Linearization of structured matrix polynomials

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Structured matrix polynomials

Topic: matrix polynomials

$$P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0, \qquad A_0, \ldots, A_k \in \mathbb{C}^{n \times n}$$

"**structured**": symmetries in the matrix coefficients

$\rightsquigarrow$ symmetries in the spectrum, eigenvectors, and invariant subspaces

Typical structures:

- palindromic structure (*symplectic* spectrum).
- even/odd structure (*Hamiltonian* spectrum);

Palindromic matrix polynomials

Definition: A matrix polynomial $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^k A_k$ is called T-palindromic (in short: **palindromic**) if

$$P(\lambda) = \mathrm{rev}\, P(\lambda)^T := \sum_{j=0}^{k} \lambda^{k-j} A_j^T.$$

Examples:

- $P(\lambda) = A + \lambda B + \lambda^2 B^T + \lambda^3 A^T$;
- $P(\lambda) = A_2^T + \lambda A_1^T + \lambda^2 A_0 + \lambda^3 A_1 + \lambda^4 A_2$, where A_0 is symmetric;
- palindromic pencils $\lambda Z + Z^T$.

Formal resemblance with linguistic palindroms like "**I prefer pi**".

Palindromic matrix polynomials

General assumption: all matrix polynomials under consideration are regular, i.e., $\det P(\lambda) \not\equiv 0$.

Spectral symmetry: Palindromic matrix polynomials have a symplectic spectrum.

- if λ_0 is an eigenvalue of $P(\lambda)$, then so is λ_0^{-1};
- pairing occurs also in algebraic, geometric, and partial multiplicities;
- symmetry degenerates for $\lambda_0 = 1$ and $\lambda_0 = -1$;

Other structured matrix polynomials

Definition: A matrix polynomial $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^k A_k$ is called:

T-palindromic		$P(\lambda) = \text{rev}\, P(\lambda)^T$
$*$-palindromic		$P(\lambda) = \text{rev}\, P(\lambda)^*$
T-anti-palindromic		$P(\lambda) = -\text{rev}\, P(\lambda)^T$
$*$-anti-palindromic	if	$P(\lambda) = -\text{rev}\, P(\lambda)^*$
T-even		$P(\lambda) = P(-\lambda)^T$
T-odd		$P(\lambda) = -P(-\lambda)^T$
$*$-even		$P(\lambda) = P(-\lambda)^*$
$*$-odd		$P(\lambda) = -P(-\lambda)^*$

Example: $\lambda^2 A + \lambda B + C$ is **T-even** if $A = A^T, B = -B^T, C = C^T$

(coefficients alter between symmetric and skew-symmetric)

Other structured matrix polynomials

Spectral symmetry:

Structure of $P(\lambda)$	eigenvalue pairing	pairing degenerates for
T-palindromic	$\lambda,\ 1/\lambda$	$+1, -1$
T-anti-palindromic	$\lambda,\ 1/\lambda$	$+1, -1$
$*$-palindromic	$\lambda,\ 1/\overline{\lambda}$	unit circle
$*$-anti-palindromic	$\lambda,\ 1/\overline{\lambda}$	unit circle
T-even	$\lambda, -\lambda$	$0, \infty$
T-odd	$\lambda, -\lambda$	$0, \infty$
$*$-even	$\lambda, -\overline{\lambda}$	imaginary axis and ∞
$*$-odd	$\lambda, -\overline{\lambda}$	imaginary axis and ∞

How to solve structured eigenvalue problems

Given: structured eigenvalue problem $(\lambda^m A_m + \cdots + \lambda A_1 + A_0)x = 0.$

Standard approach: linearize the problem; use the companion form

$$\lambda \begin{bmatrix} A_m & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix} \begin{bmatrix} \lambda^{m-1}x \\ \vdots \\ \lambda x \\ x \end{bmatrix} = \begin{bmatrix} -A_{m-1} & \dots & \dots & -A_0 \\ I & & & \\ & \ddots & & \\ & & I & 0 \end{bmatrix} \begin{bmatrix} \lambda^{m-1}x \\ \vdots \\ \lambda x \\ x \end{bmatrix}$$

Drawbacks:

- symmetry structure destroyed
- numerical solution: roundoff errors may destroy eigenvalue symmetry

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Advantages:

- easy to construct from original data
- strong linearization (properly reflects eigenvalue ∞)
- eigenvector x of $P(\lambda)$ can be easily be recovered

How to solve structured eigenvalue problems

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Task: find a linearization having the same symmetry structure as the original matrix polynomial that still has the nice properties of the companion form

An ansatz space for linearizations

Given: structured eigenvalue problem $(\lambda^m A_m + \cdots + \lambda A_1 + A_0)x = 0$.

