§ 12 SET - POLICIES

Generalizes priority and preselective policies, uses only robust information, still reasonably stable

Π is a <u>set policy</u> for $[G, \mathcal{F}]$ if it is elementary and decisions at time t are only based on

- the <u>set</u> $C(t)$ of jobs completed before t
- the <u>set</u> $B(t)$ of jobs busy at t

so only the <u>sets</u>, not the processing times etc

12.1 REMARK:

Static priority policies, ES policies, and preselective policies are set policies (not the only ones)

12.2 THEOREM [Representation Theorem]

Let Π be a set policy for $[G, \mathcal{F}]$. Then there exists a partition of $\mathbb{R}^n_>$ into finitely many sets $Z_1, \dots, Z_m$ and finitely many interval orders $G_1, \dots, G_m$ on V such that

(1) Every Z_i is a polyhedral (convex) cone

(2) $\Pi[x] = ES_{G_i}[x]$ for all $x \in Z_i$

i.e. Set policies behave locally as ES policies

12.3 COROLLARY

(1) Graham anomalies occur only across the boundaries of the cones

(2) Every set-policy is almost everywhere continuous

(3) The class of set policies is stable for continuous probability distributions

Proof of Thm. 12.2:

Expl: G 1 2 3 $\mathcal{F}$: $\{1,2,3\}$ Π: priority policy for List L: $1 < 2 < 3$

Consider a fixed vector $\bar{x}$ of processing times

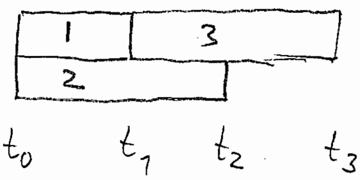
$\Rightarrow$ Π makes a sequence of decisions for $\bar{x}$ that depend only on sets.

Let S be the sequence of these sets $(S = S_{\bar{x}})$

$$S = [\underbrace{(C(0), B(0))}_{t_0 = 0}, (C(t_1), B(t_1)), \ldots, \underbrace{(C(t_k), B(t_k))}_{V \quad \emptyset}]$$

$\emptyset \quad \emptyset$

Expl: $\bar{x} = (1, 2, 2)$ $\Pi[\bar{x}] =$



$$S = [(\emptyset, \emptyset)\ (\{1\}, \{2\}), (\{1,2\}, \{3\})\ (\{1,2,3\}, \emptyset)\ ]$$

Now, given such a sequence S, define

$Z_S := \{ x \in \mathbb{R}^n_{\geq} \mid x \text{ induces the sequence } S, \text{ i.e. } S_x = S \}$

Expl:

$Z_S = \{ x \in \mathbb{R}^m_{\geq} \mid x_1 < x_2 ,\ x_2 < x_1 + x_3 \}$

Claim: The sets Z_S form a (finite) partition of $\mathbb{R}^n_{\geq}$

[Clear since every x leads to exactly one S, and there are only finitely many such sequences]

Show (1) and (2) for these sets:

<u>Proof of (1):</u>

Proceed inductively along the sequence. Every pair (C^i, s^i) represents the state of a decision at a time t_i that depends on x, but the state is the same for all $x \in Z_S$.

Claim (A): decision times t_i = sum of some x_τ with $\tau \in G(t_i)$

and every such x_τ occurs only once

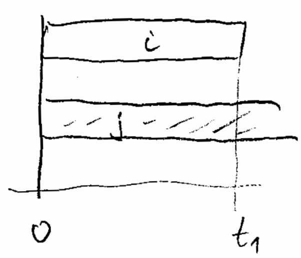
i.e. $t_i = \sum_{\tau \in G(t_i)} \alpha_\tau x_\tau$ with $\alpha_\tau \in \{0,1\}$

and the jobs with $\alpha_\tau = 1$ are sequential in $\Pi[x]$

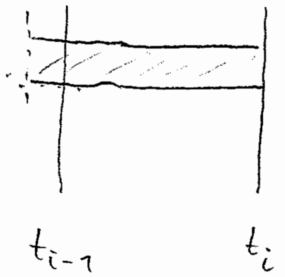
proof by induction:

$t_0 = 0 \implies$ all $\alpha_\tau = 0$

t_1:



$t_1 = x_i$ for $i \in G(t_1) = G^1$

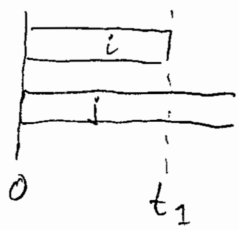


Some job must complete at t_i, say j (since Π elementary)

