§9 PRESELECTIVE POLICIES

Generalize ES-planning rules by relaxing the "rule" for solving the conflict on a forbidden set F

<u>ES-policies:</u>

for every F, choose $i_F, j_F \in F$ and add $i_F < j_F$ to G

i.e. choose a <u>waiting job</u> j_F and a <u>job</u> i_F <u>for which</u> j_F <u>must wait</u>

<u>Preselective planning rules</u>

for every F, choose a <u>waiting job</u> $j_F \in F$ that must wait for any job $\in F$, i.e. j_F can start after the first job in $F \setminus \{j_F\}$ completes

The sequence $s = (j_{F_1}, j_{F_2}, \dots, j_{F_k})$ of waiting jobs $j_{F_r} \in F_r$ for $\mathcal{F} = \{F_1, \dots, F_k\}$ is called a <u>selection</u> for $\mathcal{F}$.

9.1 EXAMPLE

G

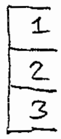
$$\begin{tikzcd} 1 \arrow[r] & 4 \\ 2 & \\ 3 \arrow[r] & 5 \end{tikzcd}$$

$\mathcal{F}$: $\{1,5\}$ $\{2,3,4\}$ $\{2,4,5\}$

defines selection $s = (5,4,4)$

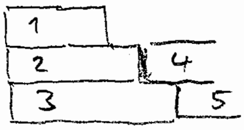
What are possible schedules when we do <u>early start scheduling</u> with respect to G and the <u>waiting conditions</u> coming from selection s?

$t = 0$



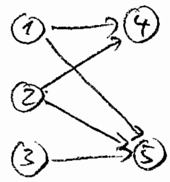
consider possible completions

(1) $x_1 < x_2 < x_3$

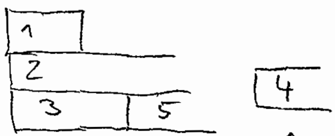


$\Rightarrow$ leads to ES of feasible order

H_1 :

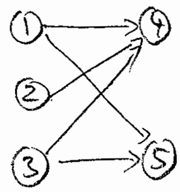


(2) $x_1 < x_3 < x_2$

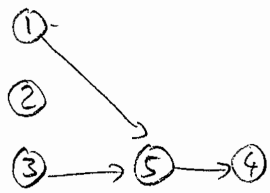


$\uparrow$
waits for 2 or 5

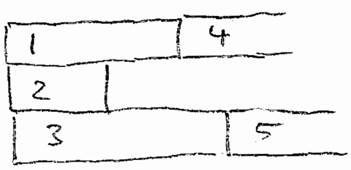
H_{21}



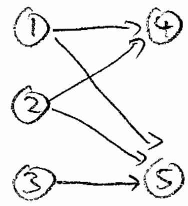
H_{22}



(3) $x_2 < x_1 < x_3$

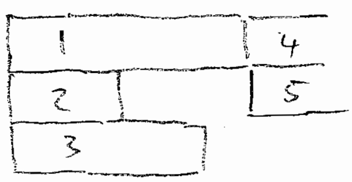


H_3

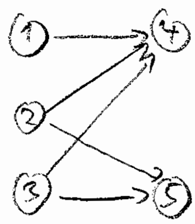


$H_3 = H_1$

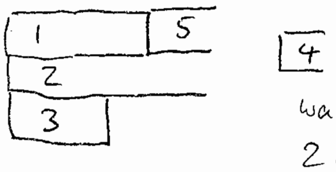
(4) $x_2 < x_3 < x_1$



H_4

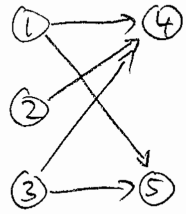


(5) $x_3 < x_1 < x_2$

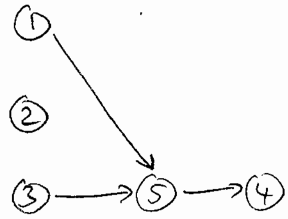


waits for 2 or 5

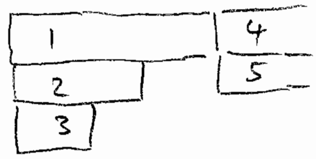
$H_{51} = H_{21}$



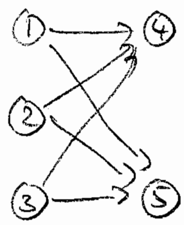
$H_{52} = H_{22}$



(6) $x_3 < x_2 < x_1$



H_6



$H_6 \succ H_1$

Cases with equality are subsumed by (1)-(6)

