Idea for approximation algorithm:

A: Solve (LP) → optimal solution C_j^{LP}

[can be done in polynomial time although we have exponentially many inequalities]

B: Use the ordering $C_{j_1}^{LP} \leq C_{j_2}^{LP} \leq \dots \leq C_{j_n}^{LP}$ as

job-based priority list $j_1 < j_2 < \dots < j_n$

[different from list scheduling as considered before:

may start j_k only after all $j_1, \dots, j_{k-1}$ have been started,

i.e. list scheduling with condition $S_{j_1} \leq S_{j_2} \leq \dots \leq S_{j_n}$] (*)

C: Use Lemma 14.2 to prove a performance guarantee.

$\boxed{\text{More on B}}$

14.3 THEOREM

(1) (*) defines a policy (called a job-based list scheduling policy)

(2) Every such policy is dominated by a preselective policy

Proof:

(1) (*) only uses info from the past at each decision point

⟹ it defines a policy Π

(2) The (*) condition implies that, for every forbidden set F,

the last job in the list from F is a waiting job

(it is started last from F and thus must wait for the completion of any other job from F)

⟹ the preselective policy Π^* with the same selection of waiting jobs dominates Π (it may start jobs earlier and thus violate $S_{j1} \leq S_{j2} \leq .. \leq S_{jn}$) □

More on $C\ell$

14.4 Lemma

Let Π be a job-based list scheduling policy with list $L = 1 < 2 < .. < n$
Let $C_j^{\Pi}(x) := \Pi[x](j) + x_j$ denote the completion time of job j w.r.t. Π, x
Then

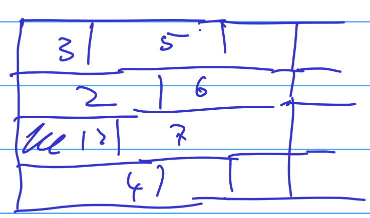
$$C_j^{\Pi}(x) \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k + x_j \quad \forall \; x$$

and thus

$$\mathbb{E}[C_j^{\Pi}] \leq \frac{1}{m} \sum_{k=1}^{j-1} \mathbb{E}[X_k] + \mathbb{E}[X_j]$$

in the stochastic case

Proof: L implies that jobs $1, 2, 3 \ldots j-1$ are started before j can start



The latest time at which a machine becomes available for j is $\frac{1}{m} \sum_{k=1}^{j-1} x_j \Rightarrow S_j \leq \frac{1}{m} \sum_{k=1}^{j-1} x_j \quad C_j \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k + x_j$

Taking expectations gives the second inequality

14.5 THEOREM

Let $C_1^{LP} \leq C_2^{LP} \leq \ldots \leq C_n^{LP}$ be an optimal solution of (LP)
and let, for fixed x, $C_1, .., C_n$ be the completion times obtained by the job-based priority policy Π with list $L = 1 < 2 < .. n$ (LP-ordering)

Then $\sum_j w_j C_j \le \left(3-\frac{1}{m}\right) \text{OPT}$

So: the algorithm "LP-guided job-based priority scheduling" is a $\left(3-\frac{1}{m}\right)$-approximation algorithm for the deterministic case

Proof:

$$C_j \underset{\uparrow \text{ Lemma 14.4}}{\le} \frac{1}{m}\sum_{k=1}^{j-1} x_k + x_j = \frac{1}{m}\sum_{k=1}^{j} x_k + \frac{m-1}{m} x_j$$

La 14.2

$$\underset{\uparrow \text{ Lemma 14.2}}{\le} 2\cdot C_j^{LP} + \frac{m-1}{m}\underbrace{x_j}_{\le C_j^{LP}} \le \left(3-\frac{1}{m}\right) C_j^{LP}$$

$$\Rightarrow \sum_j w_j C_j \le \left(3-\frac{1}{m}\right) \underbrace{\sum_j w_j C_j^{LP}}_{\text{LP objective}} \le \left(3-\frac{1}{m}\right) \text{OPT}$$

LP is a relaxation

from LP

14.2 LEMMA: If numbers $C_1 \le C_2 \le \dots \le C_n$ fulfill (1), then

$$2C_j \ge \frac{1}{m}\sum_{k\in J} p_k \quad \text{with } J = \{1,2,\dots,j\}$$

$\boxed{\text{More on } A}$

can solve the (LP) in polynomial time if the separation problem for (1) and (2) can be solved in polynomial time

Given $(C_1,\dots,C_n) \in \mathbb{R}^n_{\ge 0}$, $A \subset V$ define violation

$$v(A) := \text{rhs of (1)} - \text{lhs (1)} = \underbrace{\frac{1}{2m}\left(\sum_{j\in A} p_j\right)^2 + \frac{1}{2}\sum_{j\in A} p_j^2}_{\text{rhs}} - \underbrace{\sum_{j\in A} p_j C_j}_{\text{lhs}}$$

