Three numerical methods for the palindromic eigenvalue problem

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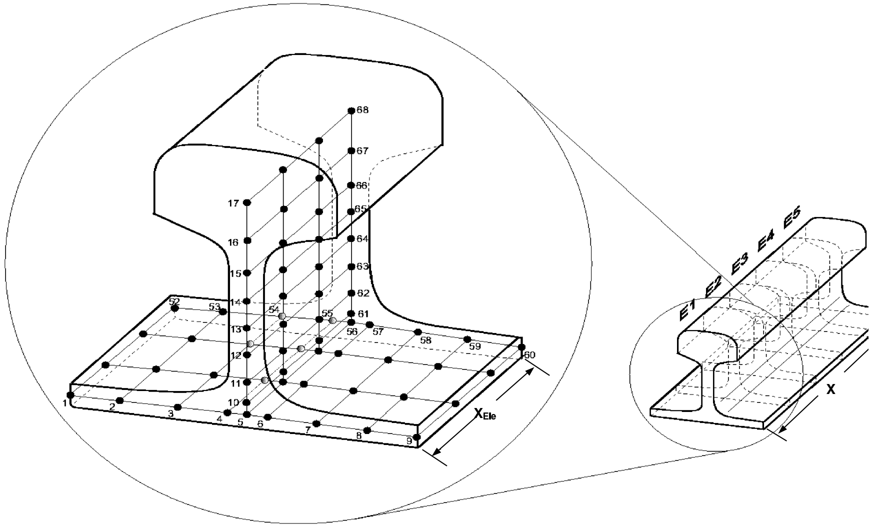
MATHEON – Mathematics for key technologies

Structured polynomial eigenvalue problems

Application: vibration analysis of rail tracks excited by high speed trains



Finite element discretization leads to the **palindromic eigenvalue problem**

$$\left(\lambda^2 A_0^T + \lambda A_1 + A_0\right) x = 0,$$

where $A_0, A_1 \in \mathbb{C}^{n\times n}$, and $A_1^T = A_1$.

Palindromic matrix polynomials

Definition: A matrix polynomial $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^k A_k$ is called T-palindromic (in short: **palindromic**) if

$$P(\lambda) = \sum_{j=0}^{k} \lambda^{k-j} A_j^T.$$

Examples:

- $P(\lambda) = A + \lambda B + \lambda^2 B^T + \lambda^3 A^T$;
- $P(\lambda) = A_2^T + \lambda A_1^T + \lambda^2 A_0 + \lambda^3 A_1 + \lambda^4 A_2$, where A_0 is symmetric;
- palindromic pencils $\lambda Z + Z^T$.

Formal resemblance with linguistic palindroms like "**I prefer pi**".

Properties of palindromic matrix polynomials

General assumption: all matrix polynomials under consideration are regular, i.e., $\det P(\lambda) \not\equiv 0$.

Spectral symmetry: Palindromic matrix polynomials have a symplectic spectrum.

- if λ_0 is an eigenvalue of $P(\lambda)$, then so is λ_0^{-1};
- pairing occurs also in algebraic, geometric, and partial multiplicities;
- symmetry degenerates for $\lambda_0 = 1$ and $\lambda_0 = -1$;

"Palindromic matrix polynomials generalize symplectic matrices".

How to solve palindromic eigenvalue problems

Linearization: Mackey, Mackey, M., Mehrmann: linearization theory for general and structured matrix polynomials

- Under modest assumptions, any polynomial palindromic eigenvalue problem can be transformed to a linear palindromic eigenvalue problem.

Example: $P(\lambda) = \lambda^2 A_0^T + \lambda A_1 + A_0$. Then

$$\lambda Z + Z^T := \lambda \begin{bmatrix} A_0^T & A_1 - A_0 \\ A_0^T & A_0^T \end{bmatrix} + \begin{bmatrix} A_0 & A_0 \\ A_1 - A_0^T & A_0 \end{bmatrix}$$

is a linearization for $P(\lambda)$ if -1 is not an eigenvalue of $P(\lambda)$.

Benefit: Symplectic spectrum preserved in finite precision arithmetic.

