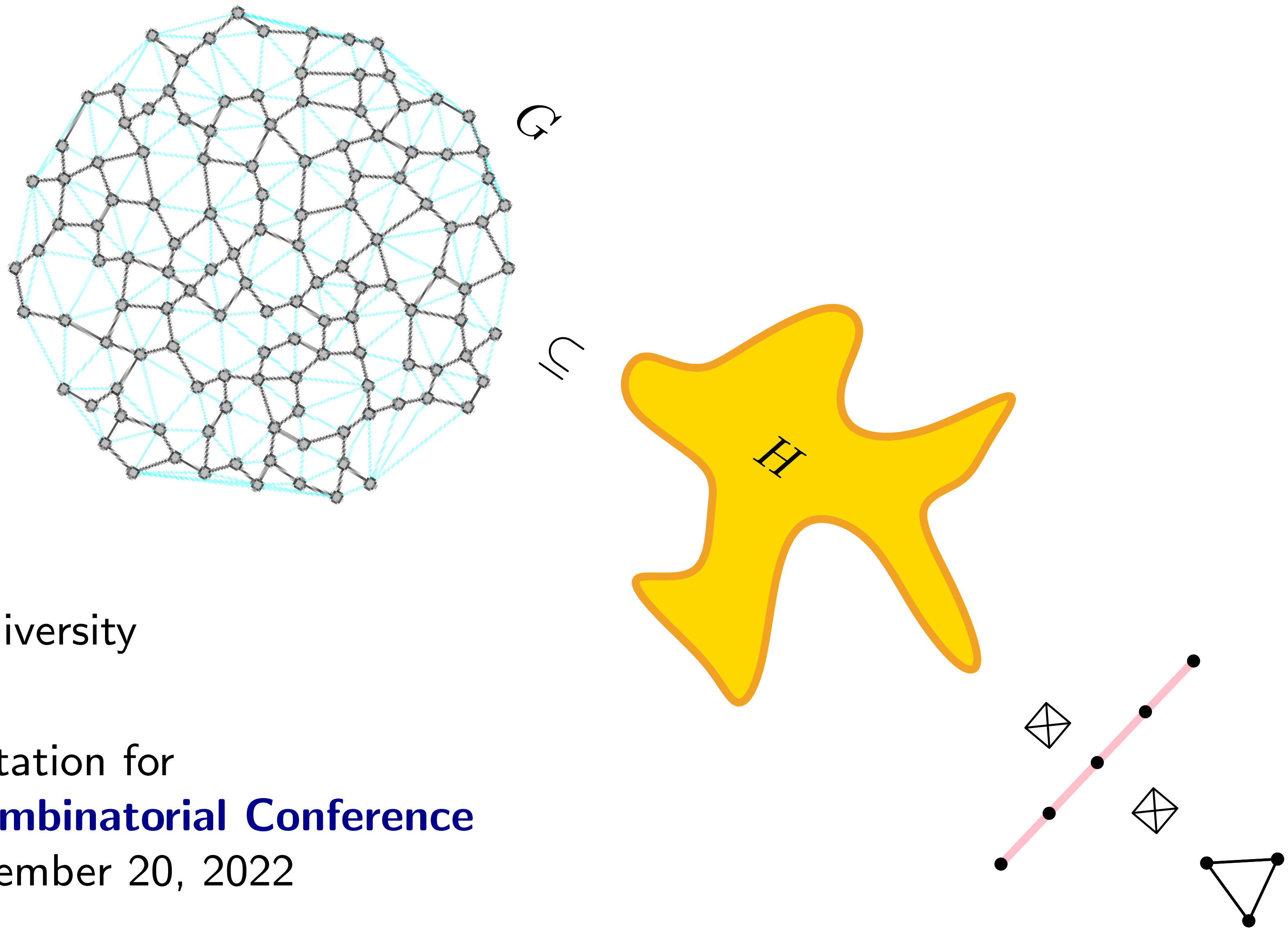
Product structure of planar graphs



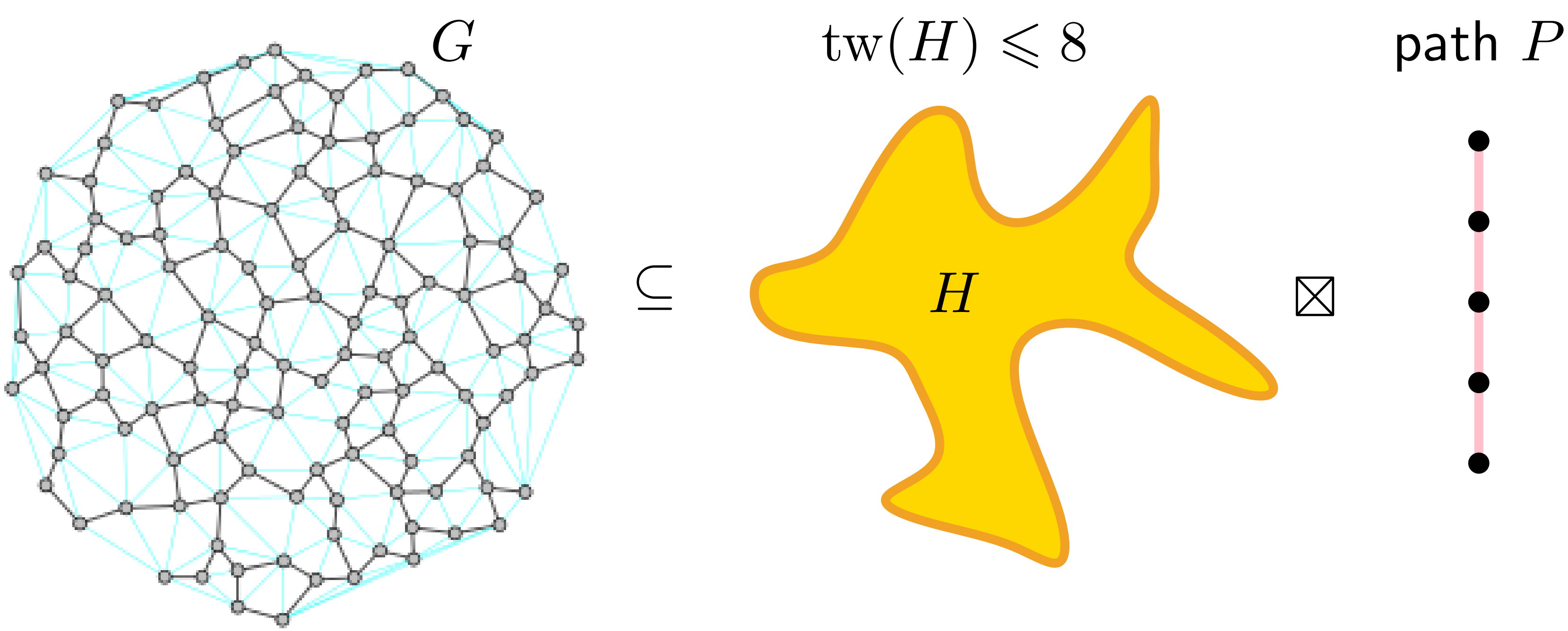
Piotr Micek
Jagiellonian University

tutorial presentation for
9th Polish Combinatorial Conference
Będlewo, September 20, 2022

product structure theorem

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.



product structure theorem

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.

product structure theorem

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

queue layouts of planar graphs with 49 queues

product structure theorem

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

queue layouts of planar graphs with 49 queues

(Dujmović, Esperet, Joret, Walczak, Wood 2020)

nonrepetitive colorings of planar graphs with 768 colors

product structure theorem

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

queue layouts of planar graphs with 49 queues

(Dujmović, Esperet, Joret, Walczak, Wood 2020)

nonrepetitive colorings of planar graphs with 768 colors

(Dębski, Felsner, PM, Schröder 2020)

p-center colorings of planar graphs with $\mathcal{O}(p^3 \log p)$ colors

product structure theorem

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

queue layouts of planar graphs with 49 queues

(Dujmović, Esperet, Joret, Walczak, Wood 2020)

nonrepetitive colorings of planar graphs with 768 colors

(Dębski, Felsner, PM, Schröder 2020)

p-center colorings of planar graphs with $\mathcal{O}(p^3 \log p)$ colors

(Dvořák, Sereni 2020)

planar graphs are fractionally td-fragile at rate $\mathcal{O}(a^3 \log a)$

product structure theorem

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

queue layouts of planar graphs with 49 queues

(Dujmović, Esperet, Joret, Walczak, Wood 2020)

nonrepetitive colorings of planar graphs with 768 colors

(Dębski, Felsner, PM, Schröder 2020)

p-center colorings of planar graphs with $\mathcal{O}(p^3 \log p)$ colors

(Dvořák, Sereni 2020)

planar graphs are fractionally td-fragile at rate $\mathcal{O}(a^3 \log a)$

(Dujmović, Esperet, Gavoillle, Joret, PM, Morin 2020)

planar graphs have a $(1 + o(1)) \log n$-bit adjacency labelling scheme

plan of tutorial

Part I	statements
	background
	proof
	variants / generalizations
Part II	quick applications
Part III	application: adjacency labelling scheme
	open problems / further research