Standard approach: linearize the problem; use the companion form

$$\lambda \underbrace{\begin{bmatrix} A_m & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} \lambda^{m-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}}_{=:\Lambda} x = \underbrace{\begin{bmatrix} -A_{m-1} & \dots & -A_1 & -A_0 \\ I & & & \\ & \ddots & & \\ & & I & 0 \end{bmatrix}}_{\mathcal{B}} \underbrace{\begin{bmatrix} \lambda^{m-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}}_{=\Lambda} x$$

The companion form $C(\lambda) = \lambda\mathcal{A} - \mathcal{B}$ satisfies

$$C(\lambda)\Lambda x = \begin{bmatrix} P(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} x = e_1 \otimes P(\lambda)x$$

An ansatz space for linearizations

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$$C(\lambda)\Lambda x = \begin{bmatrix} P(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} x = e_1 \otimes P(\lambda)x$$

Idea: Allow linearizations $L(\lambda)$ satisfying

$$L(\lambda)\Lambda x = \begin{bmatrix} v_1 P(\lambda) \\ v_2 P(\lambda) \\ \vdots \\ v_m P(\lambda) \end{bmatrix} x = v \otimes P(\lambda)x$$

Then $L(\lambda)$ still has the eigenvector recovery property as $C(\lambda)$.

We will call v an **ansatz vector**.

An ansatz space for linearizations

This can be said differently. Consider again $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} = \begin{bmatrix} A_m & 0 & \cdots & 0 \\ 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{bmatrix}, \quad -\mathcal{B} = \begin{bmatrix} A_{m-1} & \dots & A_1 & A_0 \\ -I & & & 0 \\ & \ddots & & \vdots \\ & & -I & 0 \end{bmatrix}$$

Introduce the so-called **shifted sum**:

$$nm \overset{\begin{matrix} n & n(m-1) \end{matrix}}{\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix}} \boxplus \overset{\begin{matrix} n(m-1) & n \end{matrix}}{\begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \end{bmatrix}} := \overset{\begin{matrix} n & n(m-1) & n \end{matrix}}{\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 + \mathcal{B}_1 & \mathcal{B}_2 \end{bmatrix}}$$

An ansatz space for linearizations

This can be said differently. Consider again $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} = \begin{bmatrix} A_m & 0 & \cdots & 0 \\ 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{bmatrix}, \quad -\mathcal{B} = \begin{bmatrix} A_{m-1} & \dots & A_1 & A_0 \\ -I & & & 0 \\ & \ddots & & \vdots \\ & & -I & 0 \end{bmatrix}$$

We obtain:

$$\mathcal{A} \boxplus (-\mathcal{B}) = \begin{bmatrix} A_m & A_{m-1} & \dots & A_1 & A_0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The first block row consists of the coefficients of the polynomial $P(\lambda)$.

An ansatz space for linearizations

Ansatz: Look for linearizations $L(\lambda)$ in the vector space

$$\begin{aligned}\mathbb{L}_1(P) &= \left\{\lambda\mathcal{A} - \mathcal{B} \,:\, (\lambda\mathcal{A} - \mathcal{B})\Lambda = v \otimes P(\lambda), v = (v_1, \ldots, v_m)^T \in \mathbb{C}^m\right\} \\ &= \left\{\lambda\mathcal{A} - \mathcal{B} \,:\, \mathcal{A} \boxplus (-\mathcal{B}) = \begin{bmatrix} v_1 A_m & \cdots & v_1 A_1 & v_1 A_0 \\ \vdots & \ldots & \vdots & \vdots \\ v_m A_m & \cdots & v_m A_1 & v_m A_0 \end{bmatrix}, v_i \in \mathbb{C}\right\}\end{aligned}$$

Theorem (Mackey, Mackey, M., Mehrmann, 2004):
$L(\lambda) \in \mathbb{L}_1(P)$ is a linearization of the regular matrix polynomial $P(\lambda)$ if and only if $L(\lambda)$ is regular, i.e., $\det L(\lambda) \not\equiv 0$.