How to solve linear palindromic eigenvalue problems

Task: Solve the generalized eigenvalue problem for $\lambda Z + Z^T$.

- T-congruence transformations preserve the structure:

$$(\lambda Z + Z^T) \mapsto P^T(\lambda Z + Z^T)P, \quad P \text{ invertible}$$

- Numerical stability: Choose $P = U$ unitary if possible.

- Look for condensed forms under simultaneous unitary consimilarity:

$$(\lambda Z + Z^T) \mapsto \overline{U}^{-1}(\lambda Z + Z^T)U, \quad U \text{ unitary}$$

Advantage: We have to store and work on Z only.

The anti-triangular Schur form

Theorem: Let $Z \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$U^T Z U = \begin{bmatrix} 0 & \dots & 0 & z_{1n} \\ \vdots & \iddots & z_{2,n-1} & \vdots \\ 0 & \iddots & \iddots & \vdots \\ z_{n1} & \dots & \dots & z_{nn} \end{bmatrix}$$

is in **anti-triangular Schur form**.

Consequence: If $\det(\lambda Z + Z^T) \not\equiv 0$ then the eigenvalues of $\lambda Z + Z^T$ are

$$-\frac{z_{n1}}{z_{1n}}, \dots, -\frac{z_{1n}}{z_{n1}}, \qquad (\text{where } \frac{z}{0} := \infty).$$

Question: How do we compute the anti-triangular form numerically?

Method 1: The Laub-trick method

Theorem: (generalizes a trick by A. Laub for the computation of the Hamiltonian Schur form) Let $\lambda Z + Z^T \in \mathbb{C}^{2n\times 2n}$ be regular and let

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

be its generalized Schur decomposition, where $X_{11}, Y_{11} \in \mathbb{C}^{n\times n}$. If

$$\mu \in \sigma(\lambda X_{11} + Y_{11}) \implies \frac{1}{\mu} \notin \sigma(\lambda X_{11} + Y_{11})$$

then

$$U = \begin{bmatrix} W_{11} & Q_{11}^T R_n \\ W_{21} & Q_{12}^T R_n \end{bmatrix}, \qquad \left(R_n := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \right)$$

is unitary and

$$U^T Z U = \begin{bmatrix} 0 & Y_{11}^T R_n \\ R_n X_{11} & * \end{bmatrix}.$$

is in anti-triangular form.

Method 1: The Laub-trick method

Algorithm: (for regular $\lambda Z + Z^T$ not having eigenvalues with modulus 1)

1. Compute the generalized Schur decomposition

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

2. Reorder the eigenvalues such that $\lambda X_{11} + Y_{11}$ contains all eigenvalues with $|\lambda| > 1$.

3. Set $U = \begin{bmatrix} W_{11} & Q_{11}^T R_n \\ W_{21} & Q_{12}^T R_n \end{bmatrix}$.

4. Compute $Z_{22} = \begin{bmatrix} R_n Q_{11} & R_n Q_{12} \end{bmatrix} Z \begin{bmatrix} Q_{11}^T R_n \\ Q_{12}^T R_n \end{bmatrix}$.

5. Set $\widetilde{Z} := \begin{bmatrix} 0 & Y_{11}^T R_n \\ R_n X_{11} & Z_{22} \end{bmatrix}$.

Method 1: The Laub-trick method

Properties:

+ cost is essentially the cost of `QZ` with reordering;

− only applicable if Z has even dimension and if $\lambda Z + Z^T$ does not have eigenvalues with modulus 1;

− problems if there are eigenvalues with modulus close to ± 1; $\rightsquigarrow$ `QZ` might detect more or less than n eigenvalues λ with $|\lambda| > 1$.

Questions: Are there other methods?

Method 2: A Jacobi-like method

Idea: Annihilate one diagonal or two off diagonal pivot elements in the strict upper anti-triangular part of Z in each Jacobi-step:

$$
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \bullet & \cdot & \cdot & \cdot & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & * & * & * & * & * & * \\
\cdot & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
\qquad
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & \bullet & \cdot & \cdot & \cdot & * & * \\
\cdot & \bullet & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & * & * & * & * & * & * \\
\cdot & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
$$

This can always be achieved via a unitary consimilarity transformation.