Product structure of planar graphs

Part I statements

background

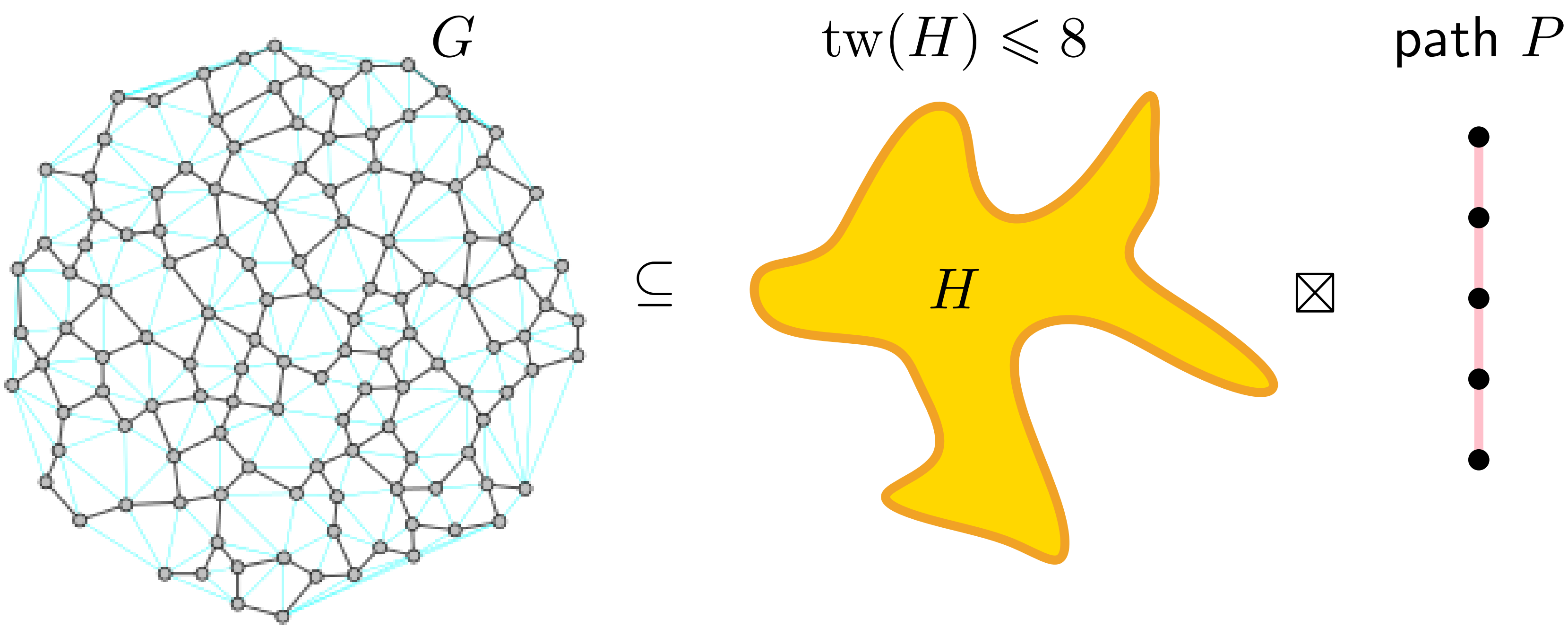
proof

variants / generalizations

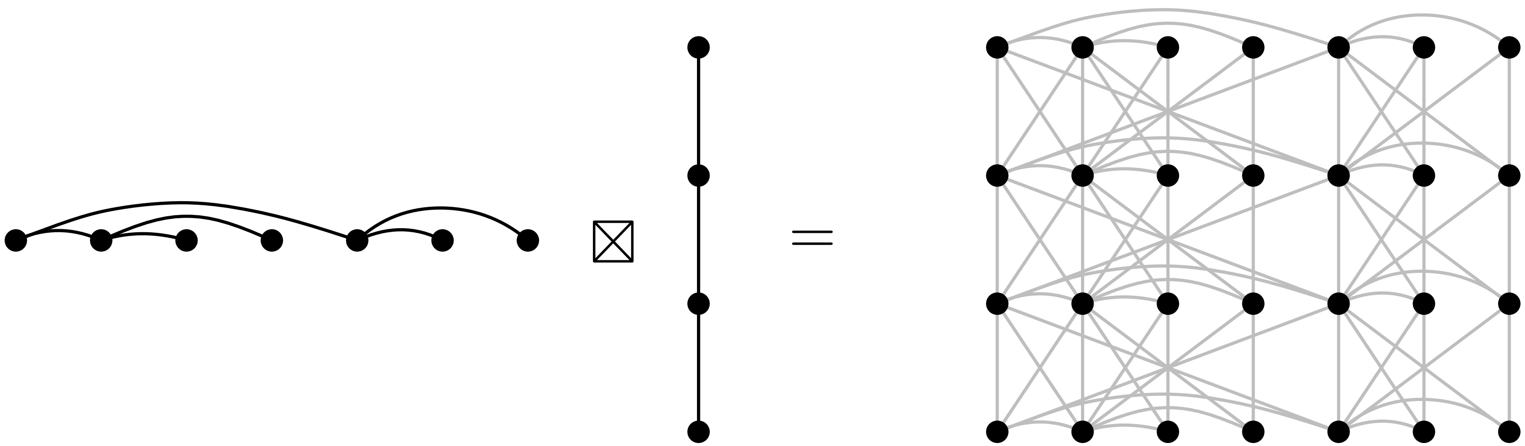
product structure theorem

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every **planar graph** G is a subgraph of a **strong product** $H \boxtimes P$ where H is a graph of **treewidth** at most 8 and P is a path.



strong product of graphs



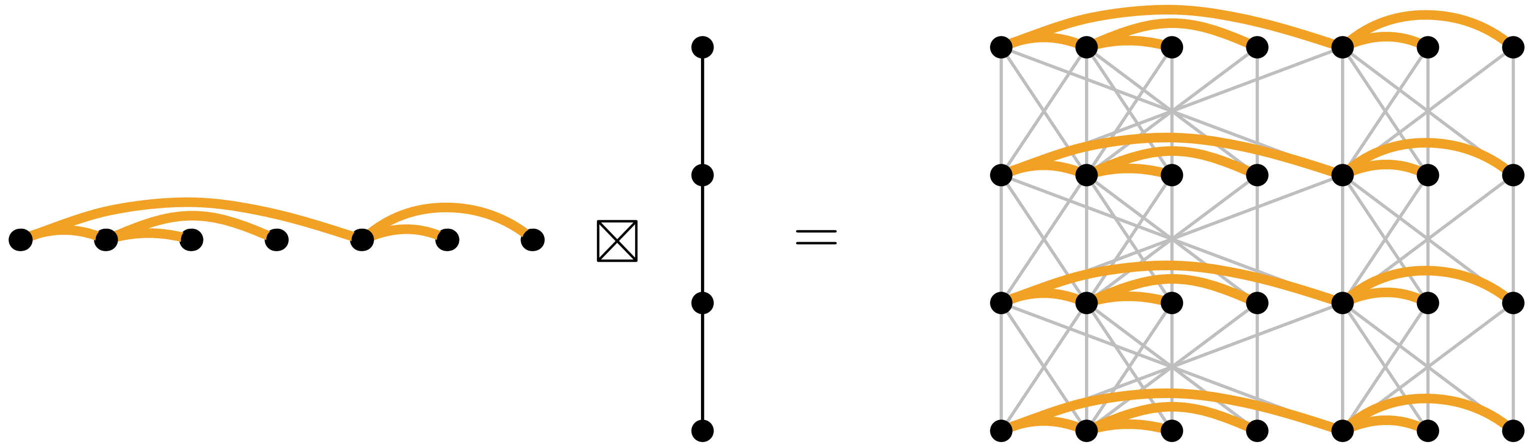
The **strong product** $H \boxtimes P$ of two graphs H and P is the graphs whose vertex set is the Cartesian product $V(H \boxtimes P) = V(H) \times V(P)$ and in which two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if

$x_1, x_2 \in E(H)$ and $y_1 = y_2$ or $x_1 = x_2$ and $y_1 y_2 \in E(P)$

or

$x_1 x_2 \in E(H)$ and $y_1 y_2 \in E(P)$

strong product of graphs



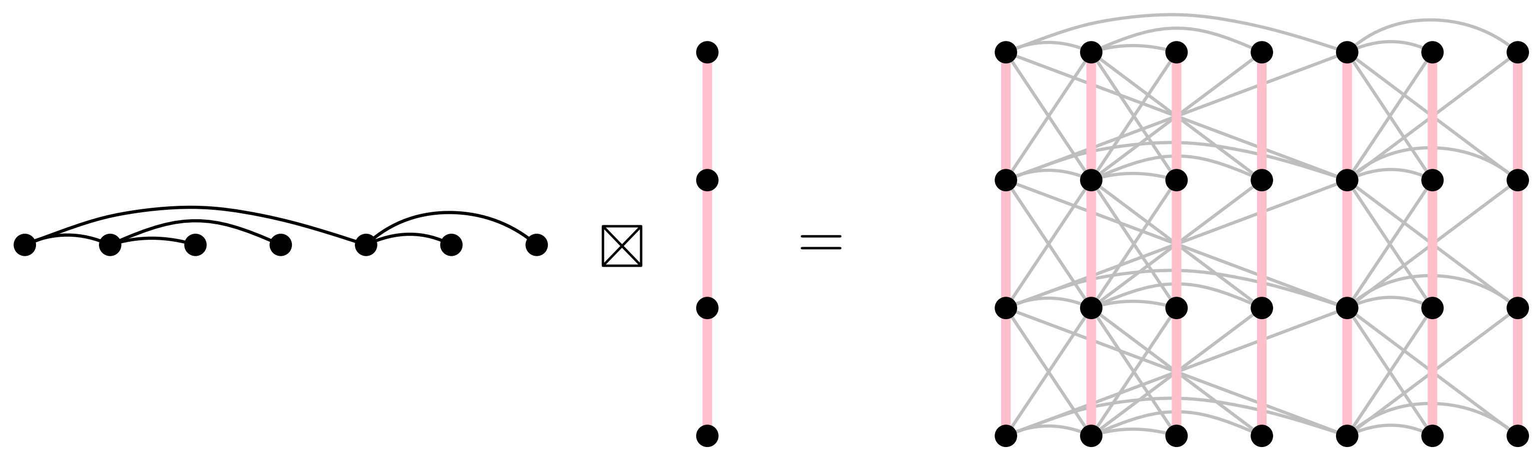
The **strong product** $H \boxtimes P$ of two graphs H and P is the graphs whose vertex set is the Cartesian product $V(H \boxtimes P) = V(H) \times V(P)$ and in which two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if

$x_1, x_2 \in E(H)$ and $y_1 = y_2$ or $x_1 = x_2$ and $y_1 y_2 \in E(P)$

or

$x_1 x_2 \in E(H)$ and $y_1 y_2 \in E(P)$

strong product of graphs



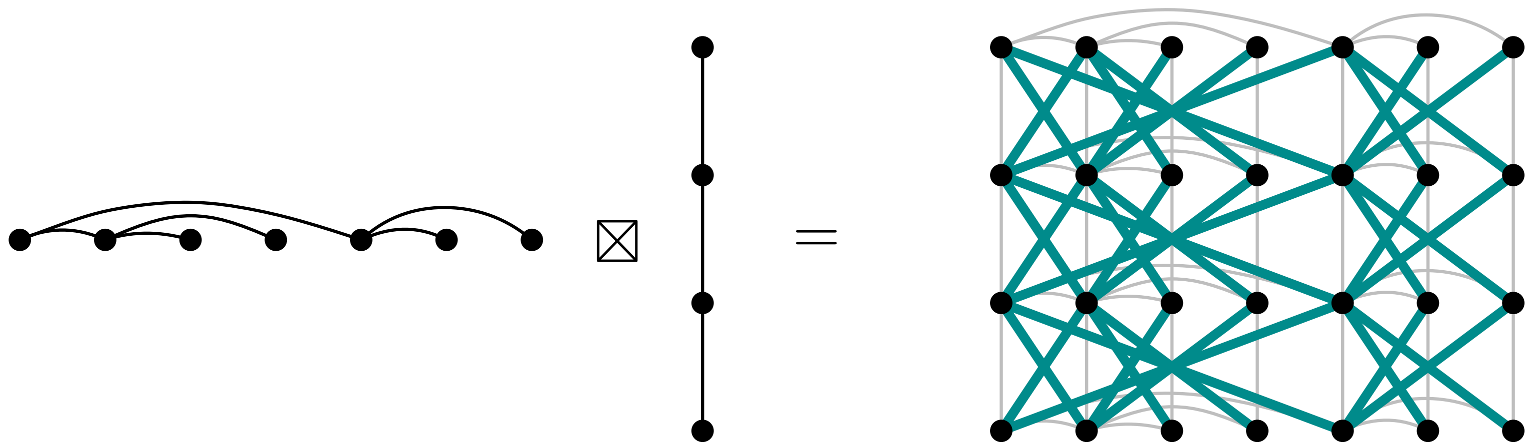
The **strong product** $H \boxtimes P$ of two graphs H and P is the graphs whose vertex set is the Cartesian product $V(H \boxtimes P) = V(H) \times V(P)$ and in which two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if

