§ 18 Bounds for dependent processing times and C_{max}

difficult to specify $\Rightarrow$ worst case approach

Worst case approach for stochastic dependencies

[Meilijson & Nadas '79, Klein-Haneveld '86]

Consider **expected tardiness**

$$\mathbb{E}_Q[(C_{\max} - t)^+] = \mathbb{E}_Q[\max\{0, C_{\max} - t\}]$$

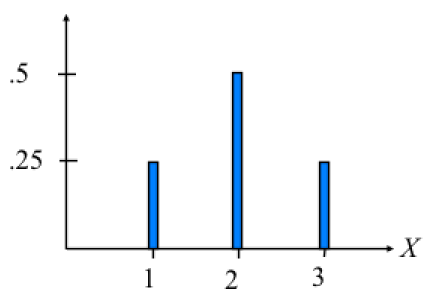
of makespan C_{max} in the worst case, i.e.

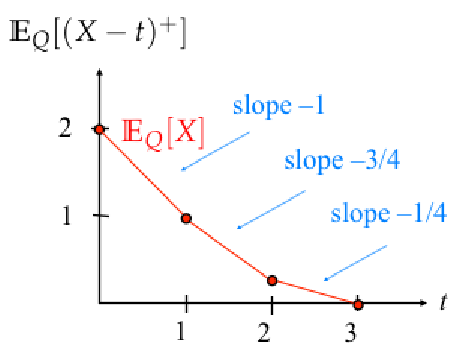
$$\psi(t) := \sup_Q \mathbb{E}_Q[(C_{\max} - t)^+]$$

ranges over all joint distributions with the given job processing time distributions as marginals

are distr. Q_j of job j

Properties of expected tardiness





$\mathbb{E}_Q[(X-t)^+]$ is piecewise linear and convex for discrete random variables X $\quad = \int_t^\infty (1 - F_X(\xi))\, d\xi$

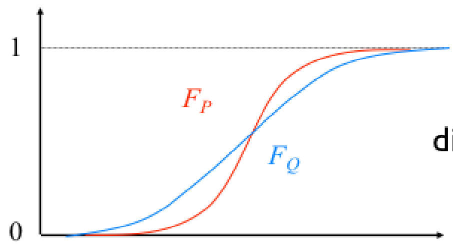
The stochastic ordering in the convex sense

P is stochastically smaller than Q in the convex sense if

$$P \leq_c Q \Leftrightarrow \int f dP \leq \int f dQ$$

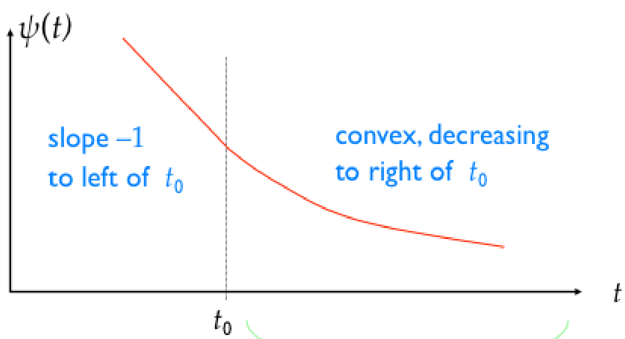
for all monoton convex functions $f : \mathbb{R}^1 \to \mathbb{R}^1$

$\Leftrightarrow \mathbb{E}[(X-t)^+] \leq \mathbb{E}[(Y-t)^+]$ for all t if X, Y are real-valued random variables with distributions P, Q, respectively



distribution functions may cross

Properties of $\psi(t) = \sup_Q \mathbb{E}_Q[(C_{\max} - t)^+]$



18.1 Theorem:

∃ t_0 such that ...

$$\psi(t) = \min_{(x_1,\ldots,x_n)} \sum_j \mathbb{E}[(X_j - x_j)^+]$$

such that $C_{\max}(x_1,\ldots,x_n) \leq t$

special convex separable optimization problem

In more detail:

18.2 Theorem (Meilijson & Nadas 1971)

Let $\mathcal{P}$ be the class of joint distributions Q whose marginals Q_j equal the processing time distributions. Then

a) $\mathbb{E}_Q[(C_{\max} - t)^+] \leq \psi(t)$ for all $Q \in \mathcal{P}$

as in the min. problem

b) $\exists$ random variable Z with $\psi(t) = \mathbb{E}[(Z-t)^+]$

c) If P_Z is the distribution of Z, then $Q_{C_{max}} \leq_c P_Z \quad \forall Q \in \mathcal{P}$

d) If G is series-parallel, then $P_Z = Q_{C_{max}^G}$ for some $Q \in \mathcal{P}$

e) $\psi(t)$ is a tight upper bound for $\mathbb{E}_Q[(C_{max}-t)^+]$ in the sense

as min problem

that, for every t $\exists Q_t \in \mathcal{P}$ s.t. $\psi(t) = \mathbb{E}_{Q_t}[(C_{max}-t)^+]$

This shows equality in Thm. 18.1

Proof: Show only a)

Proof of bounding property

Consider chain C of N and processing time vector $x = (x_1,\dots,x_n)$

$$\sum_{j\in C} X_j - t = \underbrace{\sum_{j\in C} x_j}_{\leq C_{\max}(x)} - t + \underbrace{\sum_{j\in C}(X_j - x_j)}_{\leq \sum_{j=1}^n \cancel{(X_j - x_j)} \leq \sum_{j=1}^n (X_j - x_j)^+}$$

$$\Downarrow$$

$$\sum_{j\in C} X_j - t \leq C_{\max}(x) - t + \sum_j (X_j - x_j)^+$$

$$\leq (C_{\max}(x) - t)^+ + \sum_j (X_j - x_j)^+ \quad \text{all } C, \text{ all } x$$

$$\Downarrow$$

$$\underbrace{\max_C \sum_{j\in C} X_j}_{C_{\max}(X),\ X = (X_1,\dots,X_n)} - t \leq \dots$$

$$\Downarrow$$

$$C_{\max}(X) - t \leq \underbrace{(C_{\max}(x) - t)^+ + \sum_j (X_j - x_j)^+}_{\geq 0}$$

$$\Downarrow$$

$$(C_{\max}(X) - t)^+ \leq \dots \quad \text{for all } x$$

$$\Downarrow$$

$$E_Q[\ \dots\ ] \leq E_Q[\ \dots\ ]$$

$$= (C_{\max}(x) - t)^+ + \sum_j E_Q\left[(X_j - x_j)^+\right] \quad \text{for all } x, Q$$

$$\Downarrow$$

$$E_Q\left[(C_{\max}(X) - t)^+\right] \leq \underbrace{\inf_x \left\{(C_{\max}(x) - t)^+ + \boxed{\sum_j E_Q\left[(X_j - x_j)^+\right]}\right\}}_{\text{optimization problem}} \leq \sum_j \mathbb{E}_{Q_j}[(X_j - x_j)^+] \text{ s.t. } C_{max}(x) \leq t$$

independent of Q

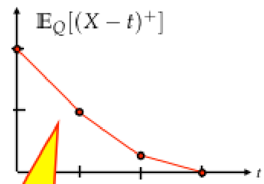
$$\Downarrow$$

$$\underbrace{\sup_Q E_Q\left[(C_{\max}(X) - t)^+\right]}_{\psi(t)} \leq \dots$$

Solving the convex optimization problem

$\psi(t) = \min_{(x_1,\dots,x_n)} \sum_j \boxed{\mathbb{E}[(X_j - x_j)^+]}$ such that $C_{\max}(x_1,\dots,x_n) \le t$

