§18 Bounds for dependent processing times and C_{max}

Previous bounds require the processing times to be stochastically independent.
Since dependencies are difficult to specify, we follow a worst case approach.

Worst case approach for stochastic dependencies

[Meilijson & Nadas '79, Klein-Haneveld '86]

Consider expected tardiness

$$\mathbb{E}_Q[(C_{\max} - t)^+] = \mathbb{E}_Q[\max\{0, C_{\max} - t\}]$$

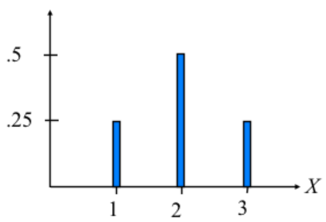
of makespan C_{max} in the worst case, i.e.

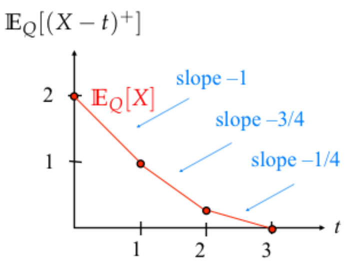
$$\psi(t) = \sup_Q \mathbb{E}_Q[(C_{\max} - t)^+]$$

ranges over all joint distributions with the given job processing time distributions as marginals

i.e. Q_j = distribution of X_j

Properties of expected tardiness

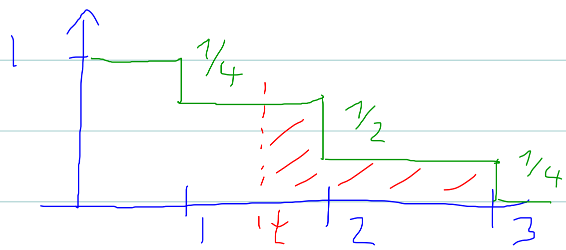




$\Rightarrow$ easy to calculate for discrete distributions

$\mathbb{E}_Q[(X-t)^+]$ is piecewise linear and convex for discrete random variables X

To see this, consider $1 - F_X(t)$:



$\Rightarrow E[(X-t)^+] =$

$(3-t) \cdot 1/4 + (2-t) \cdot 1/2$

$=$ area

$\Rightarrow E[(X-t)] = \int_t^\infty (1-F_X(t))\,dt$

Expected tardiness is closely related to another stochastic ordering (weaker than $\leq_{st}$).

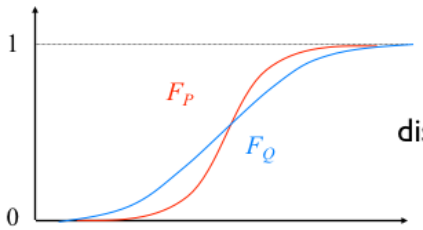
The stochastic ordering in the convex sense

P is stochastically smaller than Q in the convex sense if

$$P \leq_c Q \Leftrightarrow \int f dP \leq \int f dQ$$

for all monoton convex functions $f : \mathbb{R}^1 \to \mathbb{R}^1$

$\Leftrightarrow \mathbb{E}[(X-t)^+] \leq \mathbb{E}[(Y-t)^+]$ for all t if X, Y are real-valued random variables with distributions P, Q, respectively



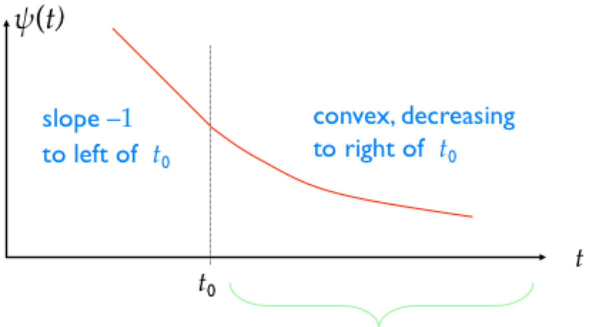
distribution functions may cross

Properties of $\psi(t) = \sup_Q \mathbb{E}_Q[(C_{\max} - t)^+]$

18.1 Theorem

main theorem about $\psi(t)$:

$\exists\, t_0$ such that ...



$$\psi(t) = \min_{(x_1,\ldots,x_n)} \textstyle\sum_j \mathbb{E}[(X_j - x_j)^+]$$

such that $C_{\max}(x_1,\ldots,x_n) \leq t$

special convex separable optimization problem

In more detail:

18.2 Theorem (Melijson & Nadas 1971)

Let $\mathcal{P}$ be the class of joint distributions Q whose marginal distributions equal the processing time distributions. Then:

a) $E_Q[(C_{max}-t)^+] \le \psi(t)$ for all $Q \in \mathcal{P}$

b) $\exists$ random variable Z with $\psi(t) = E[(Z-t)^+]$

c) If P_Z is the distribution of Z, then $Q_{C_{max}} \le_c P_Z$ for all $Q \in \mathcal{P}$

d) If G is series-parallel, then $P_Z = Q_{C^G_{max}}$ for some $Q \in \mathcal{P}$

e) $\psi(t)$ is a tight upper bound for $E_Q[(C_{max}-t)^+]$ in the sense that, for every $t \ge 0$ there exists $Q_t \in \mathcal{P}$ s.t. $\psi(t) = E_{Q_t}[(C_{max}-t)^+]$.

