§ 14 Performance guarantees of simple policies for the expected weighted sum of completion times

Now $\kappa = \sum_j w_j C_j$ weighted sum of completion times

need more complicated methods: LP-guided construction of policy

First consider fixed vector $x = (p_1, \dots, p_n)$ of processing times (use p_j to avoid confusion with LP-variables)

Model: $\boxed{\text{no precedence constraints, } m \text{ machines}}$

(can be generalized to release dates $r_j \geq 0$)

Consider LP in completion time variables C_j^{LP}

$$(LP) \begin{cases} \min \sum_j w_j C_j^{LP} \quad \text{such that} \\ (1) \quad \sum_{j \in A} p_j \cdot C_j^{LP} \geq \frac{1}{2m} \left(\sum_{j \in A} p_j \right)^2 + \frac{1}{2} \sum_{j \in A} p_j^2 \quad \forall A \subseteq V \\ (2) \quad C_j^{LP} \geq p_j \quad \forall j \end{cases}$$

14.1. LEMMA: The completion times of every feasible schedule for x fulfill (1) and (2)

Proof: Let S be a feasible schedule for x with completion times $C_1, \dots, C_n$. Show that the C_j fulfill (1)

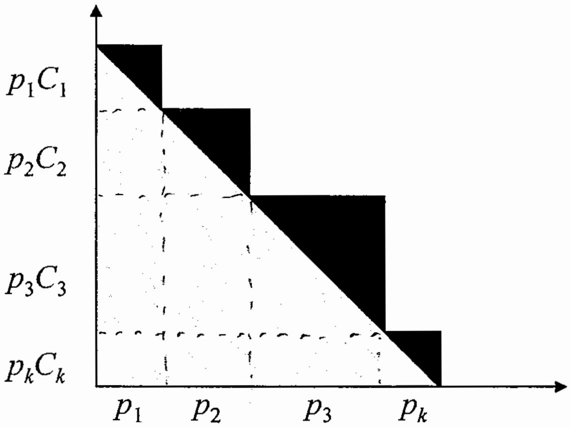
(1) Consider case $m = 1$ first

Valid inequalities for 1 machine

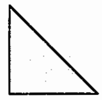
$$\sum_{j\in A} p_j C_j \ge \frac{1}{2}\left(\sum_{j\in A} p_j\right)^2 + \frac{1}{2}\sum_{j\in A} p_j^2 \qquad A = \{1,\dots,k\}$$

$\forall A \subseteq V$

lhs is smallest when jobs from A come first



$\sum_{j\in A} p_j C_j \ge$ 

$\frac{1}{2}\left(\sum_{j\in A} p_j\right)^2 =$ 

$\frac{1}{2}\sum_{j\in A} p_j^2 =$ 

$A = \{1,..,k\}$

Valid inequalities for m machines

$$\sum_{j\in A} p_j C_j \ge \frac{1}{2m}\left(\sum_{j\in A} p_j\right)^2 + \frac{1}{2}\sum_{j\in A} p_j^2 \qquad \forall A$$

use inequality for 1 machine

$$\sum_{j\in A} p_j C_j = \sum_{i=1}^{m}\sum_{j\in A\cap M_i} p_j C_j \ge \sum_{i=1}^{m}\left(\frac{1}{2}\left(\sum_{j\in A\cap M_i} p_j\right)^2 + \frac{1}{2}\sum_{j\in A\cap M_i} p_j^2\right)$$

$$= \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j\in A\cap M_i} p_j\right)^2 + \frac{1}{2}\sum_{j\in A} p_j^2$$

$$\ge \frac{1}{2m}\left(\sum_{j\in A} p_j\right)^2 + \frac{1}{2}\sum_{j\in A} p_j^2$$

Cauchy-Schwarz inequality

Cauchy-Schwarz inequality:

$$a, b \in \mathbb{R}^m \Rightarrow \langle a,b\rangle \leq \|a\| \cdot \|b\|$$

scalar product $\leq$ product of norms

$$\Leftrightarrow \left(\sum_i a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

Special case: $b = (1, \dots, 1) \in \mathbb{R}^m$

$$\Rightarrow \left(\sum_i a_i\right)^2 \leq \left(\sum_i a_i^2\right) \cdot m \quad \Rightarrow \quad \frac{1}{m}\left(\sum_i a_i\right)^2 \leq \sum_i a_i^2$$