Example shows:

- depending on x we obtain a different feasible order H with $\Pi[x] = ES_H[x]$ (holds for every elementary policy)
- $\Pi = \min\{ES_H \mid H \text{ induced by } \Pi[x] \text{ for some } x\}$

  i.e. $\Pi[x] = \min\{ES_H[x] \mid H \dots\}$ (does not hold for every elementary policy)

Questions:

(1) When is a selection contradictory? Can this be easily checked?

(2) Does a "feasible" selection define a policy?
I.e. is a preselective planning rule non-anticipative

(3) What is the relationship with ES-policies?
Is $\Pi = \min\{ES_H \mid H \dots\}$ for some set of feasible ES-policies?
(as in the example)

(4) How to calculate the start times of a preselective policy?

(5) Do preselective policies still have nice properties?

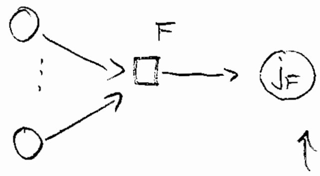
$\boxed{\text{Preselective planning rules and AND/OR - networks}}$

combinatorial representation of preselective planning rules

Consider a selected job j_F for $F \in \mathcal{F}$

$\Rightarrow$ introduce an <u>OR - precedence constraint</u>

all $j \in F$
$j \neq j_F$

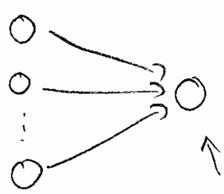


waiting job for F

interpretation: j_F may start after <u>one</u> of the direct predecessors of $\square$ is completed

OR node

ordinary precedence constraints $\hat{=}$ AND - precedence constraints



may start after <u>all</u> of the direct predecessors are completed

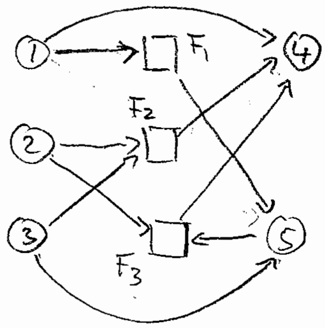
AND node

<u>AND/OR network</u> = network with AND/OR nodes

OBSERVATION: Every selection defines an AND/OR network

9.1 EXAMPLE (continued)

$s = (5,4,4)$ defines



$F_1 = \{1,5\}$

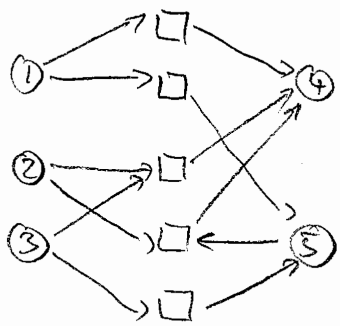
$F_2 = \{2,3,4\}$

$F_3 = \{2,4,5\}$

OBSERVATION: Every precedence constraint $i < j$ can be represented as OR-prec. constraint

$$(i) \to \square \to (j)$$

9.1. EXAMPLE (continued) Represent all given prec. constraints by OR-prec. constraints



CONSEQUENCES

(1) Resulting AND/OR network is bipartite

(2) May just speak of system of waiting conditions $\mathcal{W}$

$(X, j) \;\hat{=}\; X$ 

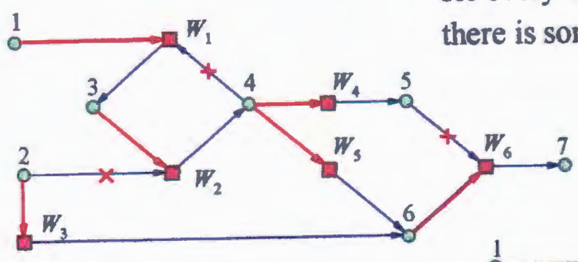
so system of waiting conditions $\Longleftrightarrow$ AND/OR network

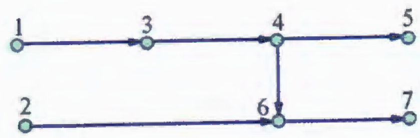
Question (1): When is a system of waiting conditions feasible