This shows equality in Theorem 18.1

Proof of bounding property

only proof of a)

Consider chain C of N and processing time vector $x = (x_1,\dots,x_n)$

$$\sum_{j\in C} X_j - t = \underbrace{\sum_{j\in C} x_j}_{\le C_{\max}(x)} - t + \underbrace{\sum_{j\in C}(X_j - x_j)}_{\le \sum_{j=1}^{n}(X_j - x_j) \le \sum_{j=1}^{n}(X_j-x_j)^+}$$

$\Downarrow$

$$\sum_{j\in C} X_j - t \le C_{\max}(x) - t + \sum_j (X_j - x_j)^+$$

$$\le (C_{\max}(x) - t)^+ + \sum_j (X_j - x_j)^+ \quad \text{all } C, \text{ all } x$$

$\Downarrow$

$$\underbrace{\max_C \sum_{j\in C} X_j}_{C_{\max}(X),\ X = (X_1,\dots,X_n)} - t \le \dots$$

$$\Downarrow$$

$$C_{\max}(X) - t \le \underbrace{(C_{\max}(x) - t)^+ + \sum_j (X_j - x_j)^+}_{\ge 0}$$

$$\Downarrow$$

$$(C_{\max}(X) - t)^+ \le \dots \qquad \text{for all } x$$

$$\Downarrow$$

$$E_Q[\ \dots\ ] \le E_Q[\ \dots\ ]$$

$$= (C_{\max}(x) - t)^+ + \sum_j E_Q\left[(X_j - x_j)^+\right] \qquad \text{for all } x, Q$$

$$\Downarrow$$

$$E_Q\left[(C_{\max}(X) - t)^+\right] \le \underbrace{\inf_x \left\{ (C_{\max}(x) - t)^+ + \boxed{\sum_j E_Q\left[(X_j - x_j)^+\right]} \right\}}_{\text{optimization problem}} \le \min_x \sum \mathbb{E}\left[(X_j - x_j)^+\right] \text{ s.t. } C_{\max}(x) \le t$$

independent of Q

$$\Downarrow$$

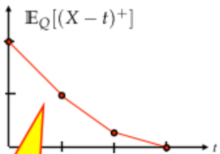
$$\underbrace{\sup_Q E_Q\left[(C_{\max}(X) - t)^+\right]}_{\psi(t)} \le \dots$$

□

Solving the convex optimization problem

$$\psi(t) = \min_{(x_1,\dots,x_n)} \sum_j \boxed{\mathbb{E}[(X_j - x_j)^+]} \text{ such that } C_{\max}(x_1,\dots,x_n) \le t$$

- piecewise linear, convex, decreasing function $\boxed{f(x_j)}$ for every job j in objective
- Interprete $f(x_j)$ as cost for excecuting job j with processing time x_j
- Side constraints:
  Find processing times x_j that
  - minimize the total cost
  - do not exceed the deadline t on the makespan



time-cost tradeoff problem

this interpretation leads to time-cost tradeoff problems

Time-cost tradeoff problems

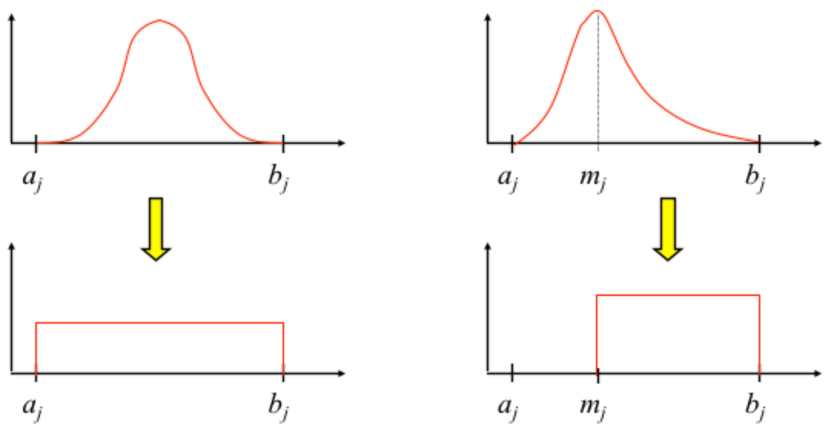
- classical network problem
- is the dual of a min-cost-flow problem for fixed t
  [Fulkerson '61]
- can be solved parametrically in t by a sequence of max-flow problems
  [Kelley '61]
- very efficient in practice

will be treated in § 19

Compatibility with incomplete information

Incomplete information about X_j [Cipra '78, Zackova '66]

unimodal & symmetric   only unimodal



These uniform distributions are larger in the convex sense

$\Rightarrow$ we still obtain upper worst case bounds if we use these uniform distributions when we only have incomplete information about the real distributions.

Exercises

$\boxed{18.1}$ $\psi(t)$ for series-parallel networks

a) Consider a 2-element chain $\bullet \xrightarrow{X_1} \xrightarrow{X_2} \bullet$ and let F_1, F_2 be the distribution functions of X_1, X_2.

Show that the distribution of the 2-dimensional random variable $(F_1^{-1}(U), F_2^{-1}(U))$, U uniformly distributed on $[0,1]$, is the worst case distribution in the convex sense.

b) Consider two jobs in parallel $\overset{X_1}{\underset{X_2}{\rightrightarrows}}$

With the notation of a), show that the distribution of $(F_1^{-1}(U), F_2^{-1}(1-U))$ is the worst case distribution for $\rightrightarrows$ in the convex sense

c) Use a) and b) to compute the worst case expected tardiness of a series-parallel network