$\Rightarrow$ statement with $a_i := \sum_{j \in A \cap M_i} p_j$

(2) is trivial $\square$

14.2 LEMMA: If numbers $C_1 \leq C_2 \leq \dots \leq C_n$ fulfill (1), then

$$2C_j \geq \frac{1}{m} \sum_{k \in J} p_k \quad \text{with } J = \{1, 2, \dots, j\}$$

[This will be the form how we will use inequality (1)]

Proof:
$$C_j \cdot \sum_{k \in J} p_k = \sum_{k=1}^{j} p_k C_j \geq \sum_{k=1}^{j} p_k C_k$$

$$\overset{(1)}{\geq} \frac{1}{2m}\left(\sum_{k \in J} p_k\right)^2 + \frac{1}{2}\sum_{k \in J} p_k^2$$

$$\geq \frac{1}{2m}\left(\sum_{k \in J} p_k\right)^2$$

$$\Rightarrow 2C_j \geq \frac{1}{m}\sum_{k \in J} p_k \quad \square$$

Idea for approximation algorithm:

A: Solve (LP) $\rightarrow$ optimal solution C_j^{LP}

[can be done in polynomial time although we have exponentially many inequalities]

B: Use the ordering $C_{j_1}^{LP} \leq C_{j_2}^{LP} \leq \ldots \leq C_{j_n}^{LP}$ for a

<u>job-based priority list</u> $j_1 < j_2 < \ldots < j_n$

[different from list scheduling as considered before: may start j_k only after all $j_1, \ldots, j_{k-1}$ have been started, i.e. list scheduling with condition $S_{j_1} \leq S_{j_2} \leq \ldots \leq S_{j_n}$]

C: Use Lemma 14.2 to prove a performance guarantee.

$\boxed{\text{More on B:}}$

14.3 THEOREM

(1) Every job-based list scheduling rule defines a policy [called <u>job-based list scheduling policy</u> hereafter]

(2) Every job-based list scheduling policy is dominated by a preselective policy

Proof: (1) A job based list scheduling rule is clearly non-anticipative $\Rightarrow$ policy

(2) Let $1 < 2 < \ldots < n$ be the priority list L, and Π be the job-based priority policy. The condition

$S_1 \leq S_2 \leq \dots \leq S_n$ implies that, for every forbidden set F, the last job of F in L is selected as waiting job.

The preselective policy Π^* with that selection s dominates Π since it does early start scheduling w.r.t. s (it may violate $S_1 \leq \dots \leq S_n$). □

$\boxed{\text{More on } C}$

14.4. LEMMA Let Π be a job-based priority policy with List $L = 1 < 2 < \dots < n$. Let $C_j^\Pi(x) := \Pi[x](j) + x_j$ denote the completion time of job j w.r.t. Π and x. Then

$$C_j^\Pi(x) \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k + x_j \quad \text{for every } x$$

and thus

$$\mathbb{E}[C_j^\Pi] \leq \frac{1}{m} \sum_{k=1}^{j-1} E(X_k) + \mathbb{E}(X_j)$$

in the stochastic case

Proof: Consider x fixed. When j is started, jobs $1, \dots, j-1$ have already been started. The latest time by which a machine becomes available for j is $\frac{1}{m} \sum_{k=1}^{j-1} x_k$ (all machines are busy as long as possible).

$$\Rightarrow S_j \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k \Rightarrow C_j \leq \frac{1}{m} \sum_{k=1}^{j-1} x_k + x_j.$$

Taking expectations gives the second inequality □

14.5 THEOREM

Let $C_1^{LP} \le C_2^{LP} \le \dots C_n^{LP}$ be an optimal solution of (LP) and let, for fixed x, $C_1, \dots, C_n$ be the vector of completion times obtained by job-based list scheduling according to the list $L: 1 < 2 < \dots < n$. Then

$$\sum_j w_j C_j \le \left(3 - \frac{1}{m}\right) OPT$$

i.e. the algorithm "LP-guided job-based priority scheduling" is a $(3-\frac{1}{m})$-approximation algorithm for the deterministic case.