- piecewise linear, convex, decreasing function $\boxed{f(x_j)}$ for every job j in objective
- Interprete $f(x_j)$ as cost for excecuting job j with processing time x_j
- Side constraints:
  Find processing times x_j that
  - minimize the total cost
  - do not exceed the deadline t on the makespan



time-cost
tradeoff
problem

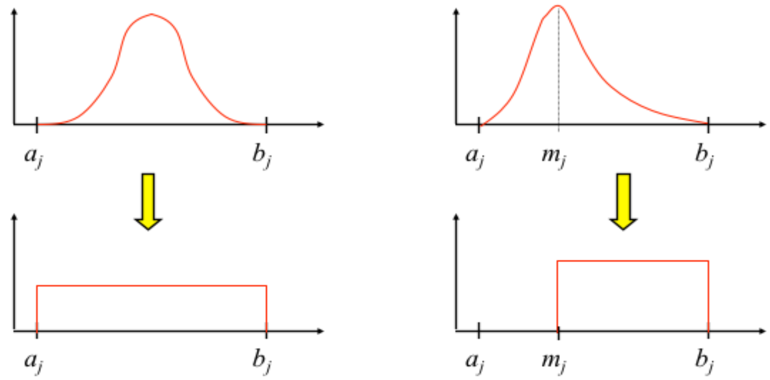
Time-cost tradeoff problems

- classical network problem
- is the dual of a min-cost-flow problem for fixed t
  [Fulkerson '61]
- can be solved parametrically in t by a sequence of max-flow problems
  [Kelley '61]
- very efficient in practice

Compatibility with incomplete information

Incomplete information about X_j [Cipra '78, Zackova '66]

unimodal & symmetric — only unimodal

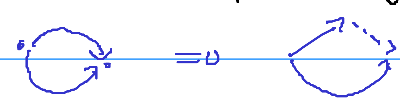


These uniform distributions are larger in the convex sense

§ 19 Time-cost tradeoff problems

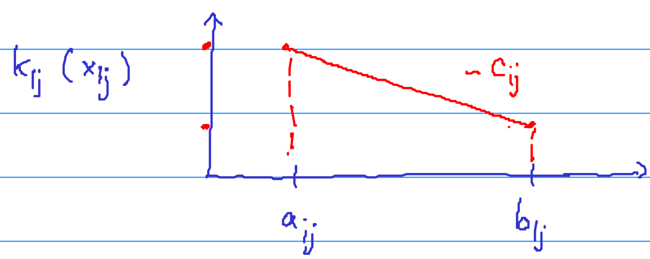
The linear case

Given: • project network as arc diagram without parallel jobs



• for every job (i,j) an interval $I_{ij} = [a_{ij}, b_{ij}]$ of possible job durations

• for every job (i,j) a cost function k_{ij} with slope $-c_{ij}$, $c_{ij} > 0$



$k_{ij}(x_{ij})$ denotes the cost of processing job (i,j) with processing time $x_{ij} \in I_{ij}$

• time limit t for the makespan

Goal: Execute the project at minimum cost within the given time limit,

i.e. $\min \; k(x) := \sum_{\text{jobs}(i,j)} k_{ij}(x_{ij})$

s.t. x = vector of chosen x_{ij}

and $C_{max}(x) \le t$

$H(t) :=$ min cost for t

$H(t)$ is called the project cost curve

This problem is called the (linear) time-cost tradeoff problem (tcfo)

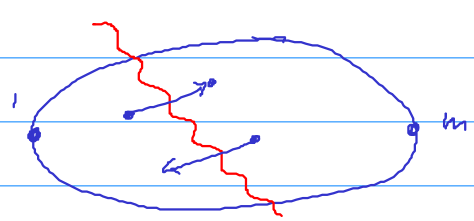
Note: $k_{ij}(x_{ij}) = \underbrace{k_{ij}(b_{ij})}_{\text{constant}} + \underbrace{(b_{ij} - x_{ij})}_{\text{shortening from } b_{ij}} \cdot c_{ij}$

$\Rightarrow$ we shorten jobs at a cost rate c_{ij}
and want to find the right shortenings