Method 2: A Jacobi-like method

Diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & {\color{red}\bullet} & \cdot & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & {\color{blue}\bullet} & * & * & * & * & {\color{blue}\bullet} & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Consider the colored 2×2 subproblem:

$$\begin{bmatrix} {\color{red}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}$$

Method 2: A Jacobi-like method

Diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & \bullet & * & * & * & * & \bullet & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Compute the anti-triangular form of the 2×2 problem:

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} \circ & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Diagonal pivots:

$$
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & & u_{21} & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & u_{12} & & & & & u_{22} & \\
 & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & {\color{red}\bullet} & \cdot & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & * & * & * & * & * & * \\
\cdot & {\color{blue}\bullet} & * & * & * & * & {\color{blue}\bullet} & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
\begin{bmatrix}
1 & & & & & & & \\
 & u_{11} & & & & & u_{12} & \\
 & & 1 & & & & & \\
 & & & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & u_{21} & & & & & u_{22} & \\
 & & & & & & & 1
\end{bmatrix}
$$

Then update the $n \times n$ matrix.

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\begin{bmatrix} {\color{red}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}
\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{red}\circ} & * \\ * & * \end{bmatrix}
$$

Method 2: A Jacobi-like method

Diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \circ & \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Then update the $n \times n$ matrix.

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} \circ & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots:

$$\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & \bullet & \cdot & \cdot & \cdot & * & * \\
\cdot & \bullet & \cdot & \cdot & \cdot & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & * & * & * & * & * & * \\
\cdot & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}$$

Question: Why consider two pivots?

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Consider the colored 2×2 problem:

$$\begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Compute the anti-triangular form of the 2×2 problem:

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

We may use different unitary transformation from the left and the right,

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$
\begin{bmatrix}
1 &&&&&&& \\
& u_{11} &&&& u_{21} && \\
&& v_{11} &&&& v_{21} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
& u_{12} &&&& u_{22} && \\
&& v_{12} &&&& v_{22} & \\
&&&&&&& 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ast \\
\cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & \ast \\
\cdot & \cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast \\
\cdot & \cdot & \cdot & \cdot & \ast & \ast & \ast & \ast \\
\cdot & \cdot & \cdot & \ast & \ast & \ast & \ast & \ast \\
\cdot & \cdot & {\color{blue}\bullet} & \ast & \ast & \ast & {\color{blue}\bullet} & \ast \\
\cdot & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
1 &&&&&&& \\
& u_{11} &&&& u_{12} && \\
&& v_{11} &&&& v_{12} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
& u_{21} &&&& u_{22} && \\
&& v_{21} &&&& v_{22} & \\
&&&&&&& 1
\end{bmatrix}
$$

because of the position of the subproblem. $\rightsquigarrow$ more freedom in the parameters

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}
\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{green}\circ} & \ast \\ \ast & \ast \end{bmatrix}
$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$\begin{bmatrix} 1 &&&&&&& \\ & u_{11} &&&& u_{21} && \\ && v_{11} &&&& v_{21} & \\ &&& 1 &&&& \\ &&&& 1 &&& \\ & u_{12} &&&& u_{22} && \\ && v_{12} &&&& v_{22} & \\ &&&&&&& 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & \bullet & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\ \cdot & \bullet & * & * & * & \bullet & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix} \begin{bmatrix} 1 &&&&&&& \\ & u_{11} &&&& u_{12} && \\ && v_{11} &&&& v_{12} & \\ &&& 1 &&&& \\ &&&& 1 &&& \\ & u_{21} &&&& u_{22} && \\ && v_{21} &&&& v_{22} & \\ &&&&&&& 1 \end{bmatrix}$$

Simultaneously, a second 2×2 system – marked by $\bullet$ – will be transformed.