Proof:

$$C_j \overset{\text{La } 14.4}{\le} \frac{1}{m} \sum_{k=1}^{j-1} x_k + x_j = \frac{1}{m} \sum_{k=1}^{j} x_k + \frac{m-1}{m} x_j$$

$$\overset{\text{La } 14.2}{\le} 2\, C_j^{LP} + \frac{m-1}{m} \underbrace{x_j}_{\le C_j^{LP}} \le \left(3 - \frac{1}{m}\right) C_j^{LP}$$

$$\Rightarrow \sum w_j C_j \le \left(3 - \frac{1}{m}\right) \sum w_j C_j^{LP} \le \left(3 - \frac{1}{m}\right) OPT \qquad \square$$

La 14.1 shows that (LP) is a relaxation of the original problem

$\boxed{\text{More on A:}}$

can solve (LP) in polynomial time if the separation problem for (1) and (2) can be solved in polynomial time.

This is trivial for (2), so show it for (1)

Given $(C_1, \dots, C_n) \in \mathbb{R}^n$, $A \subseteq V$, define <u>violation</u>

$$v(A) := \underbrace{\frac{1}{2m}\Big(\sum_{j\in A} p_j\Big)^2 + \frac{1}{2}\sum_{j\in A} p_j^2}_{\text{rhs of (1)}} - \underbrace{\sum_{j\in A} p_j C_j}_{\text{lhs of (1)}}$$

14.6 LEMMA: Let A maximize the violation. Then

$$\boxed{k \in A \iff C_k - \frac{1}{2} p_k < \frac{1}{m} \sum_{j\in A} p_j}$$

Proof: <u>Let $k \in A$.</u> Then

$$v(A \setminus \{k\}) = v(A) - p_k \Big(\frac{1}{m} \sum_{j\in A} p_j + \frac{m-1}{2m} p_k - C_k\Big)$$

<u>Let $k \notin A$</u>. Then

$$v(A \cup \{k\}) = v(A) + \frac{1}{m} p_k \Big(\sum_{j\in A} p_j\Big) + \frac{1}{2m} p_k^2 + \frac{1}{2} p_k^2 - p_k C_k$$

$$= v(A) + p_k \Big[\frac{1}{m} \sum_{j\in A} p_j + \frac{m+1}{2m} p_k - C_k\Big]$$

Let A maximize the violation. Then

$\underline{k \in A} \Rightarrow v(A\setminus\{k\}) \le v(A)$

$\Rightarrow p_k\Big(\frac{1}{m} \sum_{j\in A} p_j + \frac{m-1}{2m} p_k - C_k\Big) \geq 0$

$\Rightarrow C_k \le \frac{1}{m} \sum_{j\in A} p_j + \frac{m-1}{2m} p_k < \frac{1}{m} \sum_{j\in A} p_j + \frac{1}{2} p_k$

$\underline{k \notin A} \Rightarrow v(A\cup\{k\}) \le v(A) \Rightarrow p_k[\dots] \le 0$

$\Rightarrow \frac{1}{m} \sum_{j\in A} p_j + \frac{m+1}{2m} p_k \leq C_k$

$\Rightarrow \frac{1}{m} \sum_{j\in A} p_j \leq C_k - \frac{1}{2} p_k$ $\square$

14.7 SEPARATION ALGORITHM:

(1) Sort jobs w.r.t. increasing $C_j - \frac{1}{2}p_j$ values. Let $1, 2, \ldots, n$ be this ordering

(2) The set A with maximum violation is an initial segment $J = \{1, 2, \ldots, j\}$ of this ordering

(3) Check initial segments of the ordering for violation

Proof of (2): Let A maximize the violation and $i \in A$

Show that $k \in A$ for every $k \le i$.

$$i \in A \underset{\text{La 14.6}}{\Rightarrow} \quad C_i - \frac{1}{2}p_i < \frac{1}{m}\sum_{j \in A} p_j$$

$$k \le i \overset{\text{ordering}}{\Rightarrow} C_k - \frac{1}{2}p_k \le C_i - \frac{1}{2}p_i < \frac{1}{m}\sum_{j \in A} p_j$$

$$\underset{\text{La 14.6.}}{\Rightarrow} \quad k \in A \quad \Rightarrow \quad (2)$$

