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Depth, Width, and Duality in Structural Graph and Matroid Theory

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Abstract

This thesis studies results in structural matroid theory and structural graph theory, with particular emphasis on how results in one setting inspire analogous results in the other. Its three main results develop this interaction in both directions.

The first result is directly inspired by one of the equivalent definitions of graph tree-depth: the minimum height of a rooted forest whose ancestor–descendant closure contains the graph. It establishes a matroid counterpart of this statement via extensions: the contraction*-depth of a matroid containing a non-loop circuit is one less than the smallest contraction-depth of any matroid containing it as a restriction. Thus, adding elements to make a shallow contraction procedure possible parallels adding edges to obtain a supergraph with a shallow depth-first search spanning forest. This theorem applies without any representability assumption.

The second mirrors the connection between rank-depth and tree-depth: any graph of small rank-depth is a vertex-minor of a graph of small tree-depth. We give a construction showing that any $\mathbb{F}$ -representable matroid of small branch-depth is a minor of a matroid of small contraction-deletion-depth. The construction preserves $\mathbb{F}$ -representability, and the bounds do not depend on the field $\mathbb{F}$.

The third reverses the direction of the flow of ideas. The bound on tree-depth by circumference for 2-connected graphs suggests a conjectural linear bound on matroid contraction*-depth in terms of maximum circuit size. This would imply the corresponding linear bounds on matroid branchwidth and pathwidth. Applying the pathwidth conjecture to the bond matroid of a graph, and using matroid duality, suggests the following question: is the pathwidth of a 2-connected graph bounded by a linear function of its cocircumference? We prove a stronger statement: for every 2-connected graph G , each edge belongs to a bond F that satisfies $\text{pw}(G) \leq 3|F| - 2$. The assumption of 2-connectivity cannot be replaced by connectedness. Additionally, we prove that every graph with an edge has treewidth at most its cocircumference, and complement these results by examples showing that linear dependence is necessary, and that no such bound can apply to tree-depth.

Source works, contributions, and use of AI

Source works and attribution

This thesis is based on the following joint works. All main results were obtained collaboratively with the coauthors listed below. The mathematical chapters adapt the original manuscripts' arguments to common notation, with no substantial changes to the underlying mathematical ideas. They also include exposition explaining the connections between the results.

1. Marcin Briański, Daniel Král', and Ander Lamaison, *Closure property of contraction-depth of matroids*, Journal of Combinatorial Theory, Series B **175** (2025), 240–265 [9]. Adapted in Chapter 3.
2. Marcin Briański, Daniel Král', and Kristýna Pekárková, *Branch-depth is minor closure of contraction-deletion-depth*, preprint [10]. Adapted in Chapter 4.
3. Marcin Briański, Gwenaél Joret, and Michał T. Seweryn, *Pathwidth vs cocircumference*, SIAM Journal on Discrete Mathematics **38** (2024), 857–866 [7]. Adapted in Chapter 5.

Specific contributions of the thesis author

The author took an active part in deriving the results in each of these works. Individual contributions cannot always be separated precisely in work developed through joint discussions. The following declarations describe the author's contributions to each work.

Contraction-depth and extensions

The extension theorem relating contraction*-depth to contraction-depth (Theorem 3.1) was developed jointly with Daniel Král' and Ander Lamaison during a research visit in Kraków. The tamed-set construction and the proof were derived from joint discussions, to which all three authors contributed in equal measure.

Branch-depth and minor containers

Daniel Král' conjectured the subspace statement underlying Lemma 4.4, without specifying quantitative bounds. The author found the proof of this lemma, which constructs an auxiliary subspace of dimension at most $3r$ and bounds the sum of the dimensions of the quotient images of the given subspaces. The application of the lemma to construct minor containers of bounded contraction-deletion-depth, and thereby obtain the branch-depth

characterisation and its explicit bound, was joint work with Daniel Král' and Kristýna Pekárková, with equal contributions from all three authors.

Treewidth, pathwidth, and cocircumference

This work was developed jointly with Gwenaél Joret and Michał T. Seweryn. The author took the leading role in developing the technical proof of Theorem 5.4, which shows that every edge of a 2-connected graph G belongs to a bond F satisfying $\text{pw}(G) \leq 3|F| - 2$. The mechanics of the argument were inspired by joint discussions and by the earlier work on tree-depth versus circumference [6].

Use of AI

AI tools were used as drafting and proofreading aids in preparing this thesis. This assistance included organising the material, drafting the connecting exposition, and unifying the notation across the three original works. The research results contained in this thesis were obtained without any AI involvement.

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Chapter 1

Introduction

1.1 The art of decomposition

Decomposing combinatorial structures is a central theme in combinatorics. A suitable decomposition can reveal features of an object that its size alone does not capture. This idea takes different forms in the Robertson–Seymour graph minor structure theorem [35], Seymour’s decomposition of regular matroids [37], and Szemerédi’s regularity lemma [38], as well as many other results. The pieces and the rules for combining them differ, but each theorem makes a complicated object accessible through a structured decomposition.

Treewidth and pathwidth are two flagship examples of these ideas. A tree-decomposition organises overlapping vertex sets, called bags, in a tree, with their intersections controlling how different parts interact. Its width measures the largest bag, with a conventional subtraction of one, and captures how “tree-like” a given graph is. The modern framework was developed in Robertson and Seymour’s Graph Minors series [32, 33], with antecedents in Halin’s work on S -functions [19]. Pathwidth, introduced in *Graph Minors I* [31], requires the bags to occur in a linear order. This can require much larger bags than a tree arrangement, and gives a quantitative measure of how “path-like” a given graph is.

These parameters show how excluding a local object can lead to a global structure. For each fixed forest H , graphs with no H minor have bounded pathwidth [31]; for each fixed planar graph H , graphs with no H minor have bounded treewidth [34]. In each case the bound depends only on H .

Decompositions also serve as a powerful tool in algorithm design: a computation on the whole object can often be assembled from computations on its constituent parts while recording their interactions. This is especially useful in parameterised complexity. A problem is *fixed-parameter tractable* (FPT) with parameter k if it admits an algorithm with running time $f(k)n^a$, where f is computable, n is the input size, and the constant a is independent of k . Structural parameters are natural candidates for FPT algorithms for many computationally hard problems.

Courcelle’s theorem is a central example of this principle. For each fixed treewidth bound and each fixed monadic second-order sentence, allowing quantification over vertex sets and edge sets, the property can be decided in linear time [12, 3]. A different example comes from the theory of sparsity. Dvořák, Král’, and Thomas showed that each fixed first-order property can be decided in linear time on any fixed class of bounded expansion [16]. Their argument uses colourings that break the graph into subgraphs of bounded tree-depth. These results exemplify the connection between structural graph theory and broader questions in theoretical computer science about algorithmic tractability.

Whereas treewidth and pathwidth measure the size of overlapping bags, *tree-depth* mea-

sure the height of a recursive elimination structure. Deleting a vertex separates the remaining components, which can then be treated independently; the depth records the longest chain of successive deletions. Earlier formulations include node rankings and separator trees, studied for trees by Iyer, Ratliff, and Vijayan [22]. The modern treatment under the name tree-depth is due to Nešetřil and Ossona de Mendez [27]. The three parameters satisfy the inequalities $\text{tw}(G) \leq \text{pw}(G) \leq \text{td}(G) - 1$, but each parameter captures something different about the graph in question: trees have treewidth at most 1, but can have unbounded pathwidth; paths have pathwidth at most 1, but can have unbounded tree-depth. Their formal definitions and comparisons are given in Chapter 2.

Tree-depth also plays a central role in the sparsity theory of Nešetřil and Ossona de Mendez [29]. Low tree-depth colourings are one of the many equivalent ways of defining bounded expansion and nowhere dense graph classes, and are particularly useful in algorithmic treatment of these classes of graph [27, 28, 16]. For a graph class, bounded tree-depth has a particularly simple interpretation: it is equivalent to a uniform bound on the number of vertices in a path occurring as a subgraph [29].

The richness of this structural theory suggests a natural question: can its ideas be transplanted into matroid theory and applied there with similar success? How far can this analogy be taken?

The first two works in this thesis approach that question by asking how *enlarging* an object can make its decomposition more explicit. One way is through *extension*: adding elements and studying a matroid that contains the original as a restriction. The other is through *minor containment*, allowing the original matroid to be recovered from a larger one by both contraction and deletion.

The third work asks about the reverse direction. Can matroid theory suggest graph-theoretic statements that are less apparent from a purely graph-theoretic viewpoint? Matroid duality supplies one route. Every matroid has a dual; the dual of a graph's cycle matroid is again graphic precisely when the graph is planar [30]. Applying duality to the cycle matroid exchanges graph cycles with bonds. This allows conjectures about matroid width to suggest graph questions about cuts, even when the graph has no planar embedding.

1.2 From graph tree-depth to matroid extensions

The tree-depth $\text{td}(G)$ of a graph G can be defined recursively:

- if $|V(G)| \leq 1$, then $\text{td}(G) = |V(G)|$;
- if G is disconnected with components $C_1, \dots, C_k$, then $\text{td}(G) = \max_{i \in [k]} \text{td}(C_i)$;
- otherwise $\text{td}(G) = 1 + \min_{v \in V(G)} \text{td}(G - v)$.

An equivalent definition uses closures of rooted forests [27]. The closure $\text{cl}(F)$ of a rooted forest F joins every vertex to its ancestors, and the height of F is the largest number of vertices on a root-to-leaf path. An *elimination forest* for G is a rooted forest on $V(G)$ such that $G \subseteq \text{cl}(F)$. Then

$$\text{td}(G) = \min\{\text{height}(F) : V(F) = V(G), G \subseteq \text{cl}(F)\}.$$

Crucially, the edges of F need not belong to G .

In matroid theory there are several parameters that can be considered analogues of tree-depth. One of them is the *contraction*-depth*, denoted $\text{cd}^*(M)$, originally introduced under the name of branch-depth [23]. We follow the terminology of [8], which distinguishes

it from the connectivity-based branch-depth of DeVos, Kwon, and Oum [13]. However, before formally defining contraction*-depth, we will first introduce *contraction-depth*, denoted $\text{cd}(M)$, which mirrors the recursive definition of tree-depth [13]:

- if M has at most one element, then $\text{cd}(M) = |E(M)|$;
- if M is disconnected with components $C_1, \dots, C_k$, then $\text{cd}(M) = \max_{i \in [k]} \text{cd}(C_i)$;
- otherwise $\text{cd}(M) = 1 + \min_{e \in E(M)} \text{cd}(M/e)$.

This definition suggests a generalisation of graph tree-depth, but contraction-depth is not minor-monotone: deletion can increase its value [13]. There is a useful parallel with requiring an elimination forest to use only edges of the graph. For this discussion, write $\text{td}_{\text{DFS}}(G)$ for the minimum height of a depth-first search spanning forest of G . In an undirected graph, such forests are exactly the rooted spanning forests satisfying

$$F \subseteq G \subseteq \text{cl}(F).$$

Deleting edges can increase this minimum. For example, the graph obtained by adding a diagonal to a 4-cycle has a DFS tree of height three, whereas deleting that diagonal leaves a cycle whose DFS trees all have height four. Requiring the elimination forest to be a subgraph of the graph parallels the requirement that contraction-depth use only elements already present in the matroid. For a represented matroid, these contractions use directions spanned by existing elements. This is exactly the condition relaxed by contraction*-depth.

Contraction*-depth is defined for all matroids by a rooted tree and rank inequalities, as given in Section 2.6. However, for an $\mathbb{F}$ -represented matroid, it has the following equivalent recursive description [11]:

- if M has at most one element, then $\text{cd}^*(M) = \text{rank}(M)$;
- if M is disconnected with components $C_1, \dots, C_k$, then $\text{cd}^*(M) = \max_{i \in [k]} \text{cd}^*(C_i)$;
- otherwise $\text{cd}^*(M) = 1 + \min_L \text{cd}^*(M/L)$.

Here L ranges over the one-dimensional subspaces of the span K of the representing vectors. The matroid M/L is obtained by projecting every labelled vector in the representation of M to the quotient space K/L ; all element labels are retained. In particular, the line L need not be spanned by an element of M . Allowing these “hidden” directions gives a minor-monotone parameter [23].

This parameter also connects the structural question with integer programming. The *dual graph* of a matrix has one vertex for each row, where two rows are adjacent whenever there is a column where they both have a nonzero entry. The matrix’s dual tree-depth is the tree-depth of this graph. For a nonzero matrix, the smallest dual tree-depth among all row-equivalent matrices equals the contraction*-depth of its column matroid [11]. Thus row operations can expose a useful graph decomposition that is not visible in the original arrangement of nonzero entries. For rational matrices, this leads to an FPT algorithm for integer programming in standard equality form, with constraints $Ax = b$ and variable bounds, parameterised jointly by contraction*-depth and the entry complexity of A [11]. Here entry complexity means the maximum binary encoding length of an entry, including its numerator and denominator.

Returning to graphs, the freedom to choose an elimination forest can be expressed by first passing to a supergraph and then choosing a shallow DFS forest. More precisely,

$$\text{td}(G) = \min\{\text{td}_{\text{DFS}}(H) : G \subseteq H\}. \quad (1.1)$$

Restricting the ancestor order of a DFS forest of H to $V(G)$ gives an elimination forest for G of no greater height, so $\text{td}_{\text{DFS}}(H) \geq \text{td}(G)$. Conversely, an optimal elimination forest F for G is a DFS forest of $\text{cl}(F)$, giving equality. The analogous matroid question is whether adding elements can supply the directions needed for an optimal contraction*-decomposition, so that ordinary element contractions can realise it.

Chapter 3, based on joint work with Daniel Král' and Ander Lamaison [9], turns this intuition into an extension theorem. Write

$$e_{\text{cd}}(M) = \min\{\text{cd}(N) : N|E(M) = M\}.$$

Then, for any M containing a non-loop circuit, we have

$$e_{\text{cd}}(M) = \text{cd}^*(M) + 1.$$

The same identity holds for a nonempty matroid whose every element is a loop. If every element is a loop or a coloop and at least one is a coloop, then $e_{\text{cd}}(M) = \text{cd}^*(M) = 1$. The empty matroid has both values zero. These cases are collected in Theorem 3.1.

The linear case can be illuminating here: directions selected in the recursive quotients can be lifted to vectors and supplied as new elements. Equivalently, the edges of a contraction*-decomposition can be labelled by vectors so that the elements assigned to each leaf are spanned by the labels on its root path [11]. For arbitrary matroids, the construction still introduces one new element per tree edge. The difficulty is in defining independence on the enlarged ground set without having access to a representation to rely on. This is done via tamed sets and a token-counting argument.

1.3 From recursive operations to minor closures

The graph-theoretic analogue for the second result is the relationship between rank-depth and tree-depth. Graph rank-depth measures the radius of a decomposition tree and the binary ranks of adjacency matrices across its cuts. A vertex-minor is obtained by vertex deletions and local complementations, where local complementation at a vertex toggles adjacency between each pair of its neighbours. A class of graphs has bounded rank-depth if and only if it is contained in the vertex-minor closure of a class of bounded tree-depth [21, 13]. Thus, a description in terms of cut ranks can be realised by a shallow elimination structure in a larger graph. This suggests seeking an analogous connection between matroid depth parameters.

Contraction-deletion-depth, denoted $\text{cdd}(M)$, permits both contractions and deletions in a recursive decomposition. Branch-depth instead controls the radius of a decomposition tree and the connectivity across unions of the parts at each internal node [13]. A bound on the contraction-deletion-depth gives a bound on the branch-depth, but bounded branch-depth does not bound contraction-deletion-depth. The cycle matroids of thickened cycles give examples with bounded branch-depth and arbitrarily large contraction-deletion-depth.

Chapter 4 explains how allowing a larger containing matroid repairs this failure for representable matroids. For a fixed field $\mathbb{F}$ and an $\mathbb{F}$ -representable matroid M , put

$$m_{\text{cdd},\mathbb{F}}(M) = \min\{\text{cdd}(N) : N \text{ is } \mathbb{F}\text{-representable and } M \text{ is a minor of } N\}.$$

In joint work with Daniel Král' and Kristýna Pekárková [10], we proved

$$\text{bd}(M) \leq m_{\text{cdd},\mathbb{F}}(M) \leq 2 \text{bd}(M)(4^{\text{bd}(M)} - 1) + 1.$$

After [10] appeared, Jim Geelen (personal communication) pointed out that the qualitative characterisation for representable matroids can also be derived from the proof of [17,

Theorem 1.3], although it is not stated there in terms of branch-depth. He also indicated that this approach should yield the qualitative statement for arbitrary matroids, formulated as a conjecture in [10]. Chapter 4 presents a self-contained linear-algebraic proof with the explicit bound above. Its subspace lemma, Lemma 4.4, may also be of independent interest.

1.4 From matroid duality back to graph width

The third result brings the depth and width viewpoints together. For 2-connected graphs, the bound $\text{td}(G) \leq \text{circ}(G)$ [6] controls the height of an elimination forest by the length of a largest cycle. This naturally suggest a question whether matroidal equivalent of this phenomenon is also true. The following conjecture is implicit in the concluding discussion on contraction depth in [8, full version, Section 7], where it is shown that a quadratic upper bound in terms of the matroid circumference is optimal for contraction-depth, but it leaves open whether one can do better for contraction*-depth. It arose in discussions with Daniel Král', and we state it explicitly here.

Conjecture 1.1. *There is an absolute constant $C > 0$ such that, for every positive integer k and every finite matroid M whose circuits have at most k elements,*

$$\text{cd}^*(M) \leq Ck.$$

The quadratic bound $\text{cd}^*(M) \leq k^2$ is known [23]; the conjecture asks whether this dependence can be made linear. This would parallel the improvement from the earlier quadratic bound on tree-depth in terms of circumference for 2-connected graphs [25] to the linear bound of [6]. Since for any graph G we have the inequality $\text{cd}^*(M(G)) \leq \text{td}(G)$ [23], it also follows from [6] that Conjecture 1.1 holds for graphic matroids. A linear upper bound is also the best one can hope for: for $r \geq 2$, $\text{cd}^*(U_{r,r^2}) = r$ and every circuit of U_{r,r^2} has $r+1$ elements, where $U_{r,n}$ is the uniform matroid of rank r on n elements (a matroid in which every subset of the ground set of size at most r is independent). To see that $\text{cd}^*(U_{r,r^2}) = r$ consider any contraction*-decomposition; this is a rooted tree with r edges, so it has at most r leaves; by pigeonhole principle there is a leaf that is assigned at least r elements of the matroid: such a set has full rank, and it follows that the decomposition is a path (rooted at one of the ends), and so has depth r (see Subsection 2.6.2 for a formal definition of a contraction*-decomposition).

The conjecture also links matroid depth with the width parameters. Proposition 2.2 gives

$$\text{pw}(M) \leq \text{cd}^*(M), \quad \text{bw}(M) \leq \text{cd}^*(M) + 1.$$

Consequently, the conjecture would imply $\text{pw}(M) \leq Ck$ and $\text{bw}(M) \leq Ck + 1 \leq (C+1)k$. It therefore implies both linear width conjectures stated in the joint graph manuscript [7], recorded as Conjectures 5.11 and 5.12.

The pathwidth conjecture motivates a graph question through matroid duality. Matroid pathwidth is self-dual, and the circuits of the bond matroid $M(G)^*$ are precisely the bonds of G . The cographic case therefore asks whether $\text{pw}(M(G)^*)$ can be bounded by a linear function of the largest bond. Passing to graph pathwidth requires care: the cycle matroid of a tree is free and has pathwidth zero, while trees can have arbitrarily large graph pathwidth and all their bonds have size one. For a connected simple graph with at least three vertices, matroid connectivity corresponds to 2-connectivity of the graph. This suggests the following question: does every 2-connected graph have pathwidth bounded linearly by its cocircumference? Graph pathwidth and matroid pathwidth are in general different parameters, but an affirmative answer implies the cographic case of the matroid pathwidth conjecture.

Chapter 5, based on joint work with Gwenaël Joret and Michał T. Seweryn [7], is motivated by these questions. We prove that for every graph G with at least one edge,

$$\text{tw}(G) \leq \text{cocirc}(G).$$

Further, for a 2-connected graph G , every edge belongs to a bond F satisfying

$$\text{pw}(G) \leq 3|F| - 2, \quad \text{and hence} \quad \text{pw}(G) \leq 3 \text{cocirc}(G) - 2.$$

These results are complemented by examples showing that linear dependence is necessary. Furthermore, the connection to bond size cannot be strengthened to tree-depth. Long cycles have cocircumference two and unbounded tree-depth.

1.5 How to read the thesis

Chapter 2 fixes the shared terminology. It distinguishes height from depth, graph connectivity from matroid connectivity, and the two historically different parameters called branch-depth. The notation c^*d always means contraction*-depth, whereas bd always means branch-depth defined from the connectivity function.

The next three chapters retain the full arguments of the source manuscripts, with common definitions moved to the preliminaries and local constructions introduced where they are used. Chapter 6 focuses on the circuit-size conjecture for contraction*-depth and the optimal quantitative bounds for minor containers.

Chapter 2

Common preliminaries and notation

This chapter fixes the conventions and notation used throughout the thesis. Graph decompositions are followed by matroid operations, representations, and decomposition parameters. We specify the height, depth, and connectivity conventions that determine the additive constants in later statements.

2.1 Basic conventions

We assume familiarity with the standard notation and terminology of graph and matroid theory; see Diestel [14] and Oxley [30], respectively.

All graphs and matroids are finite. Graphs are undirected and simple unless parallel edges or loops are explicitly allowed. The notation $[k]$ denotes the set $\{1, \dots, k\}$, with $[0] = \emptyset$. $\dot{\cup}$ denotes a union of pairwise disjoint sets. We write $\mathbb{N} = \{0, 1, 2, \dots\}$; a positive integer is specified as such.

2.2 Graphs, decompositions, and bonds

2.2.1 Graph notation and elementary operations

The vertex and edge sets of G are denoted $V(G)$ and $E(G)$; their cardinalities are its *order* and *size*, respectively. For $X \subseteq V(G)$, $G[X]$ is the *induced subgraph* with vertex set X and all edges of G whose ends lie in X . For $S \subseteq V(G)$, $G - S = G[V(G) \setminus S]$, whereas for $F \subseteq E(G)$, $G - F$ deletes only the edges in F and retains all vertices. For a vertex v of G we write $G - v$ as a shorthand for $G - \{v\}$. Thus $G - v$ deletes a vertex and all its incident edges, while $G - xy$ deletes the edge with ends x, y . For distinct existing vertices x, y , $G + xy$ adds that edge if it is absent; it leaves G unchanged if it is already present.