Feasibility of waiting conditions $\mathcal{W}$

A realization of $\mathcal{W}$ is an acyclic graph $R = (V, A)$ on V s.t. for every waiting condition (X, j), there is some $i \in X$ preceding j

$\mathcal{W}$ feasible $:\Leftrightarrow$ $\mathcal{W}$ has a realization $\Leftrightarrow$ $\mathcal{W}$ has a linear realization





red arcs define a realization

special case: linear realization

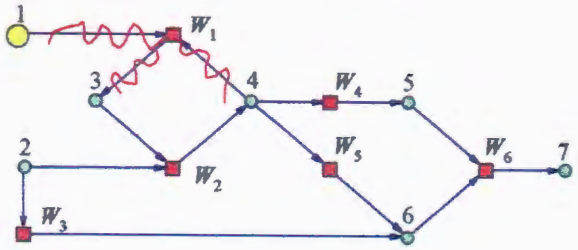
9.2 ALGORITHM

An algorithm for testing feasibility

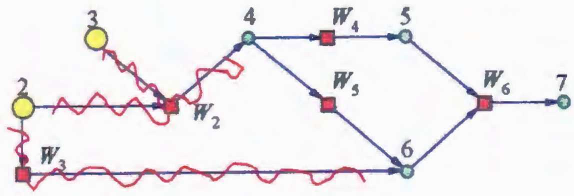
- Tries to construct a linear realization for $\mathcal{W}$
- Imitates topological sort for digraphs

List L := [ ]
while (there is a job $i \in V$ that is not a waiting job of a condition in $\mathcal{W}$)
    begin
        insert i at the end of L
        if (some waiting condition (X, j) becomes satisfied)
            delete (X, j) from $\mathcal{W}$
    end
return L

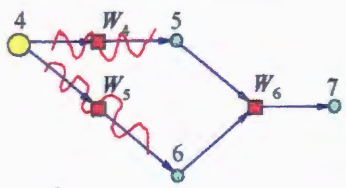
Example:



$L = [\ 1, \qquad\qquad]$



$L = [\ 1, 2, 3, \qquad\qquad]$



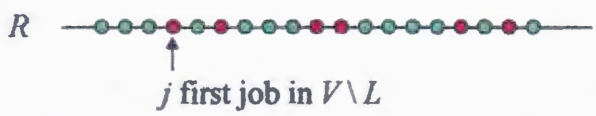
$L = [\ 1, 2, 3, 4, 5, 6, 7\ ]$

Correctness of the algorithm

$\mathcal{W}$ feasible $\Leftrightarrow$ L contains all jobs

9.3 LEMMA

"$\Rightarrow$": Let R be a linear realization but $L \neq V$



red jobs
= jobs in $V \setminus L$

j not in L $\Rightarrow$ there is a waiting condition (X, j) with $X \subseteq V \setminus L$

$\Rightarrow$ all jobs in X are red

but in R, there must be a job $i \in X$, i.e., a red job, before j

$\Rightarrow$ contradiction

Consequences

9.4. CONSEQUENCES

- $V \setminus L$ corresponds to a generalized cycle
  i.e., a set C fulfilling

  $j \in C \Rightarrow$ there is a waiting condition (X, j) with $X \subseteq C$
- The list L is an $\subseteq$-maximal feasible subset of V (1)
- Every linear realization can be constructed by the algorithm (2)
- The linear realizations form the basic words of an antimatroid (3)

Korte & Lovasz [1984]

Proof of (1): Let $V \setminus L \neq \emptyset$. Consider $C := V \setminus L$ and

$W_C :=$ all waiting conditions (X, j) with $X \subseteq C$, $j \in C$

$\Rightarrow$ W_C is a relaxation of W

No job of C can be found in any linear realization of W_C

$\Rightarrow$ no job of C can be in any linear realization of W

$\Rightarrow$ L is $\subseteq$-maximal set of jobs that can be in a linear realization $\square$

(2): choose jobs in the algorithm according to a given linear realization R $\Rightarrow$ algorithm produces R $\square$

Question (2): $\boxed{\text{Does a feasible selection define a policy?}}$

9.5 THEOREM: Every feasible selection s for $[G, \mathcal{F}]$ defines a preselective policy (i.e. the planning rule is non-anticipative)

Proof needs following lemma:

9.6. LEMMA: Let s be a feasible selection for $[G, \mathcal{F}]$. Then for every vector $x \in \mathbb{R}^n_{>}$ of processing times, the earliest start $ES_W[x]$ w.r.t. the system W of waiting conditions given by the selection s and G is well defined.