$x_1, x_2 \in E(H)$ and $y_1 = y_2$ or $x_1 = x_2$ and $y_1 y_2 \in E(P)$

or

$x_1 x_2 \in E(H)$ and $y_1 y_2 \in E(P)$

strong product of graphs



The **strong product** $H \boxtimes P$ of two graphs H and P is the graphs whose vertex set is the Cartesian product $V(H \boxtimes P) = V(H) \times V(P)$ and in which two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if

$x_1, x_2 \in E(H)$ and $y_1 = y_2$ or $x_1 = x_2$ and $y_1 y_2 \in E(P)$

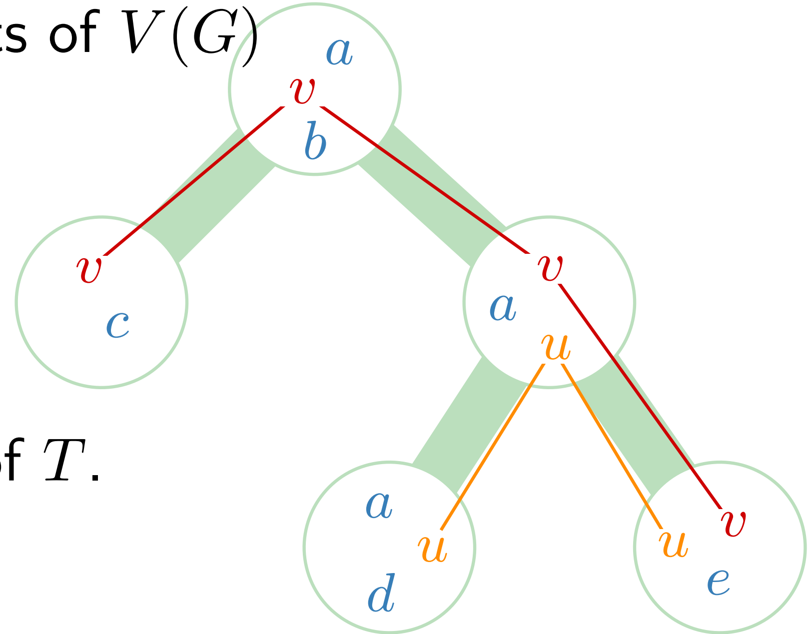
or

$x_1 x_2 \in E(H)$ and $y_1 y_2 \in E(P)$

tree-decomposition and treewidth

A **tree-decomposition** of G is a pair $(T, \mathcal{B})$ where

- ▷ T is a tree;
- ▷ $\mathcal{B} = (B_t \mid t \in V(T))$ is a family of subsets of $V(G)$



such that

- ▷ $\displaystyle\forall_{uv \in E(G)} \exists_{t \in V(T)} \; u, v \in B_t$;
- ▷ $\displaystyle\forall_{v \in V(G)}$ the set $\{t \mid v \in B_t\}$ is a subtree of T.

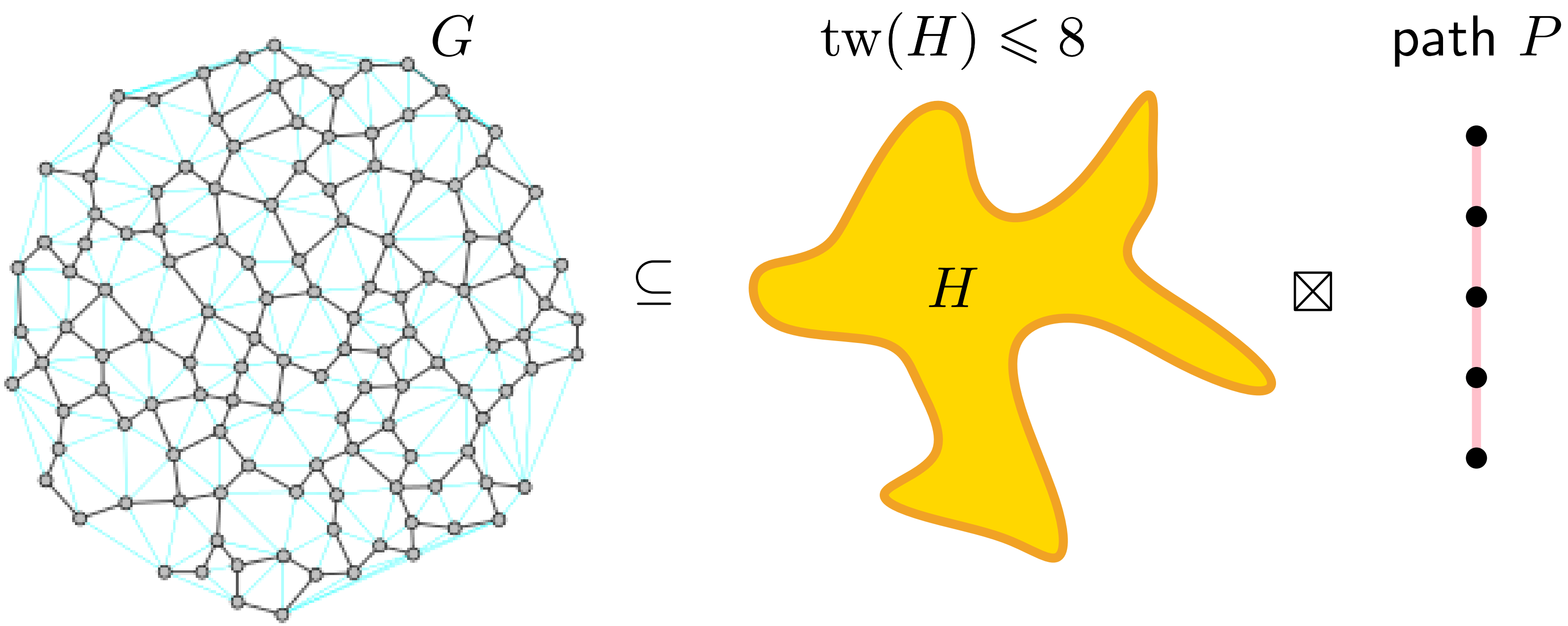
width of $(T, \mathcal{B})$: $\displaystyle\max_{t \in V(T)} |B_t| - 1$

treewidth of G: $\displaystyle\text{tw}(G) = \min_{(T,\mathcal{B})} \max_{t \in V(T)} |B_t| - 1$

product structure theorems

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every **planar graph** G is a subgraph of a **strong product** $H \boxtimes P$ where H is a graph of **treewidth** at most 8 and P is a path.



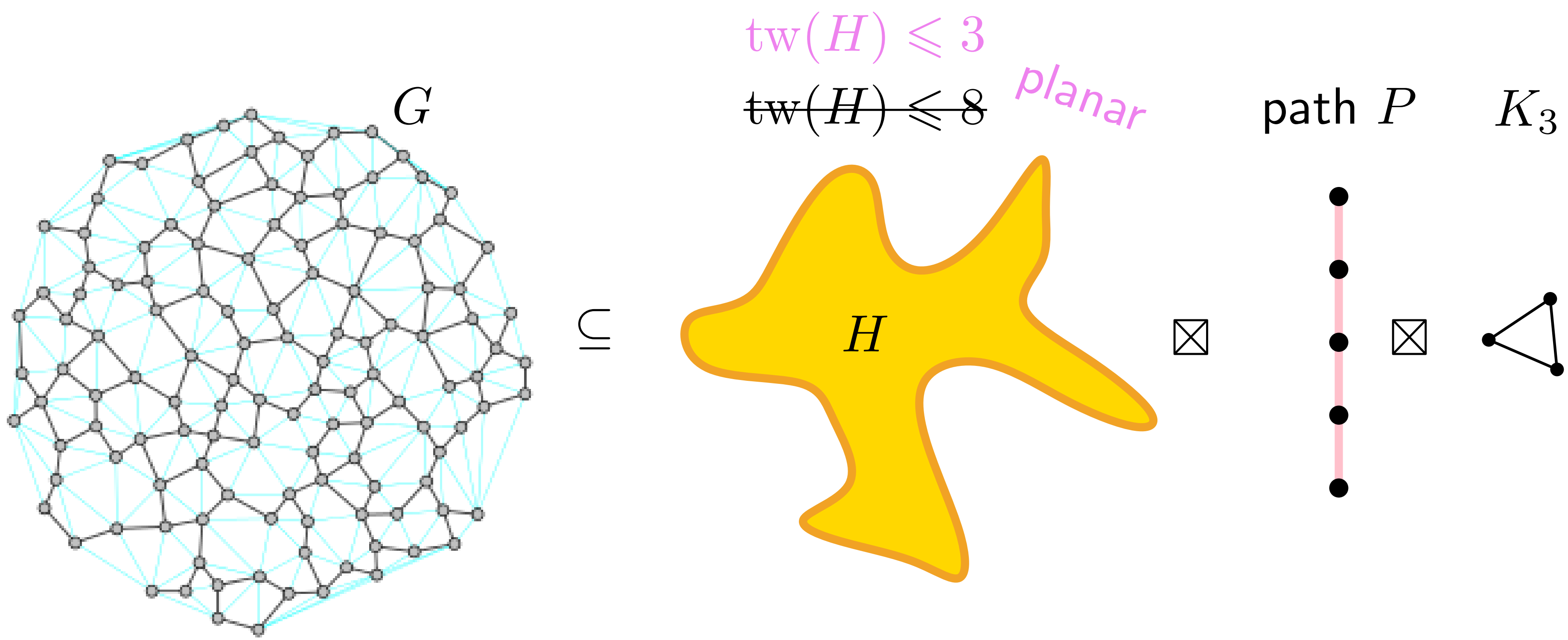
product structure theorems

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every **planar graph** G is a subgraph of a **strong product** $H \boxtimes P$ where H is a graph of **treewidth** at most 8 and P is a path.