A *k-connected graph* has more than k vertices and remains connected after deletion of any set of fewer than k vertices.

A *path* has no repeated vertices, and its *length* counts edges. We use P_n for a path on n vertices and K_n for a complete graph on n vertices. When a path is explicitly identified with its edge set P , as in Chapter 3, $|P|$ counts edges. A *bridge* is an edge whose deletion increases the number of components, and a *cutvertex* is a vertex with the analogous property. A *block* is a maximal connected subgraph with no cutvertex; bridges with their ends and isolated vertices are included as blocks.

A graph *minor* is obtained by deleting vertices or edges and contracting edges. For simple graphs, loops created by contraction are discarded and parallel edges are merged. When edge identities matter, particularly for cycle matroids and plane duality, we retain the resulting multigraph. A *plane graph* is a graph with a fixed embedding in the plane. Its *plane dual* is G^* , with a vertex for each face and a dual edge e^* crossing each edge e . In a multigraph, a loop is a cycle of length one and two parallel edges form a cycle of length two. The vertex-decomposition parameters below are evaluated on the underlying simple graph when loops or parallel edges occur.

2.2.2 Graph width parameters: treewidth and pathwidth

A *tree-decomposition* of G [32, 33] is a pair $(T, \{X_t\}_{t \in V(T)})$, where T is a nonempty tree and each *bag* X_t is a subset of $V(G)$, satisfying:

- (i) $\bigcup_{t \in V(T)} X_t = V(G)$;
- (ii) for each $uv \in E(G)$ there is $t \in V(T)$ with $\{u, v\} \subseteq X_t$;
- (iii) for each $v \in V(G)$, the set $\{t : v \in X_t\}$ induces a connected subtree of T .

The *width* of the decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and the *treewidth* $\text{tw}(G)$ of G are

$$\text{width}(\mathcal{T}) = \max_{t \in V(T)} |X_t| - 1 \quad \text{and} \quad \text{tw}(G) = \min_{\mathcal{T}} \text{width}(\mathcal{T}),$$

where the minimum ranges over all tree-decompositions of G . Empty bags are allowed, so $\text{tw}(\emptyset) = -1$.

A *path-decomposition* [31] is a tree-decomposition whose indexing tree is a path. We write it as a nonempty sequence $(X_1, \dots, X_m)$. The connectedness condition says that the indices of the bags containing any fixed vertex form an interval. Its width is $\max_i |X_i| - 1$, and the *pathwidth* $\text{pw}(G)$ is the minimum width of a path-decomposition. In particular, $\text{pw}(\emptyset) = -1$. Chapter 5 also uses $\text{pw}(G; x)$ and $\text{pw}(G; x, y)$: these minimise width subject, respectively, to x belonging to the first bag, and to x belonging to the first bag and y to the last.

2.2.3 Bonds, separations, and branch-decompositions

A *bond* in a graph G is a nonempty inclusion-wise minimal edge set whose deletion increases the number of components of G . Equivalently, it consists of all edges between two nonempty connected sides of a partition of the vertex set of one component. The *circumference* $\text{circ}(G)$ is the maximum size of a cycle, when a cycle exists. The *cocircumference* $\text{cocirc}(G)$ is the maximum size of a bond, when a bond exists. Thus the latter is defined for every simple graph with at least one edge. We do not assign a maximum to an empty family of cycles or bonds.

We use the separation convention in Chapter 5: a *separation* (G_1, G_2) of a graph G is a pair of edge-disjoint subgraphs with $G_1 \cup G_2 = G$, and its *order* is $|V(G_1) \cap V(G_2)|$. It is *nontrivial* when both sides contain an edge. This convention allows one side to consist of an edge whose two ends both lie in the other side.

For a graph G with at least two edges, a *branch-decomposition* is a *subcubic tree*, that is, a tree of maximum degree at most three, with its leaves labelled bijectively by $E(G)$. Removing an edge of the tree partitions $E(G)$ into two sets. The subgraphs consisting of these edge sets and their incident vertices form a separation (G_1, G_2) of G , after assigning each isolated vertex of G to exactly one side. Its order is the number of vertices incident

with edges on both sides. The *width* of the branch-decomposition is the maximum order of these separations, and the *branchwidth* $\text{bw}(G)$ is the minimum width over all branch-decompositions. Set $\text{bw}(G) = 0$ when $|E(G)| \leq 1$.

2.3 Rooted forests and graph tree-depth

2.3.1 Rooted-tree notation

The *height* of a rooted tree counts vertices on a longest root-to-leaf path; its *depth* counts edges. Height is therefore depth plus one. A *leaf* has no children, so a tree consisting of its root has depth zero. An ancestor or descendant is strict unless equality is explicitly allowed. A *rooted forest* is a disjoint union of rooted trees; its height is the maximum height of its components, with value zero for the empty forest. A *branching vertex* has at least two children; an *internal branching vertex* is a branching vertex other than the root.

Write $u \preceq v$ if $u = v$ or u is a descendant of v . Thus the root is the greatest element in this order. An edge's *bottom endpoint* is its descendant endpoint, and its *top endpoint* is its ancestor endpoint. The rooted subtree at v , including v , is $T[v]$. For a vertex set A , $T\langle A \rangle$ is the union of paths from the root to A ; for $A = \emptyset$ it has no edges. The tree $T[v]$ consists of $T[v]$ together with the path to the root. If elements of a matroid are assigned to leaves by f , then $T(v) = f^{-1}(V(T[v]))$ denotes the assigned matroid elements, not the vertices of a subtree.

2.3.2 Tree-depth: the height of recursive elimination

The *closure* $\text{cl}(F)$ of a rooted forest F is the simple graph on $V(F)$ in which distinct vertices are adjacent exactly when one is an ancestor of the other. A *tree-depth decomposition*, also called an *elimination forest*, of G is a rooted forest F with $V(F) = V(G)$ and $G \subseteq \text{cl}(F)$. Thus the ends of every edge of G must be comparable in the ancestor order. The edges of F need not be edges of G . The *tree-depth* $\text{td}(G)$ is the minimum height of such a forest [27], and $\text{td}(\emptyset) = 0$. Here $\text{cl}(F)$ denotes graph closure; the ground-set closure $\text{cl}_M(X)$ of a matroid is a different operation.

Equivalently, if $G_1, \dots, G_s$ are the components of G , then

$$\text{td}(G) = \begin{cases} 0, & V(G) = \emptyset, \\ \max_{1 \leq i \leq s} \text{td}(G_i), & s \geq 2, \\ 1 + \min_{v \in V(G)} \text{td}(G - v), & G \text{ is connected and nonempty.} \end{cases} \quad (2.1)$$

For a connected nonempty graph, the last case follows by choosing the root and constructing elimination forests for the components of $G - v$.

2.3.3 Comparing the three graph parameters

Proposition 2.1 ([5]). *For every graph G ,*

$$\text{tw}(G) \leq \text{pw}(G) \leq \text{td}(G) - 1. \quad (2.2)$$

Proof. A path-decomposition is a tree-decomposition, proving the first inequality. For the second, suppose G is nonempty and let F be an elimination forest of height h . Order its vertices in depth-first preorder, completing each descendant subtree before moving to the next, and process the components consecutively. For each vertex v take a bag consisting of v and all its ancestors. Every vertex appears in its own bag. Each edge of G has comparable

ends, so its ends occur together in the bag of the lower endpoint. The bags containing a fixed vertex u are exactly those indexed by vertices in $F[u]$, which form an interval in preorder. These bags give a path-decomposition of width at most $h - 1$. Minimise over F . For the empty graph both widths are -1 and the depth is zero. $\square$

The following examples show that boundedness of these parameters gives successively stronger conditions. For paths, we have [29]

$$\text{tw}(P_n) = \text{pw}(P_n) = 1, \quad \text{td}(P_n) = \lceil \log_2(n+1) \rceil \quad (n \geq 2).$$

Treewidth does not bound pathwidth either. Let T_h be the complete ternary tree of depth h . Then the following bounds are well known

$$\text{tw}(T_h) = 1, \quad \text{pw}(T_h) = h, \quad \text{td}(T_h) = h + 1 \quad (h \geq 1).$$

Here is a short verification of the less immediate pathwidth lower bound. Induct on h , with the singleton case $h = 0$ as the base. In any path-decomposition choose vertices x and y from its first and last nonempty bags. One of the three subtrees below the root, say S , contains neither x nor y . Restricting the bags to S gives, by induction, a bag containing at least h vertices of S . The complement of S is connected and contains x, y , so the indices of bags meeting it form an interval covering every nonempty bag. The same bag therefore contains a vertex outside S and has size at least $h + 1$. The given rooting and (2.2) supply the matching upper bound and the tree-depth equality. These trees also explain the connectivity hypotheses in the pathwidth results of Chapter 5. All three graph parameters are non-increasing under taking minors [14, 27].

2.4 Matroids, minors, and duality

A *matroid* $M = (E, \mathcal{I})$ consists of a finite *ground set* E and a nonempty hereditary family $\mathcal{I} \subseteq 2^E$ of *independent sets* satisfying augmentation: if $I, J \in \mathcal{I}$ and $|I| < |J|$, there is $e \in J \setminus I$ with $I \cup \{e\} \in \mathcal{I}$. A *basis* is a maximal independent set, and a *circuit* is a minimal dependent set. We also write $E(M)$ and $\mathcal{I}(M)$ when the matroid needs to be specified. We write $|M| = |E(M)|$ and occasionally $e \in M$ or $X \subseteq M$ for $e \in E(M)$ or $X \subseteq E(M)$. A matroid is *free* if every subset of its ground set is independent. The *rank* $\text{rank}_M(X)$ is the maximum size of an independent subset of X ; $\text{rank}(M) = \text{rank}_M(E(M))$. We omit the subscript M when the matroid is clear from the context. Rank is submodular:

$$\text{rank}_M(X \cup Y) + \text{rank}_M(X \cap Y) \leq \text{rank}_M(X) + \text{rank}_M(Y).$$

The *closure* of X is $\text{cl}_M(X) = \{e \in E : \text{rank}_M(X \cup \{e\}) = \text{rank}_M(X)\}$. An element of rank zero is a *loop*. An element in every basis is a *coloop*. Distinct nonloops e, f are *parallel* when $\text{rank}_M(\{e, f\}) = 1$.

For $X \subseteq E(M)$, we write $\overline{X} = E(M) \setminus X$ when the ambient matroid M is clear.

The *restriction* $M|X$ has independent sets those of M contained in X . *Deletion* is $M \setminus X = M|(E \setminus X)$. *Contraction* is determined by

$$\text{rank}_{M/X}(Y) = \text{rank}_M(X \cup Y) - \text{rank}_M(X) \quad (Y \subseteq E \setminus X). \quad (2.3)$$

For a loop, contraction and deletion coincide. A *minor* is any $M/C \setminus D$ with $C \cap D = \emptyset$; its ground set is $E(M) \setminus (C \cup D)$. Deletion and contraction of disjoint sets commute. We abbreviate $M/\{e\}$ and $M \setminus \{e\}$ to M/e and $M \setminus e$. An *extension* N of M satisfies

$N|E(M) = M$, after identifying the original labels. We use $N \supseteq M$ only for this restriction relation, not for general minor containment.

The bases of the *dual* M^* are the complements of bases of M ; hence

$$\text{rank}_{M^*}(X) = |X| + \text{rank}_M(E \setminus X) - \text{rank}(M).$$

In particular, $(M^*)^* = M$, $(M \setminus X)^* = M^*/X$, and $(M/X)^* = M^* \setminus X$: duality interchanges deletion and contraction, as well as loops and coloops. A *cocircuit* of M is a circuit of M^* . The *circumference* $\text{circ}(M)$ of a matroid containing a circuit is its maximum circuit size; its *cocircumference* $\text{cocirc}(M)$ is its maximum cocircuit size when a cocircuit exists. These quantities are left undefined when the respective family is empty. The *components* are the equivalence classes of the relation in which e and f are related when $e = f$ or they lie in a common circuit. Singleton loops and singleton coloops are components. A nonempty matroid is *connected* when it has one component. For matroids $M_1, \dots, M_k$ on pairwise disjoint labelled ground sets, their *direct sum* $M_1 \oplus \dots \oplus M_k$ has as independent sets the sets whose intersection with each $E(M_i)$ is independent in M_i . Writing M for this direct sum, for every $X \subseteq E(M)$ we have

$$\text{rank}_M(X) = \sum_{i=1}^k \text{rank}_{M_i}(X \cap E(M_i)).$$

Every matroid is the direct sum of its component restrictions. Conversely, if a partition $E(M) = X_1 \dot{\cup} \dots \dot{\cup} X_k$ satisfies $\sum_i \text{rank}_M(X_i) = \text{rank}(M)$, then each X_i is a union of components.

The *cycle matroid* $M(G)$ of a graph G has ground set $E(G)$, with the acyclic edge sets as its independent sets. Its circuits are the cycles of G , and its cocircuits are the bonds of G . Its dual is the *bond matroid*. A matroid is *graphic* if it is isomorphic to $M(G)$ for some graph G , and *cographic* if its dual is graphic. For a plane graph, the identification of each edge with its dual edge gives $M(G)^* \cong M(G^*)$. Thus, when defined,

$$\text{circ}(M(G)) = \text{circ}(G), \quad \text{circ}(M(G)^*) = \text{cocirc}(G).$$

For a connected simple graph with at least three vertices, matroid connectedness of $M(G)$ is equivalent to 2-connectedness of G . This statement needs those qualifications: isolated vertices are invisible to a cycle matroid, and a singleton coloop is a connected matroid.

2.5 Representations and connectivity

A *representation* of a matroid M over the field $\mathbb{F}$ is an assignment f of a vector in $\mathbb{F}^d$ to each element of M such that a set $X \subseteq M$ is independent if and only if the matrix obtained by concatenating the vectors $f(x)$ over $x \in X$ has rank $|X|$. The vectors retain their element labels, so distinct elements may be assigned the same vector. Zero vectors represent loops; two distinct elements are parallel exactly when their vectors are nonzero scalar multiples of one another.

A matroid is $\mathbb{F}$ -*representable* if it admits a representation over $\mathbb{F}$, and $\mathbb{F}$ -*represented* when a representation is fixed. We may take its ambient space to be $\mathbb{F}^{\text{rank}(M)}$. Invertible row operations preserve the represented matroid. Deletion removes a labelled column. To contract a nonzero column, apply invertible row operations to make it a unit vector, then delete that column and the row containing its nonzero entry. More generally, contraction of a set is represented by projecting the remaining labelled vectors into the quotient by the span of that set.

For a set X of vectors over $\mathbb{F}$, $\langle X \rangle_{\mathbb{F}}$ denotes its *linear span*. When a representation is fixed and X is a set of matroid elements, we use $\langle X \rangle_{\mathbb{F}}$ for the span of their representing vectors. For vector subspaces X, Y , $X + Y = \{x + y : x \in X, y \in Y\}$ is their *subspace sum*; $\sum_i X_i$ denotes an iterated sum, with the empty sum equal to the zero subspace. A union of vector sets and a sum of subspaces are thus different operations, though $\langle A \cup B \rangle_{\mathbb{F}} = \langle A \rangle_{\mathbb{F}} + \langle B \rangle_{\mathbb{F}}$.

For subspaces X, A of a common vector space, the notation X/A in Chapter 4 means the image of X in the ambient quotient by A , namely $(X + A)/A \cong X/(X \cap A)$. Thus

$$\dim(X/A) = \dim X - \dim(X \cap A).$$

It does not require $A \subseteq X$, and a choice of complementary subspace is a model for the quotient, not a canonical identification.

The *connectivity function* is normalised without an additive one:

$$\lambda_M(X) = \text{rank}_M(X) + \text{rank}_M(E(M) \setminus X) - \text{rank}(M).$$

It is symmetric, submodular, and self-dual. For disjoint sets we also use the *local connectivity*

$$\lambda_M(X, Y) = \text{rank}_M(X) + \text{rank}_M(Y) - \text{rank}_M(X \cup Y).$$

For represented matroids this is $\dim(\langle X \rangle_{\mathbb{F}} \cap \langle Y \rangle_{\mathbb{F}})$. These two-argument and one-argument notations agree when the sets partition the whole ground set. Define

$$\lambda_M^*(X_1, \dots, X_k) = \max_{I \subseteq [k]} \lambda_M \left(\bigcup_{i \in I} X_i, \bigcup_{i \notin I} X_i \right)$$

for disjoint ground-set subsets. For subspaces use the same expression with rank of a union replaced by dimension of a sum. At a partition of the entire ground set this measures the largest connectivity of a union of parts. The star in λ_M^* is part of this notation and does not indicate duality. Self-duality of the one-argument function means $\lambda_{M^*}(X) = \lambda_M(X)$; the two-argument local connectivity need not be preserved by duality.

2.6 Matroid depth parameters

2.6.1 Recursive matroid depths

For the empty matroid set $\text{cd} = \text{dd} = \text{cdd} = 0$. Each of these parameters is one on a singleton, whether a loop or a coloop, and is the maximum over components on a disconnected nonempty matroid. For a connected matroid with at least two elements, define

$$\begin{aligned} \text{cd}(M) &= 1 + \min_{e \in E(M)} \text{cd}(M/e), \\ \text{dd}(M) &= 1 + \min_{e \in E(M)} \text{dd}(M \setminus e), \\ \text{cdd}(M) &= 1 + \min_{e \in E(M)} \min\{\text{cdd}(M/e), \text{cdd}(M \setminus e)\}. \end{aligned}$$

These are *contraction-depth*, *deletion-depth*, and *contraction-deletion-depth*, respectively. They satisfy $\text{cd}(M) = \text{dd}(M^*)$ and $\text{cdd}(M) = \text{cdd}(M^*)$. Contraction-depth is non-increasing under contractions, and deletion-depth is non-increasing under deletions. None of the three is minor-monotone in general [13].

2.6.2 Contraction*-depth

A *contraction*-decomposition* of M is a pair (T, f) , where T is a rooted tree with $\text{rank}(M)$ edges and f maps the ground set to its leaves, such that

$$\text{rank}_M(X) \leq |E(T\langle f(X) \rangle)| \quad \text{for every } X \subseteq E(M).$$

Its *depth* is the depth of T . The *contraction*-depth* $\text{c}^*\text{d}(M)$ is the minimum depth of such a decomposition. For a rank-zero matroid, including the empty matroid, the one-vertex tree gives value zero. This parameter is minor-monotone [23].

For a represented matroid the equivalent recursion assigns zero to the empty matroid, assigns rank to a singleton, and takes the maximum over components for a disconnected nonempty matroid. For a connected matroid with at least two elements one minimises $1 + \text{c}^*\text{d}(M/L)$ over one-dimensional subspaces L of the span. Here M/L retains all element labels and projects their vectors into the quotient. The line L need not be represented by an element of M [11]. We use the term contraction*-depth as in [8]; this parameter is called branch-depth in [23].

2.6.3 Branch-depth

For $|E(M)| \geq 2$, a *branch-depth decomposition* is a tree with at least one internal vertex and a bijection from its leaves to $E(M)$. Its *radius* is the minimum d such that there is a vertex of the underlying tree such that every vertex is at distance at most d from it. At an internal vertex v , the components of $T - v$ induce a partition $(X_1, \dots, X_k)$ of $E(M)$; the *width at v* is $\lambda_M^*(X_1, \dots, X_k)$. A (d, r) -*decomposition* has radius at most d and all these widths at most r . The *branch-depth* $\text{bd}(M)$ is the least k for which a (k, k) -decomposition exists. Set $\text{bd}(M) = 0$ for $|E(M)| \leq 1$. In the two-element case an internal vertex is retained, so an unsubdivided single edge is not a permissible decomposition. This prevents its width constraints from disappearing.

A *rooted (d, r) -decomposition* has depth at most d , labelled leaves, and width at most r at every vertex, measured using the sets below its children. The part towards the parent is omitted. The one-vertex tree is allowed for a singleton. Rooting a branch-depth decomposition at a centre gives such a decomposition, as proved in Proposition 4.3.

Branch-depth is minor-monotone and satisfies the following inequality [13]:

$$\text{bd}(M) \leq \text{cdd}(M). \quad (2.4)$$

Unlike contraction*-depth, it is self-dual. These facts follow from the connectivity formulation [13].

2.7 Matroid width and its graph counterparts

A matroid *branch-decomposition* is a subcubic tree whose leaves are bijectively labelled by the ground set of the decomposed matroid. Thus, an edge of the tree induces a partition $E(M) = X \dot{\cup} Y$, and the *width* of the edge is defined as $\lambda_M(X) + 1$. Minimising the maximum width of an edge gives the *branchwidth* $\text{bw}(M)$. For a matroid M with $|E(M)| = 1$ set $\text{bw}(M) = 1$, and for the empty matroid set $\text{bw}(M) = 0$. For an ordering $e_1, \dots, e_m$, its width is

$$\max_{0 \leq i \leq m} \lambda_M(\{e_1, \dots, e_i\}).$$

The minimum width over orderings is the matroid *pathwidth* $\text{pw}(M)$; its value on the empty matroid is zero. It uses no additive one. Consequently $\text{bw}(M) \leq \text{pw}(M) + 1$, and self-duality

of connectivity gives

$$\text{bw}(M) = \text{bw}(M^*), \quad \text{pw}(M) = \text{pw}(M^*). \quad (2.5)$$

Matroid pathwidth is the maximum of the pathwidths of its direct summands: concatenate optimal orderings for one inequality, and restrict an ordering to a summand for the other.