Checking (3) clearly requires only polynomial time. $\square$

Consider now the stochastic case with independent processing times

The LP-based approach

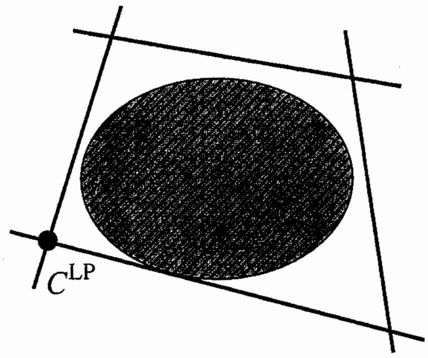
Consider the achievable region
$\{ (E[C_1^{\Pi}],\ldots,E[C_n^{\Pi}]) \in \mathbb{R}^n \mid \Pi \text{ policy} \}$

Find a polyhedral relaxation ***P***

Solve the linear program

(LP) $\min\{ \sum_j w_j C_j^{\mathrm{LP}} \mid C^{\mathrm{LP}} \in P \}$

Use the list L: $i_1 \leq i_1 \leq \ldots \leq i_n$
defined by $C_{i_1}^{\mathrm{LP}} \leq C_{i_2}^{\mathrm{LP}} \leq \ldots \leq C_{i_n}^{\mathrm{LP}}$
as list for priority/lin. pres. policies

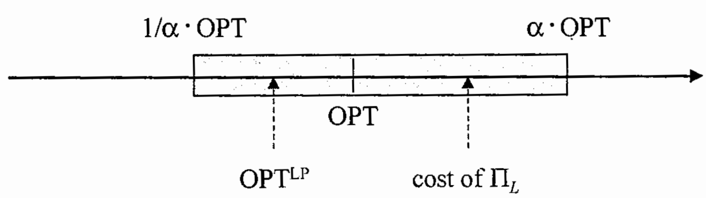


Bounds

Performance guarantees from the LP

Let Π_L be the policy induced by L: $i_1 \leq i_2 \leq \ldots \leq i_n$

Hope that $\boxed{E[\kappa^{\Pi_L}] \leq \alpha \cdot \mathrm{OPT}^{\mathrm{LP}},\ \alpha \geq 1}$



The tasks

- Find the relaxation P
- Solve the LP optimally
- Prove $E[\kappa^{\Pi_L}] \le \alpha \cdot \mathrm{OPT}^{\mathrm{LP}}$, $\alpha \ge 1$

The polyhedral relaxation I

Generalize valid inequalities from deterministic scheduling
Hall, Shmoys, Schulz & Wein '97

$\hat{=}$ Lemma 14.1

$$\left.\begin{array}{l} \sum_{j\in A} E[X_j]E[C_j^\Pi] \ge \frac{1}{2m}\left(\sum_{j\in A} E[X_j]\right)^2 + \frac{1}{2}\sum_{j\in A} E[X_j]^2 \\ \qquad - \frac{m-1}{2m}\sum_{j\in A} Var[X_j] \\ \text{for all } A \subseteq \{1,\dots,n\} \text{ and all policies } \Pi \end{array}\right\} \quad (3)$$

14.7 LEMMA: Every policy Π (for an m-machine problem without precedence constraints) fulfills (3).

Proof: Consider a fixed realization $x = (x_1,\dots,x_n)$. Then Lemma 14.1 gives

$$\sum_{j\in A} x_j \cdot \underbrace{C_j^\Pi(x)}_{\text{completion time of } j \text{ w.r.t. to } \Pi \text{ and } x} \geq \frac{1}{2m}\Big(\sum_{j\in A} x_j\Big)^2 + \frac{1}{2}\sum_{j\in A} x_j^2 \qquad \forall\, A \subseteq V$$

Rewriting in terms of start times $S_j^\pi(x) = C_j^\pi - x_j$

$$\sum_{j \in A} x_j S_j^\pi(x) \geq \frac{1}{2m} \Big(\sum_{\substack{i,j \in A \\ i \neq j}} x_i x_j \Big) - \frac{m-1}{2m} \sum_{j \in A} x_j^2 \qquad (4)$$