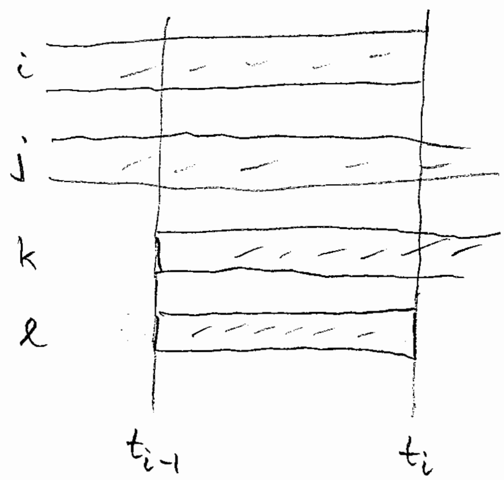
$\Rightarrow$ start of j is decision time $t \le t_{i-1}$ (since Π elementary)

$$\Rightarrow \quad t_i = \underbrace{t^{\text{start of } j}}_{\text{involves only other } x_r,\text{ sequential}} + x_j \quad \Rightarrow \text{claim}$$

Use claim (A) to express conditions on $x \in Z_S$ by homogeneous inequalities and equations



$x_j > x_i$ for all $j \in B^1$, $i \in C^1$



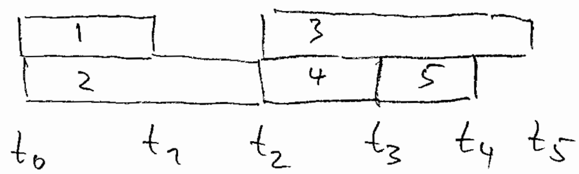
$i \in C(t_i) \setminus C(t_{i-1})$

$x_i + t^{\text{start of } i} = t_i$

$j \in B(t_i)$

$x_j + t^{\text{start of } j} > t_i$

Expl:



$t_0 = 0 \quad t_1 = x_1 \quad t_2 = x_2$

$t_3 = x_2 + x_4 \;,\; t_4 = x_2 + x_4 + x_5$

$t_5 = x_2 + x_3$

$$S = [\underset{t_0}{(0,0)}\ \underset{t_1}{(\{1\},\{2\})}\ \underset{t_2}{(\{1,2\},\emptyset)}\ \underset{t_3}{(\{1,2,4\},\{3\})}\ \underset{t_4}{(\{1,2,4,5\},\{3\})}\ \underset{t_5}{(V,\emptyset)}]$$

$Z_S = \{ x_1 < x_2 \,,\ x_4 + x_5 < x_3 \}$

Π set policy $\Rightarrow$ equations and inequalities induce exactly the sequence S

[every state is uniquely obtained from the previous]

$\Rightarrow$ Z_S is the intersection of finitely many open halfspaces and linear spaces, all containing 0

$\Rightarrow$ Z_S is a polyhedral cone

<u>Proof of Claim (2):</u>

S defines an "abstract" interval order G_S

$$i \rightarrow [\, t^{\text{start } i} \,,\ t^{\text{finish } i} \,]$$

both abstracts points given by sum of x_r of sequential jobs

the order of these is the same for all $x \in Z_S$

$\Pi[x] = ES_{G_S}[x]$ for $x \in Z_S$, follows from

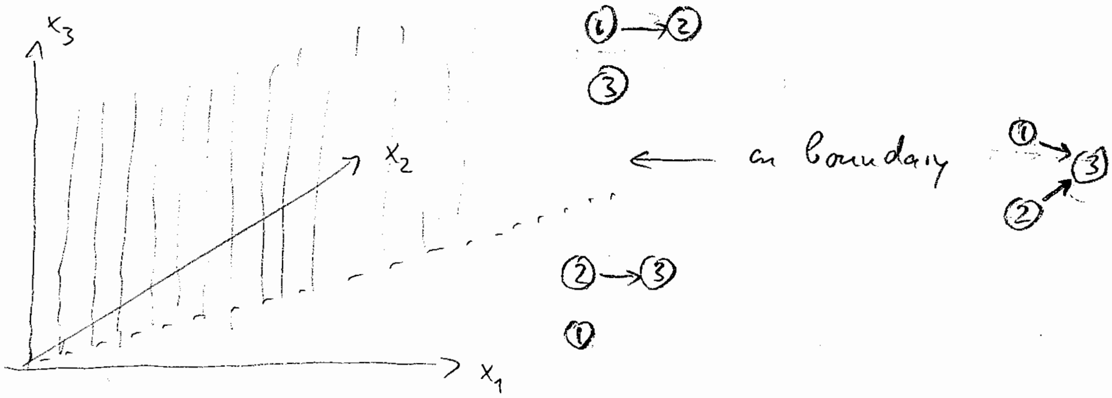
Note: $G_S = G_{S'}$ is possible

Expl:

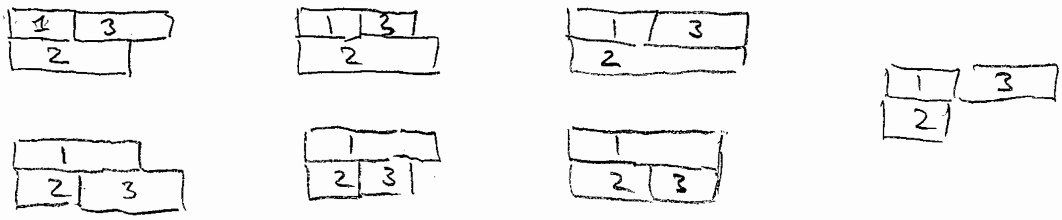
1

2 $\quad m = 2$

3



in total 7 cases Z_S corresponding to the following interval repr.



$\Rightarrow$ only 3 different interval orders

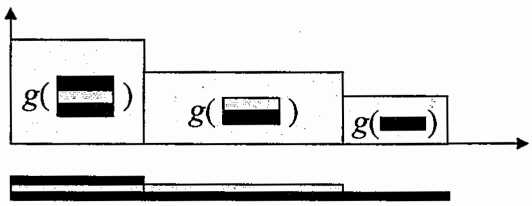
Optimality of set policies

12.4 THEOREM

If • all jobs are exponentially distributed and independent
• the cost function κ is *additive*
then there is an optimal set policy Π (among all policies).

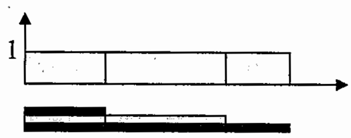
M., Radermacher, Weiß 1985

κ is *additive* if there is a set function $g : 2^V \to \mathbb{R}$ (the *cost rate*)
with $\kappa(C_1,\ldots,C_n) = \int g(U(t))dt$ $\quad U(t) =$ set of uncompleted jobs at t



Special cases of optimal set policies

$\kappa = C_{\max}$

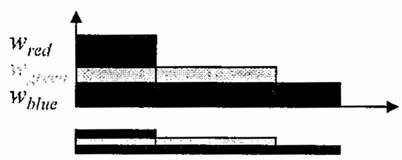


$$g(U) = \begin{cases} 1 \text{ if } U \neq \emptyset \\ 0 \text{ if } U = \emptyset \end{cases}$$

LEPT is optimal for $P \mid p_j \sim \exp \mid C_{\max}$

Weiss '82

$\kappa = \sum w_j C_j$



$$g(U) = \sum_{j \in U} w_j$$

SEPT is optimal for $P \mid p_j \sim \exp \mid \sum C_j$

Weiss & Pinedo '80

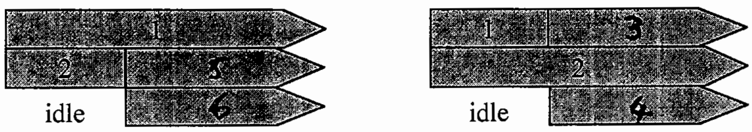
12.5 THEOREM

Set policies may be involved

No precedence constraints, $m = 3$ identical machines, 6 jobs
$X_j \sim \exp(a_j)$ with $a_1 = a_2 = a > 1,\ a_3 = a_4 = a_5 = a_6 = 1$

$$g(U) = \begin{cases} w \gg 1 & \text{if } 1 \in U \text{ and } 2 \in U \\ 1 & \text{if } 1 \notin U \text{ and } (2,5 \in U \text{ or } 2,6 \in U) \\ 1 & \text{if } 2 \notin U \text{ and } (1,3 \in U \text{ or } 1,4 \in U) \\ 0 & \text{otherwise} \end{cases}$$

Optimal set policy involves deliberate idleness



Non-idleness problem

Under which conditions on $g: 2^V \to \mathbb{R}$ does there exist an optimal set policy without idleness?
(m-machine problem without precedence constraints)