14.6 Lemma: Let A maximize the violation. Then

$$k \in A \iff C_k - \frac{1}{2} p_k < \frac{1}{m}\sum_{j\in A} p_j$$

Proof: Let $k \notin A$. Then

$$v(A \cup \{k\}) = v(A) + \frac{1}{m} p_k \left(\sum_{j \in A} p_j\right) + \frac{1}{2m} p_k^2 + \frac{1}{2} p_k^2 - p_k C_k$$

$$= v(A) + p_k \left[\frac{1}{m} \sum_{j \in A} p_j + \frac{m+1}{2m} p_k - C_k\right]$$

Let $k \in A$. Then

$$v(A \setminus \{k\}) = v(A) - p_k \left[\frac{1}{m} \sum_{j \in A} p_j + \frac{m-1}{2m} p_k - C_k\right]$$

Let A maximize the violation. Then

$k \in A$: $v(A \setminus \{k\}) \leq v(A) \Rightarrow \frac{1}{m} \sum_{j \in A} p_j + \frac{m-1}{2m} p_k - C_k \geq 0$

$$\Rightarrow C_k \leq \frac{1}{m} \sum_{j \in A} p_j + \frac{m-1}{2m} p_k < \frac{1}{m} \sum_{j \in A} p_j + \frac{1}{2} p_k$$

$$\Rightarrow \boxed{C_k - \frac{1}{2} p_k < \frac{1}{m} \sum_{j \in A} p_j}$$

$k \notin A$ $\Rightarrow v(A \cup \{k\}) \leq v(A) \Rightarrow p_k [\dots] \leq 0$

$$\Rightarrow \frac{1}{m} \sum_{j \in A} p_j + \frac{m+1}{2m} p_k \leq C_k$$

$$\Rightarrow \frac{1}{m} \sum_{j \in A} p_j + \frac{1}{2} p_k < C_k$$

$$\Rightarrow \boxed{C_k - \frac{1}{2} p_k > \frac{1}{m} \sum_{j \in A} p_j}$$

14.7 Separation algorithm

(1) Sort jobs w.r.t. increasing $C_j - \frac{1}{2} p_j$ values. Let $1, 2, 3 \dots$ be this ordering

(2) The set A with maximum violation is an initial segment $J = \{1, 2, \dots, j\}$ of this ordering

(3) Check initial segments of this ordering for violation

Proof (2): let A max the violation and $i \in A$

Show $k \in A$ for every $k < i$

$i \in A \underset{\text{La 14.6.}}{\Rightarrow} C_i - \frac{1}{2} p_i < \frac{1}{m} \sum_{j \in A} p_j$

$k < i \underset{\text{ordering}}{\Rightarrow} C_k - \frac{1}{2} p_k \leq C_i - \frac{1}{2} p_i < \frac{1}{m} \sum_{j \in A} p_j \underset{\text{La 14.6}}{\Rightarrow} k \in A$

(1), (2) can be done in polyn. time $\square$

Consider now the stochastic counterpart of this problem

(m machines, no prec, random independent processing times X_j

minimize $\sum_j w_j \mathbb{E}[C_j^\Pi]$ over all policies Π)

The LP-based approach

similar in queuing theory:
Bertsimas, Glazebrook, Nino-Mora

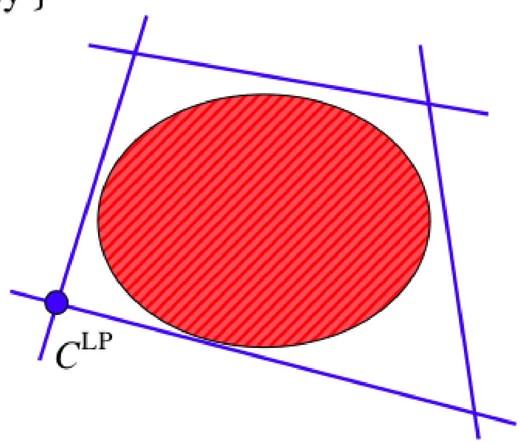
Consider the achievable region

$\{ (E[C_1^\Pi], \ldots, E[C_n^\Pi]) \in \mathbb{R}^n \mid \Pi \text{ policy} \}$

Find a polyhedral relaxation ***P***

Solve the linear program

(LP) $\min\{ \sum_j w_j C_j^{LP} \mid C^{LP} \in P \}$



we do not know properties of the achievable region (only boundedness if $\mathbb{E}(X_j)$ exist)