An ansatz space for linearizations

Ansatz: Look for linearizations $L(\lambda)$ in the vector space

$$\begin{aligned}
\mathbb{L}_1(P) &= \left\{\lambda\mathcal{A} - \mathcal{B} \;:\; (\lambda\mathcal{A} - \mathcal{B})\Lambda = v \otimes P(\lambda), v = (v_1, \ldots, v_m)^T \in \mathbb{C}^m\right\} \\
&= \left\{\lambda\mathcal{A} - \mathcal{B} \;:\; \mathcal{A} \boxplus (-\mathcal{B}) = \begin{bmatrix} v_1 A_m & \cdots & v_1 A_1 & v_1 A_0 \\ \vdots & \ldots & \vdots & \vdots \\ v_m A_m & \cdots & v_m A_1 & v_m A_0 \end{bmatrix}, v_i \in \mathbb{C}\right\}
\end{aligned}$$

Note: These linearizations share nice properties with the companion form:

- eigenvectors have the form $\Lambda = \begin{bmatrix} \lambda^{m-1}x \\ \vdots \\ x \end{bmatrix}$, where $P(\lambda)x = 0$;
- information on eigenvalue ∞ is properly reflected (strong linearization)

Analogous generalization of 2nd companion form

row shifted sum:

$$\begin{array}{c} \\ n \\ n(k-1) \end{array}\overset{nk}{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}} \;\underset{\downarrow}{\boxplus}\; \begin{array}{c} \\ n(k-1) \\ n \end{array}\overset{nk}{\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}} := \begin{array}{c} n \\ n(k-1) \\ n \end{array}\begin{bmatrix} X_1 \\ X_2 + Y_1 \\ Y_2 \end{bmatrix}$$

$$\mathbb{L}_2(P) := \left\{ \lambda\mathcal{A} + \mathcal{B} \;:\; \mathcal{A} \underset{\downarrow}{\boxplus} \mathcal{B} = w^T \otimes \begin{bmatrix} A_k \\ \vdots \\ A_0 \end{bmatrix}, \; w \in \mathbb{C}^k \right\}$$

Example: Second companion form

$$\begin{bmatrix} A_k & 0 & \cdots & 0 \\ 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{bmatrix} \underset{\downarrow}{\boxplus} \begin{bmatrix} A_{k-1} & -I & & 0 \\ \vdots & & \ddots & \\ A_1 & 0 & & -I \\ A_0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} A_k & 0 & \cdots & 0 \\ A_{k-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & 0 & \cdots & 0 \\ A_0 & 0 & \cdots & 0 \end{bmatrix}$$

Vector spaces of linearizations

Define: $\mathbb{DL}(P) := \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$

Properties of $\mathbb{DL}(P)$:

- pencils only exists if the ansatz vectors v in $\mathbb{L}_1(P)$ and w in $\mathbb{L}_2(P)$ are equal: $v = w$;
- pencils are uniquely determined, once the ansatz vector v is specified;
- pencils in $\mathbb{DL}(P)$ are block-symmetric;
- left **and** right eigenvectors of $P(\lambda)$ can easily be recovered;
- $L(\lambda) \in \mathbb{DL}(P)$ is a (strong) linearization for $P(\lambda)$ if and only if $L(\lambda)$ is regular.

Vector spaces of linearizations

Define: $\mathbb{DL}(P) := \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$

Question: How do we check regularity of $L(\lambda)$?

v-**polynomial Theorem**: Let $v = (v_1, \ldots, v_m)$ be the ansatz vector for $L(\lambda) \in \mathbb{DL}(P)$. Define the v-polynomial

$$\mathbf{p}(t; v) = v_m t^{m-1} + \cdots + v_2 t + v_1.$$

Then $L(\lambda)$ is regular (and thus a linearization for $P(\lambda)$) if and only if no root of the v-polynomial $\mathbf{p}(t; v)$ is an eigenvalue of $P(\lambda)$.

Structure linearizations

Task: given a structured polynomial $P(\lambda) = \lambda^m A_m + \cdots + \lambda A_1 + A_0$, find a structured linearization $L(\lambda)$ that still has the nice properties of the companion form.

But: Expecting to find a palindromic $L(\lambda) \in \mathbb{DL}(P)$ would be too much.