Proposition 2.2. *For every finite matroid M ,*

$$\text{pw}(M) \leq \text{cd}^*(M), \quad \text{bw}(M) \leq \text{cd}^*(M) + 1. \quad (2.6)$$

Proof. The empty matroid is immediate. Let (T, f) be a contraction*-decomposition of a nonempty matroid M , of depth d . Order the leaves of T in depth-first order and list the elements assigned to each leaf consecutively. For a prefix X of this ordering, put $Y = E(M) \setminus X$, and let $T_X = T\langle f(X) \rangle$ and $T_Y = T\langle f(Y) \rangle$. The rank inequality for the whole ground set implies that $T\langle f(E(M)) \rangle$ uses all $\text{rank}(M)$ edges of T . Thus the edge sets of T_X and T_Y together cover $E(T)$. Their intersection lies on a single root-to-leaf path: only the path through the boundary between the two consecutive groups of leaves can belong to both subtrees. This also holds if the ordering is cut within the elements assigned to one leaf. Consequently,

$$\lambda_M(X) \leq |E(T_X)| + |E(T_Y)| - |E(T)| = |E(T_X) \cap E(T_Y)| \leq d.$$

Minimising over (T, f) gives the pathwidth bound. The branchwidth bound follows from $\text{bw}(M) \leq \text{pw}(M) + 1$. $\square$

The graph and matroid widths satisfy the following comparisons [36, 26, 24]:

$$\text{bw}(G) - 1 \leq \text{tw}(G) \leq \lfloor \frac{3}{2} \text{bw}(G) \rfloor - 1 \quad \text{if } \text{bw}(G) \geq 2, \quad (2.7)$$

$$\text{bw}(G) = \text{bw}(M(G)) \quad \text{if } G \text{ is simple and has a cycle}, \quad (2.8)$$

$$\text{pw}(M(G)) \leq \text{pw}(G) + 1. \quad (2.9)$$

In particular, these comparisons do not identify graph pathwidth with matroid pathwidth. A tree has a free cycle matroid of pathwidth zero, even though trees can have arbitrarily large graph pathwidth.

2.8 Notation at a glance

Notation	Meaning
$V(G), E(G)$	Vertex set and edge set of a graph.
$G[X]$	Subgraph induced by $X \subseteq V(G)$.
$G - S, G - F$	Deletion of vertices S and their incident edges; deletion of edges F only.
$G + xy$	Add the edge xy if it is absent.
P_n, K_n	Path and complete graph on n vertices.
G^*, e^*	Plane dual graph and the edge dual to e , for a fixed embedding.
$\text{circ}(G), \text{cocirc}(G)$	Maximum cycle size and maximum bond size, when the respective family is nonempty.

Notation	Meaning (continued)
$\text{td}(G)$	Graph tree-depth, measured in vertices of a root-to-leaf path.
$\text{td}_{\text{DFS}}(G)$	Minimum height of a DFS spanning forest, as used in the introduction.
$\text{tw}(G), \text{pw}(G)$	Graph treewidth and pathwidth, measured as bag size minus one.
$\text{pw}(G; x), \text{pw}(G; x, y)$	Minimum path-decomposition width with the indicated vertices prescribed in the first, or first and last, bags.
$\text{cl}(F)$	Closure of a rooted forest: join each vertex to its ancestors.
$u \preceq v$	$u = v$ or u is a descendant of v .
$T[v], T(v)$	Descendant subtree rooted at v ; set of matroid elements assigned below v .
$T\langle A \rangle, T\llbracket v \rrbracket$	Union of root paths to A ; $T[v]$ together with its path to the root.
$E(M), \mathcal{I}(M), M $	Ground set, independent sets, and ground-set cardinality.
$\text{rank}_M(X), \text{rank}(M)$	Rank of a subset and rank of the whole matroid.
$\text{cl}_M(X)$	Matroid closure of X , consisting of the elements whose addition does not increase its rank.
$M X, M \setminus X, M/X$	Restriction to X , deletion of X , and contraction of X .
$M^*, M_1 \oplus M_2$	Dual matroid; direct sum on disjoint ground sets.
$M(G)$	Cycle matroid with ground set $E(G)$.
$N \sqsupseteq M$	M is a restriction of N .
$\overline{X}$	Complement in the current matroid ground set.
$\langle X \rangle_{\mathbb{F}}$	Linear span over the fixed field $\mathbb{F}$.
$X + Y, X/A$	Subspace sum; image of X in the ambient quotient by A .
$\lambda_M(X)$	Connectivity across $(X, E(M) \setminus X)$, with no additive one.
$\lambda_M(X, Y)$	Local connectivity between disjoint ground-set subsets.
$\lambda_M^*(X_1, \dots, X_k)$	Maximum local connectivity across unions of the indicated parts; the star does not denote duality.
$\text{cd}^*(M)$	Contraction*-depth, measured in edges of a rooted rank decomposition.
$\text{cd}(M), \text{dd}(M), \text{cdd}(M)$	Recursive depths with singleton value one.
$\text{bd}(M)$	Connectivity-based branch-depth; singleton value zero.
$\text{bw}(M), \text{pw}(M)$	Matroid branchwidth (cut order $\lambda + 1$) and pathwidth (cut order λ).

Chapter 3

Contraction-depth and matroid extensions

This chapter proves an extension characterisation of contraction*-depth, based on joint work with Daniel Král' and Ander Lamaison [9]. The graph-theoretic motivation is that adding edges can make an optimal elimination forest available as a depth-first search forest. We ask whether adding matroid elements can similarly realise an optimal contraction*-decomposition through ordinary element contractions.

The main difficulty is to construct such an extension for arbitrary matroids. In a represented matroid, additional elements can supply the directions used by the decomposition. Without a representation, their dependencies must be specified directly. The proof below does this through a definition of independence expressed in terms of matroid ranks.

3.1 The extension theorem

We use $\text{cd}^*(M)$ for contraction*-depth and $\text{cd}(M)$ for contraction-depth, with the definitions and conventions of Section 2.6.

Theorem 3.1. *Let M be a finite matroid. If M is empty, or if every element of M is a loop or a coloop and M has at least one coloop, then*

$$\min_{M' \supseteq M} \text{cd}(M') = \text{cd}^*(M) = \text{cd}(M) \in \{0, 1\}.$$

In all other cases,

$$\min_{M' \supseteq M} \text{cd}(M') = \text{cd}^*(M) + 1,$$

where the minimum ranges over finite matroids containing M as a restriction.

In particular, a nonempty rank-zero matroid satisfies the additive-one formula: its contraction*-depth is zero, whereas its contraction-depth is one. Section 3.6 discusses an alternative singleton normalisation that removes the exception.

The cases in which every element is a loop or a coloop are immediate. For all other matroids, there are two parts to the proof. First, we prove $\text{cd}^*(M) + 1 \leq \text{cd}(M)$. Every extension of M also lies outside the exceptional cases, so minor-monotonicity of contraction*-depth gives the required lower bound for every such extension.

For the upper bound, fix a contraction*-decomposition $\mathcal{T} = (T, f)$ and introduce one new element for each edge of T . Think of a root-to-leaf path as specifying a “subspace” in which the original elements assigned to that leaf must lie. The guiding idea is to make the

new elements behave generically subject to this path structure and the dependencies already present in M : additional dependencies should arise only when forced by the corresponding rank constraints. Thus a candidate independent set must have its original elements independent in M , and both its original and its new elements must fit within the rank available along the tree.

The definition of *tamed sets* formalises this intuition using a token distribution procedure. Ranks in contractions of M determine where the original elements contribute tokens, while each selected new element contributes one further token. Tokens move towards the root, with each non-root vertex retaining at most one and passing any excess to its parent. A set is tamed when its original elements are independent in M and no token reaches the root. The central difficulty is to verify the matroid independence axioms, particularly augmentation, and to show that the resulting matroid restricts to M .

The new elements form a basis of the resulting extension $M^\mathcal{T}$, which has the same rank as M . Contracting along a root-to-leaf path separates the corresponding branches and eventually turns the original elements assigned to that leaf into loops. Consequently,

$$\begin{aligned} |E(M^\mathcal{T})| &= |E(M)| + \text{rank}(M), & \text{rank}(M^\mathcal{T}) &= \text{rank}(M), \\ \text{cd}(M^\mathcal{T}) &\leq \text{height}(T). \end{aligned}$$

Taking a contraction*-decomposition of minimum depth proves the upper bound. Section 3.5 develops the construction and its linear-algebraic motivation in detail.

3.2 Notation for the construction

We use the matroid terminology of Section 2.4 and the rooted-tree notation of Section 2.3.1. For a fixed contraction*-decomposition (T, f) , the notation $T[v]$ denotes the subtree rooted at v , whereas $T(v)$ denotes the set of elements of M mapped to leaves of that subtree. The tree $T[v]$ consists of $T[v]$ together with the path from v to the root. An overline on a subset of $E(M)$ denotes its complement in $E(M)$. The order $u \preceq v$ means that $u = v$ or u is a descendant of v ; in particular, the root is the greatest vertex. Tree height counts vertices and tree depth counts edges on a longest root-to-leaf path. We retain both notions because the difference of one is exactly the normalisation appearing in Theorem 3.1.

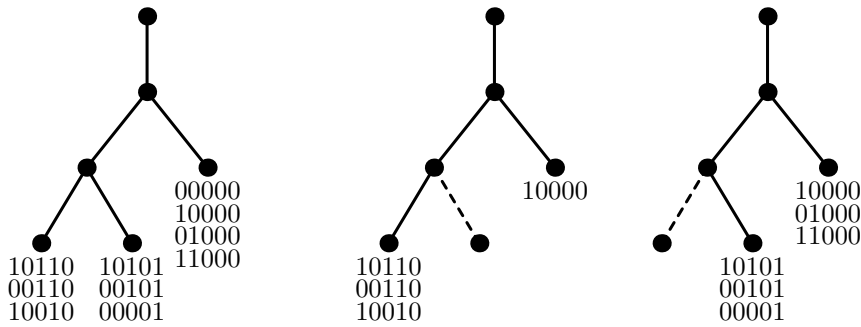


Figure 3.1: An example of a contraction*-decomposition (T, f) of a matroid M (on the left); the matroid M is a 10-element vector matroid of rank five represented over the binary field. The vectors assigned by the function f to each of the three leaves are listed as row vectors by the leaves. The subtrees $T[f(X)]$ for $X = \{10000, 10110, 00110, 10010\}$ and for $X = \{10000, 01000, 11000, 10101, 00101, 00001\}$ are depicted on the middle and on the right.

3.3 The lower bound for extensions

For a nonempty matroid M containing an element that is neither a loop nor a coloop, we prove that $\text{cd}^*(M) \leq \text{cd}(M) - 1$. The same inequality holds for a nonempty matroid consisting only of loops. To do so, we need the following lemma.

Lemma 3.2. *Let M be a matroid and X an arbitrary set of elements of M . If an element $e \in M \setminus X$ is not a coloop of M , then e is also not a coloop in M/X .*

Proof. Recall that an element e is a coloop in M if and only if e is a loop in the dual matroid M^* . Since the element e is not a loop in M^* , e is also not a loop in $M^* \setminus X$ and so in $(M/X)^*$. It follows that e is not a coloop in M/X . $\square$

We are now ready to prove the upper bound on contraction*-depth in terms of contraction-depth.

Theorem 3.3. *Let M be a nonempty matroid. Then $\text{cd}^*(M) \leq \text{cd}(M) - 1$ unless every element of M is either a loop or a coloop and M does not consist solely of loops.*

Proof. The proof proceeds by induction on the number of elements of M . We first analyse the case when $\text{rank}(M) = 0$. If $\text{rank}(M) = 0$, then the matroid M consists of loops only. It follows that $\text{cd}(M) = 1$ and $\text{cd}^*(M) = 0$ (the unique contraction*-decomposition of M is the rooted tree consisting of the root only and all loop elements of M assigned to the root).

Suppose that $\text{rank}(M) = 1$. Since the exceptional case is excluded, M consists of two or more parallel elements and some (possibly zero) number of loops. It follows that $\text{cd}(M) = 2$ (the contraction of any non-loop element results in a non-empty matroid consisting of loops only and so of contraction-depth 1). The contraction*-depth of M is equal to one and the unique contraction*-decomposition of M is the rooted tree with a single edge and all elements of M mapped to its only leaf.

Suppose that $\text{rank}(M) \geq 2$. If M is not connected, then let $M_1, \dots, M_k$ be the components of M . We may assume that the first $\ell > 0$ components do not consist of a loop or a coloop only while the remaining $k - \ell$ ones do (note that ℓ can be equal to k). Since the contraction-depth of the components $M_{\ell+1}, \dots, M_k$ is equal to one, the contraction-depth of M is the maximum of the contraction-depths of $M_1, \dots, M_\ell$. By induction, there exist contraction*-decompositions $\mathcal{T}_1, \dots, \mathcal{T}_\ell$ of the matroids $M_1, \dots, M_\ell$ respectively, such that the depth of $\mathcal{T}_i = (T_i, f_i)$ is at most $\text{cd}(M_i) - 1$ for each $i \in [\ell]$. Let T be the rooted tree obtained by identifying the roots of $T_1, \dots, T_\ell$ and adding a new leaf adjacent to the root for every component $M_{\ell+1}, \dots, M_k$ that consists of a coloop only. We define the function f from the elements of M to the leaves of T as follows. If e is an element of M_i for $i \in [\ell]$, then $f(e) = f_i(e)$. If e is a coloop, then e is mapped by f to the newly added leaf corresponding to the component consisting of e . Finally, if e is a loop, then $f(e)$ is an arbitrary leaf of T . Observe that (T, f) is a contraction*-decomposition of M . The depth of T is the maximum depth of T_i where $i \in [\ell]$, thus it is at most

$$\max_{i \in [\ell]} \text{cd}(M_i) - 1 = \text{cd}(M) - 1.$$

It follows that $\text{cd}^*(M) \leq \text{cd}(M) - 1$.

If $\text{rank}(M) \geq 2$ and the matroid M is connected, there exists an element e such that $\text{cd}(M/e) = \text{cd}(M) - 1$. By Lemma 3.2, the matroid M/e has no coloops (it may have loops however). In particular, it is possible to apply induction. Let (T, f) be a contraction*-decomposition of M/e with depth at most $\text{cd}(M/e) - 1 = \text{cd}(M) - 2$. We obtain a contraction*-decomposition of M by creating a new root v_0 and making v_0 adjacent to

the root of T ; the element e is mapped to an arbitrary leaf of T . The obtained rooted tree is a contraction*-decomposition of M since $\text{rank}_M(X) \leq \text{rank}_{M/e}(X \setminus \{e\}) + 1$ and any path from the root to any leaf contains the edge between v_0 and the root of T . Since the depth of the obtained contraction*-decomposition of M is at most $\text{cd}(M) - 1$, we conclude that $\text{cd}^*(M) \leq \text{cd}(M) - 1$. $\square$

3.4 Auxiliary results

In this section we prove several auxiliary results used in the proof of our main theorem. We start with the following lemma, which (informally) says that if the root of a contraction*-decomposition of a matroid M has degree at least two, then M is not connected and the elements of each component of M are mapped to the leaves of the same subtree of a child of the root. We refer to Figure 3.2 for illustration of the notation used in the statement of the lemma.

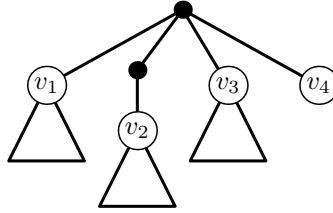


Figure 3.2: Illustration of the notation used in the statement of Lemma 3.4.

Lemma 3.4. *Let (T, f) be a contraction*-decomposition of a matroid M and let $v_1, \dots, v_k$ be the $\preceq$ -maximal descendants of the root that are branching vertices or leaves. Then each set $T(v_i)$, $i \in [k]$, is a union of components of the matroid M . In particular, the following holds for every subset X of elements of M :*

$$\text{rank}_M(X) = \sum_{i \in [k]} \text{rank}_M(X \cap T(v_i)).$$

Proof. Fix a matroid M of positive rank and a contraction*-decomposition (T, f) of M . Let $v_1, \dots, v_k$ be the vertices of T as in the statement of the lemma and let t_i be the number of edges of the subtree $T[v_i]$, $i \in [k]$. Note that the tree T has exactly $t_1 + \dots + t_k$ edges.

We will show that $\text{rank}(T(v_i)) = t_i$ for every $i \in [k]$. The definition of a contraction*-decomposition yields that $\text{rank}(T(v_i)) \leq t_i$ for every $i \in [k]$. By symmetry, we may assume for contradiction that $\text{rank}(T(v_1)) < t_1$. It follows that $\text{rank}(M \setminus T(v_1)) > t_2 + \dots + t_k$. However, the number of edges from the leaves of $T[v_2], \dots, T[v_k]$ to the root is only $t_2 + \dots + t_k$, which is impossible since (T, f) is a contraction*-decomposition of M . We conclude that $\text{rank}(T(v_i)) = t_i$ for every $i \in [k]$.

Since the rank of the matroid M is $t_1 + \dots + t_k$, the sets $T(v_1), \dots, T(v_k)$ partition the elements of M , and $\text{rank}(T(v_i)) = t_i$ for every $i \in [k]$, it must be the case that each set $T(v_i)$ is a union of components of M and so the lemma follows. $\square$

To prove the final lemma of this section, we need the following auxiliary lemma.

Lemma 3.5. *Let M be a matroid of positive rank, and let X and $Z_1 \supseteq Z_2 \supseteq \dots \supseteq Z_k$ be subsets of elements of M . Then for any element $e \notin X \cup Z_1$ there exists $i_0 \in [k + 1]$ such*

that

$$\text{rank}_{M/Z_i}((X \setminus Z_i) \cup \{e\}) = \begin{cases} \text{rank}_{M/Z_i}(X \setminus Z_i) & \text{if } i < i_0, \text{ and} \\ \text{rank}_{M/Z_i}(X \setminus Z_i) + 1 & \text{if } i \geq i_0 \end{cases}$$

for all $i \in [k]$.

Proof. Recall that the *span* of a subset A of the elements of a matroid is the inclusion-wise maximal superset of A with the same rank, and if $A \subseteq B$, then the span of A is a subset of the span of B . For $x \notin A \cup B$, observe that x is contained in the span of $A \cup B$ if and only if $\text{rank}_M(A \cup B) = \text{rank}_M(A \cup B \cup \{x\})$, which is equivalent to $\text{rank}_{M/B}(A \setminus B) = \text{rank}_{M/B}((A \setminus B) \cup \{x\})$.

If the element e is not contained in the span of $X \cup Z_1$, then e is not contained in the span of $X \cup Z_i$ for all $i \in [k]$ and so $\text{rank}_{M/Z_i}((X \setminus Z_i) \cup \{e\}) = \text{rank}_{M/Z_i}(X \setminus Z_i) + 1$ for all $i \in [k]$; in particular, the statement of the lemma holds with $i_0 = 1$. If the element e is contained in the span of $X \cup Z_1$, set $i_0 = j_0 + 1$ where $j_0 \in [k]$ is the largest integer such that e is contained in the span $X \cup Z_{j_0}$. Since it holds that $\text{rank}_{M/Z_i}((X \setminus Z_i) \cup \{e\}) = \text{rank}_{M/Z_i}(X \setminus Z_i)$ for $i \in [j_0]$ and $\text{rank}_{M/Z_i}((X \setminus Z_i) \cup \{e\}) = \text{rank}_{M/Z_i}(X \setminus Z_i) + 1$ for $i \in [k] \setminus [j_0]$, the statement of the lemma follows. $\square$

We conclude this section with the next lemma concerning ranks after contracting a set defined by a contraction*-decomposition; we refer to Figure 3.3 for illustration of the notation.

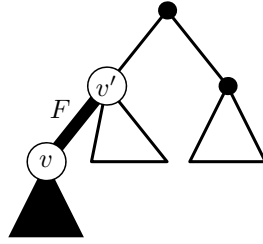


Figure 3.3: Illustration of the notation used in the statement of Lemma 3.6.

Lemma 3.6. *Let M be a matroid of positive rank and (T, f) a contraction*-decomposition of M . Let v be a leaf or an internal branching vertex and v' its $\preceq$ -minimal ancestor that is a branching vertex or the root. Let F be the union of the set of edges contained inside $T[v]$ and the set of edges on the path from v to v' . Then the size of the set F is at most $\text{rank}_{M/\overline{T(v)}}(T(v))$.*

Proof. Observe that $\text{rank}(M) = \text{rank}_M(\overline{T(v)}) + \text{rank}_{M/\overline{T(v)}}(T(v))$. Since (T, f) is a contraction*-decomposition of M , the number of edges not contained in F must be at least $\text{rank}_M(\overline{T(v)})$. Since the total number of edges of T is $\text{rank}(M)$, it follows that the size of F is at most $\text{rank}(M) - \text{rank}_M(\overline{T(v)}) = \text{rank}_{M/\overline{T(v)}}(T(v))$. $\square$

3.5 Constructing a matroid extension

This section constructs the extension needed for Theorem 3.1. The construction is carried out for positive-rank matroids; rank-zero matroids are handled directly in the final proof.

We start with an informal description of the proof idea. Our goal is to use a contraction*-decomposition $\mathcal{T} = (T, f)$ of a matroid M to construct a suitable extension of M . Suppose that v is an internal branching vertex of T . Intuitively, the path from v to the root provides rank that can accommodate dependencies between the elements assigned to different branches below v . This intuition manifests in the case of vector matroids as follows (see [11]). If $\mathcal{T} = (T, f)$ is a contraction*-decomposition of a vector matroid M , it is possible to label the edges of T with vectors in a way that any subset X of the elements of M is contained in the linear hull of the labels of the edges of $T \setminus f(X)$ (note that this implies that $\text{rank}(X)$ is at most the number of the edges of $T \setminus f(X)$). In particular, in the case of vector matroids, adding the vectors that label the edges yields the sought extension of M . In the general case, we will also introduce a new element for each edge of the tree T and our aim will be to do so in a way that the new elements behave generically subject to the rank constraints imposed by M and the paths in T .

3.5.1 Tamed sets

In this subsection, we define the notion of tamed sets. In the following subsections we show that tamed sets form a matroid which contains the given matroid as a restriction, and we then analyse the contraction-depth of the matroid formed by tamed sets. For the rest of this subsection, fix a matroid M of positive rank and a contraction*-decomposition $\mathcal{T} = (T, f)$ of M . Let X_0 be the set containing all elements of M and all edges of T , i.e., $X_0 = E(M) \dot{\cup} E(T)$; we regard the edges of T as new elements disjoint from $E(M)$. Note that X_0 has $|M| + \text{rank}(M)$ elements. The rest of the subsection is devoted to defining when a set $X \subseteq X_0$ is $\mathcal{T}$ -tamed.

Token assignment. For a set $X \subseteq X_0$, we define an assignment of tokens to the vertices of the tree T , which is next referred to as the *token assignment with respect to the set X* . Every vertex that is neither a leaf nor an internal branching vertex receives zero tokens. Let v be a vertex that is either a leaf or an internal branching vertex, and let v' be the $\preceq$ -minimal ancestor of v that is a branching vertex or the root. Observe that all internal vertices of the path from v to v' have degree two, and let P be the set of all edges on the path from v to v' in T .