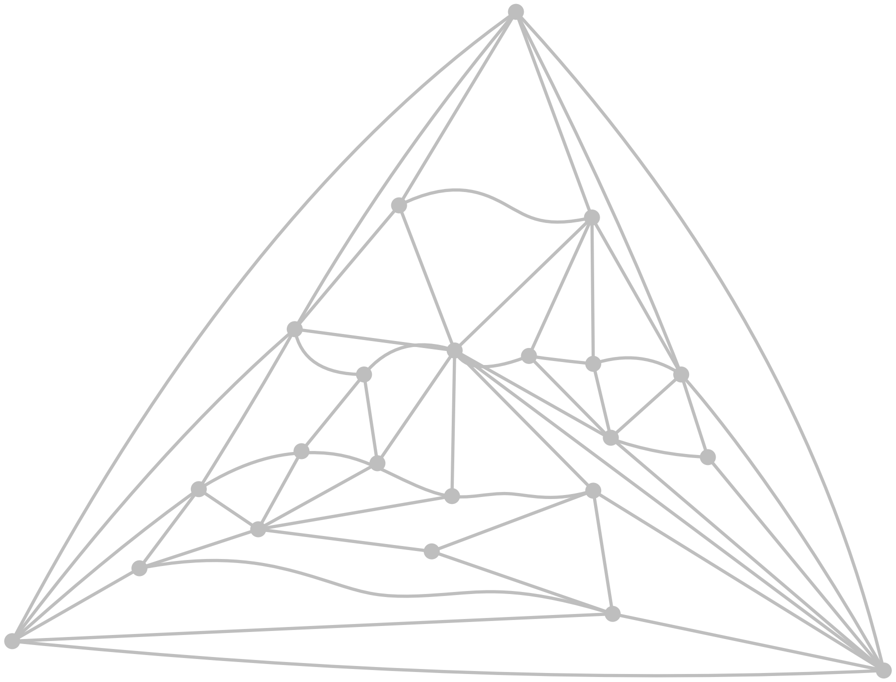
(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every **planar graph** G is a subgraph of a **strong product** $H \boxtimes P \boxtimes K_3$ where H is a planar graph of **treewidth** at most 3 and P is a path.



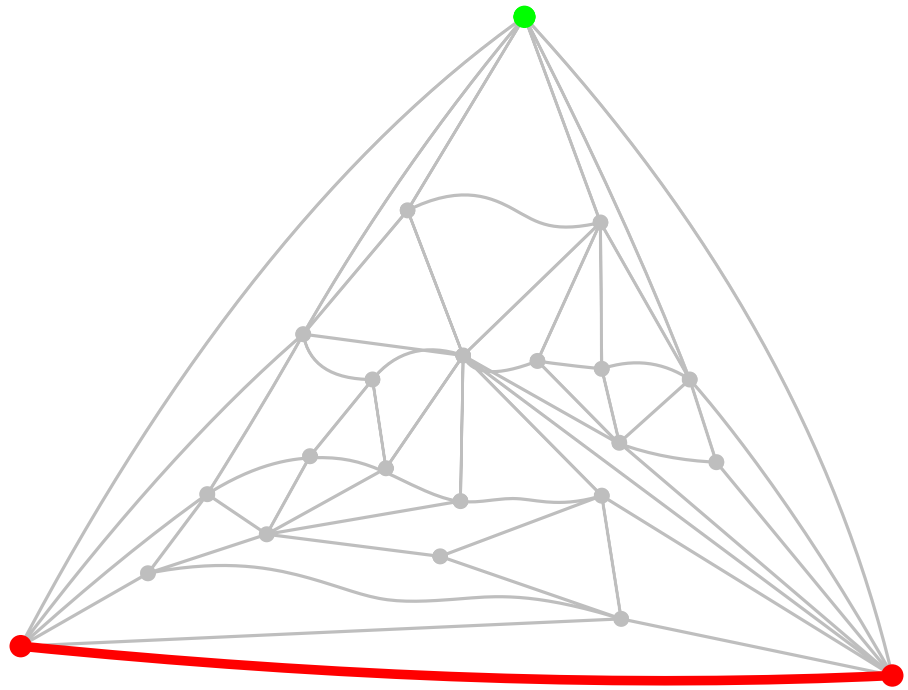
collecting insights

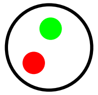
Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\mathrm{tw}(G/\mathcal{P}) \leqslant 2.$



collecting insights

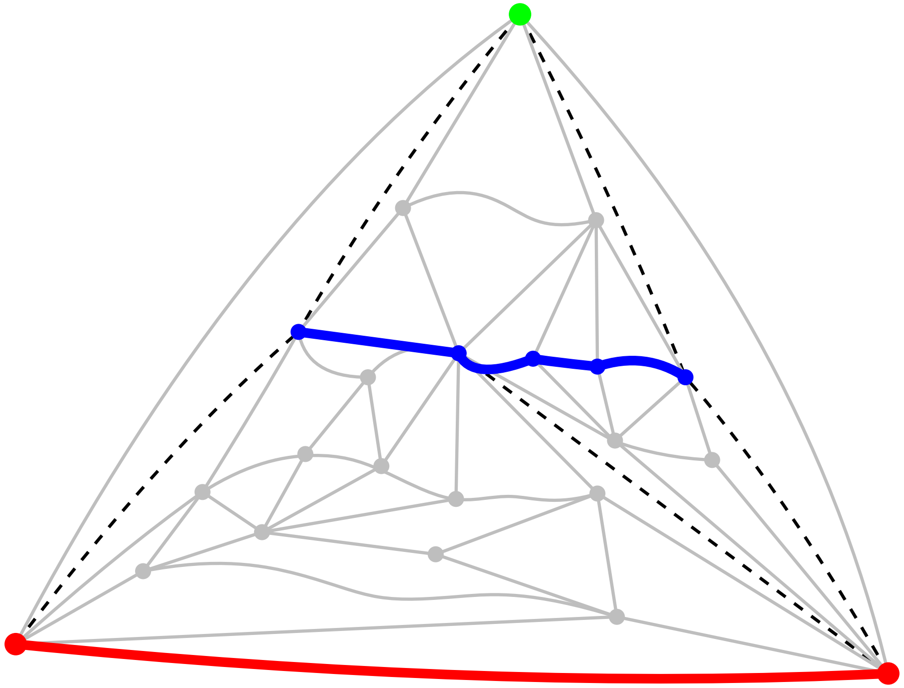
Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\mathrm{tw}(G/\mathcal{P}) \leqslant 2$.

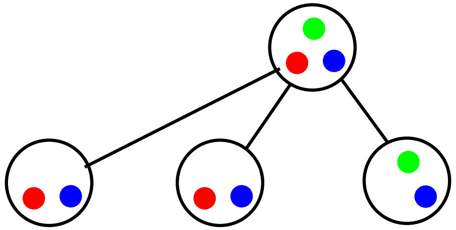




collecting insights

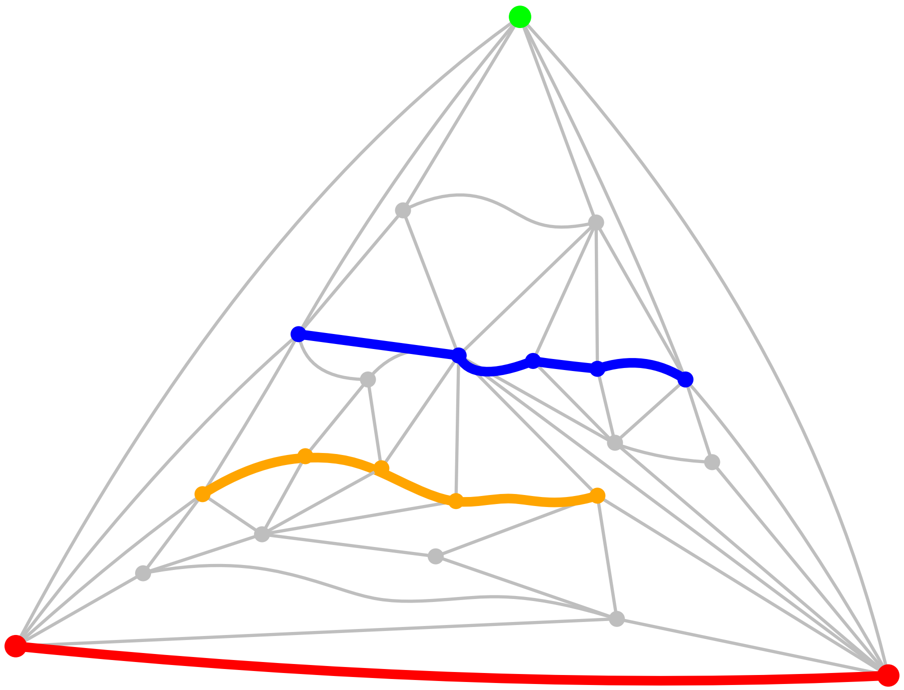
Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\operatorname{tw}(G/\mathcal{P}) \leqslant 2.$