Idea: Hunt for palindromic $L(\lambda)$ in $\mathbb{L}_1(P)$ only.

Example: palindromic matrix polynomials

Palindromic linearizations

Example: $P(\lambda) = \lambda^2 A_0^T + \lambda A_1 + A_0$, where $A_1^T = A_1$.

Ansatz: $L(\lambda) = \lambda Z + Z^T =: \lambda \begin{bmatrix} D & E \\ F & G \end{bmatrix} + \begin{bmatrix} D^T & F^T \\ E^T & G^T \end{bmatrix}$

If $L(\lambda) \in \mathbb{L}_1(P)$ then for some $(v_1, v_2)^T \in \mathbb{C}^2$:

$$Z \boxplus Z^T = \begin{bmatrix} D & E + D^T & F^T \\ F & G + E^T & G^T \end{bmatrix} = \begin{bmatrix} v_1 A_0^T & v_1 A_1 & v_1 A_0 \\ v_2 A_0^T & v_2 A_1 & v_2 A_0 \end{bmatrix}.$$

$$\Rightarrow \; v_2 = v_1 \text{ and } \lambda Z + Z^T = v_1 \left(\lambda \begin{bmatrix} A_0^T & A_1 - A_0 \\ A_0^T & A_0^T \end{bmatrix} + \begin{bmatrix} A_0 & A_0 \\ A_1 - A_0^T & A_0 \end{bmatrix} \right).$$

Palindromic linearizations

Example: $P(\lambda) = \lambda^2 A_0^T + \lambda A_1 + A_0$, where $A_1^T = A_1$.

Observations:

- the set of vectors v such that a palindromic $L(\lambda) \in \mathbb{L}_1(P)$ exists is restricted;

- these vectors $v = (v_1, v_1)^T$ are "palindromic";

- a palindromic $L(\lambda) \in \mathbb{L}_1(P)$ is uniquely determined, once a "palindromic" v has been specified;

Definition: $v \in \mathbb{C}^k$ is **palindromic** if $R_k v = v$, where $R_k = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$.

Palindromic linearizations

Consider the palindromic $L(\lambda) = \lambda \begin{bmatrix} A_0^T & A_1 - A_0 \\ A_0^T & A_0^T \end{bmatrix} + \begin{bmatrix} A_0 & A_0 \\ A_1 - A_0^T & A_0 \end{bmatrix}$

$L(\lambda)$ is not in $\mathbb{DL}(P)$, but a related pencil is:

$$(R_2 \otimes I)L(\lambda) = \lambda \begin{bmatrix} A_0^T & A_0^T \\ A_0^T & A_1 - A_0 \end{bmatrix} + \begin{bmatrix} A_1 - A_0^T & A_0 \\ A_0 & A_0 \end{bmatrix} \in \mathbb{DL}(P)$$

We still have: $\begin{bmatrix} A_0^T & A_0^T \\ A_0^T & A_1 - A_0 \end{bmatrix} \boxplus \begin{bmatrix} A_1 - A_0^T & A_0 \\ A_0 & A_0 \end{bmatrix} = \begin{bmatrix} A_0^T & A_1 & A_0 \\ A_0^T & A_1 & A_0 \end{bmatrix}$

Palindromic linearizations

Consider the palindromic $L(\lambda) = \lambda \begin{bmatrix} A_0^T & A_1 - A_0 \\ A_0^T & A_0^T \end{bmatrix} + \begin{bmatrix} A_0 & A_0 \\ A_1 - A_0^T & A_0 \end{bmatrix}$

$L(\lambda)$ is not in $\mathbb{DL}(P)$, but a related pencil is:

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We also have: $\begin{bmatrix} A_0^T & A_0^T \\ A_0^T & A_1 - A_0 \end{bmatrix} \boxplus\!\!\downarrow \begin{bmatrix} A_1 - A_0^T & A_0 \\ A_0 & A_0 \end{bmatrix} = \begin{bmatrix} A_0^T & A_0^T \\ A_1 & A_1 \\ A_0 & A_0 \end{bmatrix}$

Consequence: The theory of linearizations in $\mathbb{DL}(P)$ applies.

Palindromic linearizations

Example: $P(\lambda) = \lambda^2 A_0^T + \lambda A_1 + A_0$, where $A_1^T = A_1$.