π policy $\Rightarrow$ non-anticipative

$\Rightarrow$ S_j^π and X_j are stochastically independent

$\Rightarrow$ $E[X_j \cdot S_j^\pi] = E[X_j] \cdot E[S_j^\pi]$ $\qquad (*)$

Furthermore, $VAR[X_j] = E[X_j^2] - E[X_j]^2$ $\qquad (**)$

Take expectations in (4) $\Rightarrow$ (linearity of expectations)

$$\sum_{j \in A} \underbrace{E[X_j \cdot S_j^\pi]}_{\overset{(*)}{=} E[X_j] E[S_j^\pi]} \geq \frac{1}{2m} \sum_{\substack{i,j \in A \\ i \neq j}} \underbrace{E[X_i X_j]}_{= E[X_i] \cdot E[X_j]} - \frac{m-1}{2m} \sum_{j \in A} E[X_j^2]$$

$\uparrow$ independent proc. times

$\Downarrow$

$$\sum_{j \in A} E[X_j] \cdot E[S_j^\pi] \geq \frac{1}{2m} \sum_{\substack{i,j \in A \\ i \neq j}} E[X_i] \cdot E[X_j] - \frac{m-1}{2m} \sum_{j \in A} E[X_j^2]$$

$$= \frac{1}{2m} \Big(\sum_{j \in A} E[X_j] \Big)^2 - \underbrace{\frac{1}{2m} \sum_{j \in A} E[X_j]^2}_{} - \frac{m-1}{2m} E[X_j^2]$$

add $\frac{1}{2m} \sum_{j \in A} E[X_j]^2$ to this sum

subtract it here

$$= -\frac{m-1}{2m} \sum_{j \in A} E[X_j^2] \overbrace{+ \frac{m-1}{2m} \sum_{j \in A} E[X_j]^2 - \frac{m-1}{2m} \sum_{j \in A} E[X_j]^2}^{=0} - \frac{1}{2m} \sum E[X_j]^2$$

$$\overset{(**)}{=} -\frac{m-1}{2m} \sum_{j \in A} VAR[X_j] - \frac{1}{2} \sum_{j \in A} E[X_j]^2$$

So $\sum_{j\in A} E[X_j]\cdot E[S_j^\pi] \geq \frac{1}{2m}\left(\sum_{j\in A} E[X_j]\right)^2 - \frac{1}{2}\sum_{j\in A} E[X_j]^2 - \frac{m-1}{2m}\sum_{j\in A} VAR[X_j]$

adding $\sum_{j\in A} E[X_j]^2$ on both sides gives

$$\sum_{j\in A} E[X_j]\cdot E[C_j^\pi] \geq \frac{1}{2m}\left(\sum_{j\in A} E[X_j]\right)^2 + \frac{1}{2}\sum_{j\in A} E[X_j]^2 - \frac{m-1}{2m}\sum_{j\in A} VAR[X_j] \qquad \square$$

$\uparrow$

$E[C_j^\pi] = E[S_j^\pi] + E[X_j]$

NOTE: Similar inequality to deterministic case, except for the variances.

The term $\sum_j Var[X_j]$

Coefficient of variation $CV[X_j] = \dfrac{Var[X_j]}{(E[X_j])^2}$

≤ 1 for all distributions that are NBUE
New Better than Used in Expectation $\leftarrow$ e.g. exponential, Erlang, uniform distributions; not: discrete distribution, multi-modal distr.

$$E[X_j - t \mid X_j > t] \leq E[X_j] \text{ for all } t > 0$$

Assume $CV[X_j] \leq \Delta$

The polyhedral relaxation II

Assume $CV[X_j] \leq \Delta$

$$\left.\begin{aligned}\sum_{j\in A} E[X_j]E[C_j^\Pi] \geq \frac{1}{2m}\left(\left(\sum_{j\in A} E[X_j]\right)^2 + \sum_{j\in A} E[X_j]^2\right) \\ -\frac{(m-1)(\Delta-1)}{2m}\sum_{j\in A} E[X_j]^2\end{aligned}\right\} (4)$$

for all $A \subseteq \{1,\ldots,n\}$ and all policies Π

RHS depends only on $E[X_j]$ and Δ

14.8 LEMMA

Proof Lemma 14.8: just calculation $\square$

14.9. LEMMA: Let $C = (C_1, \ldots, C_n) \in \mathbb{R}^n$, satisfy (4) and $C_j \geq E[X_j]$. Assume $C_1 \leq \ldots \leq C_n$. Then

$$\frac{1}{m}\sum_{k=1}^{j} E[X_k] \leq \left(1 + \max\left\{1, \frac{m-1}{m}\Delta\right\}\right) C_j \qquad \forall j$$