- If v is a leaf, then the number of tokens assigned to the leaf v is equal to

$$|X \cap P| + \text{rank}_{M/\overline{T(v)}}(X \cap T(v)). \quad (3.1)$$

- If v is an internal branching vertex, then we proceed as follows. Let $v_1, \dots, v_k$ be all $\preceq$ -maximal descendants of v that are either leaves or branching vertices. The number of tokens assigned to v is equal to

$$|X \cap P| + \text{rank}_{M/\overline{T(v)}}(X \cap T(v)) - \sum_{i \in [k]} \text{rank}_{M/\overline{T(v_i)}}(X \cap T(v_i)). \quad (3.2)$$

This concludes the definition of the token assignment with respect to X .

Each selected new edge element contributes one token at the first branching vertex or leaf below that edge. This token can then move towards the root. In particular, the new elements on each such path P have the same starting point; their positions within that path are not distinguished by the token assignment. The rank terms account for the contribution of the selected original elements after contracting all original elements outside the relevant subtree. At a branching vertex, subtracting the contributions of the descendant branches avoids counting them again.

In the next lemma, we show that the notion of a token assignment is well-defined in the sense that the number of tokens assigned to each vertex is non-negative, i.e., the expression (3.2) is non-negative for every $X \subseteq X_0$ and every internal branching vertex v .

Lemma 3.7. *Let X be a subset of the ground set of M . Let v be an internal branching vertex of T , and let $v_1, \dots, v_k$ be all $\preceq$ -maximal descendants of v that are either branching vertices or leaves. Then*

$$\text{rank}_{M/\overline{T(v)}}(X \cap T(v)) \geq \sum_{i \in [k]} \text{rank}_{M/\overline{T(v_i)}}(X \cap T(v_i)).$$

Proof. For every $i \in [k]$, let Y_i be a subset of $X \cap T(v_i)$ of size $\text{rank}_{M/\overline{T(v_i)}}(X \cap T(v_i))$ that is independent in $M/\overline{T(v_i)}$. Set $Y = Y_1 \cup \dots \cup Y_k$. Since Y is a subset of $X \cap T(v)$, the lemma will be proved once we establish that Y is independent in $M/\overline{T(v)}$.

We prove the following claim by induction on ℓ :

For every $\ell \in [k]$, the set $Y_1 \cup \dots \cup Y_\ell$ is independent in $M/\overline{T(v)}$.

The base case of the induction holds as Y_1 is independent in $M/\overline{T(v_1)}$ and $\overline{T(v)} \subseteq \overline{T(v_1)}$, therefore Y_1 is independent in $M/\overline{T(v)}$. Suppose we have shown that $Y_1 \cup \dots \cup Y_\ell$ is independent in $M/\overline{T(v)}$ for some $\ell < k$. Let Z be an independent superset of $Y_1 \cup \dots \cup Y_\ell$ contained in $T(v) \setminus T(v_{\ell+1})$ of size $\text{rank}_{M/\overline{T(v)}}(T(v) \setminus T(v_{\ell+1}))$. As $Y_{\ell+1}$ is independent in $M/\overline{T(v_{\ell+1})} = (M/\overline{T(v)})/(T(v) \setminus T(v_{\ell+1}))$, it must be the case that $Z \cup Y_{\ell+1}$ is independent in $M/\overline{T(v)}$. In particular, $Y_1 \cup \dots \cup Y_{\ell+1}$ is independent in $M/\overline{T(v)}$, and the claim follows. $\square$

Before proceeding further, we compute the total number of tokens assigned to the vertices of T . To do so, we will need the following lemma. Recall that the matroid M and the contraction*-decomposition (T, f) of M are fixed.

Lemma 3.8. *Let X be any subset of the set $M \cup E(T)$ such that $X \cap M$ is independent, and consider the token assignment with respect to the set X . Let v be a leaf or an internal branching vertex of T , and let F be the union of the set of all edges contained in $T[v]$ and the set of edges on the path from v to the $\preceq$ -minimal ancestor of v that is a branching vertex or the root. Then the total number of tokens assigned to all vertices of the subtree $T[v]$ is equal to $|X \cap F| + \text{rank}_{M/\overline{T(v)}}(X \cap T(v))$.*

Proof. If v is a leaf, then the quantity in the statement of the lemma matches (3.1). If v is not a leaf, then let $v_1, \dots, v_k$ be the $\preceq$ -maximal descendants of v that are branching vertices or leaves. For every $i \in [k]$, let F_i be the union of the set of edges contained in $T[v_i]$ and the set of edges on the path from v_i to v . Note that the set F from the statement of the lemma is the union of the sets F_i for $i \in [k]$ and the set of edges of the path from v to its $\preceq$ -minimal ancestor that is a branching vertex or the root. It follows that $F \setminus \bigcup_{i \in [k]} F_i$ is the set of edges of the path from v to its $\preceq$ -minimal ancestor that is a branching vertex or the root. Using (3.2), we obtain that the number of tokens assigned to v itself is equal to

$$\left| X \cap \left(F \setminus \bigcup_{i \in [k]} F_i \right) \right| + \text{rank}_{M/\overline{T(v)}}(X \cap T(v)) - \sum_{i \in [k]} \text{rank}_{M/\overline{T(v_i)}}(X \cap T(v_i)).$$

By induction, it holds that the number of tokens assigned to the vertices of $T[v_i]$ is equal to

$$|X \cap F_i| + \text{rank}_{M/\overline{T(v_i)}}(X \cap T(v_i))$$

for every $i \in [k]$. Since the sets $F_1, \dots, F_k$ are disjoint, we obtain that the number of tokens assigned to the vertices of $T[v]$ is equal to

$$|X \cap F| + \text{rank}_{M/\overline{T(v)}}(X \cap T(v))$$

as desired. $\square$

We are now ready to compute the number of tokens assigned with respect to a set X .

Lemma 3.9. *Let X be any subset of the set $M \cup E(T)$ such that $X \cap M$ is independent. Then the token assignment with respect to the set X contains exactly $|X|$ tokens.*

Proof. Let $v_1, \dots, v_k$ be $\preceq$ -maximal descendants of the root that are branching vertices or leaves, and let F_i be the set of edges contained in $T[v_i]$. If $k = 1$, then Lemma 3.8 implies that the number of tokens assigned to the vertices of T (note that $T(v_1) = M$ and F_1 are all edges of T) is equal to

$$|X \cap F_1| + \text{rank}_M(X \cap T(v_1)) = |X \cap E(T)| + |X \cap M| = |X|.$$

Hence, the number of tokens assigned to the vertices of T is equal to $|X|$.

If $k \geq 2$, then Lemma 3.8 implies that the number of tokens assigned to the vertices of T is equal to

$$\sum_{i \in [k]} \left(|X \cap F_i| + \text{rank}_{M/\overline{T(v_i)}}(X \cap T(v_i)) \right). \quad (3.3)$$

Note that each $T(v_i)$ is a union of components of the matroid M by Lemma 3.4. It follows that $\text{rank}_{M/\overline{T(v_i)}}(X \cap T(v_i)) = \text{rank}_M(X \cap T(v_i))$ for every $i \in [k]$. Using that $X \cap M$ is independent in M and that the sets $F_1, \dots, F_k$ partition the edge set of T , we obtain that (3.3) is equal to

$$\begin{aligned} \sum_{i \in [k]} (|X \cap F_i| + \text{rank}_M(X \cap T(v_i))) &= \sum_{i \in [k]} |X \cap F_i| + \sum_{i \in [k]} \text{rank}_M(X \cap T(v_i)) \\ &= |X \cap (F_1 \cup \dots \cup F_k)| + \text{rank}_M(X \cap M) \\ &= |X \cap E(T)| + |X \cap M| = |X|. \end{aligned}$$

We conclude that the number of tokens assigned to the vertices of T is equal to $|X|$. $\square$

Token distribution. Once the tokens have been assigned to the leaves and internal branching vertices of T , they are distributed within the tree T in a bottom up fashion using a procedure that we now describe (also see Figure 3.4 for an example). Vertices that have already sent (some of) their tokens up will be *marked*. At the beginning no vertex is marked. As long as there is an unmarked non-root vertex v such that all of its descendants are marked we do the following. Let k be the number of tokens held by v . If $k = 0$, then v becomes marked (and sends no tokens as it has no tokens). If $k > 0$, then v sends $k - 1$ tokens to its parent and v then becomes marked. Observe that the procedure terminates when all vertices other than the root are marked.

Definition of tamed sets. A set $X \subseteq X_0$ is $\mathcal{T}$ -tamed if $X \cap M$ is independent and the procedure for the token distribution starting with the token assignment with respect to X yields an assignment such that the root has zero tokens at the end of the procedure. The set of $\mathcal{T}$ -tamed subsets will be denoted by $\mathcal{I}_M^{\mathcal{T}}$; if the contraction*-decomposition $\mathcal{T}$ is clear from the context, we will simply write $\mathcal{I}_M$.

Observe that if the matroid M is given by the independence oracle or by the rank oracle, then it can be decided in polynomial time whether a subset of $M \cup E(T)$ is $\mathcal{T}$ -tamed. Also

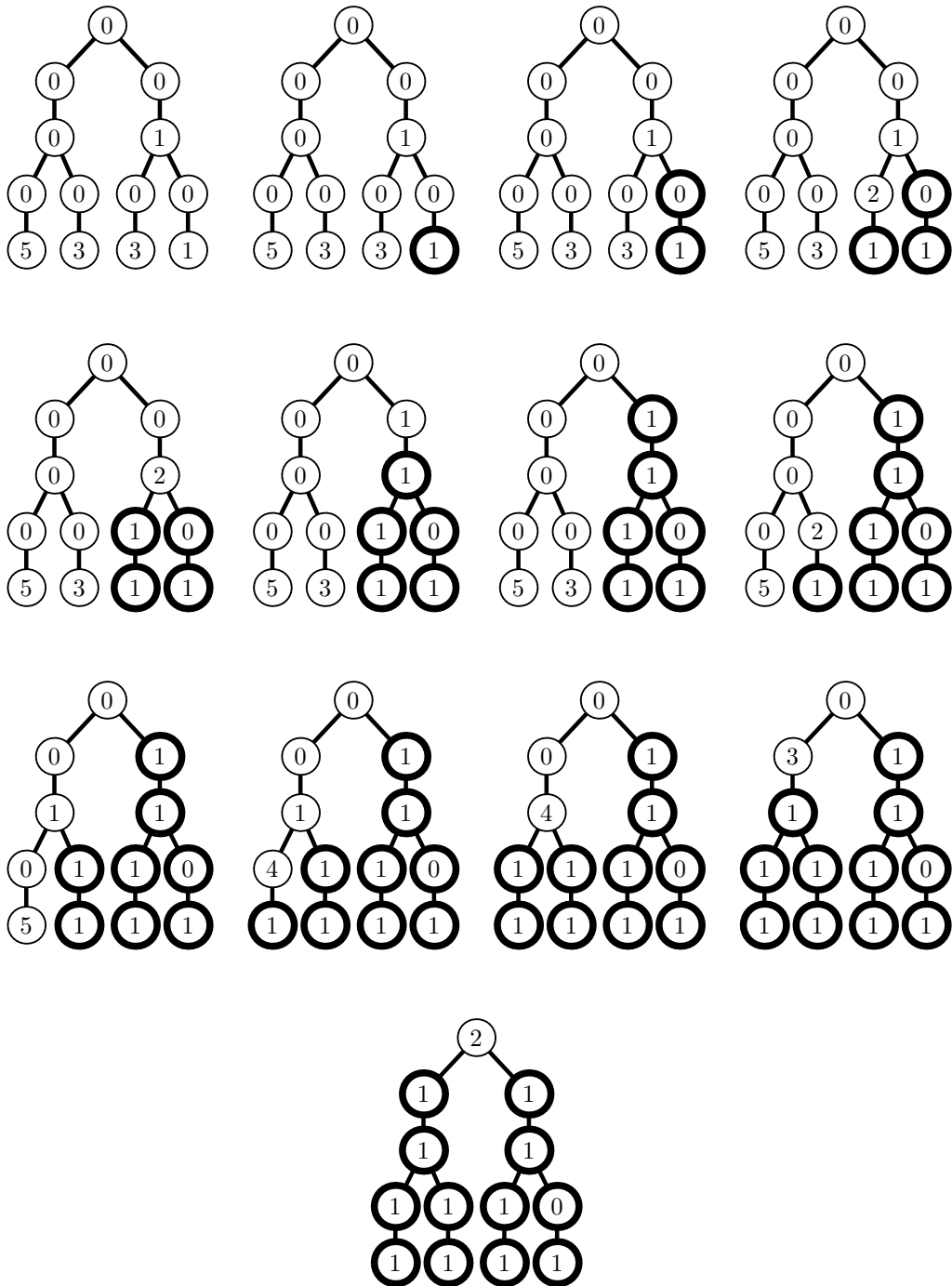


Figure 3.4: Illustration of the steps of the procedure for token distribution. Marked vertices are denoted by bold circles.

observe that the set $E(T)$, i.e., the set formed by all edges of the tree T , is always $\mathcal{T}$ -tamed. Indeed, if v is a vertex that is either a leaf or an internal branching vertex, and v' is the $\preceq$ -minimal ancestor of v that is a branching vertex or the root, the tokens assigned to v in the token assignment with respect to $E(T)$ are distributed exactly to the bottom vertices of the edges of the path from v to v' .

3.5.2 Tamed sets define a matroid

The goal of this subsection is to show that if M is a matroid of positive rank and $\mathcal{T}$ is a contraction*-decomposition of M , then the collection of all $\mathcal{T}$ -tamed sets is hereditary and satisfies the augmentation axiom, i.e., the $\mathcal{T}$ -tamed sets form a collection of independent sets of a matroid.

We start with showing that the collection of all $\mathcal{T}$ -tamed sets is hereditary.

Lemma 3.10. *Let M be a matroid of positive rank and $\mathcal{T} = (T, f)$ a contraction*-decomposition of M . Then the collection $\mathcal{I}_M^{\mathcal{T}}$ of all $\mathcal{T}$ -tamed sets is hereditary.*

Proof. Consider a non-empty set X contained in $\mathcal{I}_M^{\mathcal{T}}$ and an element $e \in X$. Our goal is to show that $X \setminus \{e\}$ is also $\mathcal{T}$ -tamed.

Suppose that e is an edge of the tree T and let u be the bottom vertex of e . If u is a leaf or a branching vertex, then we set $v = u$. Otherwise we define v to be the $\preceq$ -maximal descendant of u that is either a leaf or a branching vertex (note that the vertex v is uniquely defined). Observe that every vertex of T except for v is assigned the same number of tokens with respect to the set X and with respect to the set $X \setminus \{e\}$, and the vertex v is assigned exactly one token less. It follows that if the root of T gets no tokens during the procedure for the token distribution starting with the token assignment with respect to X , then the root of T gets no tokens during the procedure for the token distribution starting with the token assignment with respect to $X \setminus \{e\}$. In particular, the set $X \setminus \{e\}$ is $\mathcal{T}$ -tamed.

It remains to deal with the case when e is an element of the matroid M . Let v_0 be the leaf such that $f(e) = v_0$, and let $v_1, \dots, v_k$ be all internal branching vertices on the path from v_0 to the root listed in the ascending order by $\preceq$. We apply Lemma 3.5 with the matroid M , the set $(X \setminus \{e\}) \cap M$ and $\overline{T(v_0)} \supseteq \overline{T(v_1)} \supseteq \dots \supseteq \overline{T(v_k)}$. The lemma yields that there exists $i_0 \in \{0\} \cup [k+1]$ such that

$$\text{rank}_{M/\overline{T(v_i)}}(X \cap T(v_i)) = \begin{cases} \text{rank}_{M/\overline{T(v_i)}}((X \setminus \{e\}) \cap T(v_i)) & \text{if } i < i_0, \text{ and} \\ \text{rank}_{M/\overline{T(v_i)}}((X \setminus \{e\}) \cap T(v_i)) + 1 & \text{if } i \geq i_0. \end{cases}$$

Note that $i_0 \leq k$, as otherwise we would have $i_0 = k+1$, which implies that the number of tokens assigned with respect to the set $X \setminus \{e\}$ and with respect to the set X would be the same, which is impossible by Lemma 3.9. It follows that the vertex v_{i_0} is assigned one token less with respect to the set $X \setminus \{e\}$ compared to the number of tokens assigned with respect to the set X ; all other vertices are assigned the same number of tokens. We conclude that if the root of T gets no tokens during the distribution phase with respect to the set X , then the root of T gets no tokens during the distribution phase with respect to the set $X \setminus \{e\}$ and so the set $X \setminus \{e\}$ is $\mathcal{T}$ -tamed. $\square$

We are now ready to prove that the collection of $\mathcal{T}$ -tamed sets is a collection of independent sets of a matroid.

Theorem 3.11. *Let M be a matroid of positive rank and $\mathcal{T} = (T, f)$ a contraction*-decomposition of M . Then the collection $\mathcal{I}_M^{\mathcal{T}}$ is non-empty, hereditary and satisfies the augmentation axiom.*

Proof. The collection $\mathcal{I}_M^\mathcal{T}$ is non-empty. Indeed, there are no tokens assigned with respect to the empty set, and so the empty set is $\mathcal{T}$ -tamed. By Lemma 3.10, the collection $\mathcal{I}_M^\mathcal{T}$ is hereditary. Thus, we need to prove that the collection $\mathcal{I}_M^\mathcal{T}$ satisfies the augmentation axiom. Consider $\mathcal{T}$ -tamed sets X and X' such that $|X| < |X'|$. We will establish that there exists an element $e \in X' \setminus X$ such that $X \cup \{e\}$ is $\mathcal{T}$ -tamed.

For a vertex v of the tree T that is either a leaf or an internal branching vertex, we define $d(v)$ to be the number of tokens kept by vertices on the path from v (including v) to the $\preceq$ -minimal ancestor of the vertex v that is a branching vertex or the root (excluding this ancestor) during the procedure for the token distribution starting with the token assignment with respect to X . Analogously, we define $d'(v)$ for the token assignment with respect to X' . Note that the sum of all $d(v)$ (the sum is taken over all leaves and internal branching vertices) is the total number of tokens, which, by Lemma 3.9, equals $|X|$. Similarly, the sum of all $d'(v)$ is $|X'|$. Let v be a $\preceq$ -minimal vertex such that $d(v) < d'(v)$, and let P be the set of edges of the path from v to the $\preceq$ -minimal ancestor of v that is a branching vertex or the root; see Figure 3.5 for illustration of the notation. Since $d'(v)$ cannot exceed $|P|$, $d(v)$ is strictly smaller than $|P|$ and so at least one of the bottom vertices of the edges in P kept no tokens during the procedure for the token distribution starting with the token assignment with respect to X .

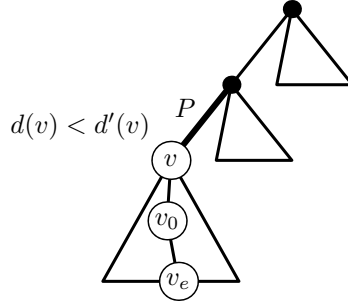


Figure 3.5: Illustration of the notation used in the statement of Theorem 3.11.

If the set X' contains an edge $e \in E(T[v]) \cup P$ that is not contained in X , then the set $X \cup \{e\}$ is $\mathcal{T}$ -tamed and so the sought element has been identified. Indeed, the number of tokens assigned to the vertices increases by one exactly for one vertex of $T[v]$, and the additional token is either kept by a vertex of $T[v]$ or by one of the bottom vertices of the edges in P .

If there is no such edge, i.e., every edge of $E(T[v]) \cup P$ contained in X' is also contained in X , Lemma 3.8 yields that

$$\text{rank}_{M/\overline{T(v)}}(X \cap T(v)) < \text{rank}_{M/\overline{T(v)}}(X' \cap T(v)). \quad (3.4)$$

Let e be any element of $X' \cap \overline{T(v)}$ such that $(X \cap T(v)) \cup \{e\}$ has larger rank than the set $X \cap T(v)$ in the matroid $M/\overline{T(v)}$; the element e exists by (3.4). In the rest of the proof, we will show that the set $X \cup \{e\}$ is $\mathcal{T}$ -tamed.

The choice of e and the contraction rank formula imply that e is not in the span in M of $(X \cap T(v)) \cup \overline{T(v)}$. Since $X \cap M$ is a subset of this set, monotonicity of span implies that e is not in the span of $X \cap M$. The set $X \cap M$ is independent because X is $\mathcal{T}$ -tamed, and therefore $(X \cap M) \cup \{e\}$ is independent in M . We next analyse the assignment of tokens and their distribution. Let v_e be the leaf of T such that $f(e) = v_e$. Note that the number of tokens assigned to the vertices that are not on the path from v_e to the root has not changed as neither the formula (3.1) nor the formula (3.2) is affected for such vertices by

adding e to the set X . Let v_0 be the first vertex on the path from v_e to the root such that $\text{rank}_{M/\overline{T(v_0)}}((X \cap T(v_0)) \cup \{e\}) = \text{rank}_{M/\overline{T(v_0)}}(X \cap T(v_0)) + 1$. The choice of e implies that the vertex v_0 exists and is on the path from v_e to v (inclusive of the end vertices of the path). The choice of v_0 yields that

$$\text{rank}_{M/\overline{T(v')}}((X \cap T(v')) \cup \{e\}) = \text{rank}_{M/\overline{T(v')}}(X \cap T(v'))$$

for any vertex v' on the path from v_e to v_0 (inclusive of v_e and exclusive of v_0). Lemma 3.5 now yields that

$$\text{rank}_{M/\overline{T(v')}}((X \cap T(v')) \cup \{e\}) = \text{rank}_{M/\overline{T(v')}}(X \cap T(v')) + 1$$

for any vertex v' on the path from v_0 to the root (inclusive of v_0 and exclusive of the root).