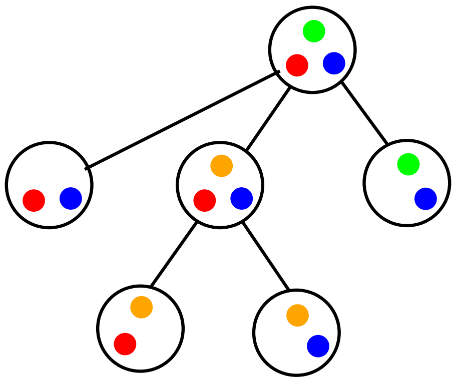




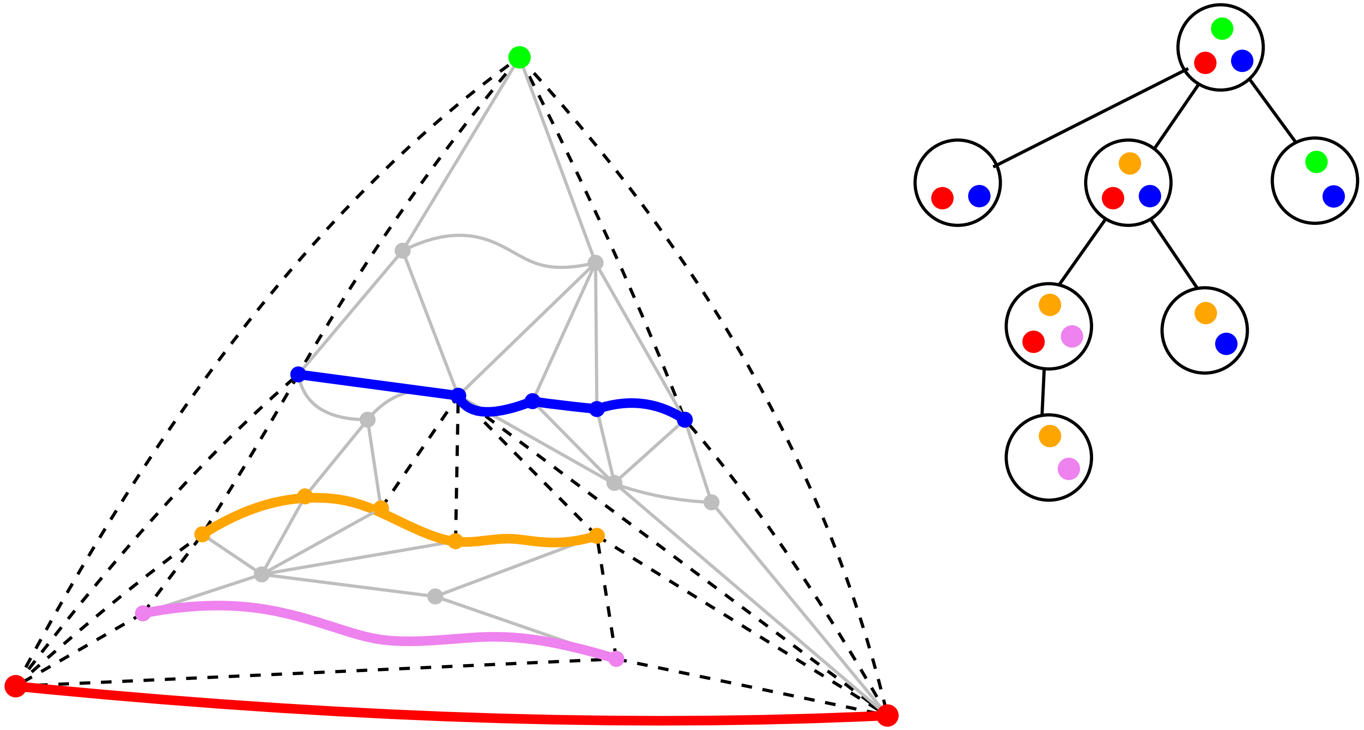
collecting insights

Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\operatorname{tw}(G/\mathcal{P}) \leqslant 2$.



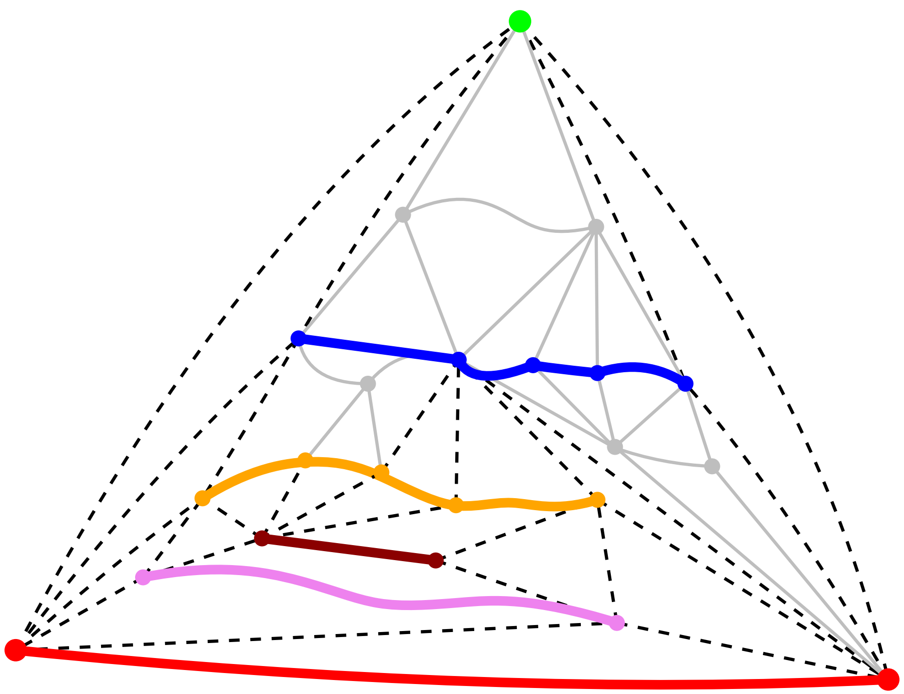


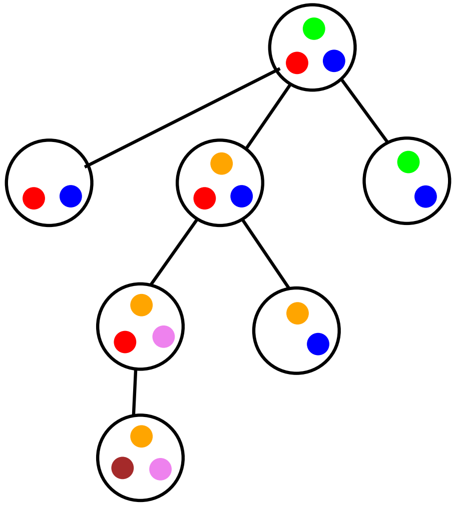
Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\operatorname{tw}(G/\mathcal{P}) \leqslant 2$.



collecting insights

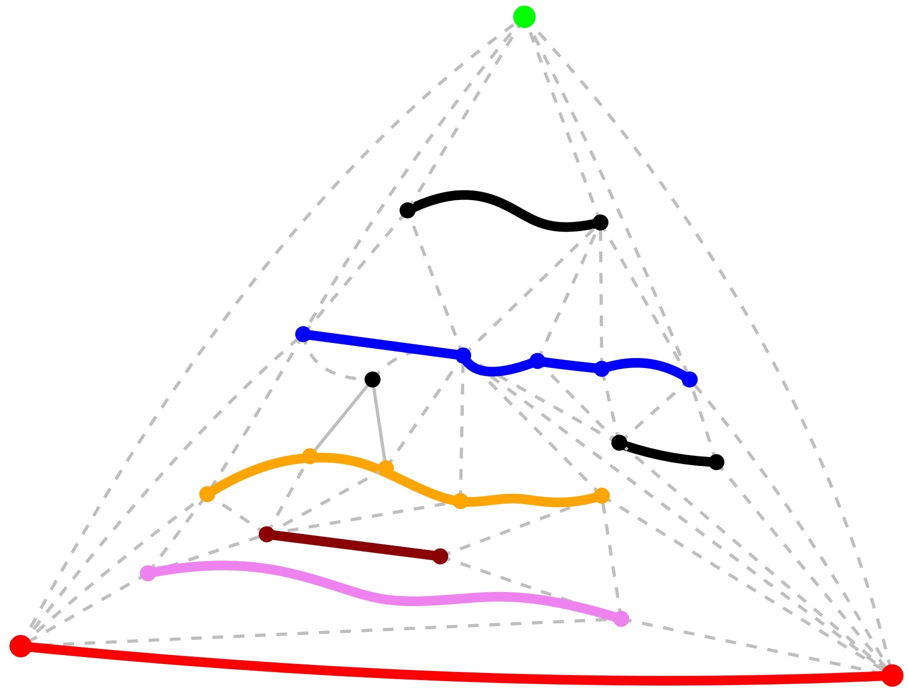
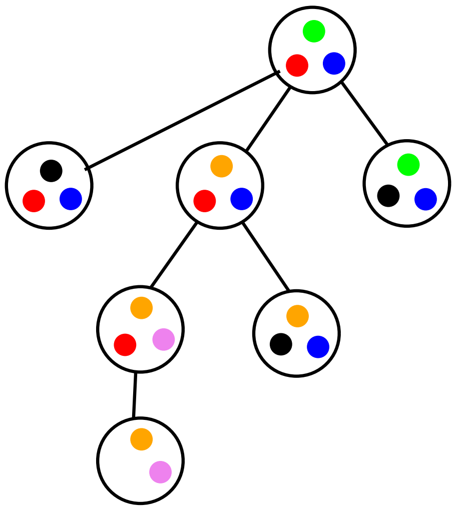
Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\mathrm{tw}(G/\mathcal{P}) \leqslant 2$.





collecting insights

Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\operatorname{tw}(G/\mathcal{P}) \leqslant 2.$

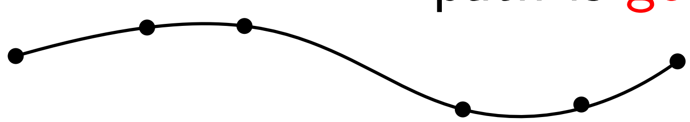



collecting insights

Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\quad \mathrm{tw}(G/\mathcal{P}) \leqslant 2$

(Mi. Pilipczuk, Siebertz 2019)
Every planar graph G has
a vertex partition $\mathcal{P}$ into **geodesics** in G such that $\quad \mathrm{tw}(G/\mathcal{P}) \leqslant 8$

path is geodesic



if it is a shortest path
between its endpoints

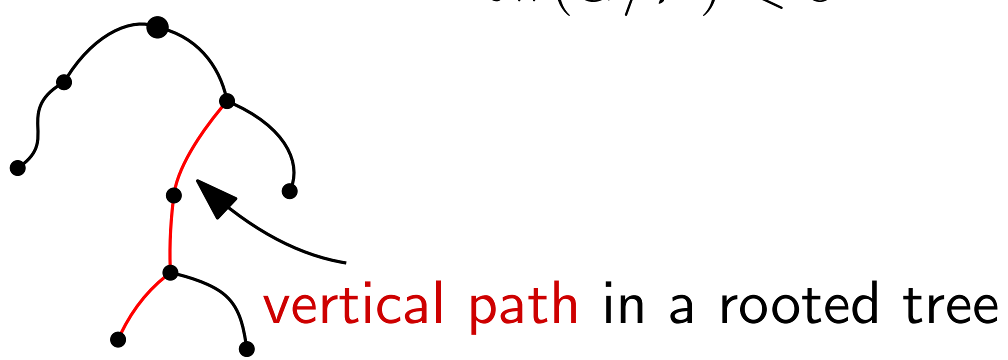
collecting insights

Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\mathrm{tw}(G/\mathcal{P}) \leqslant 2$

(Mi. Pilipczuk, Siebertz 2019)
Every planar graph G has
a vertex partition $\mathcal{P}$ into **geodesics** in G such that $\mathrm{tw}(G/\mathcal{P}) \leqslant 8$

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)
Every connected planar graph G with a rooted spanning tree T has
a vertex partition $\mathcal{P}$ into **vertical paths** in T such that

$\mathrm{tw}(G/\mathcal{P}) \leqslant 8$



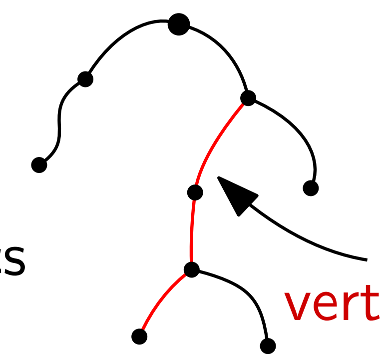
collecting insights

Observation Let G be a triangulation.
Then G has a vertex-partition $\mathcal{P}$ into **paths** such that $\mathrm{tw}(G/\mathcal{P}) \leqslant 2$