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix}$$

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} {\color{green}\bullet} & \bullet \\ \bullet & \bullet \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} {\color{green}*} & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots: assume, we only consider one off-diagonal pivot;

$$
\begin{bmatrix}
1 &&&&&&& \\
& u_{11} &&&& u_{21} && \\
&& v_{11} &&&& v_{21} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
& u_{12} &&&& u_{22} && \\
&& v_{12} &&&& v_{22} & \\
&&&&&&& 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\
\cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & \bullet & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\
\cdot & \bullet & * & * & * & \bullet & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
\begin{bmatrix}
1 &&&&&&& \\
& u_{11} &&&& u_{12} && \\
&& v_{11} &&&& v_{12} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
& u_{21} &&&& u_{22} && \\
&& v_{21} &&&& v_{22} & \\
&&&&&&& 1
\end{bmatrix}
$$

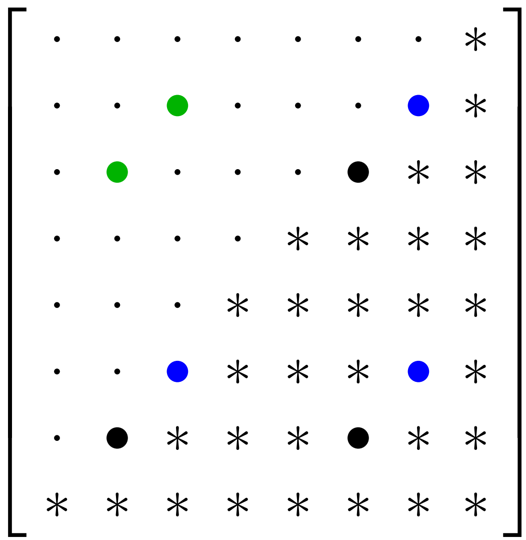
Use the freedom in the parameters to anti-triangularize the second system as well.

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix}
\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix}
$$

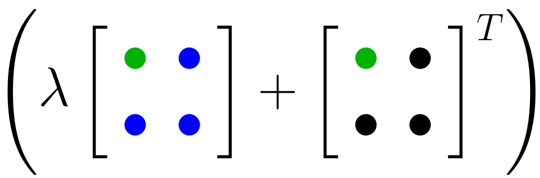
$$
\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}
\begin{bmatrix} {\color{green}\bullet} & \bullet \\ \bullet & \bullet \end{bmatrix}
\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}
=
\begin{bmatrix} {\color{green}*} & * \\ * & * \end{bmatrix}
$$

Method 2: A Jacobi-like method

Off-diagonal pivots:



Anti-triangularize the colored/black generalized 2×2 problem:



Method 2: A Jacobi-like method

Off-diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\ \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & \bullet & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\ \cdot & \bullet & * & * & * & \bullet & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Anti-triangularize the colored/black generalized 2×2 problem:

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \left(\lambda \begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix} + \begin{bmatrix} {\color{green}\bullet} & \bullet \\ \bullet & \bullet \end{bmatrix}^T \right) \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \lambda \begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix} + \begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Off-diagonal pivots:

$$
\begin{bmatrix}
1 &&&&&&& \\
& u_{11} &&&& u_{21} && \\
&& v_{11} &&&& v_{21} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
& u_{12} &&&& u_{22} && \\
&& v_{12} &&&& v_{22} & \\
&&&&&&& 1
\end{bmatrix}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\
\cdot & \cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & {\color{blue}\bullet} & * \\
\cdot & {\color{green}\bullet} & \cdot & \cdot & \cdot & \bullet & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & {\color{blue}\bullet} & * & * & * & {\color{blue}\bullet} & * \\
\cdot & \bullet & * & * & * & \bullet & * & * \\
* & * & * & * & * & * & * & *
\end{bmatrix}
\begin{bmatrix}
1 &&&&&&& \\
& u_{11} &&&& u_{12} && \\
&& v_{11} &&&& v_{12} & \\
&&& 1 &&&& \\
&&&& 1 &&& \\
& u_{21} &&&& u_{22} && \\
&& v_{21} &&&& v_{22} & \\
&&&&&&& 1
\end{bmatrix}
$$

Update the $n \times n$ matrix.