Use the list L: $i_1 \leq i_1 \leq \ldots \leq i_n$

defined by $C_{i_1}^{LP} \leq C_{i_2}^{LP} \leq \ldots \leq C_{i_n}^{LP}$

as list for priority/lin. pres./other policy

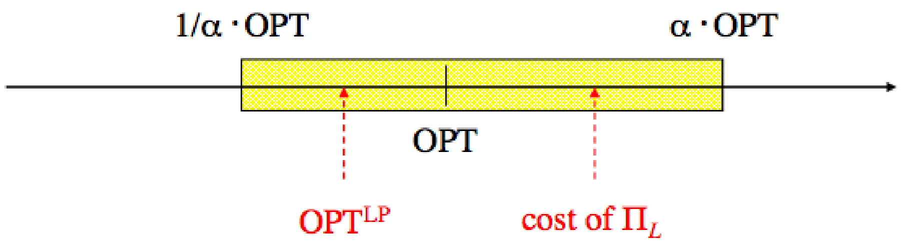
will be job based priority policy

Bounds

Performance guarantees from the LP

Let Π_L be the policy induced by L: $i_1 \le i_1 \le \ldots \le i_n$ ← job-based priority policy

Hope that $\boxed{E[\kappa^{\Pi_L}] \le \alpha \cdot \mathrm{OPT}^{\mathrm{LP}},\ \alpha \ge 1}$



The tasks

- Find the relaxation P
- Solve the LP optimally in polynomial time to obtain list L
- Prove $E[\kappa^{\Pi_L}] \le \alpha \cdot \mathrm{OPT}^{\mathrm{LP}},\ \alpha \ge 1$

The polyhedral relaxation

Lemma 14.7 ($\hat{=}$ Lemma 14.1 in the deterministic case)

Every policy Π (for our problem) fulfills

$$\sum_{j\in A} E[X_j]\cdot E[C_j^{\Pi}] \ge \frac{1}{2m}\Big(\sum_{j\in A} E[X_j]\Big)^2 + \frac{1}{2}\sum_{j\in A} E[X_j]^2$$ ← as in det. case

$$- \frac{m-1}{2m}\sum_{j\in A} \mathrm{Var}[X_j]$$ ← new

$\forall A \subseteq V$

Proof: Consider fixed $x = (x_1, \dots, x_n)$. Lemma 14.1 gives

$$\sum_{j\in A} x_j\, C_j^{\pi}(x) \geq \frac{1}{2m}\Big(\sum_{j\in A} x_j\Big)^2 + \frac{1}{2}\sum_{j\in A} x_j^2 \qquad \forall A \subseteq V$$

Rewriting in terms of start times $S_j^{\pi}(x) = C_j^{\pi} - x_j$

$$\sum_{j\in A} x_j\, S_j^{\pi}(x) \geq \frac{1}{2m}\Big(\sum_{\substack{i,j\\ i\neq j}} x_i \cdot x_j\Big) - \frac{m-1}{2m}\sum_{j\in A} x_j^2 \qquad (1)$$

S_j^{π} and X_j are stoch indep $\Rightarrow$ $\mathbb{E}(X_j \cdot S_j^{\pi}) = \mathbb{E}(X_j)\cdot \mathbb{E}(S_j^{\pi})$ (*)

will also use $VAR[X_j] = \mathbb{E}[X_j^2] - \mathbb{E}(X_j)^2$ (**)

Take expectations in (1) $\Rightarrow$

$$\sum_{j\in A} \underbrace{\mathbb{E}[X_j \cdot S_j^{\pi}]}_{\substack{\mathbb{E}[X_j]\cdot \mathbb{E}[S_j^{\pi}] \\ \text{indep.}}} \geq \frac{1}{2m}\sum_{\substack{i,j\\ i\neq j}} \underbrace{\mathbb{E}[X_i \cdot X_j]}_{\substack{\mathbb{E}[X_i]\,\mathbb{E}[X_j] \\ \text{indep.}}} - \frac{m-1}{2m}\sum_{j\in A} \mathbb{E}[X_j^2]$$

$$\sum_{j\in A} \mathbb{E}[X_j]\cdot \mathbb{E}[S_j^{\pi}] \geq \frac{1}{2m}\Big(\sum_{j\in A} \mathbb{E}(X_j)\Big)^2 - \frac{1}{2m}\sum_{j\in A} \mathbb{E}[X_j]^2 - \frac{m-1}{2m}\sum_{j\in A} \mathbb{E}[X_j^2]$$

add $\frac{1}{2m}\sum_{j\in A} \mathbb{E}[X_j]^2$ to first term and subtract

reformulate this part by adding to it a "nice" $0 = \frac{m-1}{2m}\sum_{j} \mathbb{E}[X_j]^2 - \frac{m-1}{2m}\sum_{j} \mathbb{E}[X_j]^2$

$$= -\frac{m-1}{2m}\sum_{j\in A} VAR[X_j] - \frac{1}{2}\sum_{j\in A} \mathbb{E}[X_j]^2$$

Adding $\sum_{j \in A} \mathbb{E}[X_j]^2$ on both sides gives

$$\sum_{j \in A} \mathbb{E}[X_j] \cdot \mathbb{E}[C_j^\pi] \geq \underbrace{\frac{1}{2m} \quad \cdots\cdots}_{\text{statement of Lemma}} \qquad \square$$