Definite versus Indefinite Linear Algebra

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Indefinite Linear Algebra

- name **Indefinite Linear Algebra** invented by Gohberg, Lancaster, Rodman in 2005;

- $\pm H^\star = H \in \mathbb{F}^{n\times n}$ invertible defines an **inner product** on $\mathbb{F}^n$:

$$[x, y]_H := y^\star H x \quad \text{for all } x, y \in \mathbb{F}^n;$$

  Here, * either denotes the transpose ^{T} or the conjugate transpose *;

$H = H^*$	Hermitian sesquilinear form
$H = -H^*$	skew-Hermitian sesquilinear form
$H = H^T$	symmetric bilinear form
$H = -H^T$	skew-symmetric bilinear form

- the inner product **may be indefinite** (needs not be positive definite).

Indefinite inner products

The adjoint: For $X \in \mathbb{F}^{n\times n}$ let $X^\star$ be the matrix satisfying

$$[v, Xw]_H = [X^\star v, w]_H \quad \text{for all } v, w \in \mathbb{F}^n.$$

We have $X^\star = H^{-1}X^T H$ resp. $X^\star = H^{-1}X^* H$.

Matrices with symmetries in indefinite inner products:

	adjoint	$y^T Hx$	$y^* Hx$
A H-**selfadjoint**	$A^\star = A$	$A^T H = HA$	$A^* H = HA$
S H-**skew-adjoint**	$S^\star = -S$	$S^T H = -HS$	$S^* H = -HS$
U H-**unitary**	$U^\star = U^{-1}$	$U^T HU = H$	$U^* HU = H$

Indefinite inner products in applications

The Linear Quadratic Optimal Control Problem: minimize the cost functional

$$\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

subject to the dynamics

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \in [0, \infty),$$

where $x(t), x_0 \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A, Q \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$, $R \in \mathbb{R}^{m \times m}$, $\left[\begin{smallmatrix} Q & S \\ S^T & R \end{smallmatrix}\right] \geq 0$, $R > 0$.

The solution can be obtained by solving the eigenvalue problem for the **Hamiltonian matrix**

$$\mathcal{H} := \begin{bmatrix} A - BR^{-1}S^T & -BR^{-1}B^T \\ SR^{-1}S^T - Q & -A^T + SR^{-1}B^T \end{bmatrix}.$$

Indefinite inner products in applications

- A matrix $\mathcal{H} \in \mathbb{F}^{2n\times 2n}$ is called **Hamiltonian** if

$$\mathcal{H}^T J = -J\mathcal{H}, \quad \text{where } J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

- Hamiltonian matrices are skew-adjoint with respect to the skew-symmetric bilinear form induced by J.

J-selfadjoint	$\mathcal{N}^T J = J\mathcal{N}$	**skew-Hamiltonian**
J-skew-adjoint	$\mathcal{H}^T J = -J\mathcal{H}$	**Hamiltonian**
J-unitary	$\mathcal{S}^T J\mathcal{S} = J$	**symplectic**

- **Symplectic matrices** occur in **discrete optimal control problems**.

Indefinite inner products in applications

Classical Mechanics: **vibration analysis of structural systems**: solve the second order system

$$M\ddot{x} + C\dot{x} + Kx = 0.$$

- $M \in \mathbb{R}^{n\times n}$ symmetric pos.def.: **mass matrix**;
- $C \in \mathbb{R}^{n\times n}$ symmetric: **damping matrix**;
- $K \in \mathbb{R}^{n\times n}$ symmetric pos.def.: **stiffness matrix**;

The ansatz $x(t) = x_0 e^{\lambda t}$ leads to the **quadratic eigenvalue problem**

$$(\lambda^2 M + \lambda C + K)x_0 = 0.$$

Linearization leads to an equivalent **generalized symmetric eigenvalue problem**

$$\left(\lambda \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} - \begin{bmatrix} -C & -K \\ -K & 0 \end{bmatrix}\right) \begin{bmatrix} \lambda x_0 \\ x_0 \end{bmatrix} = 0.$$

Indefinite inner products in applications

There is **Indefinite Linear Algebra** in generalized symmetric eigenvalue problems:

If H is invertible, then the **generalized symmetric eigenvalue problem**

$$(\lambda H - G)x = 0$$

is equivalent to the standard eigenvalue problem

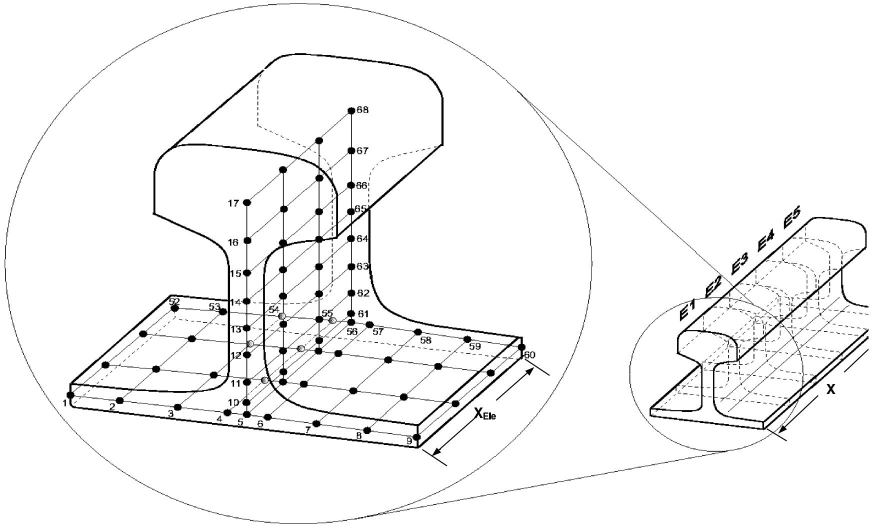
$$\lambda x = H^{-1}Gx.$$

$H^{-1}G$ is **selfadjoint** with respect to the inner product induced by H:

$$(H^{-1}G)^T H = G^T = G = H(H^{-1}G)$$

Indefinite inner products in applications

Vibration analysis of rail tracks excited by high speed trains



Finite element discretization of rail leads to a **palindromic eigenvalue problem**

$$\left(\lambda^2 A_0^T + \lambda A_1 + A_0\right) x = 0,$$

where $A_0, A_1 \in \mathbb{C}^{n\times n}$, and $A_1^T = A_1$.

Indefinite inner products in applications

Palindromic eigenvalue problems are equivalent to standard eigenvalue problems with **symplectic matrices** if not both 1 and -1 are eigenvalues.

There are many more applications with **Indefinite Linear Algebra** inside!

Indefinite Linear Algebra is everywhere!

Outline for the remainder of the talk

1) Canonical forms

2) Normal matrices

3) Polar decompositions

4) Singular value decompositions

1) Canonical forms

Canonical forms

Definite Linear Algebra: Any Hermitian matrix is unitarily diagonalizable and all its eigenvalues are real.

Indefinite Linear Algebra: Selfadjoint matrices with respect to indefinite inner products may have complex eigenvalues and need not be diagonalizable.

Example:

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

A_1 and A_2 are H-selfadjoint, i.e., $A_i^* H = H A_i$.

Canonical forms

Transformations that preserve structure:

- for bilinear forms: $(H, A) \mapsto (P^T HP,\ P^{-1}AP)$, $\quad P$ invertible;
- for sesquilinear forms: $(H, A) \mapsto (P^* HP,\ P^{-1}AP)$, $\quad P$ invertible;

$$A \text{ is} \left\{ \begin{array}{l} H\text{-selfadjoint} \\ H\text{-skew-adjoint} \\ H\text{-unitary} \end{array} \right\} \Leftrightarrow P^{-1}AP \text{ is} \left\{ \begin{array}{l} P^\star HP\text{-selfadjoint} \\ P^\star HP\text{-skew-adjoint} \\ P^\star HP\text{-unitary} \end{array} \right\}$$

Here $P^\star = P^T$ or $P^\star = P^*$, respectively.

Canonical forms

Theorem (Gohberg, Lancaster, Rodman, 1983, Thompson, 1976)
Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Then there exists $P \in \mathbb{C}^{n\times n}$ invertible such that

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_k, \quad P^*HP = H_1 \oplus \cdots \oplus H_k,$$

where either

1) $A_i = \mathcal{J}_{n_i}(\lambda)$, and $H_i = \varepsilon F_{n_i}$, where $\lambda \in \mathbb{R}$ and $\varepsilon = \pm 1$; or

2) $A_i = \begin{bmatrix} \mathcal{J}_{n_i}(\mu) & 0 \\ 0 & \mathcal{J}_{n_i}(\overline{\mu}) \end{bmatrix}$, $H_i = \begin{bmatrix} 0 & F_{n_i} \\ F_{n_i} & 0 \end{bmatrix}$, where $\mu \notin \mathbb{R}$.