Lemma 3.8 yields that if v' is a leaf or an internal branching vertex, then the number of tokens assigned to the vertices of the subtree $T[v']$ with respect to the set $X \cup \{e\}$ is one larger than the number of tokens assigned with respect to the set X if and only if v' lies on the path from v_0 to the root (inclusive); otherwise, the numbers of tokens assigned are the same. Hence, all vertices except for v_0 are assigned the same number of tokens with respect to X and with respect to $X \cup \{e\}$, and the vertex v_0 is assigned one additional token. During the procedure for the token distribution starting with the token assignment with respect to $X \cup \{e\}$, this additional token is either kept by a vertex on the path from v_0 to v or by one of the bottom vertices of the edges in P . Hence, the set $X \cup \{e\}$ is $\mathcal{T}$ -tamed as desired. This completes the proof that the collection of $\mathcal{T}$ -tamed sets satisfies the augmentation axiom. $\square$

In the rest of this construction, if M is a matroid of positive rank and $\mathcal{T}$ is a contraction*-decomposition of M , then $M^\mathcal{T}$ denotes the matroid with the ground set $M \cup E(T)$ and independent sets being precisely $\mathcal{T}$ -tamed subsets of $M \cup E(T)$. Note that $M^\mathcal{T}$ is indeed a matroid by Theorem 3.11. Since for any set X of size larger than $|E(T)|$, the procedure for the token distribution starting with the token assignment with respect to X results in assigning at least one token to the root of T , every $\mathcal{T}$ -tamed set has size at most $\text{rank}(M) = |E(T)|$. Recall that the set $E(T)$ is $\mathcal{T}$ -tamed and so it is a basis of $M^\mathcal{T}$.

We now show that the matroid M is a restriction of the matroid $M^\mathcal{T}$.

Lemma 3.12. *Let M be a matroid of positive rank and $\mathcal{T}$ a contraction*-decomposition of M . Then the restriction of the matroid $M^\mathcal{T}$ to the elements of M is the matroid M itself.*

Proof. Since every $\mathcal{T}$ -tamed subset of the elements of M is independent in M , it is enough to show that every basis of M is an independent set in $M^\mathcal{T}$. Let B be a basis of M and consider the token assignment with respect to B . We show that the following holds for every leaf or internal branching vertex v :

$$\text{The number of tokens assigned to the vertices of } T[v] \text{ is equal to } \text{rank}_{M/\overline{T(v)}}(T(v)).$$

Consider a leaf or an internal branching vertex v . Observe that

$$\begin{aligned} \text{rank}_{M/\overline{T(v)}}(T(v)) + \text{rank}\left(\overline{T(v)}\right) &= \text{rank}(M) = \text{rank}(B) \\ &\leq \text{rank}_{M/\overline{T(v)}}(B \cap T(v)) + \text{rank}\left(\overline{T(v)}\right). \end{aligned}$$

where ranks are calculated in M unless otherwise specified. It follows that $\text{rank}_{M/\overline{T(v)}}(B \cap T(v))$ is at least $\text{rank}_{M/\overline{T(v)}}(T(v))$ and so $\text{rank}_{M/\overline{T(v)}}(B \cap T(v))$ is equal to $\text{rank}_{M/\overline{T(v)}}(T(v))$.

The statement above now follows from Lemma 3.8, which asserts that the number of tokens assigned to the vertices of $T[v]$ is $\text{rank}_{M/\overline{T(v)}}(B \cap T(v))$.

We next show that every vertex of the tree T except for the root has kept one token during the token distribution phase. Let v be a leaf or an internal branching vertex and v' the $\preceq$ -minimal ancestor of v that is a branching vertex or the root. Lemma 3.6 yields that the path from v to v' has at most $\text{rank}_{M/\overline{T(v)}}(T(v)) - |E(T[v])|$ edges. Since the vertices of $T[v]$ are assigned exactly $\text{rank}_{M/\overline{T(v)}}(T(v))$ tokens, the number of tokens of v before it is processed is at least $\text{rank}_{M/\overline{T(v)}}(T(v)) - |E(T[v])|$. It follows that every vertex on the path from v to v' (inclusive of v but exclusive of v') keeps one token. Since this holds for every choice of the vertex v , we conclude that every vertex of the tree T except for the root has kept one token during the token distribution phase. However, the number of such vertices is $\text{rank}(M) = |B|$ and the total number of tokens assigned is also $|B|$ by Lemma 3.9. Hence, the root of the tree T has zero tokens at the end of the token distribution phase and so the basis B is $\mathcal{T}$ -tamed. $\square$

3.5.3 Contraction-depth

We now bound the contraction-depth of $M^\mathcal{T}$ by the height of T . The following three lemmata describe how contractions of the new elements separate the branches of T .

Lemma 3.13. *Let M be a matroid of positive rank, $\mathcal{T} = (T, f)$ a contraction*-decomposition of M , and v a leaf of T . Let P be the set of edges on the path from v to the root of T . Then every element of $T(v)$ is a loop in the matroid $M^\mathcal{T}/P$.*

Proof. Recall that the set $E(T)$ is a basis of $M^\mathcal{T}$. Since the set P is independent in $M^\mathcal{T}$, it is enough to show that for every $e \in T(v)$, the set $P \cup \{e\}$ is not independent in the matroid $M^\mathcal{T}$. Suppose that $P \cup \{e\}$ is independent for some $e \in T(v)$; note that e is not a loop in M (otherwise, it would be a loop in $M^\mathcal{T}$, too). Observe that with respect to the set $P \cup \{e\}$, the tokens are assigned only to the vertices on the path P . By Lemma 3.9, there are $|P| + 1$ tokens assigned, however, the number of non-root vertices on the path is only $|P|$. Hence, the set $P \cup \{e\}$ is not $\mathcal{T}$ -tamed and so it is not independent in $M^\mathcal{T}$. $\square$

Lemma 3.14. *Let M be a matroid of positive rank and $\mathcal{T} = (T, f)$ a contraction*-decomposition of M . Suppose that the root of T is a branching vertex. Let $v_1, \dots, v_k$ be all $\preceq$ -maximal descendants of the root that are branching vertices or leaves, and let F_i be the set of edges contained in $T[v_i]$, $i \in [k]$. Then for each $i \in [k]$ the set $T(v_i) \cup F_i$ is a union of components of $M^\mathcal{T}$.*

Proof. Recall that $\text{rank}(M^\mathcal{T}) = |E(T)| = \text{rank}(M)$. We show that $\text{rank}_{M^\mathcal{T}}(T(v_i) \cup F_i) = |F_i|$ for every $i \in [k]$. As $F_i \subseteq E(T)$ is independent, we get $\text{rank}_{M^\mathcal{T}}(T(v_i) \cup F_i) \geq |F_i|$. For any subset $X \subseteq T(v_i) \cup F_i$, all tokens are assigned to vertices in $T[v_i]$ and so the procedure for the token distribution starting with the token assignment with respect to X assigns the token to the vertices of the tree $T[v_i]$ only. In particular, if $X \subseteq T(v_i) \cup F_i$ is independent in $M^\mathcal{T}$, then the size of X is at most $|F_i|$.

Observe that the sets $F_1, \dots, F_k$ partition the edge set of the tree T and so it holds that $|F_1| + \dots + |F_k| = |E(T)| = \text{rank}(M^\mathcal{T})$. We conclude that since the sets $T(v_i) \cup F_i$, $i \in [k]$, partition the ground set of $M^\mathcal{T}$ and the sum of their ranks is the rank of the matroid $M^\mathcal{T}$, each set $T(v_i) \cup F_i$ is a union of components of $M^\mathcal{T}$. $\square$

We refer to Figure 3.6 for illustration of the notation used in the next lemma.

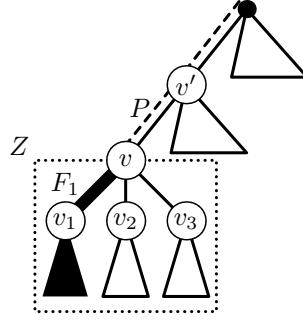


Figure 3.6: Illustration of the notation used in the proof of Lemma 3.15.

Lemma 3.15. *Let M be a matroid of positive rank and $\mathcal{T} = (T, f)$ a contraction*-decomposition of M . Let v be a branching vertex or the root of T , and P the set of edges of T contained on the path from v to the root. Moreover, let $v_1, \dots, v_k$ be the $\preceq$ -maximal descendants of v that are branching vertices or leaves, and let F_i be the union of the set of edges contained in $T[v_i]$ and the set of edges on the path from v_i to v for every $i \in [k]$. Then for each $i \in [k]$ the set $T(v_i) \cup F_i$ is a union of components of $M^\mathcal{T}/P$.*

Proof. The proof proceeds by induction on the distance of the vertex v from the root. Suppose that we are in the setting of the lemma and the statement holds for all possible choices of v larger in $\preceq$. Note that $T(v) = T(v_1) \cup \dots \cup T(v_k)$ and $E(T[v]) = F_1 \cup \dots \cup F_k$. Further, let Z be the union $T(v) \cup E(T[v])$.

We first argue that the following claim holds: *the set Z is a union of components of the matroid $M^\mathcal{T}/P$.* Let v' be the $\preceq$ -minimal ancestor of v that is a branching vertex; if v has no ancestor that is a branching vertex, then v' is the root. In particular, if v itself is the root, then v' is the root (and this is the only case when $v = v'$). We next distinguish three cases based on whether v' is the root or not and whether v' is a branching vertex or not.

- If v' is the root and it is not a branching vertex, then the set Z is the set of all elements of $M^\mathcal{T}$ except for those contained in P (note that P can be the empty set). Hence, the set Z contains all elements of the matroid $M^\mathcal{T}/P$ and the claim holds.
- If v' is the root and it is a branching vertex, then $Z \cup P$ is a union of components of the matroid $M^\mathcal{T}$ by Lemma 3.14. Hence, the set Z is a union of components of $M^\mathcal{T}/P$.
- If v' is a branching vertex but not the root, we can apply induction to v' as $v' \neq v$. Let P' be the set of edges on the path from v' to the root. The induction yields that $Z \cup (P \setminus P')$ is a union of components of the matroid $M^\mathcal{T}/P'$. Hence, Z is a union of components of the matroid $M^\mathcal{T}/P$.

The claim is now proven.

Let F_0 be the set of all edges of T not contained in $T[v]$. Note that P is a subset of F_0 , and if v is the root, then $P = F_0 = \emptyset$. Since Z is a union of components of the matroid $M^\mathcal{T}/P$ and so contracting elements not contained in Z does not affect the restriction of $M^\mathcal{T}/P$ to Z , the restriction of the matroid $M^\mathcal{T}/P$ to Z and the restriction of the matroid $M^\mathcal{T}/F_0$ to Z are the same. Hence, it is enough to prove that each $T(v_i) \cup F_i$, $i \in [k]$, is a union of components of $M^\mathcal{T}/F_0$.

Consider $i \in [k]$. Since the set F_i is independent in $M^\mathcal{T}/F_0$ (as the set $F_0 \cup F_i$ is independent in $M^\mathcal{T}$), the rank of $T(v_i) \cup F_i$ in the matroid $M^\mathcal{T}/F_0$ is at least $|F_i|$. On the

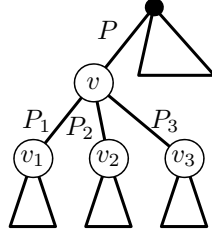


Figure 3.7: Illustration of the notation used in the proof of Theorem 3.16.

other hand, if X is a subset of $T(v_i) \cup F_i$, the token assignment with respect to $X \cup F_0$ assigns no tokens to a vertex of any tree $T[v_j]$, $j \in [k] \setminus \{i\}$. Hence, if $X \cup F_0$ is independent in $M^\mathcal{T}$, i.e., $\mathcal{T}$ -tamed, its size is at most $|F_i| + |F_0|$. It follows that the size of any independent subset of $T(v_i) \cup F_i$ in $M^\mathcal{T}/F_0$ is at most $|F_i|$. We conclude that

$$\text{rank}_{M^\mathcal{T}/F_0}(T(v_i) \cup F_i) = |F_i|$$

for every $i \in [k]$. Observe that any element of $\overline{T(v)}$ is a loop in the matroid $M^\mathcal{T}/F_0$ by Lemma 3.13. Since the sets $T(v_1) \cup F_1, \dots, T(v_k) \cup F_k$ partition the set $T(v) \cup E(T[v])$, which contains all non-loop elements of the matroid $M^\mathcal{T}/F_0$, and the sum over $i \in [k]$ of the ranks of the sets $T(v_i) \cup F_i$ is equal to the rank of $M^\mathcal{T}/F_0$, each set $T(v_i) \cup F_i$ is a union of components of $M^\mathcal{T}/F_0$. $\square$

We are now ready to prove that the contraction-depth of the matroid $M^\mathcal{T}$ is at most the height of the contraction*-decomposition $\mathcal{T}$ of a matroid M .

Theorem 3.16. *Let M be a matroid of positive rank and $\mathcal{T} = (T, f)$ a contraction*-decomposition of M . Then the contraction-depth of the matroid $M^\mathcal{T}$ is at most the height of the tree T .*

Proof. We prove the following claim: *Let v be a leaf, a branching vertex or the root of T , and let P be the set of edges on the path from v to the root. The contraction-depth of the matroid $M^\mathcal{T}/P$ restricted to $T(v) \cup E(T[v])$ is at most the height of the tree $T[v]$.* Once proven, the claim implies the statement of the theorem by choosing the vertex v to be the root of T .

We prove the claim using bottom-up induction on the tree T starting with the leaves. Let v be a leaf and let P be the set of edges on the path from v to the root. By Lemma 3.13, every element of $T(v)$ is a loop in $M^\mathcal{T}/P$. Hence, the contraction-depth of the matroid $M^\mathcal{T}/P$ restricted to $T(v)$ is one, i.e., the height of $T[v]$.

Let v be a branching vertex of the tree T or the root of T . Let P be the set of edges on the path from v to the root. Moreover, let $v_1, \dots, v_k$ be the $\preceq$ -maximal descendants of v that are branching vertices or leaves, and let P_i be the set of edges on the path from v_i to v for each $i \in [k]$. By Lemma 3.15, each set $T(v_i) \cup E(T[v_i]) \cup P_i$ is a union of components in $M^\mathcal{T}/P$. Hence, the contraction-depth of $M^\mathcal{T}/P$ restricted to $T(v) \cup E(T[v])$ is at most the maximum over $i \in [k]$ of the contraction-depth of $M^\mathcal{T}/P$ restricted to $T(v_i) \cup E(T[v_i]) \cup P_i$. Consider $i \in [k]$. The contraction-depth of $M^\mathcal{T}/P$ restricted to $T(v_i) \cup E(T[v_i]) \cup P_i$ is at most the contraction-depth of $M^\mathcal{T}/(P \cup P_i)$ restricted to $T(v_i) \cup E(T[v_i])$ increased by $|P_i|$ (the elements of P_i can be those that are first contracted in the definition of the contraction-depth). By induction, the contraction-depth of $M^\mathcal{T}/(P \cup P_i)$ restricted to $T(v_i) \cup E(T[v_i])$ is at most the height of $T[v_i]$ and so the contraction-depth of $M^\mathcal{T}/P$ restricted to $T(v_i) \cup E(T[v_i]) \cup P_i$ is at most the height of $T[v_i]$ increased by $|P_i|$, which

is at most the height of $T[v]$. It follows that the contraction-depth of $M^\mathcal{T}/P$ restricted to $T(v) \cup E(T[v])$ is at most the height of $T[v]$ as desired. $\square$

We are now ready to prove the extension theorem.

Proof of Theorem 3.1. Suppose that the matroid M consists of loops and coloops only. If M is empty, then $\text{cd}^*(M) = \text{cd}(M) = 0$. If M has at least one coloop, then the contraction*-depth $\text{cd}^*(M)$ of M is equal to 1, and if M has loops only, then the contraction*-depth $\text{cd}^*(M)$ of M is equal to 0. Note that if M is not empty, then the contraction-depth of any matroid, in particular, any matroid containing M as a restriction, is at least one, and the contraction-depth of the matroid M itself is equal to 1. Hence, the additive-one identity holds if M is nonempty and consists of loops only; the exceptional identity holds if M has at least one coloop or is empty. This establishes the theorem when M has coloops and loops only.

In the rest of the proof, we assume that the matroid M contains an element that is neither a loop nor a coloop. Consider a matroid M' that contains M as a restriction. Observe that M' does not contain coloops and loops only. Hence, Theorem 3.3 yields that the contraction-depth of such a matroid M' is at least $\text{cd}^*(M') + 1$. Since the contraction*-depth is minor-monotone [23], we obtain that $\text{cd}^*(M') \geq \text{cd}^*(M)$. Thus the contraction-depth of the matroid M' is at least $\text{cd}^*(M) + 1$. We have proven that $\text{cd}^*(M) \leq \text{cd}(M') - 1$ for any $M' \supseteq M$.

Let $\mathcal{T}$ be a contraction*-decomposition of M with depth $\text{cd}^*(M)$. The matroid $M^\mathcal{T}$, which contains M as a restriction by Lemma 3.12, has contraction-depth at most $\text{cd}^*(M) + 1$ by Theorem 3.16, i.e. $\text{cd}(M^\mathcal{T}) \leq \text{cd}^*(M) + 1$. Hence, the matroid $M^\mathcal{T}$ is a matroid $M' \supseteq M$ such that $\text{cd}(M') - 1 \leq \text{cd}^*(M)$ (in fact, the equality holds since $\text{cd}^*(M) \leq \text{cd}(M') - 1$ for any such matroid M'). The statement of the theorem now follows. $\square$

3.6 Normalisation and perspective

It is interesting to note that nonempty matroids formed by loops and coloops and containing at least one coloop, which form an exception to the additive-one identity in Theorem 3.1, also appear as exceptional cases in regard to other matroid width parameters. Most prominently, the branchwidth of a simple graph G containing a cycle equals the branchwidth of its cycle matroid, while a forest can be exceptional [20, 26], in which case the cycle matroid of G consists of coloops only. We remark that it is possible to modify the definition of contraction-depth in a way that the exceptional case in Theorem 3.1 disappears. To modify Theorem 3.1, consider the *altered contraction-depth* of a matroid M , denoted by $\text{cd}'(M)$, which is defined recursively as follows:

- If M is empty, then $\text{cd}'(M) = 0$.
- If M has one element, then $\text{cd}'(M) = \text{rank}(M)$.
- If M is not connected, then $\text{cd}'(M)$ is the maximum altered contraction-depth of a component of M .
- Otherwise $\text{cd}'(M) = 1 + \min_{e \in M} \text{cd}'(M/e)$, i.e., $\text{cd}'(M)$ is one plus the minimum altered contraction-depth of M/e where the minimum is taken over all elements e of M .

Following the proof of Theorem 3.1, using altered contraction-depth instead of contraction-depth whenever it appears, one can derive the next theorem.

Theorem 3.17. *Let M be a matroid. Then the contraction*-depth of M is equal to the minimum altered contraction-depth of a matroid M' that contains M as a restriction, i.e.,*

$$\text{cd}^*(M) = \min_{M' \supseteq M} \text{cd}'(M').$$

From the broader point of view of matroid theory, it is desirable that contraction-depth and deletion-depth are dual notions. When one attempts to define a notion dual to altered contraction-depth, which we dub here altered deletion-depth and denote by $\text{dd}'(M)$, it would necessarily be defined as follows:

- If M is empty, then $\text{dd}'(M) = 0$.
- If M consists of a single loop, then $\text{dd}'(M) = 1$.
- If M consists of a single coloop, then $\text{dd}'(M) = 0$.
- If M is not connected, then $\text{dd}'(M)$ is the maximum altered deletion-depth of a component of M .
- Otherwise $\text{dd}'(M) = 1 + \min_{e \in M} \text{dd}'(M \setminus e)$, i.e., $\text{dd}'(M)$ is one plus the minimum altered deletion-depth of $M \setminus e$ where the minimum is taken over all elements e of M .

However, it seems very unnatural that the altered deletion-depth of a matroid consisting of a single loop should be larger than that of a matroid consisting of a single coloop.

The extension theorem gives an exact interpretation of contraction*-depth as the best contraction-depth attainable after adding elements, subject to the explicit base-case convention. This is one instance of the thesis's broader theme: an operation-based recursive parameter becomes a structural parameter when the ambient object is allowed to change. The next chapter considers both deletion and contraction and replaces restriction containment by minor containment. The distinction between these two forms of containment is essential, as is the different meaning of branch-depth in that setting.

Chapter 4

Branch-depth and minor containers

4.1 Minor containers and the quantitative bound

The extension theorem of Chapter 3 permits additional elements that make a contraction strategy available. This chapter permits both deletion and contraction, and asks for containment as a minor. The depth of a recursive procedure can then be compared with the radius and connectivity of a decomposition tree.

We prove the following results from joint work with Daniel Král' and Kristýna Pekárková [10].

Theorem 4.1 (Minor-closure characterisation [10]; cf. [17] and discussion in Section 1.3). *Fix a field $\mathbb{F}$. A class $\mathcal{M}$ of $\mathbb{F}$ -representable matroids has bounded branch-depth if and only if there is a class $\mathcal{N}$ of $\mathbb{F}$ -representable matroids with bounded contraction-deletion-depth such that every member of $\mathcal{M}$ is a minor of a member of $\mathcal{N}$.*

The qualitative characterisation follows from the next bound, together with minor-monotonicity of branch-depth and the inequality $\text{bd} \leq \text{cdd}$.

Theorem 4.2 (Quantitative minor-container bound). *Every $\mathbb{F}$ -representable matroid M is a minor of an $\mathbb{F}$ -representable matroid N such that*

$$\text{cdd}(N) \leq 2 \text{bd}(M)(4^{\text{bd}(M)} - 1) + 1.$$

The containing matroid is representable over the same field as M . The field may be infinite, and the bound is independent of $\mathbb{F}$.

A family of graphic matroids shows that bounded branch-depth does not bound contraction-deletion-depth. Replace each edge of an n -cycle, for $n \geq 3$, with n parallel edges. Its cycle matroid has branch-depth two and contraction-deletion-depth n . This family is discussed in [10, 13]. For the upper bound, contract $n - 2$ successive cycle edges, leaving loops and a rank-one parallel class of depth two. For the lower bound, observe inductively that a cycle of length at least k , with every edge having multiplicity at least k , has contraction-deletion-depth at least k for $k \geq 2$. The base case is a connected nonsingleton matroid (a cycle of length two is a parallel class). When $k \geq 3$, deletion of any element or contraction of any element leaves a component of the same form for $k - 1$; loops produced by contraction form separate components. The depth recursion gives the claim.

The same example illustrates why passing to a larger matroid helps. The thickened-cycle matroid is obtained by contracting the single edge in the cycle matroid of an $(n + 1)$ -cycle in

which one edge is single and all the others have multiplicity n . The latter has contraction-deletion-depth at most three: delete the single edge to obtain a direct sum of rank-one parallel classes, and contract one element in each class. Thus one extra element can expose a short recursive decomposition.