(Mi. Pilipczuk, Siebertz 2019)
Every planar graph G has
a vertex partition $\mathcal{P}$ into **geodesics** in G such that $\mathrm{tw}(G/\mathcal{P}) \leqslant 8$

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)
Every connected planar graph G with a rooted spanning tree T has
a vertex partition $\mathcal{P}$ into **vertical paths** in T such that $\mathrm{tw}(G/\mathcal{P}) \leqslant 8$

take T to be a BFS tree
then vertical paths in T are geodesics



vertical path in a rooted tree

collecting insights

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every connected planar graph G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **vertical paths** in T such that

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 8$$

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.

collecting insights

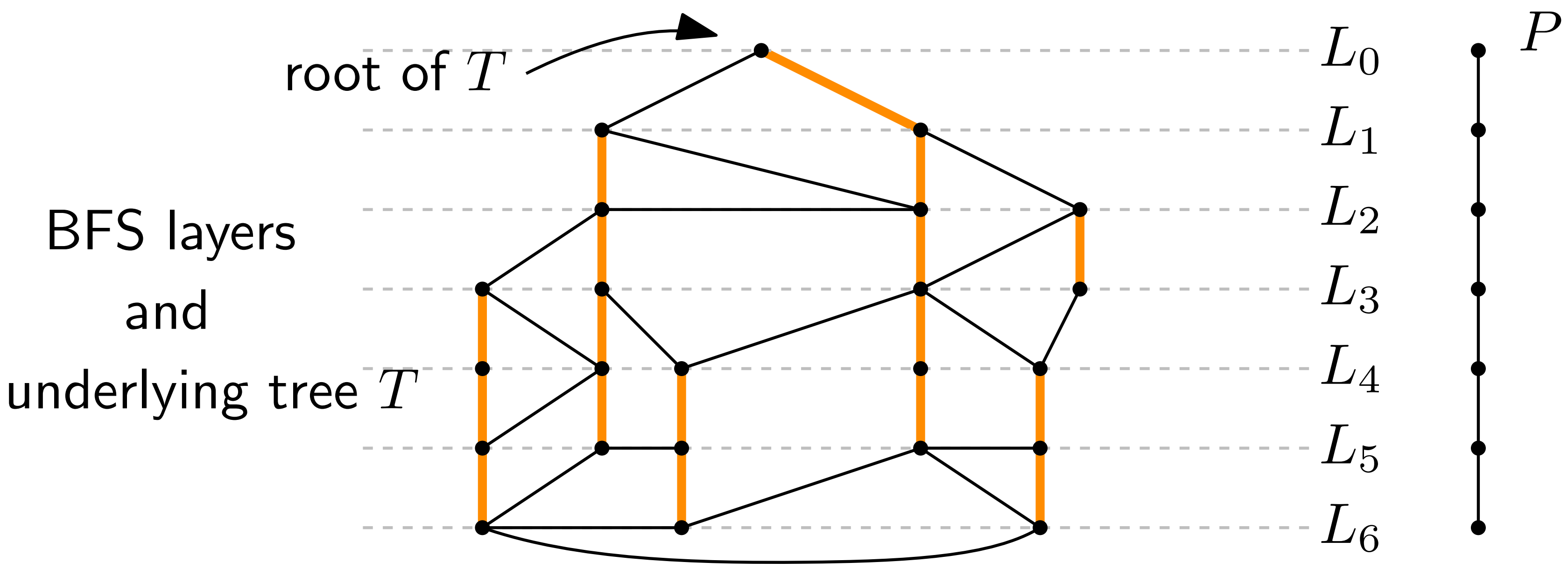
(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every connected planar graph G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **vertical paths** in T such that

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 8$$

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every **planar graph** G is a subgraph of a strong product $H \boxtimes P$ where H is a graph of treewidth at most 8 and P is a path.



$\mathcal{P}$ partition into vertical paths $\qquad H = G/\mathcal{P}$

main technical statement

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every connected planar graph G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **vertical paths** in T such that

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 8$$

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every triangulation G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **tripods** in T such that

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 3$$

(and $G/\mathcal{P}$ is planar)

main technical statement

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every connected planar graph G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **vertical paths** in T such that

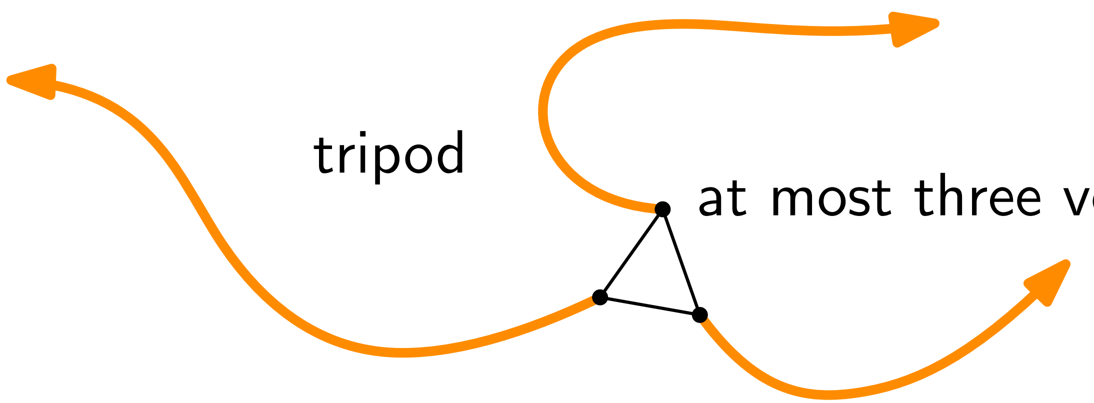
$$\mathrm{tw}(G/\mathcal{P}) \leqslant 8$$

(Dujmović, Joret, Morin, PM, Ueckerdt, Wood 2019)

Every triangulation G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **tripods** in T such that

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 3$$

(and $G/\mathcal{P}$ is planar)



at most three vertical paths in T

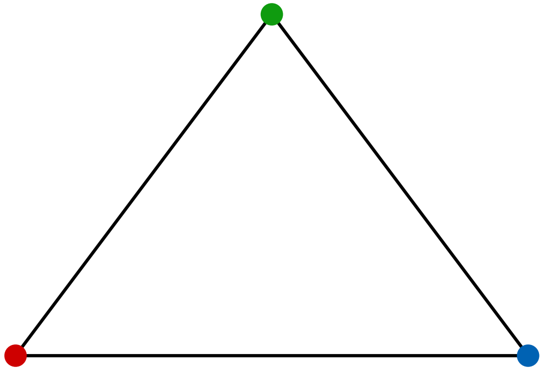
with connected starting points

proof

setting: region bounded by at most three tripods
root of T on the boundary or outside

proof

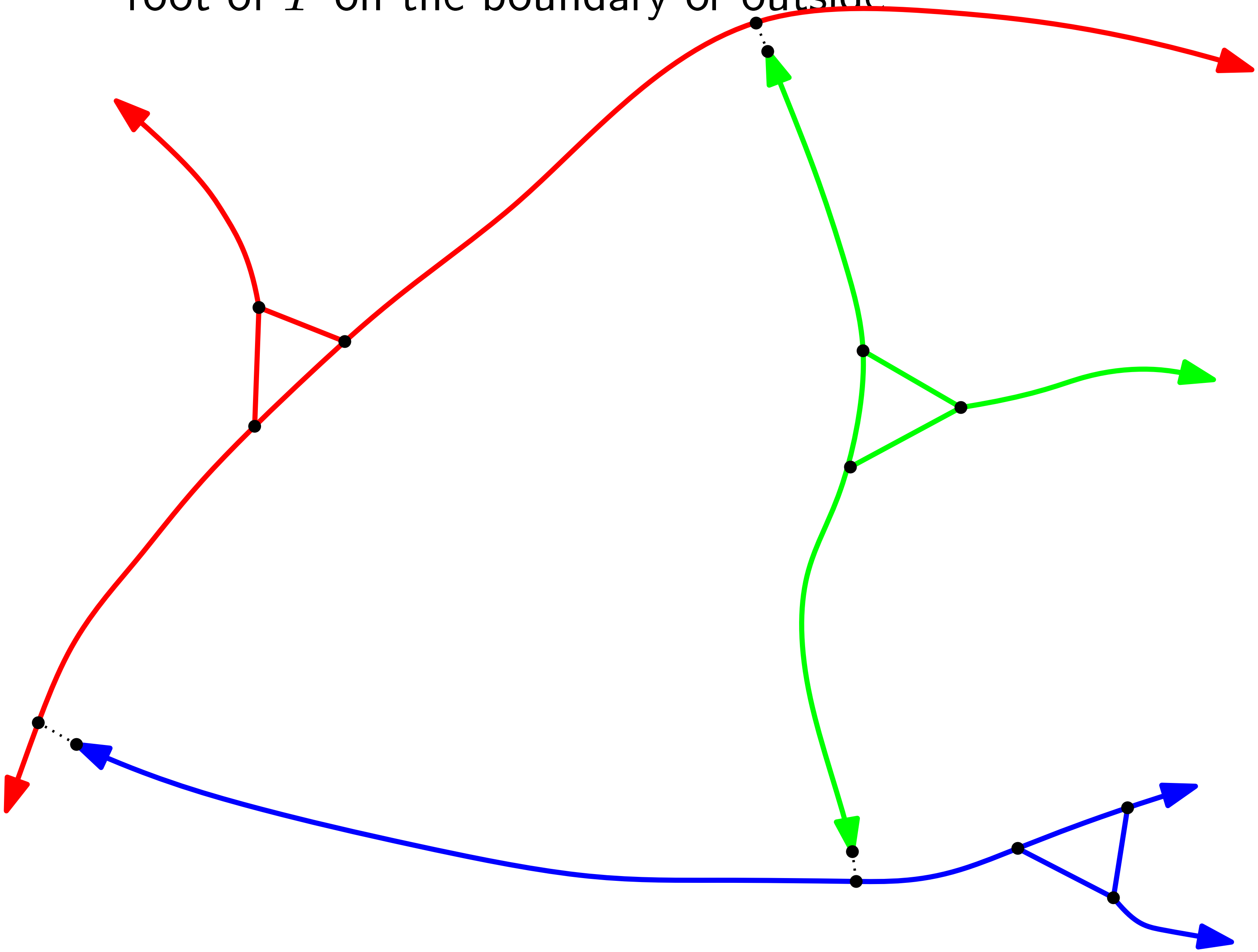
setting: region bounded by at most three tripods
root of T on the boundary or outside