[Stochastic counterpart of Lemma 14.2: $\frac{1}{m}\sum_{k=1}^{j} p_j \leq 2 \cdot C_j$]

Proof: $$C_j \sum_{k=1}^{j} E[X_k] \underset{C_1 \leq \ldots \leq C_n}{\geq} \sum_{k=1}^{j} E[X_k] C_k \underset{(4)}{\geq} \text{rhs of } (4)$$

$$\underset{\text{rewriting}}{=} \frac{1}{2m}\left(\sum_{k=1}^{j} E[X_k]\right)^2 + \frac{m - \Delta(m-1)}{2m}\sum_{k=1}^{j} E[X_k]^2$$

$$\overset{\text{division}}{\Rightarrow} \quad C_j \geq \frac{1}{2m} \sum_{k=1}^{j} E[X_k] + \frac{m-\Delta(m-1)}{2m} \cdot \frac{\sum_k E[X_k]^2}{\sum_k E(X_k)}$$

CASE 1: $\Delta \leq \frac{m}{m-1}$ (holds for NBUE and deterministic case)

$$\Rightarrow 2C_j \geq \frac{1}{2m} \sum_{k=1}^{j} E[X_k] \quad \hat{=} \text{ Lemma 14.2}$$

CASE 2: $\Delta > \frac{m}{m-1}$ $\Rightarrow$ second term in sum is negative

Use $C_j \geq C_k \geq E[X_k]$ for $k = 1, \dots, j$

$$\Rightarrow C_j \geq \max_{k=1,\dots,j} E[X_k] \geq \frac{\sum_k E[X_k]^2}{\sum_k E[X_k]}$$

let $E[X_i] = \max_k E[X_k]$

$$\Rightarrow \sum_k E[X_k]^2 \leq E[X_i] \cdot \sum_k E[X_k]$$

second term negative

$$\Rightarrow C_j \geq \frac{1}{2m} \sum_{k=1}^{j} E[X_k] + \frac{m-\Delta(m-1)}{2m} C_j$$

$\vdots$

$$\Rightarrow \frac{1}{m} \sum_{k=1}^{j} E[X_k] \leq \left(1 + \max\left\{1, \frac{m-1}{m}\Delta\right\}\right) C_j \qquad \square$$

Cases 1, 2   Case 1   Case 2

14.10 THEOREM Let Π be the priority policy induced by the (LP)

$$\begin{cases} (4) \\ C_j^{LP} \geq E[X_j] \end{cases} \qquad \min \sum_j w_j C_j^{LP}$$

Then Π is a $\left(2 + \max\left\{1, \frac{m-1}{m}\Delta\right\}\right)$ - approximation

Proof: Let $C_1^{LP} \le \dots \le C_n^{LP}$ be an optimal solution of the LP.
and let $L: 1 < 2 < \dots < n$ be the priority list. for policy Π.

Then $E[C_j^\Pi] \underset{\text{La 14.4}}{\le} \frac{1}{m} \sum_{k=1}^{j-1} E[X_k] + E[X_j]$

$$= \underbrace{\frac{1}{m} \sum_{k=1}^{j} E[X_k]}_{\underset{\text{La 14.9}}{\le} \left(1 + \max\{\dots\}\right) C_j^{LP}} + \underbrace{\frac{m-1}{m} E[X_j]}_{\underset{(LP)}{\le} \frac{m-1}{m} C_j^{LP}}$$

$$\le \underbrace{\left(\left(2 - \frac{1}{m}\right) + \max\{\dots\}\right)}_{=:\alpha} C_j^{LP}$$

$\Rightarrow \sum w_j E[C_j^\Pi] \le \alpha \cdot \sum w_j C_j^{LP} \le \alpha \cdot OPT$ □