Summary:

- the vector $v = (v_1, v_2)$ is palindromic iff $R_2 v = v$, that is, $v_2 = v_1$;
- we have $(R_2 \otimes I)L(\lambda) \in \mathbb{DL}(P)$;
- if v is the ansatz vector of $L(\lambda)$, then $R_2 v$ is the ansatz vector of $(R_2 \otimes I)L(\lambda) \in \mathbb{DL}(P)$;
- the corresponding v-polynomial is $\mathbf{p}(t; v) = v_1 t + v_1$;
- $L(\lambda)$ is a linearization for $P(\lambda)$ iff -1 is no eigenvalue of $P(\lambda)$;

The general case

Given: a palindromic matrix polynomial $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0$ which is *regular* (i.e., $\det P(\lambda) \not\equiv 0$).

- there exists a unique palindromic $L(\lambda) \in \mathbb{L}_1(P)$ if and only if the ansatz v is palindromic, i.e., $R_k v = v$;
- we have $(R_k \otimes I)L(\lambda) \in \mathbb{DL}(P)$;
- $L(\lambda)$ is a linearization for $P(\lambda)$ iff no root of $\mathbf{p}(t; R_k v)$ is an eigenvalue of $P(\lambda)$;
- there is a constructive procedure to compute $L(\lambda)$

Constructing palindromic linearizations

Example: $P(\lambda) = \lambda^3 A + \lambda^2 B + \lambda B^T + A^T$.
We construct a palindromic pencil corresponding with $v = (1, 0, 1)^T \in \mathbb{R}^3$.

- the corresponding v-polynomial is $\mathbf{p}(t; R_3 v) = t^2 + 1$;
- the roots are i and $-i$;
- the resulting pencil will be a linearization for $P(\lambda)$ iff i and $-i$ are no eigenvalues of $P(\lambda)$;

Constructive procedure: Alternatingly use properties of shifted sums and palindromic structure.

Constructing palindromic linearizations

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We construct a palindromic pencil corresponding with $v = (1, 0, 1)^T \in \mathbb{R}^3$.

We know: $Z \boxplus Z^T = \begin{bmatrix} A & B & B^T & A^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A & B & B^T & A^T \end{bmatrix}$

using properties of shifted sums:

$$Z = \begin{bmatrix} A & \mathbf{?} & \mathbf{?} \\ \mathbf{0} & \mathbf{?} & \mathbf{?} \\ A & \mathbf{?} & \mathbf{?} \end{bmatrix}, \quad Z^T = \begin{bmatrix} \mathbf{?} & \mathbf{?} & A^T \\ \mathbf{?} & \mathbf{?} & \mathbf{0} \\ \mathbf{?} & \mathbf{?} & A^T \end{bmatrix}$$

Constructing palindromic linearizations

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using palindromic structure:

$$Z = \begin{bmatrix} A & \mathbf{?} & \mathbf{?} \\ \mathbf{0} & \mathbf{?} & \mathbf{?} \\ A & \mathbf{0} & A \end{bmatrix}, \quad Z^T = \begin{bmatrix} A^T & \mathbf{0} & A^T \\ \mathbf{?} & \mathbf{?} & \mathbf{0} \\ \mathbf{?} & \mathbf{?} & A^T \end{bmatrix}$$

Constructing palindromic linearizations

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We know: $Z \boxplus Z^T = \begin{bmatrix} A & B & B^T & A^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A & B & B^T & A^T \end{bmatrix}$

using properties of shifted sums:

$$Z = \begin{bmatrix} A & B - A^T & B^T \\ \mathbf{0} & \mathbf{?} & \mathbf{?} \\ A & \mathbf{0} & A \end{bmatrix}, \quad Z^T = \begin{bmatrix} A^T & \mathbf{0} & A^T \\ \mathbf{?} & \mathbf{?} & \mathbf{0} \\ B & B^T - A & A^T \end{bmatrix}$$

Constructing palindromic linearizations

Example: $P(\lambda) = \lambda^3 A + \lambda^2 B + \lambda B^T + A^T$.
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using palindromic structure:

$$Z = \begin{bmatrix} A & B - A^T & B^T \\ \mathbf{0} & \mathbf{?} & B - A^T \\ A & \mathbf{0} & A \end{bmatrix}, \quad Z^T = \begin{bmatrix} A^T & \mathbf{0} & A^T \\ B^T - A & \mathbf{?} & \mathbf{0} \\ B & B^T - A & A^T \end{bmatrix}$$