Basic Idea

- consider an optimal proc. time vector x for t
- characterize "optimal" tradeoffs to $t - \varepsilon$, ε small, in the arc diagram
  [will show: must shorten on a cut in the network of critical jobs]

Def: Let $D = (N, A)$ be the arc diagram of G
$N = \{1, \dots, m\}$ $\quad 1 \,\hat{=}\,$ source, $m \,\hat{=}\,$ sink
A cut $[S, T]$ of D is a partition $N = S \,\dot\cup\, T$ of N
with $1 \in S$, $m \in T$



forward arc (i,j), $i \in S$, $j \in T$

backward arc (i,j) $\quad j \in S$, $i \in T$

(i,j) is *critical* if it is in a critical path (for given x)

$D_{crit} = (N_{crit}, E_{crit})$ denotes the subnetwork of critical jobs

Given x, let $\pi_i(x)$ denote the length of a longest path from 1 to i w.r.t. x

"potential" π

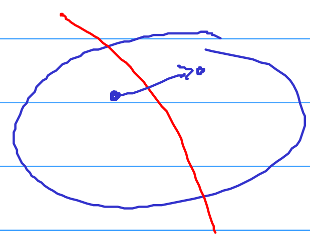
let x be optimal for t $\longrightarrow$ node potentials $\pi_i(x)$

z -"- for $t-\varepsilon$ $\longrightarrow$ node potentials $\pi_i(z)$

$\Rightarrow \pi_1(x) = \pi_1(z) = 0$

$\pi_m(x) = t \quad , \quad \pi_m(z) = t - \varepsilon$

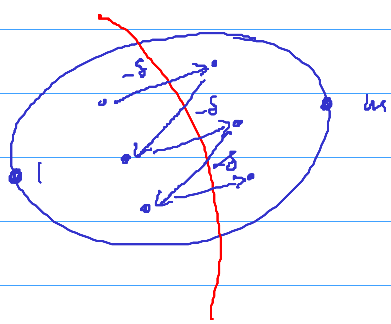
$\Rightarrow S := \{ i \in N \mid \pi_i(x) = \pi_i(z) \}$ $\quad T := N \setminus S$ is a cut



proc. times have changed on forward arcs in the cut (and maybe elsewhere)

$\Downarrow$

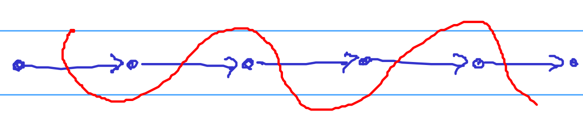
Idea: Consider what happens if we change proc. times on a cut only



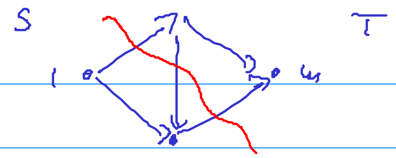
shorten forward arcs by δ
$\Rightarrow$ reduces π_m by at least δ $\quad$ (19.1)

But some $\pi_i(z)$ may be changed by a multiple of δ

A cut $[S,T]$ is *good* if every $s \in S$ can be reached from 1 by a directed path in S



not a good cut



Shortening all forward arcs in a good cut of D_{crit} by δ reduces π_n by exactly δ } (19.2)

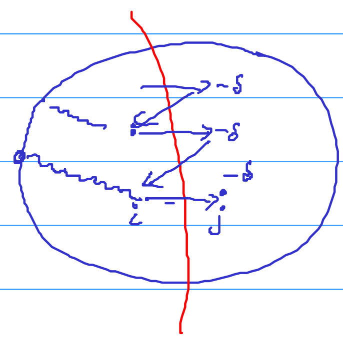
Proof: assume topological sort of nodes

let (i,j) be forward arc with largest i in S

$[S,T]$ good $\Rightarrow$ there is a path from 1 to i in S

$\Rightarrow$ this path is only shortened by δ

path ^from j can not cross to S again (top. sort)



(i,j) on critical path $\Rightarrow$ π_n reduces by exactly δ