Proof of Theorem 12.4 (Sketch)

show stronger version:

- exp. distr., independent
- κ is additive
- a prescribed set V' of jobs must start at time 0

$\Rightarrow$ $\exists$ optimal set policy among all policies

Let π be a policy for the stronger version

Claim: There is a set policy π^* for the stronger version with

$$E[\kappa^{\pi^*}] \leq E[\kappa^{\pi}]$$

Since there are only finitely many set policies, there is an optimal set policy for the stronger version and thus also for the original problem. (with $V' = \emptyset$)

Proof of claim by induction on $n = |V|$

$n = 1$: only policy: start 1 at 0 is set policy ✓

<u>inductive step</u>

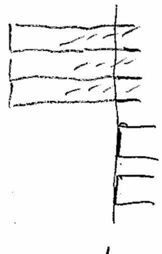
Consider problem with n jobs and start set V'

assume that there is an optimal set policy for all problems with $< n$ jobs and arbitrary start set V''.

(1) There is a policy $\hat{\pi}$ with $E(\kappa^{\hat{\pi}}) \leq E(\kappa^{\pi})$ that always waits for the first completion of a job (no tentative decision time at time 0)

Suppose π fixes a tentative decision time $t < \infty$ at time 0

Consider $X = \{x \in \mathbb{R}^n \mid$ every job started at 0 is longer than $t\}$

π: 

t

$\Rightarrow$ π starts some new jobs at t

$$\Rightarrow E(\kappa^{\pi}) = Q(X) \cdot E(\kappa^{\pi} \mid X) + Q(\bar{X}) \cdot E(\kappa^{\pi} \mid \bar{X})$$

$$E(\kappa^{\pi} \mid X) = t \cdot g(V) + E[\kappa^{\pi}]$$

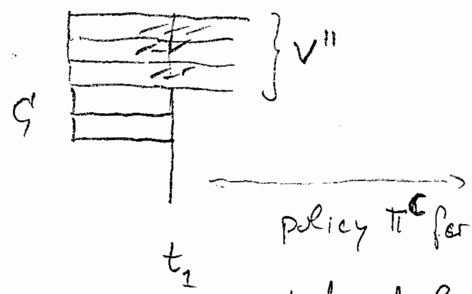
exponential distributions are memoryless

Consider $\hat{\Pi}$ that starts all jobs at 0 and otherwise behaves as Π

$$\Rightarrow \quad E(\kappa^{\Pi}) - E(\kappa^{\hat{\Pi}}) = Q(X) \cdot t \cdot g(V) > 0 \quad \Rightarrow \quad (1)$$

So w.l.o.g. let Π wait for the first completion

Consider a particular set G of jobs to complete first



policy Π^G for subproblem on $V \setminus G$ with start set V'' induced by Π

$$\Rightarrow \quad E[\kappa^{\Pi}] = \sum_{\text{all possible } G} Q(\text{jobs in } G \text{ end first})$$

$$\cdot \Big[\text{expected completion of } C \cdot g(V) + \underset{\substack{\uparrow \\ \text{memoryless property}}}{E[\kappa^{\Pi^G}]} \Big]$$

$$= \sum_{j \in V} \underbrace{Q(j \text{ ends first})}_{\substack{\uparrow \\ \frac{\lambda_j}{\lambda_1 + \dots + \lambda_n}}} \cdot \Big[\underbrace{E(j \text{ completes})}_{\frac{1}{\lambda_j}} \cdot g(V) + \underbrace{E[\kappa^{\Pi^j}]}_{\text{inductive hypothesis}} \Big] \qquad (*)$$

double completions have prob. 0

inductive hypothesis $\Downarrow$ Π^j may be replaced by optimal set policy for subproblem

combining this for all subproblems gives set policy Π^* for given problem with $E[\kappa^{\Pi^*}] \le E[\kappa^{\Pi}]$ $\square$

12.6 REMARK:

Formula (*) in the proof of Thm 12.4 gives an algorithm for computing the expected cost of a set policy for an additive cost function κ

Exercises

12.1 Calculate the expected makespan by the algorithm in Remark 12.6 for the ES-policy $\Pi = ES_H$ with

H: $$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & 3 \\ 2 \arrow[r] & 4 \end{tikzcd}$$ $X_j \sim \exp(\lambda_j)$ with $\lambda_1 = 1 \quad \lambda_2 = 2$

$\lambda_3 = 2 \quad \lambda_4 = 1$