Constructing palindromic linearizations

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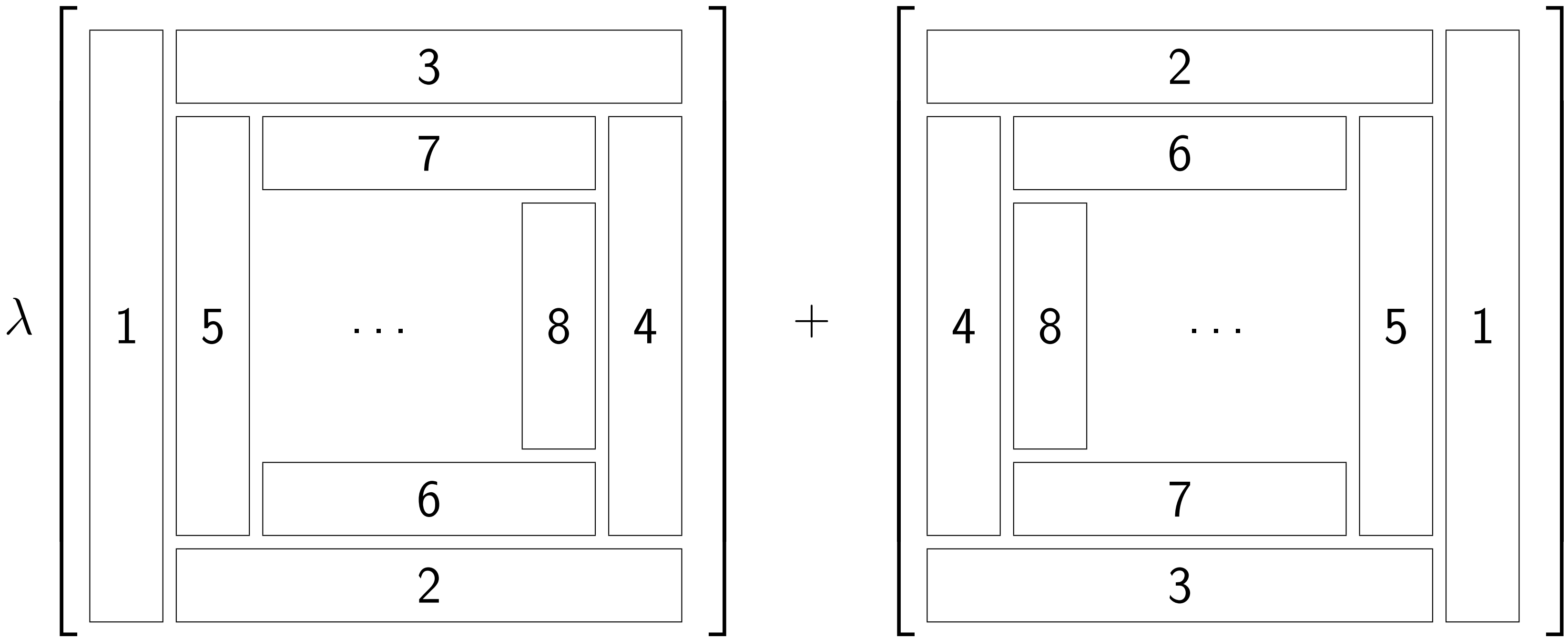
using properties of shifted sums:

$$Z = \begin{bmatrix} A & B - A^T & B^T \\ \mathbf{0} & A - B^T & B - A^T \\ A & \mathbf{0} & A \end{bmatrix}, \quad Z^T = \begin{bmatrix} A^T & \mathbf{0} & A^T \\ B^T - A & A^T - B & \mathbf{0} \\ B & B^T - A & A^T \end{bmatrix}$$

Constructing palindromic linearizations

The general case: construct $\lambda Z + Z^T$ via the following pattern:



Conclusions

- Vector space $\mathbb{L}_1(P)$ of potential linearizations that share nice properties with the companion form:
  - easy to construct from original data
  - eigenvectors of original problem can be easily recovered
- theory on existence of palindromic linearizations in $\mathbb{L}_1(P)$
- algorithm for constructing palindromic linearizations
- results can be generalized to other structures (even/odd/symmetric)
- linearization uniquely determined by structure and ansatz vector
- theory on optimally conditioned linearizations (Mackey, Mackey, Higham, Tisseur, 2006)