Here $\mathcal{J}_m(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \lambda & 1 \\ 0 & \dots & 0 & \lambda \end{bmatrix} \in \mathbb{C}^{m\times m}$ and $F_m = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix} \in \mathbb{C}^{m\times m}$.

Canonical forms

There are similar results for H-skewadjoint and H-unitary matrices and for real or complex bilinear forms.

Spectral symmetries:

	$y^T Hx$	$y^* Hx$	$y^T Hx$
field	$\mathbb{F} = \mathbb{C}$	$\mathbb{F} = \mathbb{C}$	$\mathbb{F} = \mathbb{R}$
H-selfadjoints	λ	$\lambda, \overline{\lambda}$	$\lambda, \overline{\lambda}$
H-skew-adjoints	$\lambda, -\lambda$	$\lambda, -\overline{\lambda}$	$\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$
H-unitaries	λ, λ^{-1}	$\lambda, \overline{\lambda}^{-1}$	$\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$

The sign characteristic

Sign characteristic: There are additional invariants for real eigenvalues of H-selfadjoint matrices: **signs** $\varepsilon = \pm 1$.

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad H_\varepsilon = \begin{bmatrix} \varepsilon & 0 \\ 0 & -1 \end{bmatrix}, \quad \varepsilon = \pm 1;$$

- There is no transformation $P^{-1}AP = A$, $P^*H_{+1}P = H_{-1}$, because of Sylvester's Law of Inertia;

- each Jordan block associated with a real eigenvalue of A has a corresponding sign $\varepsilon \in \{+1, -1\}$;

- the collection of all the signs is called the **sign characteristic** od A;

The sign characteristic

Interpretation of the sign characteristic for simple eigenvalues:

- let (λ, v) be an eigenpair of the selfadjoint matrix A, where $\lambda \in \mathbb{R}$:
- let ε be the sign corresponding to λ;
- the inner product $[v, v]_H$ is positive if $\varepsilon = +1$;
- the inner product $[v, v]_H$ is negative if $\varepsilon = -1$.

Analogously:

- purely imaginary eigenvalues of H-skew-adjoint matrices have signs;
- unimodular eigenvalues of H-unitary matrices have signs.

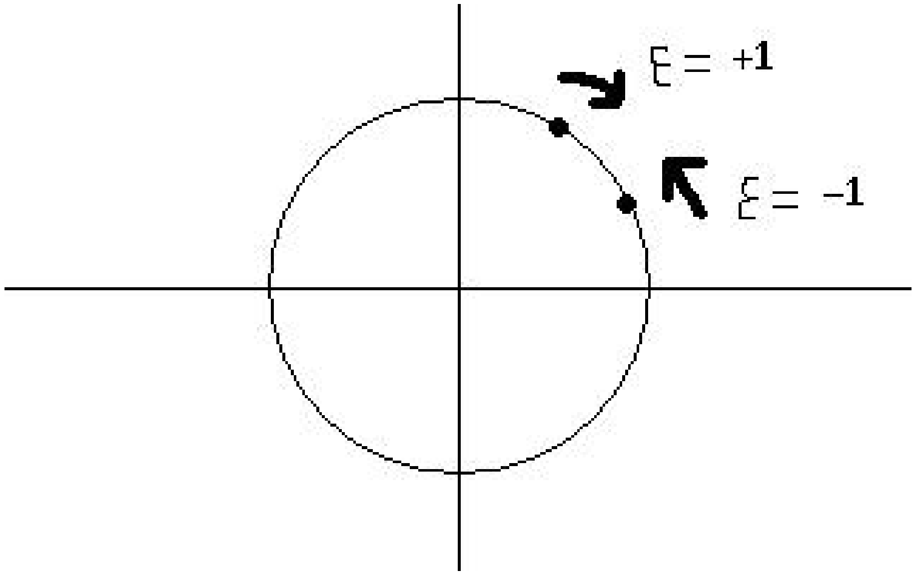
What happens under structured perturbations?