proof

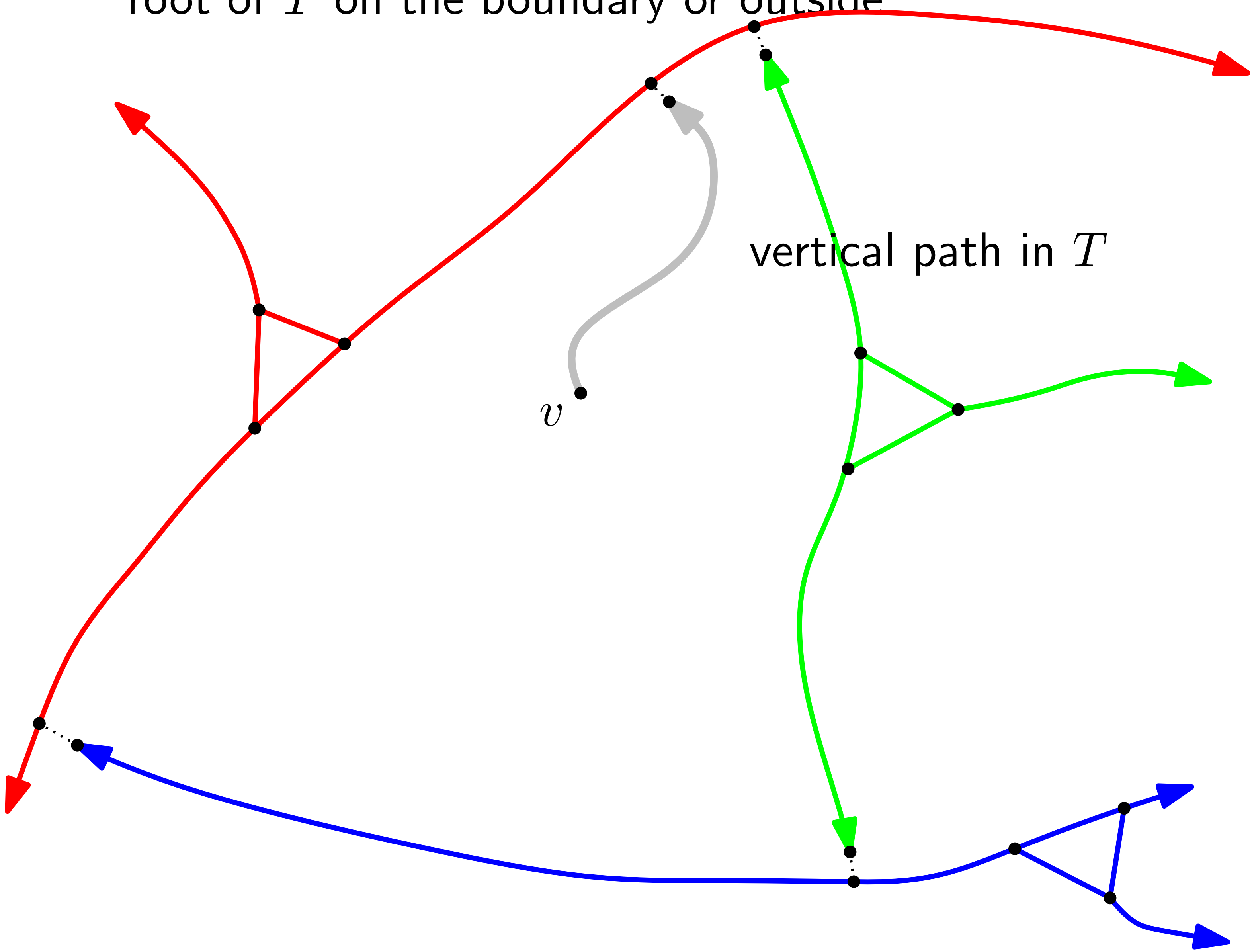
setting: region bounded by at most three tripods

root of T on the boundary or outside



proof

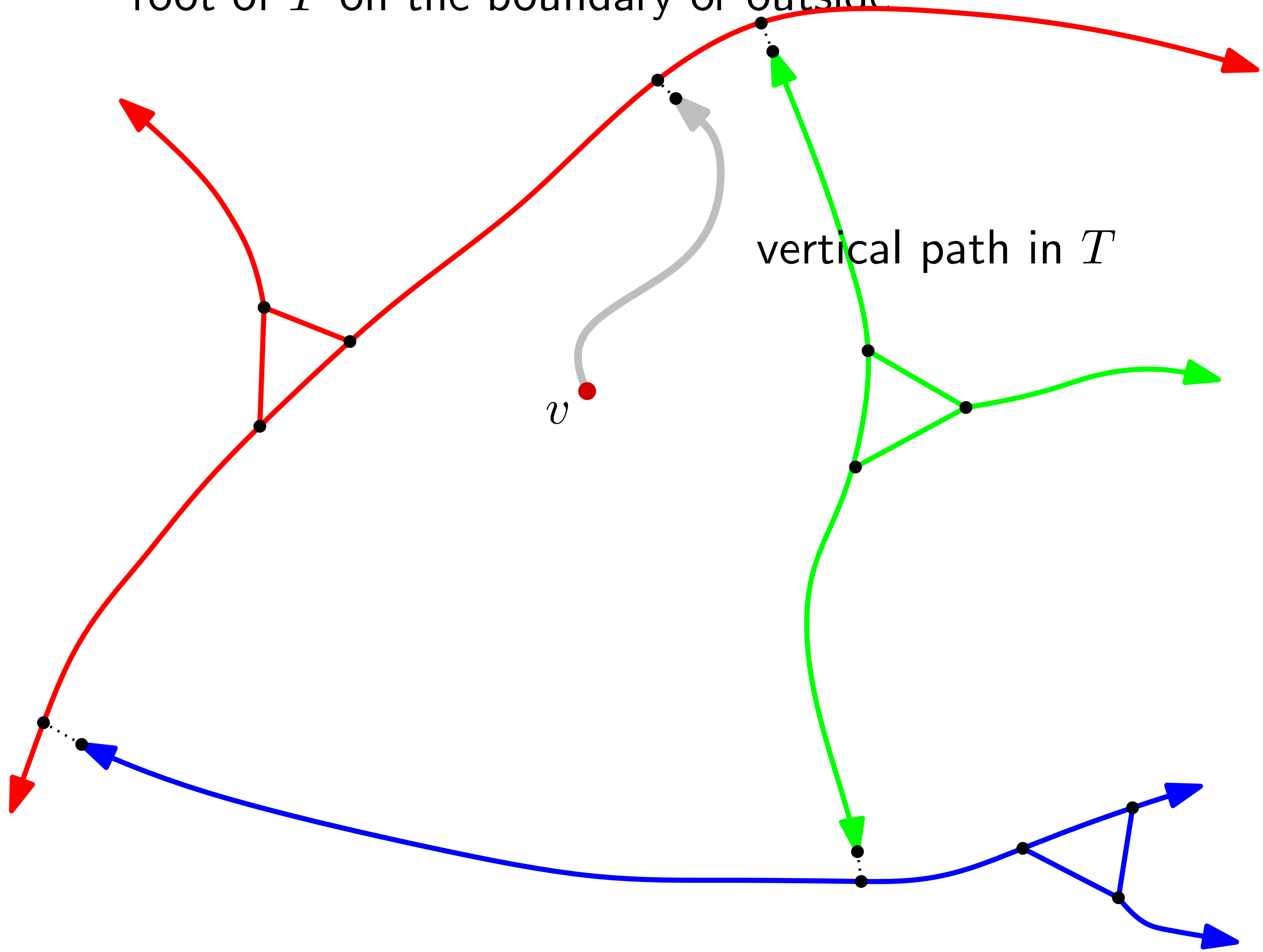
setting: region bounded by at most three tripods
root of T on the boundary or outside



proof

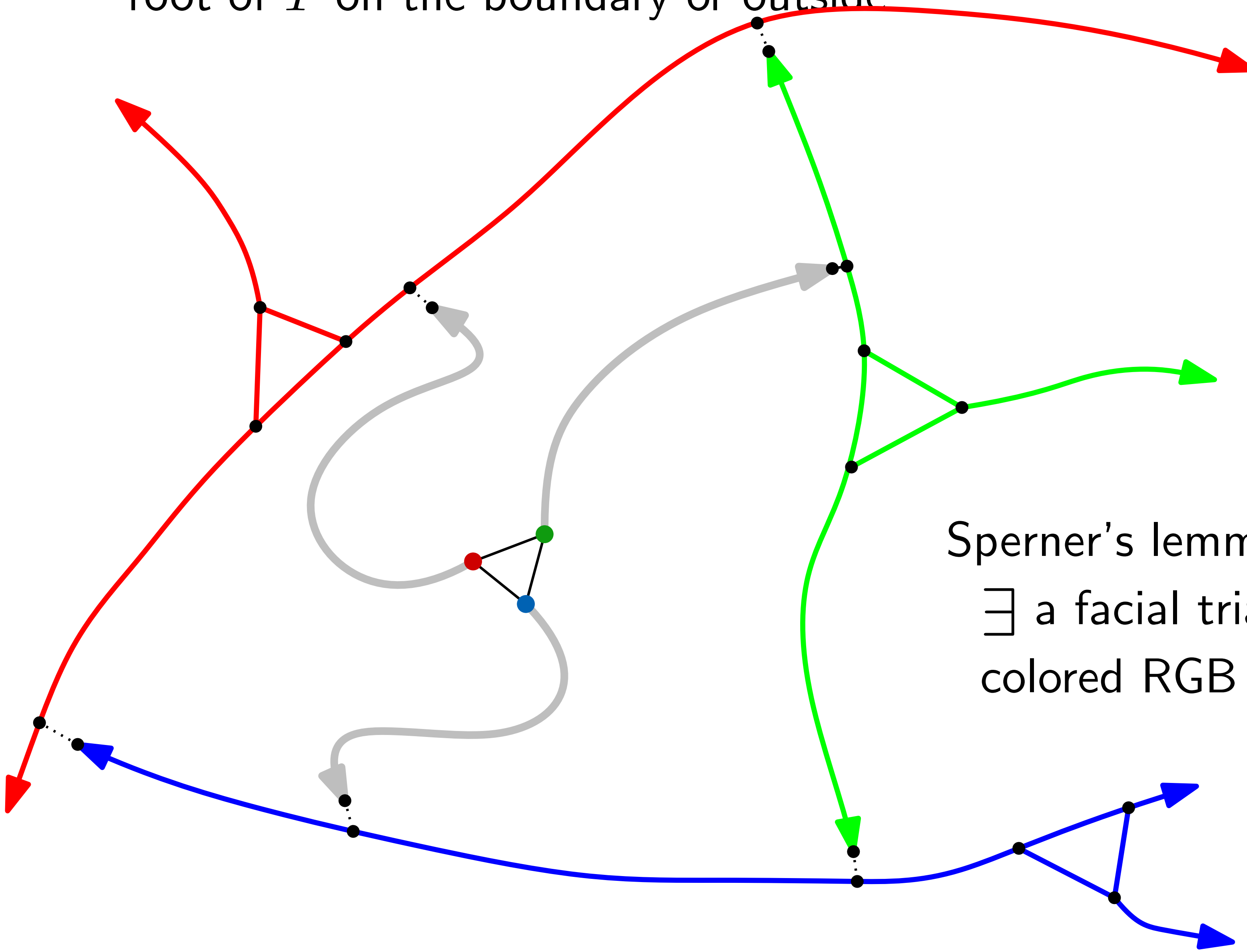
setting: region bounded by at most three tripods

root of T on the boundary or outside



proof

setting: region bounded by at most three tripods
root of T on the boundary or outside