Proof: s feasible $\Rightarrow$ W is feasible.

Consider the OR-nodes $\square$ as dummy jobs

Let $S = (S_1, \dots, S_N)$ be a vector of times associated with the given jobs $j \in V$ and the dummy jobs $w \in W$.

S is <u>feasible</u> for W if all waiting conditions of W are fulfilled

$$\Leftrightarrow \left\| \begin{array}{lll} S_w \geq \min\limits_{\text{arcs } (j) \to \boxed{w}} [S_j + x_j] & \text{for OR-nodes } w \\ S_j \geq \max\limits_{\text{arcs } \boxed{w} \to (j)} S_w & \text{for AND-nodes } j \end{array} \right.$$

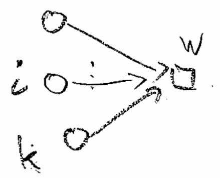
special case of a system of min-max inequalities for arbitrary arc weights in bipartite AND-OR networks:

$$
\text{MIN-MAX}\quad
\left\|
\begin{array}{lll}
S_w \geq \min\limits_{(j) \to [w]} \left[S_j + d_{jw} \right] & & w \text{ OR-node} \\[2ex]
S_j \geq \max\limits_{[w] \to (j)} \left[S_w + d_{wj} \right] & & j \text{ AND-node}
\end{array}
\right.
$$

feasible for $\mathcal{W}$ $\iff$ solution of the special MIN-MAX system.

9.7 LEMMA: Let $(S_1, \dots, S_N)$ and $(T_1, \dots, T_N)$ solutions of a MIN-MAX system. Then $(\min\{S_1, T_1\}, \dots, \min\{S_N, T_N\})$ is also a solution.

Proof: Consider OR-node w



$\Rightarrow \exists\, i$ with $S_w \geq S_i + d_{iw}$

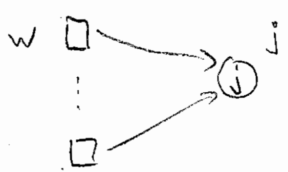
$\exists\, k$ with $T_w \geq T_k + d_{kw}$

$\Rightarrow \quad S_w, T_w \geq \min\{ S_i + d_{iw}, T_k + d_{kw} \}$

$\Rightarrow \quad \min\{S_w, T_w\} \geq \min\{\dots\} \underset{\substack{\uparrow \\ \text{w.o.l.g.}}}{=} S_i + d_{iw} \geq \min\{S_i, T_i\} + d_{iw}$

$\Rightarrow$ OR-inequality for node w fulfilled

Consider AND-node j



$\Rightarrow \quad S_j \geq S_w + d_{wj}$ for all w

$T_j \geq T_w + d_{wj}$ for all w

$\Rightarrow \quad \min\{S_j, T_j\} \geq$ the smaller of the two $= \min\{S_w, T_w\} + d_{wj}$ $\square$

for all w   for all w

Proof Lemma 9.6 continued:

$\mathcal{W}$ feasible $\Rightarrow$ every linear realization R of $\mathcal{W}$ defines a solution of MIN-MAX system

$\Rightarrow$ (Lemma 9.6) $\exists$ unique minimal feasible solution $S = (S_1, \ldots, S_N) \geq 0$

clearly, every S_j in S, $j \in V$, is the earliest start w.r.t. $\mathcal{W}$. □

Proof of Theorem 9.5:

$\left.\begin{array}{l} s \text{ feasible, } \mathcal{W} \text{ associated waiting conditions for } s, G, \\ \Pi \text{ planning rule induced by } s \end{array}\right\} \Rightarrow \Pi = ES_{\mathcal{W}}$

Proof is now similar to non-anticipativity of ES-planning rules:

let x, y look the same at time t to Π and $\Pi[x](j) = t$.