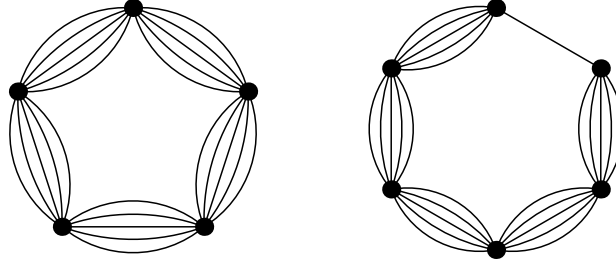


Figure 4.1: A thickened cycle and a graph whose cycle matroid contains the first cycle matroid as a minor. The added single edge exposes a shallow deletion–contraction strategy.

The proof of Theorem 4.2 uses Lemma 4.4 to control the interaction between the subspaces associated with the branches of a decomposition. This allows us to separate those branches using a bounded number of auxiliary elements.

4.2 Rooting a branch-depth decomposition

The common preliminaries define branch-depth decompositions and their rooted versions. Rooting removes the parent side from the local partition, which can only decrease the relevant local connectivity. The next proposition makes this step explicit.

Proposition 4.3. *If a matroid M has a (d, r) -decomposition, then the matroid M has a rooted (d, r) -decomposition.*

Proof. If M has a single element, the one-vertex rooted tree suffices. Otherwise, fix a (d, r) -decomposition T of M and choose a centre w of T . The tree has at least one internal vertex, so w is internal, and every vertex of T is at distance at most d from w . Let T' be the rooted tree obtained from T by rooting it at w . The depth of T' is at most d , and rooting at an internal vertex preserves the labelled leaves. Consider now an internal vertex v of T' and let k be the number of children of v . Further, let $X_1, \dots, X_k$ be the sets of elements of M assigned to the leaves of the k subtrees rooted at the children of v , and let Y be the set of elements of M not assigned to a leaf of the subtree rooted at v , namely, the elements mapped to a leaf not in any of the k subtrees. Since T is a (d, r) -decomposition and it holds that

$$\lambda_M \left(\bigcup_{i \in I} X_i, \bigcup_{i \in [k] \setminus I} X_i \right) \leq \lambda_M \left(\bigcup_{i \in I} X_i, Y \cup \bigcup_{i \in [k] \setminus I} X_i \right)$$

for any subset $I \subseteq [k]$, where the latter is at most r , we obtain that $\lambda_M^*(X_1, \dots, X_k) \leq r$. It follows that T' is a rooted (d, r) -decomposition of the matroid M . $\square$

4.3 Main lemma

For the spaces $X_i = \langle E_i \rangle_{\mathbb{F}}$ spanned by a partition of a represented matroid, two simple configurations help explain small connectivity. First, all the X_i may contain a common subspace A of small dimension, with their images in the quotient by A forming a direct

sum. All their interaction is then confined to A . Second, the spaces may themselves form an almost direct sum: the sum of their dimensions exceeds the dimension of their sum by only a small amount.

Lemma 4.4 combines these two ideas. After taking a quotient by a subspace of small dimension, the remaining excess is also small. It thus controls both the dimension removed and the remaining failure to form a direct sum. In this general form, the quotient subspace need not be contained in every X_i .

To see how these quantities control connectivity, suppose that A is a subspace of the common ambient space, with $\dim A = a$, and that

$$\sum_{i=1}^k \dim(X_i/A) \leq \dim\left(\left(\sum_{i=1}^k X_i\right)/A\right) + b. \quad (4.1)$$

Then $\lambda^*(X_1, \dots, X_k) \leq a + b$. To see this, consider the spans L and R of any two complementary groups of the spaces. Their quotient images have connectivity at most the excess b in (4.1), and passage to the quotient changes their connectivity by at most $\dim A = a$. Conversely, the main lemma provides such a quotient with both a and b at most $3r$ whenever $\lambda^*(X_1, \dots, X_k) \leq r$. Indeed, for $A \subseteq X_1 + \dots + X_k$ we have

$$\dim\left(\sum_{i=1}^k X_i\right) = \dim\left(\left(\sum_{i=1}^k X_i\right)/A\right) + \dim A,$$

so its conclusion gives (4.1) with $a = b = \dim A \leq 3r$.

The proof of Lemma 4.4 processes the subspaces $X_1, \dots, X_k$ in order, keeping the processed subspaces in two groups with span L and R , along with the sought subspace A containing the common intersection $L \cap R$. A new subspace X_m may introduce ℓ new directions of overlap with prior spaces, and these need to be added to A ; we manage this by observing that placing X_m in either L or R must increase dimension of the overlap $L \cap R$ by at least $\ell/2$. This leads to an upper bound on dimension of A : $\dim(A) \leq 3 \dim(L \cap R)$.

Lemma 4.4. *Let $X_1, \dots, X_k$ be finite-dimensional subspaces of a common vector space over a field $\mathbb{F}$, where $k \geq 1$ and $\lambda^*(X_1, \dots, X_k) \leq r$. There exists a subspace $A \subseteq X_1 + \dots + X_k$ with dimension at most $3r$ such that*

$$\sum_{i=1}^k \dim(X_i/A) \leq \dim\left(\sum_{i=1}^k X_i\right).$$

Proof. We will construct vector spaces A_m , L_m and R_m for every $m \in [k]$ with the following properties:

- (i) A_m is a subspace of $X_1 + \dots + X_m$,
- (ii) there exists $I \subseteq [m]$ such that

$$L_m = \sum_{i \in I} X_i \quad \text{and} \quad R_m = \sum_{i \in [m] \setminus I} X_i,$$

- (iii) if $m > 1$, then $A_{m-1} \subseteq A_m$, $L_{m-1} \subseteq L_m$, and $R_{m-1} \subseteq R_m$,
- (iv) $L_m \cap R_m$ is a subspace of A_m ,
- (v) $\dim A_m \leq 3 \dim(L_m \cap R_m)$, and

(vi) it holds that

$$\sum_{i=1}^m \dim(X_i/A_m) \leq \dim\left(\sum_{i=1}^m X_i\right).$$

Once we have constructed A_m , L_m and R_m for every $m \in [k]$, we set the vector space A from the statement of the lemma to be the vector space A_k . Note that since the dimension of $L_k \cap R_k$ cannot be larger than $\lambda^*(X_1, \dots, X_k) \leq r$ by the property (ii), the dimension of $A = A_k$ is at most $3r$ by the property (v), and the inequality from the statement of the lemma follows from the property (vi) above for $m = k$. Therefore, it suffices to construct the vector spaces A_m , L_m , and R_m , $m \in [k]$, in order to prove the lemma.

We now iteratively construct the vector spaces A_m , L_m , and R_m , starting with $m = 1$. We set A_1 and R_1 to be the 0-dimensional vector space and $L_1 = X_1$. It is straightforward to verify that A_1 , L_1 and R_1 satisfy all properties (i)–(vi).

Suppose that we have constructed vector spaces A_{m-1} , L_{m-1} , and R_{m-1} . Consider the vector space A'_m defined as follows:

$$A'_m = (X_m + A_{m-1}) \cap \sum_{i \in [m-1]} X_i. \quad (4.2)$$

Note that A'_m contains A_{m-1} as a subspace, since A_{m-1} is a subspace of $\sum_{i=1}^{m-1} X_i$. We next distinguish two cases: the first is that the spaces A_{m-1} and A'_m are equal, and the second is that A_{m-1} is a proper subspace of A'_m .

If the spaces A_{m-1} and A'_m are equal, we set the subspaces A_m , L_m and R_m as follows:

$$L_m = L_{m-1} + X_m, \quad R_m = R_{m-1} \quad \text{and} \quad A_m = A'_m + (L_m \cap R_m) = A_{m-1} + (L_m \cap R_m).$$

The subspaces A_m , L_m and R_m clearly satisfy the properties (i)–(iv). The property (v), which bounds the dimension of A_m , follows from the next estimates:

$$\begin{aligned} \dim A_m &= \dim(A_{m-1} + (L_m \cap R_m)) \\ &= \dim A_{m-1} + \dim(L_m \cap R_m) - \dim(A_{m-1} \cap (L_m \cap R_m)) \\ &\leq \dim A_{m-1} + \dim(L_m \cap R_m) - \dim(A_{m-1} \cap (L_{m-1} \cap R_{m-1})) \\ &= \dim A_{m-1} + \dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1}) \\ &\leq 3 \dim(L_{m-1} \cap R_{m-1}) + \dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1}) \\ &= \dim(L_m \cap R_m) + 2 \dim(L_{m-1} \cap R_{m-1}) \leq 3 \dim(L_m \cap R_m). \end{aligned}$$

We prove the property (vi) jointly for the first and the second cases later.

If the space A_{m-1} is a proper subspace of A'_m , we proceed as follows. Let $d = \dim A'_m - \dim A_{m-1}$; we remark that the previous case $A'_m = A_{m-1}$ can be viewed as the case $d = 0$ of the arguments that we next present, however, we decided to separate the two cases in the interest of the clarity of presentation. The definition of A'_m as (4.2) implies that there exist (linearly independent) vectors $x_1, \dots, x_d \in X_m$ such that any basis of A_{m-1} together with $x_1, \dots, x_d$ is a basis of A'_m . Note that none of the vectors $x_1, \dots, x_d$ belongs to A_{m-1} . Since the vector x_i belongs to A'_m , and so $x_i \in X_1 + \dots + X_{m-1} = L_{m-1} + R_{m-1}$, there exist $\ell_i \in L_{m-1}$ and $r_i \in R_{m-1}$ such that $x_i = \ell_i + r_i$. Let L' be the linear hull of the vectors $\ell_1, \dots, \ell_d$ and R' the linear hull of the vectors $r_1, \dots, r_d$. Note that the vectors $x_1, \dots, x_d$ are contained in $L' + R'$, which implies that

$$\dim(L' + R') - \dim((L' + R') \cap A_{m-1}) \geq d. \quad (4.3)$$

On the other hand, the space $L' \cap R'$ is a subspace of $L_{m-1} \cap R_{m-1}$ and so of A_{m-1} , which yields that

$$\dim(L' \cap R') - \dim((L' \cap R') \cap A_{m-1}) = 0. \quad (4.4)$$

Next observe that

$$\dim L' + \dim R' = \dim(L' + R') + \dim(L' \cap R'), \text{ and} \quad (4.5)$$

$$\dim(L' \cap A_{m-1}) + \dim(R' \cap A_{m-1}) \leq \dim((L' + R') \cap A_{m-1}) + \dim((L' \cap R') \cap A_{m-1}). \quad (4.6)$$

The inequality in (4.6) follows from the dimension formula and $(L' \cap A_{m-1}) + (R' \cap A_{m-1}) \subseteq (L' + R') \cap A_{m-1}$. We combine (4.3), (4.4), (4.5) and (4.6) to obtain that

$$(\dim L' - \dim(L' \cap A_{m-1})) + (\dim R' - \dim(R' \cap A_{m-1})) \geq d.$$

It follows that $\dim L' - \dim(L' \cap A_{m-1}) \geq d/2$ or $\dim R' - \dim(R' \cap A_{m-1}) \geq d/2$ (or possibly both). The two possibilities are symmetric, so we assume that $\dim L' - \dim(L' \cap A_{m-1}) \geq d/2$.

We define the subspaces L_m , R_m , and A_m as

$$L_m = L_{m-1}, \quad R_m = R_{m-1} + X_m \quad \text{and} \quad A_m = A'_m + (L_m \cap R_m).$$

Note that every $\ell_i = x_i - r_i$ belongs to $L_m \cap R_m$: indeed, $\ell_i \in L_{m-1} \subseteq L_m$ and $x_i - r_i \in X_m + R_{m-1} = R_m$. Hence, the space L' , which is the linear hull of the vectors $\ell_1, \dots, \ell_d$, is a subspace of $L_m \cap R_m$. Since $L_{m-1} \cap R_{m-1}$ is a subspace of A_{m-1} , it follows that

$$\begin{aligned} \dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1}) &\geq \dim(L' + (L_{m-1} \cap R_{m-1})) - \dim(L_{m-1} \cap R_{m-1}) \\ &\geq \dim(L' + A_{m-1}) - \dim A_{m-1} \\ &= \dim L' - \dim(L' \cap A_{m-1}) \\ &\geq d/2. \end{aligned}$$

Hence, it holds that $d/2 \leq \dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1})$, which is equivalent to

$$d \leq 2(\dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1})). \quad (4.7)$$

We can now estimate the dimension of A_m as follows (recall that $L_{m-1} \cap R_{m-1}$ is a subspace of $A_{m-1} \subseteq A'_m$ and also of $L_m \cap R_m$):

$$\begin{aligned} \dim A_m &\leq \dim A'_m + \dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1}) \\ &= \dim A_{m-1} + d + \dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1}) \\ &\leq \dim A_{m-1} + 3(\dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1})) \\ &\leq 3\dim(L_{m-1} \cap R_{m-1}) + 3(\dim(L_m \cap R_m) - \dim(L_{m-1} \cap R_{m-1})) \\ &= 3\dim(L_m \cap R_m), \end{aligned}$$

where the second inequality follows from (4.7). We have now verified that the properties (i)–(v) hold in the second case.

We now continue jointly the analysis of the first and second cases and establish the property (vi). We will use the definition of A'_m as (4.2) and that $A_m = A'_m + (L_m \cap R_m)$. The definition of A'_m yields that

$$\dim(X_m + A_{m-1}) + \dim\left(\sum_{i=1}^{m-1} X_i\right) = \dim\left(\sum_{i=1}^m X_i\right) + \dim A'_m; \quad (4.8)$$

indeed, (4.8) follows from the equality $\dim S + \dim T = \dim(S + T) + \dim(S \cap T)$ used for $S = X_m + A_{m-1}$ and $T = \sum_{i=1}^{m-1} X_i$. We now establish the property (vi).

$$\begin{aligned}
 \sum_{i=1}^m \dim(X_i/A_m) &\leq \sum_{i=1}^m \dim(X_i/A'_m) \\
 &= \sum_{i=1}^{m-1} \dim(X_i/A'_m) + \dim(X_m/A_{m-1}) - (\dim A'_m - \dim A_{m-1}) \\
 &= \sum_{i=1}^{m-1} \dim(X_i/A'_m) + \dim(X_m + A_{m-1}) - \dim A'_m \\
 &\leq \sum_{i=1}^{m-1} \dim(X_i/A_{m-1}) + \dim(X_m + A_{m-1}) - \dim A'_m \\
 &\leq \dim\left(\sum_{i=1}^{m-1} X_i\right) + \dim(X_m + A_{m-1}) - \dim A'_m \\
 &= \dim\left(\sum_{i=1}^m X_i\right),
 \end{aligned}$$

where the penultimate inequality is the induction hypothesis and the last equality is a restatement of (4.8). $\square$

4.4 Constructing the minor container

We apply Lemma 4.4 at the root of a rooted (d, r) -decomposition. The subspace A accounts for at most $3r$ dimensions, and at most $3r$ additional coordinates suffice to separate the remaining quotient images. Auxiliary elements record how those coordinates relate to the original representation. One choice of contractions and deletions recovers the original matroid; another produces a direct sum to which induction applies. This gives the following theorem, from which we derive the results announced in Section 4.1.

Theorem 4.5. *If an $\mathbb{F}$ -representable matroid M has a rooted (d, r) -decomposition, then there exists an $\mathbb{F}$ -representable matroid N such that M is a minor of N and $\text{cdd}(N) \leq 2r \cdot (4^d - 1) + 1$.*

Proof. The proof proceeds by induction on d . We prove the following strengthened statement for every prescribed $\mathbb{F}$ -representation of M . There are disjoint sets $S, D \subseteq E(N)$ such that $M = N/S \setminus D$ with its prescribed representation, up to row operations; the contracted set S is independent; and $E(M) \cup S$ spans N . In particular,

$$|S| = \text{rank } N - \text{rank } M,$$

and any basis of M together with S is a basis of N . Once the columns of S are made unit vectors, the represented minor is obtained by deleting their rows and columns and deleting the columns of D .

If M has a single element, set $N = M$ and $S = D = \emptyset$; the asserted depth bound and all additional properties hold. This includes the base case $d = 0$, since a rooted tree of depth zero has one leaf. For the induction step let $d \geq 1$, fix a represented matroid M with at least two elements, and fix a rooted (d, r) -decomposition T of M . Let k be the number

of children of the root of T and let $T_1, \dots, T_k$ be the (rooted) subtrees of T rooted at the children of the root of T . Further, let E_i be the elements of M assigned to the leaves of T_i , $i \in [k]$, and let X_i be the linear hull of E_i .

Since T is a rooted (d, r) -decomposition of M it holds that $\lambda^*(X_1, \dots, X_k) \leq r$. Hence, Lemma 4.4 yields that there exists a subspace A of $\mathbb{F}^{\text{rank } M}$ with dimension at most $3r$ such that

$$\sum_{i=1}^k \dim(X_i/A) \leq \dim \sum_{i=1}^k X_i. \quad (4.9)$$

Let d_A be the dimension of the vector space A and let d_i be the dimension of the vector space X_i/A , $i \in [k]$. Note that $\text{rank } M$ is equal to $\dim(X_1 + \dots + X_k)$, which is at most $d_1 + \dots + d_k + d_A$. On the other hand, the rank of M is at least $d_1 + \dots + d_k$ by (4.9).

Fix a basis $B_A = \{a_1, \dots, a_{d_A}\}$ of the vector space A , and for each $i \in [k]$ choose $B_i = \{x_1^i, \dots, x_{d_i}^i\}$ from the labelled vectors of E_i such that $a_1, \dots, a_{d_A}, x_1^i, \dots, x_{d_i}^i$ is a basis of the vector space $X_i + A$. Such a choice is possible because the images of E_i span $(X_i + A)/A$. Observe that the vector space $\mathbb{F}^{\text{rank } M} = X_1 + \dots + X_k$ is spanned by the set of vectors $B_A \cup B_1 \cup \dots \cup B_k$. It follows that there is a subset of these vectors that is a basis of the vector space $X_1 + \dots + X_k$; we may assume that this subset contains all vectors $a_1, \dots, a_{d_A}$ (as these vectors are linearly independent). Hence, after possibly permuting the vectors $x_1^i, \dots, x_{d_i}^i$, we assume that for each $i \in [k]$ there exists $d'_i \in \{0\} \cup [d_i]$, such that the set of vectors $B = B_A \cup \{x_j^i : i \in [k], j \in [d'_i]\}$ forms a basis of the vector space $X_1 + \dots + X_k$. In addition, we assume that the vectors $a_1, \dots, a_{d_A}$ are the first d_A unit vectors, $x_1^1, \dots, x_{d'_1}^1$ are the next d'_1 unit vectors, etc. Since the sum $d'_1 + \dots + d'_k + d_A$ is equal to the rank of M , which is at least $d_1 + \dots + d_k$, we obtain that

$$\sum_{i=1}^k (d_i - d'_i) \leq d_A. \quad (4.10)$$

Let d_C be the value of the sum from the left side of (4.10); note that $d_C \leq d_A \leq 3r$. We now view $X_1 + \dots + X_k$ as a subspace of $\mathbb{F}^{\text{rank } M + d_C}$, where the vectors in B are the first $\text{rank } M$ unit vectors (in the order described above). Next, let $C = \{b_j^i : i \in [k], j \in [d_i] \setminus [d'_i]\}$ be the remaining (last) d_C unit vectors of $\mathbb{F}^{\text{rank } M + d_C}$ (where b_j^1 comes before b_j^2 , which comes before b_j^3 , etc).

We next construct a vector matroid M' of rank equal to $\text{rank } M + d_C$ such that M is a minor of M' . The matroid M' contains the following $|M| + d_A + d_C$ vectors:

- For every $i \in [k]$ and every element $x \in E_i$, the matroid M' contains the vector

$$\sum_{j=1}^{d_A} \alpha_j a_j + \sum_{j=1}^{d'_i} \beta_j x_j^i + \sum_{j=d'_i+1}^{d_i} \beta_j b_j^i$$

where $\alpha_j \in \mathbb{F}$, $j \in [d_A]$, and $\beta_j \in \mathbb{F}$, $j \in [d_i]$ are the unique coefficients such that

$$x = \sum_{j=1}^{d_A} \alpha_j a_j + \sum_{j=1}^{d_i} \beta_j x_j^i.$$

In particular, the matroid M' contains the unit vectors x_j^i , $i \in [k]$ and $j \in [d'_i]$ and the unit vectors b_j^i , $i \in [k]$ and $j \in [d_i] \setminus [d'_i]$; the unit vectors x_j^i , $i \in [k]$ and $j \in [d_i] \setminus [d'_i]$ are included because of the elements x_j^i with $j \in [d_i] \setminus [d'_i]$.

$\xleftrightarrow{E_1}$					$\xleftrightarrow{E_2}$					$\xleftrightarrow{E_3}$					a_1	a_2	a_3	a_4	z_4^1	z_3^2	z_4^2	z_3^3	
x_1^1	x_2^1	x_3^1	x_4^1	y_1^1	x_1^2	x_2^2	x_3^2	x_4^2	y_1^2	x_1^3	x_2^3	x_3^3	y_1^3	y_2^3									
	*	*	*			*	*	*			*	*	*			1			*	*	*	*	
	*	*	*			*	*	*			*	*	*			1			*	*	*	*	
	*	*	*			*	*	*			*	*	*				1		*	*	*	*	
N_1	*	*	*			*	*	*			*	*	*					1	*	*	*	*	
1	*	*	*	*	*														*	*	*	*	
1	*	*	*	*	*														*	*	*	*	
1	*	*	*	*	*														*	*	*	*	
*	*	*	*	*	*	*	*	*	1	*	*								1	*	*	*	*
					N_2	1	*	*	*	*	*								*	*	*	*	
					1	*	*	*	*	*	*								*	*	*	*	
					1	*	*	*	*	*	*								1	*	*	*	
					1	*	*	*	*	*	*												
					*	*	*	*	*	*	*	1	*	*									
										N_3	1	*	*	*	*	*			*	*	*	*	
										1	*	*	*	*	*	*			*	*	*	*	
										1	*	*	*	*	*	*			*	*	*	*	
										*	*	*	*	*	*	*	1	*	*	*	*	*	
										*	*	*	*	*	*	*	1	*	*	*	*	*	

Figure 4.3: The $\mathbb{F}$ -representation of the matroid N constructed in the proof of Theorem 4.5 when $k = 3$, $d_A = 4$, $d'_1 = 3$, $d_1 = 4$, $n_1 = 1$, $d'_2 = 2$, $d_2 = 4$, $n_2 = 1$, $d'_3 = 2$, $d_3 = 3$ and $n_3 = 2$. Stars depict entries that can be arbitrary (both zero or non-zero), and the entries that are not displayed are zero. The representations of matroids N_1 , N_2 and N_3 obtained from induction are encompassed by the dashed lines.