$$
\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}
\left(\lambda \begin{bmatrix} {\color{green}\bullet} & {\color{blue}\bullet} \\ {\color{blue}\bullet} & {\color{blue}\bullet} \end{bmatrix} + \begin{bmatrix} {\color{green}\bullet} & \bullet \\ \bullet & \bullet \end{bmatrix}^T \right)
\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
= \lambda \begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix} + \begin{bmatrix} {\color{green}\circ} & * \\ * & * \end{bmatrix}
$$

Method 2: A Jacobi-like method

Off-diagonal pivots:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \circ & \cdot & \cdot & \cdot & * & * \\ \cdot & \circ & \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * & * & * & * \\ \cdot & \cdot & \cdot & * & * & * & * & * \\ \cdot & \cdot & * & * & * & * & * & * \\ \cdot & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{bmatrix}$$

Update the $n \times n$ matrix.

$$\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \left(\lambda \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}^T \right) \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \lambda \begin{bmatrix} \circ & * \\ * & * \end{bmatrix} + \begin{bmatrix} \circ & * \\ * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

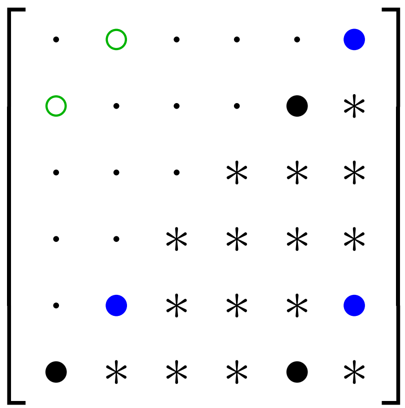
E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \circ & \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \cdot & * & * & * & * & * \\ \bullet & * & * & * & * & \bullet \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:



Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \circ & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \circ & \cdot & \cdot & \bullet & * & * \\ \cdot & \cdot & \bullet & * & * & \bullet \\ \cdot & * & * & * & * & * \\ \bullet & * & * & \bullet & * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & {\color{green}\circ} & \cdot & {\color{blue}\bullet} \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & {\color{blue}\bullet} & * & {\color{blue}\bullet} \\ {\color{green}\circ} & \cdot & \bullet & * & * & * \\ \cdot & * & * & * & * & * \\ \bullet & * & \bullet & * & * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \circ & \bullet \\ \cdot & \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * & * \\ \circ & \bullet & * & * & * & * \\ \bullet & \bullet & * & * & * & * \end{bmatrix}$$

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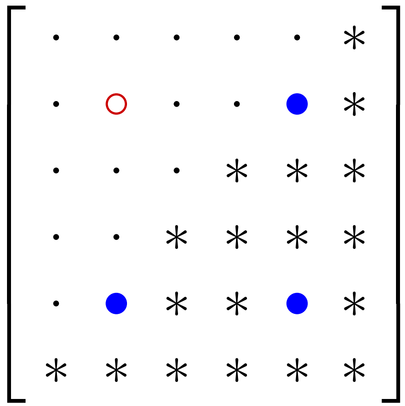
E.g., cyclic-by-row-sweep:



Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \ast \\ \cdot & \cdot & {\color{green}\circ} & \cdot & {\color{blue}\bullet} & \ast \\ \cdot & {\color{green}\circ} & \cdot & \bullet & \ast & \ast \\ \cdot & \cdot & {\color{blue}\bullet} & \ast & {\color{blue}\bullet} & \ast \\ \cdot & \bullet & \ast & \bullet & \ast & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & {\color{green}\circ} & {\color{blue}\bullet} & * \\ \cdot & \cdot & \cdot & {\color{blue}\bullet} & {\color{blue}\bullet} & * \\ \cdot & {\color{green}\circ} & \bullet & * & * & * \\ \cdot & \bullet & \bullet & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Sweep: Annihilate each pivot element at least once.