Remarks (1) LP can be solved in polynomial time (without proof)

(2) WSEPT - leads to guarantee $1 + \frac{(\Delta+1)(m-1)}{2m} \approx \alpha - 1$
(exercise)

(3) Can be generalized to release dates
(exercise)
$\Rightarrow$ guarantee $\alpha + 1$
need job-based priority policies for that

More on Remark (1):

14.11 THEOREM: Assume $w_1/E[X_1] \geq w_2/E[X_2] \geq \dots \geq w_n/E[X_n]$.

Then (LP) has the optimal solution

$$C_j^{LP} = \frac{1}{m} \sum_{k=1}^{j} E[X_k] - \frac{(\Delta-1)(m-1)}{2m} E[X_j] \qquad j=1,\dots,n$$

Proof idea

Let $f(A)$ be the r.h.s. of inequality (4)

Then: $f: 2^V \to \mathbb{R}$ is supermodular

i.e. $f(A \cup B) + f(A \cap B) \geq f(A) + f(B) \quad \forall\, A, B \in 2^V$

(*) $\Big[:\Longleftrightarrow f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B) \quad \forall\, A \subset B,\ x \in V \setminus B \Big]$

↑ "concavity"

$\Rightarrow$ The polyhedron defined by (4) is a supermodular polyhedron

$\Rightarrow$ Optimal solution can be obtained by Edmond's greedy algorithm for polyhedra

(*) is similar to Lemma 14.6

Exercises

14.1 Show that WSEPT leads to a $1+\frac{(\Delta+1)(m-1)}{2m}$ guarantee (m identical machine, no prec)

14.2 Generalize the results of this paragraph to the case with release dates (Use job-based priority rules)

14.3 Show that WSEPT may be arbitrarily bad for release dates

Consequences of 14.1 - 14.3

(A) WSEPT is an optimal policy for 1 machine

(B) The performance guarantee of WSEPT is better than that given by Theorem 14.10 (and the best known for problems without release dates)

(C) Theorem 14.10, however, can be generalized to release dates with performance guarantee

$3-\frac{1}{m}+\max\{1, \frac{m-1}{m}\Delta\}$

14.12 THEOREM:

The term Δ in the performance guarantee is essential,

i.e. there are instances $I(m,n,k)$ with $m > 1$

$\# \text{machines}$ $\# \text{jobs}$ $CV(X_j) \le k =: \Delta$

s.t. $\dfrac{WSEPT(I(m,n,k))}{OPT(I(m,n,k))} \in \Omega(\sqrt[4]{k})$

Proof: shown in Master's Thesis of B. Labonté, 2013

instance $I(m,n,k)$:

- m deterministic jobs with weight 1
  proc. time $\tau > 0$ arbitrary
- $n-m$ stochastic jobs with weight $w = \sqrt{\frac{m}{n-m}}$
  proc. time $\begin{cases} \tau w k & \text{with prob } \frac{1}{k} \\ 0 & \text{with prob } 1-\frac{1}{k} \end{cases}$

Properties of $I(m,n,k)$

- $CV(X_j) \le k-1 \quad \forall j$
- $w_j / E(X_j) = \frac{1}{\tau} \quad \forall j \quad \Rightarrow$ every order is WSEPT

Use 2 policies based on different orderings

$\Pi_{detFirst} \mathrel{\hat=}$ deterministic jobs first

$\Pi_{stocFirst} \mathrel{\hat=}$ stochastic jobs first

Then $\dfrac{E[K^{\Pi_{detFirst}}]}{E[K^{\Pi_{OPT}}]} \ge \dfrac{E[K^{\Pi_{detFirst}}]}{E[K^{\Pi_{stocFirst}}]}$ and

$$\mathbb{E}[k^{\pi_{detFirst}}] \geq \left(1-\frac{1}{k}\right)^{n-m}(m\tau + (n-m)\tau w) \in \Omega(\sqrt[4]{k})$$

$\uparrow$

for $n = m + \lfloor \sqrt{k} \rfloor$

$$\mathbb{E}[k^{\pi_{stocFirst}}] = O(1)$$

$\Rightarrow$ Result $\square$