We conclude that T_i is a rooted $(d-1, 4r)$ -decomposition of the matroid M_i'' .

By induction, the $\mathbb{F}$ -represented matroid M_i'' is a minor of an $\mathbb{F}$ -represented matroid N_i such that

$$\text{cdd}(N_i) \leq 8r \cdot (4^{d-1} - 1) + 1 = 2r \cdot (4^d - 1) + 1 - 6r.$$

Let S_i and D_i be the contracted and deleted sets supplied by the strengthened induction hypothesis. Set $n_i = |S_i| = \text{rank } N_i - \text{rank } M_i''$ and write $S_i = \{y_1^i, \dots, y_{n_i}^i\}$. The retained elements x_j^i , $j \in [d_i]$, form a basis of M_i'' ; hence they together with S_i form a basis of N_i . Represent N_i in this basis, with the x_j^i coordinates preceding the y_j^i coordinates. Contracting S_i and deleting D_i then gives precisely the prescribed coordinate representation of M_i'' .

We will now construct the sought matroid N ; the construction of the matroid N is illustrated in Figure 4.3. The matroid N is a vector matroid of rank $R = d_1 + \dots + d_k + n_1 + \dots + n_k + d_A$ with $|N_1| + \dots + |N_k| + d_A + d_C$ elements, which one-to-one correspond to the elements of the matroids $N_1, \dots, N_k$, and the elements $a_1, \dots, a_{d_A}$ and z_j^i , $i \in [k]$ and $j \in [d_i] \setminus [d'_i]$, and so we identify them with these elements. The elements $a_1, \dots, a_{d_A}$, the elements x_j^i , $i \in [k]$ and $j \in [d_i]$, and the elements y_j^i , $i \in [k]$ and $j \in [n_i]$ will form a basis of N , and so we can assume that these elements are represented by unit vectors in the vector space $\mathbb{F}^{d_1 + \dots + d_k + n_1 + \dots + n_k + d_A}$ and the elements $a_1, \dots, a_{d_A}$ are the first d_A unit vectors. The vector matroid N contains the following vectors:

- each of the unit vectors $a_1, \dots, a_{d_A}$,
- the vector z_j^i , $i \in [k]$ and $j \in [d_i] \setminus [d'_i]$ with all its additional $n_1 + \dots + n_k$ entries set to zero,

- for every $i \in [k]$ and every vector of N_i contained in M_i'' , this vector with its first d_A entries being the same as in M' and all its additional $R - d_A - d_i - n_i$ entries set to zero, and
- for every $i \in [k]$ and every vector of N_i not contained in M_i'' , this vector with all its additional $R - d_i - n_i$ entries set to zero.

Note that the matroid obtained from the matroid N by contracting the d_A elements $a_1, \dots, a_{d_A}$ and deleting the d_C elements z_j^i , $i \in [k]$ and $j \in [d_i] \setminus [d'_i]$ is the direct sum of the matroids $N_1, \dots, N_k$. Since $d_A + d_C \leq 6r$ and $\text{cdd}(N_i) \leq 2r \cdot (4^d - 1) + 1 - 6r$ for every $i \in [k]$, it follows that the contraction-deletion-depth of N is at most $2r \cdot (4^d - 1) + 1$. To recover M , first contract all y_j^i and delete all D_i . This recovers the displayed representation of M' , up to the ordering of its coordinates. Contracting all z_j^i and deleting all a_j then recovers the prescribed representation of M . Thus the contracted set in the final minor model is

$$S = \bigcup_{i=1}^k S_i \cup \{z_j^i : i \in [k], j \in [d_i] \setminus [d'_i]\},$$

and the deleted set is $D = \bigcup_i D_i \cup \{a_1, \dots, a_{d_A}\}$. The elements of S are independent: the y_j^i have their own unit coordinates, and, after contracting them, each $z_j^i = b_j^i - x_j^i$ has a distinct new b_j^i coordinate equal to one, while the original vector x_j^i has no new b coordinates. Moreover,

$$\text{rank } N - |S| = d_A + \sum_{i=1}^k d_i - d_C = d_A + \sum_{i=1}^k d'_i = \text{rank } M.$$

Since the represented minor on $E(M)$ has this rank after contracting S , $E(M) \cup S$ spans N . This verifies the additional induction properties. This completes the proof of the induction step and so of the theorem. $\square$

We immediately obtain Theorem 4.2 as a corollary.

Proof of Theorem 4.2. Let M be an $\mathbb{F}$ -representable matroid with branch-depth k . If $|E(M)| \leq 1$, take $N = M$; the claimed bound holds under our small-case conventions. We may therefore assume $|E(M)| \geq 2$. Since M has a (d, r) -decomposition for some non-negative integers $d \leq k$ and $r \leq k$, Proposition 4.3 yields that M has a rooted (d, r) -decomposition. The corollary now directly follows from Theorem 4.5. $\square$

We also obtain the characterisation of classes of representable matroids with bounded branch-depth given in Theorem 4.1.

Proof of Theorem 4.1. Fix a class $\mathcal{M}$ of $\mathbb{F}$ -representable matroids. Suppose that there exists a class $\mathcal{N}$ of matroids with contraction-deletion-depth bounded by d such that $\mathcal{M}$ is a subclass of the minor closure of the class $\mathcal{N}$. Since the contraction-deletion-depth is an upper bound on the branch-depth [13, Theorem 5.2], the branch-depth of every matroid contained in $\mathcal{N}$ is also bounded by d and so is the branch-depth of every matroid contained in the minor closure of $\mathcal{N}$ as the branch-depth is minor-monotone [13, Proposition 5.3]. Hence, if $\mathcal{M}$ is a subclass of the minor closure of $\mathcal{N}$, the branch-depth of every matroid contained in $\mathcal{M}$ is bounded by d .

We now prove the other implication. Suppose that the branch-depth of every matroid contained in $\mathcal{M}$ is bounded by d' . For every matroid $M \in \mathcal{M}$, let N_M be an $\mathbb{F}$ -representable matroid with contraction-deletion-depth bounded by $2d' \cdot (4^{d'} - 1) + 1$ such that M is a minor

of N_M ; such a matroid N_M exists by Theorem 4.2. The class $\mathcal{N}$ is formed by the matroids N_M for all $M \in \mathcal{M}$. Since the contraction-deletion-depth of every matroid contained in $\mathcal{N}$ is at most $2d' \cdot (4^{d'} - 1) + 1$ and the class $\mathcal{M}$ is a subclass of the minor closure of $\mathcal{N}$, the other implication is now proven. $\square$

4.5 Generalisation and sharper bounds

The construction uses coordinates both to separate the pieces and to combine their recursively chosen containers. The corresponding statement for arbitrary matroids was formulated as the following conjecture in [10]; see the discussion in Section 1.3 for Geelen's remarks that adapting the approach of [17] should yield it.

Conjecture 4.6. *There is a function f such that every finite matroid M is a minor of a matroid N with $\text{cdd}(N) \leq f(\text{bd}(M))$.*

Equivalently, the classes of bounded branch-depth would be precisely the subclasses of minor closures of classes of bounded contraction-deletion-depth. This is an assertion about arbitrary matroids, without a representability hypothesis.

The bound proved here is exponential in the radius of the decomposition. It appears that the natural target if one is aiming to improve this result is a bound of $Kdr + L$ for a matroid admitting a (d, r) -decomposition, with absolute constants K, L . Taking $d = r = \text{bd}(M)$ would give quadratic dependence on branch-depth; the additive constant accommodates matroids with at most one element. More generally, the quantitative relationship raises the following optimisation problem.

Problem 4.7. *Determine the optimal growth rate of a function f such that, for every field $\mathbb{F}$, every $\mathbb{F}$ -representable matroid M is a minor of an $\mathbb{F}$ -representable matroid N with $\text{cdd}(N) \leq f(\text{bd}(M))$.*

The chapter illustrates a second version of making structure explicit: the extra elements need not merely implement contractions along an existing rank decomposition. They can also record how pieces were joined. The next chapter changes perspective again, using duality to turn a question about circuits into a question about graph bonds.

Chapter 5

Bonds, duality, and graph pathwidth

5.1 From circuits to bonds

Matroid duality suggests graph-theoretic questions even when the graphs under consideration have no plane embedding. A cycle is a circuit of the cycle matroid, whereas a bond is a cocircuit. Thus the question of whether a large decomposition width forces a long cycle has a natural companion: does it force a large bond?

Conjecture 1.1 would bound matroid contraction*-depth linearly in circuit size and hence give linear width bounds. Here we study its pathwidth consequence: whether the pathwidth of a connected matroid with a circuit is bounded linearly by its largest circuit size (Conjecture 5.12). Since matroid pathwidth is invariant under duality, its cographic case suggests studying bonds in graphs. This leads us to seek a linear bound on graph pathwidth in terms of cocircumference for 2-connected graphs. Such a graph bound supplies the corresponding matroid bound through the comparison between graph and matroid pathwidth.

This chapter is based on joint work with Gwenaél Joret and Michał T. Seweryn, *Pathwidth vs cocircumference*. It establishes bounds on graph treewidth and pathwidth in terms of cocircumference. It then returns to matroids, where these graph results provide evidence for linear bounds on decomposition width in terms of the size of a largest circuit. This gives one direction of the interaction studied in the thesis: a matroid operation identifies the appropriate graph question, and the graph theorem in turn supports a matroid conjecture.

We use the graph conventions and decomposition parameters from Section 2.2. Graphs in the main theorems are finite and simple. The circumference of a graph containing a cycle is the size of its largest cycle; its cocircumference, when it has an edge, is the size of its largest bond. The identity

$$\text{cocirc}(G) = \text{circ}(M(G)^*)$$

expresses the motivation precisely. The classical starting point is the following theorem.

Theorem 5.1 (Birmelé [2]). *Every graph with at least one cycle has treewidth at most its circumference minus 1.*

For 2-connected graphs, a stronger result bounds tree-depth and therefore pathwidth. We state its edge-specific form, which will also guide our formulation for bonds.

Theorem 5.2 (Briański, Joret, Majewski, Micek, Seweryn, and Sharma [6]). *Let G be a 2-connected graph with tree-depth d and circumference k . Then every edge of G is included*

in a cycle of length at least d . Thus, G has tree-depth at most k , and in particular pathwidth at most $k - 1$.

The connectivity assumption is essential, as exemplified by a disjoint union of a complete ternary tree of depth h and a triangle. For $h \geq 1$, this graph has pathwidth h and tree-depth $h + 1$, but has circumference 3.

Our first result replaces circumference by cocircumference in the treewidth bound.

Theorem 5.3. *Every graph with at least one edge has treewidth at most its cocircumference.*

The proof associates a bond with each nonroot node of a depth-first search tree. The ends of the bond on the descendant side control the size of the corresponding bag. The pathwidth theorem requires more structure, because the decomposition must have a linear order.

Theorem 5.4. *Let G be a 2-connected graph with pathwidth p and cocircumference k . Then each edge of G is contained in a bond F such that $p \leq 3|F| - 2$. In particular, G has pathwidth at most $3k - 2$.*

Here 2-connectivity is again essential, as the ternary trees show. Moreover, the stronger conclusion for tree-depth in Theorem 5.2 has no bond analogue. Cycles have cocircumference 2 and unbounded tree-depth. Thus replacing circuits by cocircuits preserves a linear pathwidth bound, but does not preserve the corresponding bound on graph tree-depth. This distinction is relevant to the depth parameters considered elsewhere in the thesis: an analogy suggested by duality still needs to be tested against the decomposition parameter being measured.

Complete graphs show that the two quoted circumference bounds are tight. For our cocircumference bounds, we obtain optimum linear asymptotic behaviour, but the value of the leading constant is not known to be optimal. In the treewidth case, there are cubic graphs with n vertices, treewidth $\Omega(n)$, and $3n/2$ edges. These graphs rule out a sublinear bound on treewidth in terms of cocircumference; an explicit construction is discussed in [1]. For pathwidth we prove the following statement with a recursive family of outerplane graphs.

Theorem 5.5. *For every integer $k \geq 1$, there exists a 2-connected graph G_k with pathwidth at least k and cocircumference at most $2k$.*

Section 5.2 proves the three results. The key pathwidth lemma keeps track of a prescribed vertex in the first bag and constructs a bond separating two specified vertices. This additional information makes the induction compatible with decompositions along cutvertices and separations of order two. Section 5.3 then interprets the results through matroid duality. The proof of the lower bound uses plane dual multigraphs; their pathwidth is that of the underlying simple graph, and their cycles include pairs of parallel edges.

5.2 Bonds and decomposition width

We begin with the proof of Theorem 5.3.

Proof of Theorem 5.3. The treewidth of a graph is the maximum treewidth of its components, and its cocircumference is the maximum cocircumference of its components. Thus, it is enough to prove the theorem in the case where the graph G is connected.

Let T be a depth-first search tree of G rooted at some vertex r of G . Within this proof, ancestor and descendant include the vertex itself; the adjective proper excludes equality.

Let $X_r = \{r\}$, and for each $u \in V(T) \setminus \{r\}$, let X_u denote the set consisting of the parent of u in T and all descendants of u that are adjacent in G to a proper ancestor of u . In particular, since u is adjacent in G to its parent in T , we have $u \in X_u$. Let us verify that $\{X_u : u \in V(T)\}$ is a tree-decomposition of G .

Since T is a depth-first search tree of G , every edge of G is between two distinct vertices on a root-to-leaf path in T . For each $uv \in E(G)$ where v is a proper ancestor of u , if v' denotes the child of v that is an ancestor of u , then $\{u, v\} \subseteq X_{v'}$.

For $v \in V(G)$ and $u \in V(T)$, we have $v \in X_u$ if and only if one of the following holds:

- $u = v$, or
- u is a child of v , or
- u is a proper ancestor of v and there exists a proper ancestor of u that is adjacent to v in G .

Hence, for every $v \in V(G)$, the set $\{u \in V(T) : v \in X_u\}$ induces a tree in T , so $\{X_u : u \in V(T)\}$ is indeed a tree-decomposition of G .

For every vertex $u \in V(T) \setminus \{r\}$ with a parent u' in T , removing the edge uu' splits T into two trees. Let F_u denote the set of all edges of G between the vertex sets of these two trees. The two components of $T - uu'$ are spanning trees of the components of $G - F_u$, so F_u is a bond. Each vertex $v \in X_u$ except the parent of u is a descendant of u adjacent to a proper ancestor v' of u such that $vv' \in F_u$, and these edges vv' are distinct, so $|X_u| - 1 \leq |F_u| \leq \text{cocirc}(G)$. Moreover, $|X_r| - 1 = 0 \leq \text{cocirc}(G)$, so

$$\text{tw}(G) \leq \max\{|X_u| - 1 : u \in V(G)\} \leq \text{cocirc}(G). \quad \square$$

5.2.1 Path-decompositions with prescribed end vertices

Next, we introduce some lemmas that will be used in the proof of Theorem 5.4. A *separation* of a graph G is a pair (G_1, G_2) such that G_1 and G_2 are edge-disjoint subgraphs of G with $G_1 \cup G_2 = G$. A separation (G_1, G_2) is *trivial* if $E(G_1) = \emptyset$ or $E(G_2) = \emptyset$. The *order* of a separation (G_1, G_2) is $|V(G_1) \cap V(G_2)|$. For a connected graph G and a vertex x of G , there exists a nontrivial separation (G_1, G_2) with $V(G_1) \cap V(G_2) = \{x\}$ if and only if x is a cutvertex of G .

Lemma 5.6. *Let G be a connected graph without a cutvertex, and let (G_1, G_2) be a nontrivial separation of G such that $V(G_1) \cap V(G_2) = \{x, y\}$ for some vertices x and y . Then none of the graphs $G_1 + xy$ and $G_2 + xy$ has a cutvertex.*

Proof. By symmetry, we only need to show that $G_2 + xy$ does not have a cutvertex. Every component of G_2 meets $\{x, y\}$, since G is connected, so $G_2 + xy$ is connected. Suppose to the contrary that $G_2 + xy$ has a cutvertex. Let (G'_2, G''_2) be a nontrivial separation of $G_2 + xy$ of order 1. Without loss of generality, $xy \in E(G'_2)$. Let $H = G'_2$ if $xy \in E(G_2)$, and let $H = G'_2 - xy$ otherwise. Thus an original edge is retained, while an artificially added edge is removed. Then $(G_1 \cup H, G''_2)$ is a nontrivial separation of G of order 1, so G has a cutvertex, contradiction. $\square$

For a graph G and $x, y \in V(G)$, let $\text{pw}(G; x)$ denote the minimum width of a path-decomposition $(X_0, \dots, X_s)$ of G such that $x \in X_0$, and let $\text{pw}(G; x, y)$ denote the minimum width of a path-decomposition $(X_0, \dots, X_s)$ of G such that $x \in X_0$ and $y \in X_s$. Observe that $\text{pw}(G; x, y) \leq \text{pw}(G; x) + 1$, and if $x \neq y$, then $\text{pw}(G; x, y) \leq \text{pw}(G - y; x) + 1$.

Lemma 5.7. *Let G be a graph, let $x \in V(G)$, and let (G_1, G_2) be a separation of G such that $x \in V(G_1)$ and $V(G_1) \cap V(G_2) = \{x'\}$ for some $x' \in V(G)$. Then*

$$\text{pw}(G; x) \leq \max\{\text{pw}(G_1; x, x'), \text{pw}(G_2; x')\}.$$

Proof. Let $(X_0, \dots, X_s)$ be a path-decomposition of G_1 of width $\text{pw}(G_1; x, x')$ with $x \in X_0$ and $x' \in X_s$, and let $(Y_0, \dots, Y_t)$ be a path-decomposition of G_2 of width $\text{pw}(G_2; x')$ with $x' \in Y_0$. Then the sequence

$$(X_0, \dots, X_s, Y_0, \dots, Y_t)$$

is a path-decomposition of G of width at most $\max\{\text{pw}(G_1; x, x'), \text{pw}(G_2; x')\}$ and with $x \in X_0$. $\square$

Lemma 5.8. *Let G be a graph of the form $B \cup G_1 \cup \dots \cup G_m$, where $m \geq 2$, the graphs $G_1, \dots, G_m$ are pairwise vertex-disjoint and each G_i shares exactly one vertex x_i with B . Let p be an integer such that $\text{pw}(G_i; x_i) \leq p$ for $i \in \{2, \dots, m\}$. Then for every $x \in V(B)$ we have*

$$\text{pw}(G; x) \leq \max\{\text{pw}(B; x_1) + p + 1, \text{pw}(G_1; x_1)\}.$$

Proof. Let $(X_0, \dots, X_s)$ be a path decomposition of B with $x_1 \in X_0$ and width $\text{pw}(B; x_1)$, and for each $i \in \{1, \dots, m\}$, let $(Y_0^i, \dots, Y_{t_i}^i)$ be a path decomposition of G_i with $x_i \in Y_0^i$ and width $\text{pw}(G_i; x_i)$. For each $v \in V(B)$, define

$$j(v) = \min\{j \in \{0, \dots, s\} : v \in X_j\}.$$

Hence, $j(x_1) = 0$. By inserting bags that introduce one vertex at a time, beginning with $\{x_1\}$, we may assume that $j(v) \neq j(v')$ for distinct vertices $v, v' \in V(B)$, without increasing the width. We construct an appropriate path-decomposition of G in three steps. First, we join the reversed path-decomposition of B with the path decomposition of G_1 to obtain a path-decomposition of $B \cup G_1$:

$$(X_s, \dots, X_0, Y_0^1, \dots, Y_{t_1}^1).$$

Next, we add x to the bags $X_s, \dots, X_0$:

$$(X_s \cup \{x\}, \dots, X_0 \cup \{x\}, Y_0^1, \dots, Y_{t_1}^1).$$

This is a path-decomposition of $B \cup G_1$ with width at most $\max\{\text{pw}(B; x_1) + 1, \text{pw}(G_1; x_1)\}$ and with x appearing in the first bag. We transform it into a path-decomposition of G by stacking it with the path-decompositions of the graphs $G_2, \dots, G_m$. More precisely, for each $i \in \{2, \dots, m\}$, let $X'_{j(x_i)} = (X_{j(x_i)} \setminus \{x_i\}) \cup \{x\}$, and insert the new bags $X'_{j(x_i)} \cup Y_0^i, \dots, X'_{j(x_i)} \cup Y_{t_i}^i$ directly after the bag $X_{j(x_i)} \cup \{x\}$. By definition of $j(x_i)$, this is a path-decomposition of G , and the vertex x belongs to its first bag $X_s \cup \{x\}$. It remains to estimate the width. We started with a path-decomposition of $B \cup G_1$ of width at most $\max\{\text{pw}(B; x_1) + 1, \text{pw}(G_1; x_1)\}$. For $i \in \{2, \dots, m\}$ and $j \in \{0, \dots, t_i\}$, we can bound the size of the bag $X'_{j(x_i)} \cup Y_j^i$ as follows:

$$\begin{aligned} |X'_{j(x_i)} \cup Y_j^i| &\leq |X'_{j(x_i)}| + |Y_j^i| \\ &= |(X_{j(x_i)} \setminus \{x_i\}) \cup \{x\}| + |Y_j^i| \\ &\leq |X_{j(x_i)}| + |Y_j^i| \\ &\leq (\text{pw}(B; x_1) + 1) + (p + 1). \end{aligned}$$

The width of our path-decomposition of G is one less than the size of a largest bag, and thus is at most $\max\{\text{pw}(B; x_1) + p + 1, \text{pw}(G_1; x_1)\}$. This proves the lemma. $\square$

A *block* is a maximal connected subgraph without a cutvertex; in particular, an edge that is a bridge, together with its ends, is a block.

For two distinct vertices x and y of a graph G , an x - y *bond* in G is a bond F in G such that x and y are in distinct components of $G - F$. The following lemma is the heart of the proof of Theorem 5.4.