E.g., cyclic-by-row-sweep:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & * & * \\ \cdot & \cdot & \circ & \bullet & * & * \\ \cdot & \cdot & \bullet & \bullet & * & * \\ \cdot & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

Method 2: A Jacobi-like method

Properties of the algorithm:

+ locally and asymptotically quadratically convergent;

+ globally convergent in experiments;

+ converges fast for matrices Z close to anti-triangular form

– expensive in general (cost of 3 sweeps $\hat{=}$ cost of `QZ`)

– convergence problems for badly scaled problems

– convergence problems for large n

Method 3: A hybrid method

Laub-trick:

- $+$ works for moderate sizes of n;
- $+$ essentially cost of `QZ`;
- $-$ problems for eigenvalues with modulus near one;

Jacobi:

- $+$ works nicely if problem is small and eigenvalues do not differ too much in modulus;

Idea: Combine the positive properties of these two algorithms. Use the Laub-trick for getting all eigenvalues sufficiently far away from the unit circle and use Jacobi for the eigenvalues near the unit circle.

Method 3: A hybrid method

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

Step 1: Given a tolerance $\alpha > 1$ and a regular $\lambda Z + Z^T \in \mathbb{C}^{2n \times 2n}$, compute its generalized Schur decomposition, where the eigenvalues are ordered in such a way that

$$\begin{aligned} \sigma(\lambda X_{11} + Y_{11}) &\subseteq \{\lambda \in \mathbb{C} : |\lambda| \geq \alpha\}, \\ \sigma(\lambda X_{22} + Y_{22}) &\subseteq \{\lambda \in \mathbb{C} : \alpha > |\lambda| > \frac{1}{\alpha}\}, \\ \sigma(\lambda X_{33} + Y_{33}) &\subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{\alpha}\}. \end{aligned}$$

Method 3: A hybrid method

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

Step 2: By the Laub trick, the matrix

$$\begin{bmatrix} W_{11} & Q_{11}^T R_m \\ W_{21} & Q_{12}^T R_m \\ W_{31} & Q_{13}^T R_m \end{bmatrix}$$

has orthonormal columns. Extend this matrix to a unitary matrix

$$U := \begin{bmatrix} W_{11} & U_{12} & Q_{11}^T R_m \\ W_{21} & U_{22} & Q_{12}^T R_m \\ W_{31} & U_{32} & Q_{13}^T R_m \end{bmatrix}.$$

Method 3: A hybrid method

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

Step 3: Compute

$$U^T Z U = \begin{bmatrix} 0 & 0 & Y_{11}^T R_m \\ 0 & Z_{22} & Z_{23} \\ R_m X_{11} & Z_{32} & Z_{33} \end{bmatrix},$$

where $Y_{11}^T R_m \in \mathbb{C}^{m \times m}$ and $R_m X_{11} \in \mathbb{C}^{m \times m}$ are in anti-triangular form and $Z_{22} \in \mathbb{C}^{(n-2m)\times(n-2m)}$ has only eigenvalues in $\{\lambda \in \mathbb{C} : \alpha > |\lambda| > \frac{1}{\alpha}\}$.

Method 3: A hybrid method

$$\lambda Z + Z^T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \left(\lambda \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix} \right) \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

Step 3: Compute

$$U^T Z U = \begin{bmatrix} 0 & 0 & Y_{11}^T R_m \\ 0 & Z_{22} & Z_{23} \\ R_m X_{11} & Z_{32} & Z_{33} \end{bmatrix},$$

