}$
with $\leq k$ colors		Question: $\exists$? schedule with
		$C_{max} \leq k$

Then time slots $\hat{=}$ color classes

$\Rightarrow$ inapprox result for coloring applies (Zuckerman 2007) $\square$