Lemma 5.9. *Let G be a connected graph such that $E(G) \neq \emptyset$, and let $x, y \in V(G)$ be distinct vertices such that $G + xy$ does not have a cutvertex. Then there exists an x - y bond F in G such that $\text{pw}(G - y; x) \leq 3|F| - 2$, and if G does not have a cutvertex, then $\text{pw}(G - y; x) \leq 3|F| - 3$.*

Proof. We prove the lemma by induction on $|E(G)|$. In the base case, we have $|E(G)| = 1$. Since $G + xy$ does not have a cutvertex, we have $V(G) = \{x, y\}$ and $E(G) = \{xy\}$. Therefore, for the x - y bond $F = \{xy\}$, we have $\text{pw}(G - y; x) = 0 = 3|F| - 3$.

For the induction step, we assume that $|E(G)| \geq 2$. Suppose first that there exists a nontrivial separation (G_1, G_2) of G with $V(G_1) \cap V(G_2) = \{x, y\}$. Unless $xy \in E(G)$ and thus $G + xy = G$, the separation (G_1, G_2) of G can be transformed into a separation of $G + xy$ by adding the edge xy to either of the sides. Hence, by Lemma 5.6, none of the graphs $G_1 + xy$ and $G_2 + xy$ has a cutvertex. Each G_α is connected: otherwise the added edge xy would be a bridge of the connected graph $G_\alpha + xy$, which has no cutvertex and has at least two edges, a contradiction. For each $\alpha \in \{1, 2\}$, apply the induction hypothesis to G_α to obtain an x - y bond F_α in G_α such that $\text{pw}(G_\alpha - y; x) \leq 3|F_\alpha| - 2$. Then, $F = F_1 \cup F_2$ is an x - y bond in G that satisfies $|F| = |F_1| + |F_2| \geq \max\{|F_1|, |F_2|\} + 1$. By Lemma 5.7 applied to the separation $(G_1 - y, G_2 - y)$ of $G - y$, we have

$$\begin{aligned} \text{pw}(G - y; x) &\leq \max\{\text{pw}(G_1 - y; x, x), \text{pw}(G_2 - y; x)\} \\ &\leq \max\{\text{pw}(G_1 - y; x) + 1, \text{pw}(G_2 - y; x)\} \\ &\leq \max\{3|F_1| - 1, 3|F_2| - 2\} \\ &< 3 \max\{|F_1|, |F_2|\} \\ &= 3(\max\{|F_1|, |F_2|\} + 1) - 3 \\ &\leq 3|F| - 3. \end{aligned}$$

It remains to consider the case when G does not have a nontrivial separation (G_1, G_2) with $V(G_1) \cap V(G_2) = \{x, y\}$. Therefore, $xy \notin E(G)$, as otherwise we would have a nontrivial separation $(G[\{x, y\}], G - xy)$. Moreover, x is not a cutvertex of $G - y$, because otherwise there would exist a nontrivial separation (G', G'') of $G - y$ with $V(G') \cap V(G'') = \{x\}$, so the separation $(G[V(G') \cup \{y\}], G[V(G'') \cup \{y\}])$ of G would contradict our assumption. Hence, x belongs to only one block of $G - y$, which we denote by B .

Since $G + xy$ does not have a cutvertex, every component of $(G + xy) - (V(B) \cup \{y\})$ is adjacent in G to at least two vertices from $V(B) \cup \{y\}$. As B is a block of $G - y$, every component of $(G - y) - V(B)$ is adjacent in $G - y$ to at most one vertex of B . Since

$$(G + xy) - (V(B) \cup \{y\}) = (G - y) - V(B) = G - (V(B) \cup \{y\}),$$

this means that every component of $G - (V(B) \cup \{y\})$ is adjacent to y and to exactly one vertex of B .

Let $\{x_1, \dots, x_m\}$ denote the set of all vertices of B that have a neighbour outside B in G . Note that if $m = 1$, then x_1 is a cutvertex of G . For each $i \in \{1, \dots, m\}$, let H_i denote the subgraph of G induced by x_i , y , and the union of all components of $G - (V(B) \cup \{y\})$ that are adjacent to x_i . (Note that there might be no such component, in which case $H_i = G[\{x_i, y\}]$.) For each $i \in \{1, \dots, m\}$, let $H_{-i} = \bigcup_{i' \neq i} H_{i'}$. By Lemma 5.6

applied to $(B \cup H_{-i} + xy, H_i)$, the graph $H_i + x_iy$ does not have a cutvertex, so by the induction hypothesis there exists an x_i - y bond F_i in H_i such that $\text{pw}(H_i - y; x_i) \leq 3|F_i| - 2$. Observe that $\bigcup_{i=1}^m F_i$ is an x - y bond in G . Without loss of generality, we assume that $|F_1| \geq \dots \geq |F_m|$.

Suppose first that $m = 1$. Observe that $x \neq x_1$ in this case. Since B is a block of $G - y$, it does not have a cutvertex, and neither does $B + xx_1$. By the induction hypothesis, there exists an x - x_1 bond F_0 in B such that $\text{pw}(B - x_1; x) \leq 3|F_0| - 3$. Both F_0 and F_1 are x - y bonds in G ; let F be a largest one among them. By Lemma 5.7 applied to the separation $(B, H_1 - y)$ of $G - y$, we have

$$\begin{aligned} \text{pw}(G - y; x) &\leq \max\{\text{pw}(B; x, x_1), \text{pw}(H_1 - y; x_1)\} \\ &\leq \max\{\text{pw}(B - x_1; x) + 1, \text{pw}(H_1 - y; x_1)\} \\ &\leq \max\{3|F_0| - 2, 3|F_1| - 2\} \\ &= 3 \max\{|F_0|, |F_1|\} - 2 \\ &= 3|F| - 2, \end{aligned}$$

which is enough since x_1 is a cutvertex of G .

Now suppose that $m \geq 2$. Again, since B is a block of $G - y$, it does not have a cutvertex, and neither does $B + x_1x_2$. By the induction hypothesis, there exists an x_1 - x_2 bond F_0 in B such that $\text{pw}(B - x_2; x_1) \leq 3|F_0| - 3$. Let I denote the set of all $i \in \{1, \dots, m\}$ such that x_i belongs to the same component of $B - F_0$ as x . Observe that $F_0 \cup \bigcup_{i \in I} F_i$ is an x - y bond in G , and since the vertices x_1 and x_2 belong to distinct components of $B - F_0$, we have $I \cap \{1, 2\} \neq \emptyset$, and therefore

$$\left| F_0 \cup \bigcup_{i \in I} F_i \right| \geq |F_0| + \min\{|F_1|, |F_2|\} = |F_0| + |F_2|.$$

Furthermore, $\bigcup_{i=1}^m F_i$ is an x - y bond in G of size at least $|F_1| + 1$. Let F be a largest of the two x - y bonds $F_0 \cup \bigcup_{i \in I} F_i$ and $\bigcup_{i=1}^m F_i$. Therefore,

$$|F| \geq \max\{|F_0| + |F_2|, |F_1| + 1\}.$$

For each $i \in \{2, \dots, m\}$, we have $\text{pw}(H_i - y; x_i) \leq 3|F_i| - 2 \leq 3|F_2| - 2$. By Lemma 5.8 applied to $G - y = B \cup (H_1 - y) \cup \dots \cup (H_m - y)$ and $p = 3|F_2| - 2$, we have

$$\begin{aligned} \text{pw}(G - y; x) &\leq \max\{\text{pw}(B; x_1) + p + 1, \text{pw}(H_1 - y; x_1)\} \\ &\leq \max\{(\text{pw}(B - x_2; x_1) + 1) + p + 1, \text{pw}(H_1 - y; x_1)\} \\ &\leq \max\{(3|F_0| - 2) + (3|F_2| - 2) + 1, 3|F_1| - 2\} \\ &= \max\{3(|F_0| + |F_2|) - 3, 3(|F_1| + 1) - 5\} \\ &\leq 3 \max\{|F_0| + |F_2|, |F_1| + 1\} - 3 \\ &\leq 3|F| - 3. \end{aligned} \quad \square$$

We may now prove Theorem 5.4.

Proof of Theorem 5.4. For every 2-connected graph G with an edge xy , we have $G + xy = G$, so $G + xy$ is 2-connected. We have $\text{pw}(G) \leq \text{pw}(G - y; x) + 1$, and by Lemma 5.9, there exists an x - y bond F in G such that $\text{pw}(G - y; x) \leq 3|F| - 3$. Hence, $\text{pw}(G) \leq 3|F| - 2$. Since F is an x - y bond, it must contain the edge xy . $\square$

5.2.2 Outerplane examples

We end this section with the proof of Theorem 5.5. For every integer $k \geq 1$, we inductively define a 2-connected outerplane graph G_k with a distinguished *root edge* e_k incident with the outer face, as follows. For $k = 1$, the graph G_k is a cycle of length 4 and we choose any of its edges as the root edge. For $k \geq 2$, the graph G_k is obtained from a cycle with four edges e^1, e^2, e^3, e^4 by gluing to each of the edges e^1, e^2 and e^3 a copy of G_{k-1} along its root edge, and the root edge e_k of G_k is e^4 . See Figure 5.1.

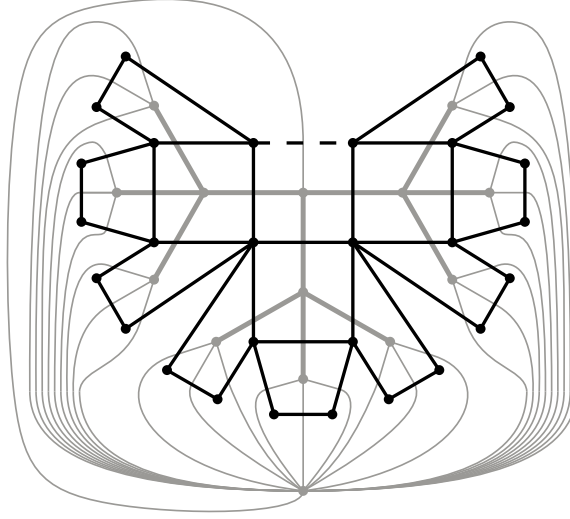


Figure 5.1: The graph G_3 and its plane dual G_3^* . The root edge e_3 is dashed.

Let G_k^* denote the plane dual of G_k , and let a_k be the vertex of G_k^* corresponding to the outer face of G_k . Observe that $T_k := G_k^* - a_k$ is a complete ternary tree of depth $k - 1$ rooted at a vertex r_k , and G_k^* is the multigraph obtained from T_k by adding the vertex a_k , an edge between a_k and r_k , and three parallel edges between a_k and each leaf of T_k . It is well known that the pathwidth of a complete ternary tree of depth h is h , thus $\text{pw}(T_k) = k - 1$. Since $T_k \subseteq G_k^*$, this implies $\text{pw}(G_k^*) \geq k - 1$. The following lemma strengthens this lower bound.

Lemma 5.10. *For every integer $k \geq 1$, $\text{pw}(G_k^*) \geq k$.*

Proof. Let e_k^* denote the edge of G_k^* between a_k and r_k which corresponds to the root edge e_k of G_k . The graph $G_k^* - e_k^*$ consists of the tree T_k and the vertex a_k attached to each leaf of T_k with three parallel edges. We prove by induction on k that $\text{pw}(G_k^* - e_k^*) \geq k$. This will imply that $\text{pw}(G_k^*) \geq k$. For $k = 1$, the graph $G_k^* - e_k^*$ consists of the vertices a_k, r_k , and three parallel edges between them, so $\text{pw}(G_k^* - e_k^*) = 1 = k$. Now, for the inductive step, suppose that $k \geq 2$, and let $(X_0, \dots, X_s)$ be a path-decomposition of $G_k^* - e_k^*$ of minimum width, and among all such path-decompositions choose one minimising s . This way, each of the bags X_0 and X_s contains at least two vertices. In particular, there exist vertices $x_0 \in X_0$ and $x_s \in X_s$ that are both distinct from a_k . The graph $G_k^* - r_k$ has three blocks B_1, B_2, B_3 , each of them is isomorphic to $G_{k-1}^* - e_{k-1}^*$, and each pair of them intersects only in a_k . Hence, each of the vertices x_0 and x_s belongs to at most one of the blocks B_1, B_2, B_3 . Without loss of generality, we assume that $V(B_2) \cap \{x_0, x_s\} = \emptyset$.

The graph B_2 is a copy of $G_{k-1}^* - e_{k-1}^*$ with a path decomposition $(X_0 \cap V(B_2), \dots, X_s \cap V(B_2))$. By the induction hypothesis, the width of this path-decomposition is at least $k - 1$, so there exists $i \in \{0, \dots, s\}$ such that $|X_i \cap V(B_2)| \geq k$. The graph $G_k^* - V(B_2)$ is a connected subgraph of $G_k^* - e_k^*$ which intersects the bags X_0 and X_s in the vertices x_0 and

x_s , respectively. Hence, it must intersect the bag X_i in some vertex x_i , and therefore,

$$|X_i| \geq |X_i \cap V(B_2)| + |\{x_i\}| \geq k + 1.$$

Therefore, the width of $(X_0, \dots, X_s)$ is at least k , and consequently, $\text{pw}(G_k^* - e_k^*) \geq k$. This completes the inductive proof. $\square$

Proof of Theorem 5.5. Bodlaender and Fomin [4] showed that for every 2-connected outerplane graph G without loops or multiple edges, we have $\text{pw}(G) \geq \text{pw}(G^*)$, where G^* denotes the plane dual of G . Hence, by Lemma 5.10, we have

$$\text{pw}(G_k) \geq \text{pw}(G_k^*) \geq k.$$

The cocircumference of G_k is the circumference of its plane dual G_k^* . Since $G_k^* - a_k$ is a tree, every cycle in G_k^* contains a_k . Therefore, the circumference of G_k^* is at most the length of a longest path in T_k plus two:

$$\text{cocirc}(G_k) = \text{circ}(G_k^*) \leq 2(k - 1) + 2 = 2k. \quad \square$$

5.3 Returning to matroid width

The definitions in Section 2.7 put the graph theorems in a common matroid framework. Branchwidth and matroid pathwidth are both invariant under duality, as in (2.5). Circumference is exchanged with cocircumference. Consequently, a theorem about bonds in graphs can establish a circuit bound for cographic matroids, including cographic matroids that are not graphic.

The following width conjectures were formulated in the joint manuscript underlying this chapter. Both follow from Conjecture 1.1 by Proposition 2.2. They express the matroid motivation for the graph questions; the graph results in turn give evidence for them by settling the cographic cases.

Conjecture 5.11. *There is an absolute constant C such that every matroid M with at least one circuit satisfies*

$$\text{bw}(M) \leq C \text{ circ}(M).$$

For graphic matroids, Theorem 5.1 and the comparisons (2.7) and (2.8) give the desired linear bound. For cographic matroids, use Theorem 5.3 and then apply branchwidth duality. The comparison between graph and cycle-matroid branchwidth is due to Mazoit and Thomassé [26]; its small forest exceptions do not affect these linear bounds. Thus the circumference and cocircumference graph theorems address two natural classes of matroids within the same conjecture.

The analogous pathwidth question is stronger. Following the usage of [18], the convention for matroid pathwidth uses the connectivity function without the additional 1 appearing in matroid branchwidth. With these conventions,

$$\text{bw}(M) \leq \text{pw}(M) + 1, \quad (5.1)$$

as follows by turning an element ordering into a caterpillar branch-decomposition; see [24]. We state the conjecture for connected matroids, where the structural issue is concentrated.

Conjecture 5.12. *There is an absolute constant C such that every connected matroid M with at least one circuit satisfies*

$$\text{pw}(M) \leq C \text{ circ}(M).$$

There is a distinction between graph connectivity and matroid connectivity here. A connected simple graph with at least three vertices is 2-connected precisely when its cycle matroid is connected. However, the graph counterexamples above do not make connectedness necessary in a matroid pathwidth bound. Indeed,

$$\text{pw}(M_1 \oplus \cdots \oplus M_t) = \max_{1 \leq i \leq t} \text{pw}(M_i). \quad (5.2)$$

For the upper bound, concatenate optimal element orderings of the summands; at each cut, only one summand contributes to the connectivity value. The reverse bound follows from minor monotonicity. This property is also recorded in [24]. A component consisting of a coloop has pathwidth zero. Therefore Conjecture 5.12 implies the same bound for arbitrary matroids having a circuit, and together with (5.1) implies Conjecture 5.11. In particular, a tree can have large graph pathwidth even though its cycle matroid is free and has matroid pathwidth zero.

For connected graphic matroids with a circuit, Theorem 5.2 and (2.9) give $\text{pw}(M) \leq \text{circ}(M)$. For connected cographic matroids with a circuit, Theorem 5.4, (2.9), and duality give $\text{pw}(M) \leq 3 \text{circ}(M) - 1$. Singleton matroids are immediate. For representations by graphs with parallel edges, simplify the graph when applying a graph pathwidth bound: this leaves graph pathwidth unchanged and cannot increase circumference or cocircumference. The two-vertex case is immediate as well. The common comparison (2.9) follows, for example, from Kashyap's construction of a supergraph whose cycle matroid has pathwidth one greater than the graph pathwidth [24].

For general matroids, the source manuscript records a quadratic upper bound. One route uses Seymour's theorem that contracting a largest circuit of a connected matroid reduces circumference, as presented by Ding, Oporowski, and Oxley [15]. Another uses Corollary 3.16 of [23]; see also Theorem 9 and the accompanying discussion in [8]. The latter route bounds contraction*-depth and then uses the width comparisons in Proposition 2.2. These connections with contraction-based decompositions link the width questions in this chapter to the depth parameters studied in the thesis. The graph results establish linear bounds in the graphic and cographic settings; the matroid questions ask how much of that control persists beyond graph representations. These width bounds do not imply the stronger depth bound in Conjecture 1.1: the inequality $\text{pw}(M) \leq c^*d(M)$ bounds pathwidth from above by depth, so a small pathwidth alone gives no corresponding upper bound on depth.

Chapter 6

Conclusions and further directions

The results of this thesis connect graph and matroid theory in both directions. Graph decompositions motivate the extension and minor-container theorems, while matroid depth and width questions lead to the pathwidth–bond theorem for graphs. Two quantitative problems bring these connections into focus: bounding contraction*-depth by circuit size, and determining the depth needed to realise a matroid of bounded branch-depth as a minor.

6.1 Contraction*-depth and circumference

The first problem, Conjecture 1.1, asks whether there is an absolute constant $C > 0$ such that, for every positive integer k and every finite matroid M whose circuits have at most k elements,

$$\text{cd}^*(M) \leq Ck.$$

The known bound is quadratic: $\text{cd}^*(M) \leq k^2$ [23]. A linear dependence would be best possible in order of magnitude. As observed in Section 1.4, for $r \geq 2$ the uniform matroid U_{r,r^2} has contraction*-depth r and circumference $r + 1$.

The extension theorem gives another way to approach the conjecture. Recall that $e_{\text{cd}}(M)$ is the minimum contraction-depth of an extension of M . Theorem 3.1 gives

$$\text{cd}^*(M) \leq e_{\text{cd}}(M) \leq \text{cd}^*(M) + 1.$$

Consequently, the conjecture is equivalent to asking whether every matroid whose circuits have at most k elements has an extension of contraction-depth $O(k)$, with an absolute implied constant. The quadratic dependence on circuit size is necessary for contraction-depth itself [8, full version, Section 7]; the question is whether allowing extra elements always makes a linear bound possible. This formulation connects the conjecture directly to Chapter 3.

For graphic matroids the conjecture follows from the bound $\text{td}(G) \leq \text{circ}(G)$ for 2-connected graphs [6], the inequality $\text{cd}^*(M(G)) \leq \text{td}(G)$ [23], and reduction to matroid components. The cographic case is a natural next test. Chapter 5 establishes the corresponding pathwidth bound, but contraction*-depth is not self-dual, so the graphic depth bound does not transfer by duality.

The cycle example illustrates why the obstruction for graph tree-depth does not carry over. The graphs C_n have unbounded tree-depth and cocircumference two. Their bond matroids have circumference two and satisfy

$$M(C_n)^* \cong U_{1,n}, \quad \text{cd}^*(M(C_n)^*) = 1 \quad (n \geq 3),$$

Thus the failure of a bound on graph tree-depth in terms of bond size does not obstruct the cographic case of Conjecture 1.1.

Finally, Proposition 2.2 gives

$$\text{pw}(M) \leq \text{cd}^*(M), \quad \text{bw}(M) \leq \text{cd}^*(M) + 1.$$

A linear bound for contraction*-depth would therefore settle both width conjectures, Conjectures 5.11 and 5.12. The circuit-size question would then provide a common explanation for the graphic and cographic width bounds discussed in Section 5.3.

6.2 The growth of minor-container bounds

The second problem concerns the optimal quantitative relationship in Chapter 4. For an $\mathbb{F}$ -representable matroid M , recall the notation

$$m_{\text{cdd},\mathbb{F}}(M) = \min\{\text{cdd}(N) : N \text{ is } \mathbb{F}\text{-representable and } M \text{ is a minor of } N\}.$$

Minor-monotonicity of branch-depth and Theorem 4.2 give

$$\text{bd}(M) \leq m_{\text{cdd},\mathbb{F}}(M) \leq 2 \text{bd}(M)(4^{\text{bd}(M)} - 1) + 1.$$

How large can the gap between these parameters be? The aim is to determine the smallest possible growth rate of a function f such that $m_{\text{cdd},\mathbb{F}}(M) \leq f(\text{bd}(M))$ for every field $\mathbb{F}$ and every $\mathbb{F}$ -representable matroid M . This is Problem 4.7.

A quadratic bound

$$m_{\text{cdd},\mathbb{F}}(M) \leq K \text{bd}(M)^2 + L,$$

with absolute constants K, L , provides a natural target for improving the exponential estimate proved here. The stronger two-parameter bound of $Kdr + L$ discussed in Section 4.5 would apply whenever M admits a (d, r) -decomposition. Taking $d = r = \text{bd}(M)$ would give the quadratic bound, with matroids on at most one element handled separately.

The proof in Chapter 4 identifies a source of the exponential dependence. At each level, the additional contraction and deletion steps cost at most $6r$, but the width passed to the recursive calls may grow from r to $4r$. A possible route to a better bound is to control how this increase accumulates along a root-to-leaf path. Merely reducing a fixed multiplicative increase that remains greater than one would still give exponential dependence on the depth.

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