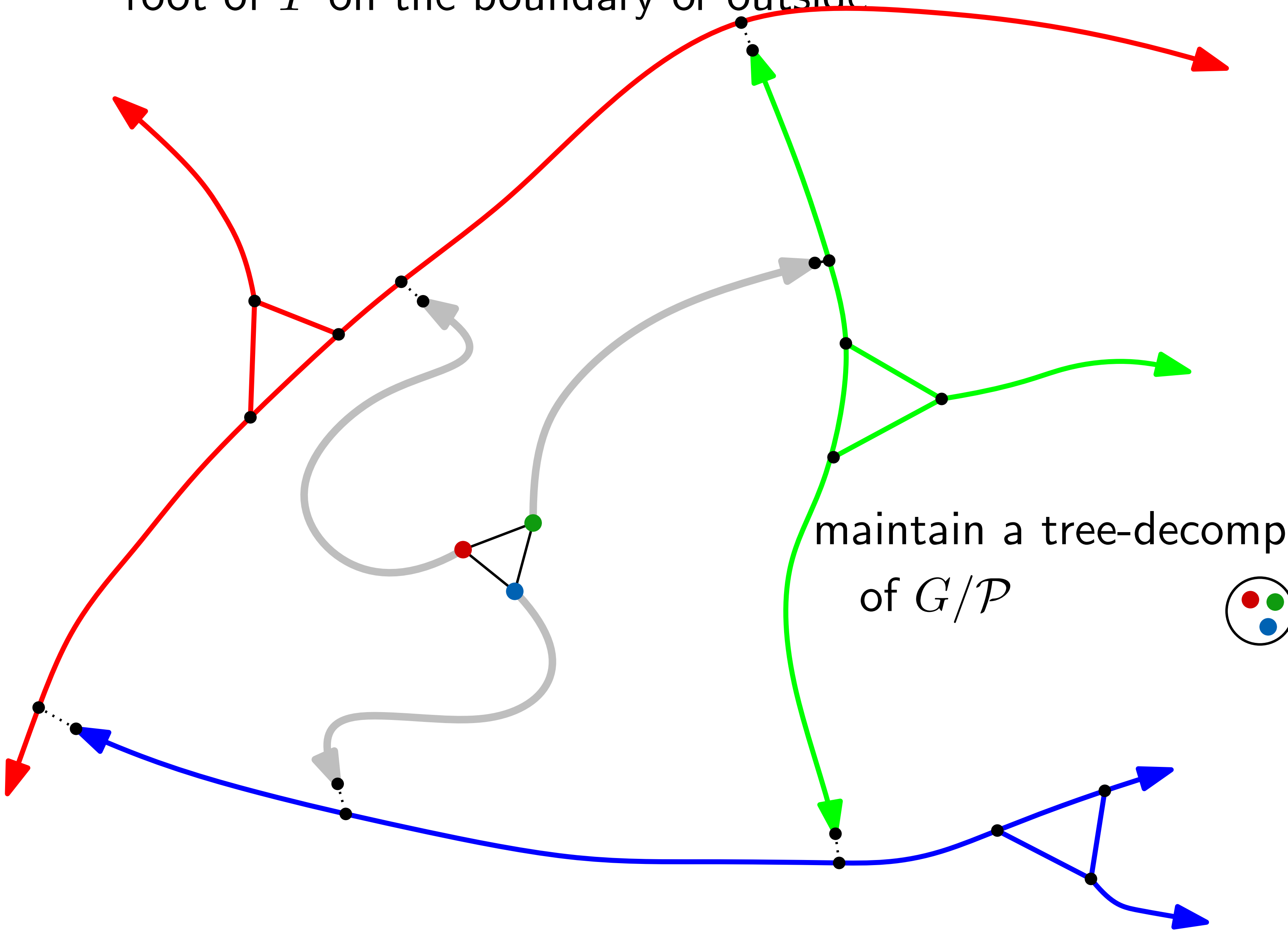


Sperner's lemma:
$\exists$ a facial triangle colored RGB

proof

setting: region bounded by at most three tripods

root of T on the boundary or outside

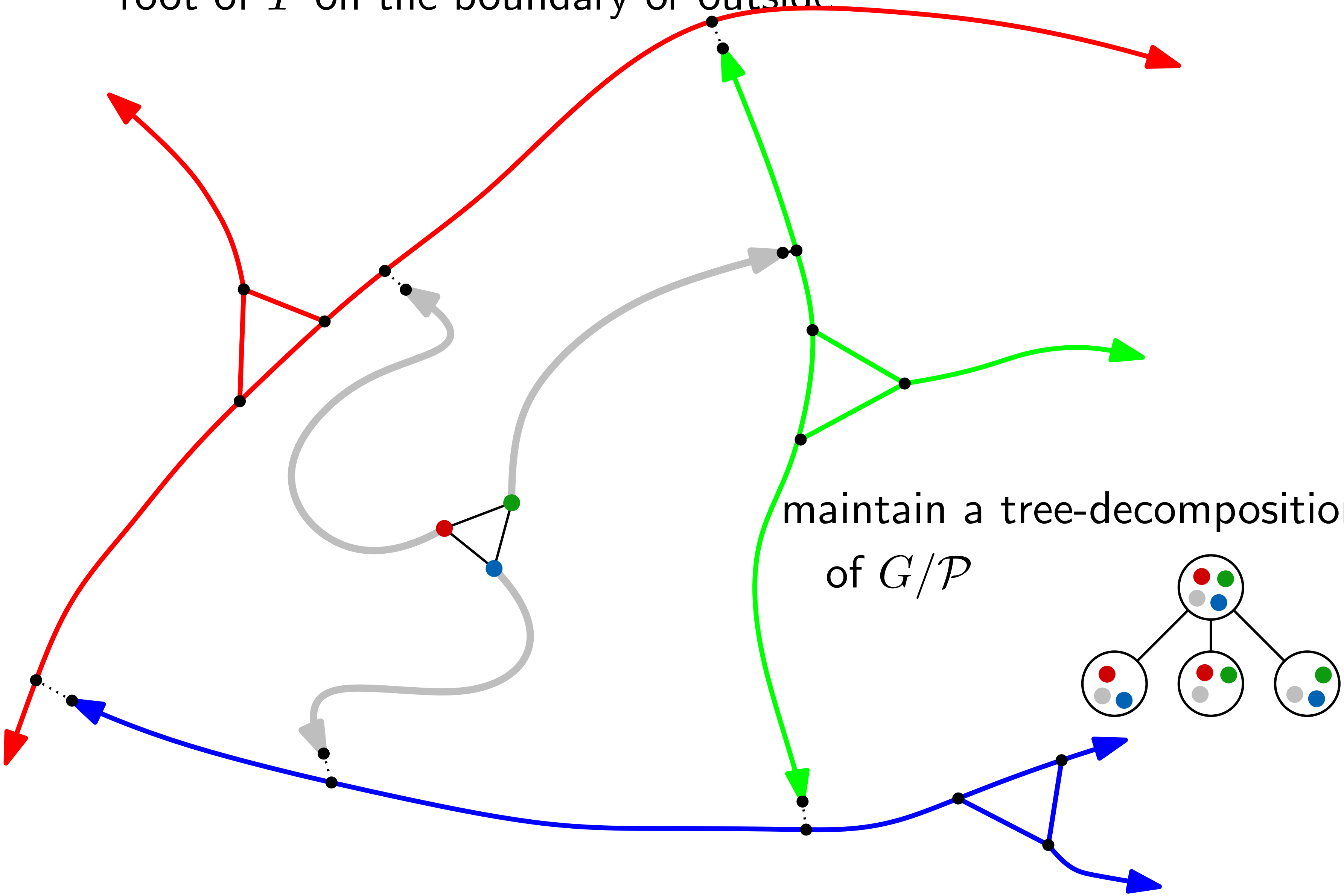


maintain a tree-decomposition of $G/\mathcal{P}$

proof

setting: region bounded by at most three tripods

root of T on the boundary or outside



maintain a tree-decomposition of $G/\mathcal{P}$

statements

Every triangulation G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **tripods** in T such that

$\mathrm{tw}(G/\mathcal{P}) \leqslant 3$
(and $G/\mathcal{P}$ is planar)

statements

Every triangulation G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **tripods** in T such that

$$\operatorname{tw}(G/\mathcal{P}) \leqslant 3$$

(and $G/\mathcal{P}$ is planar)

replace each tripod with its three legs: $4 \cdot 3 = 12$

Every connected planar graph G with a rooted spanning tree T has a vertex partition $\mathcal{P}$ into **vertical paths** in T such that

$$\operatorname{tw}(G/\mathcal{P}) \leqslant 11$$

statements

Every triangulation G with a rooted spanning tree T has
a vertex partition $\mathcal{P}$ into **tripods** in T such that

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 3$$
(and $G/\mathcal{P}$ is planar)

replace each tripod with its three legs: $4 \cdot 3 = 12$

Every connected planar graph G with a rooted spanning tree T has
a vertex partition $\mathcal{P}$ into **vertical paths** in T such that

~~$\mathrm{tw}(G/\mathcal{P}) \leqslant 11$~~

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 8$$

statements

Every triangulation G with a rooted spanning tree T has
a vertex partition $\mathcal{P}$ into **tripods** in T such that

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 3$$
(and $G/\mathcal{P}$ is planar)

replace each tripod with its three legs: $4 \cdot 3 = 12$

Every connected planar graph G with a rooted spanning tree T has
a vertex partition $\mathcal{P}$ into **vertical paths** in T such that

~~$\mathrm{tw}(G/\mathcal{P}) \leqslant 11$~~

~~$\mathrm{tw}(G/\mathcal{P}) \leqslant 8$~~

$$\mathrm{tw}(G/\mathcal{P}) \leqslant 6$$

(Ueckerdt, Wood, Yi 2022)