Then x, y have the same history up to t

$\Rightarrow$ $x_i = y_i$ of all jobs completed up to t, $\bar{x}_i = \bar{y}_i$ of all busy jobs

$\Rightarrow$ all waiting conditions fulfilled with the same times

$\Rightarrow$ $\Pi[y](j) = t$ □

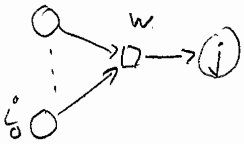
Question (3) $\boxed{\text{What is the relationship with ES-policies}}$

9.8 THEOREM: Let s be a feasible selection for $[G, \mathcal{F}]$, Π be the associated preselective policy, and $\mathcal{W}$ be the associated system of waiting conditions. Then

$$\Pi = ES_{\mathcal{W}} = \min \{ ES_R \mid R \text{ is a realizer of } \mathcal{W} \}$$

Proof:

Consider $F \in \mathcal{F}$ and the corresponding OR-node w.



Let $j \in F$ be the waiting job.

Then there is some $i_0 \in F \setminus \{j\}$ s.t. deleting all other arcs

$i \to \boxed{w}$ yields the same minimal solution S for a given x.

[S minimal $\Rightarrow$ $\exists\, i_0$ with $S_w = S_{i_0} + x_{i_0}$

deleting other arcs $i \to \boxed{w}$ $\Rightarrow$

(i) S is a solution for modified problem

(ii) modified problem is tighter

(i), (ii) $\Rightarrow$ S is unique minimal solution for modified problem]

$\Rightarrow$ F is settled by letting j wait for i_0

$\Rightarrow$ $[G + (i_0, j),\ \mathcal{F} - \{F\}]$ has same min solution S on jobs V.

Iteration $\Rightarrow$ $[\underbrace{G + \{(i_F, j_F) \mid F \in \mathcal{F}\}}_{=:R},\ \emptyset]$ has the same min. solution S

claim: $\boxed{R \text{ is a realization of } W}$

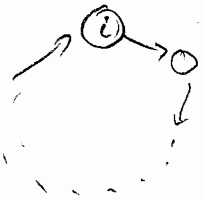
Have selected one arc (i_F, j_F) per waiting condition coming from $\mathcal{F}$

For precedence constraints $i \to \square \to j$, choice (i,j) is forced in every realizer.

Choices lead to an acyclic graph since all $x_j > 0$

if not, the cycle would have positive length and the waiting conditions are not satisfied

$S_i \geq S_i +$ length of cycle $\Rightarrow S_i > S_i$



$\Rightarrow$ Claim

Claim: $\boxed{S_j = ES_R[x](j)}$

R has same min solution S

R acyclic $\Rightarrow$ min solution is $ES_R[x]$ $\square$

Note: We have not carefully distinguished between S in AND/OR networks and S defined only on job set V,
also not between waiting conditions coming from $F \in \mathcal{F}$
and those coming from G (all these can thought of as arcs in AND/OR network).

Question (4) $\boxed{\text{How to calculate start times of a preselective policy ?}}$

$\to$ § 10

Question (5) Do preselective policies have nice properties

9.9 Let Π be a policy for $[G, \mathcal{F}]$. Then the following conditions are equivalent:

(1) Π is preselective

(2) Π is monotone $[\ x \le y \Rightarrow \Pi[x] \le \Pi[y]\ ]$

(3) Π is continuous

Remark: (2) and (3) show that Graham-anomalies of type a) occur in pairs!

Proof: here only (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3)

The other directions are shown in § 11.

$$\Pi = ES_W = \min \{ ES_R \mid R \text{ realizer of } W \}$$

Thm 9.8

is continuous, monotone, convex

preserves continuity, monotonicity, but not convexity □

Exercises

9.1 Theorem 9.8 shows that a preselective policy is the minimum of ES-policies. Consider now a set $\mathcal{H}$ of feasible orders and set $\Pi = \min \{ ES_H \mid H \in \mathcal{H} \}$. Give necessary and sufficient conditions (on $\mathcal{H}$) for Π being a policy.

9.2 Let $OPT^{PRES}(K,Q) = \min\{ E_Q(K^{\Pi}) \mid \Pi \text{ preselective} \}$ be the optimum value over the class of preselective policies.

Show that there are instances with

$$OPT^{PRES}(K,Q) < OPT^{ES}(K,Q)$$

but that OPT^{PRES} and OPT^{PRIOR} are incomparable in general

e
x