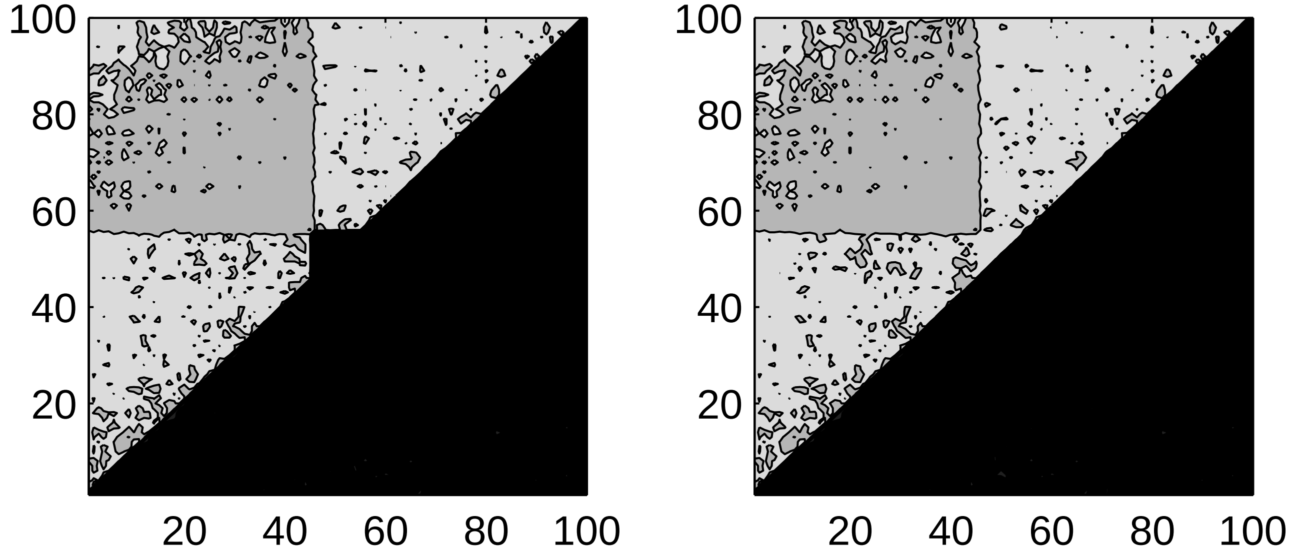
where $Y_{11}^T R_m \in \mathbb{C}^{m\times m}$ and $R_m X_{11} \in \mathbb{C}^{m\times m}$ are in anti-triangular form and $Z_{22} \in \mathbb{C}^{(n-2m)\times(n-2m)}$ has only eigenvalues in $\{\lambda \in \mathbb{C} : \alpha > |\lambda| > \frac{1}{\alpha}\}$.

Step 4: Anti-triangularize Z_{22} by some expensive, but accurate method (e.g., palindromic Jacobi algorithm, Schröder's palindromic QR algorithm).

Numerical experiments

Test: 100 random 100×100 matrices with 10 eigenvalues in an annulus in the complex plane with outer radius $1+10^{-12}$ and inner radius $1/(1+10^{-12})$.

Typical behavior: (black: elements of modulus larger than one; light grey: elements of modulus 10^{-15})



left: method 3 before applying Jacobi to the small problem
right: method 3 after applying Jacobi to the small problem

Numerical experiments

Test: 100 random 100×100 matrices with 10 eigenvalues in an annulus in the complex plane with outer radius $1+10^{-12}$ and inner radius $1/(1+10^{-12})$.

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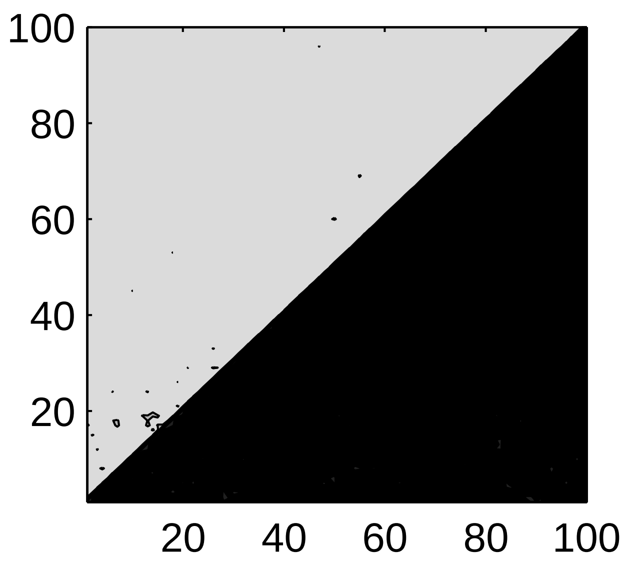

method 3 with applying Jacobi to the small submatrix followed by one full sweep of Jacobi.