The **sign characteristic** plays an important role in **perturbation theory**:

Example: symplectic matrices $S \in \mathbb{R}^{2n \times 2n}$;

- consider a slightly perturbed matrix $\widetilde{S}$ that is still symplectic;
- the behavior of the unimodular eigenvalues under perturbation depends on the sign characteristic;
- if two unimodular eigenvalues meet, the behavior is different if the corresponding signs are opposite or equal.

What happens under structured perturbations?



- let S have two close **unimodular eigenvalues** with **opposite signs**;
- if S is perturbed and the two eigenvalues meet, they generically form a Jordan block; then they may split off as a pair of nonunimodular reciprocal eigenvalues;

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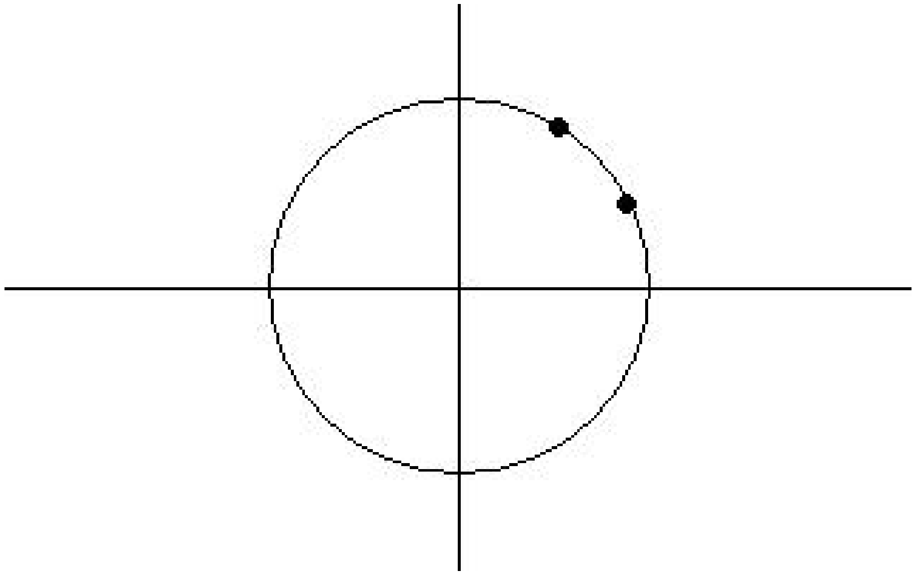

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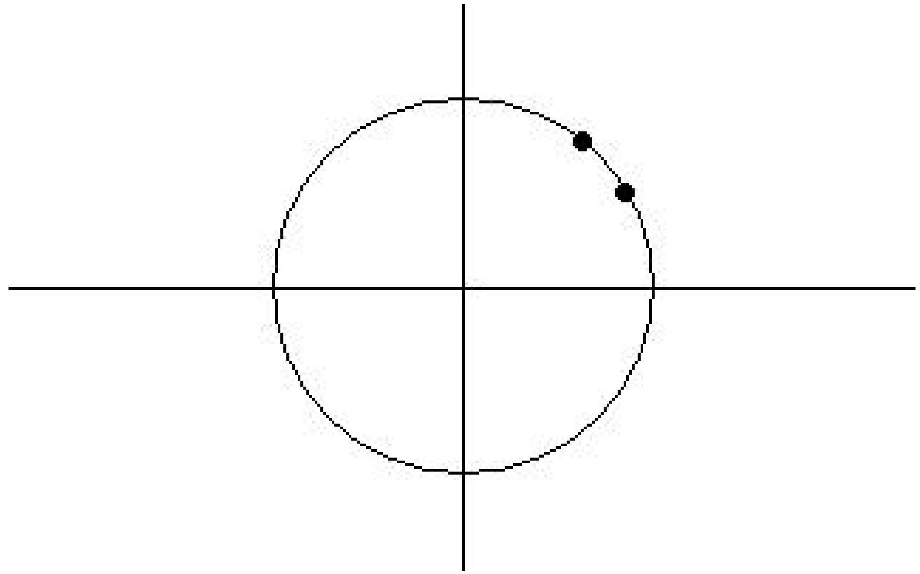

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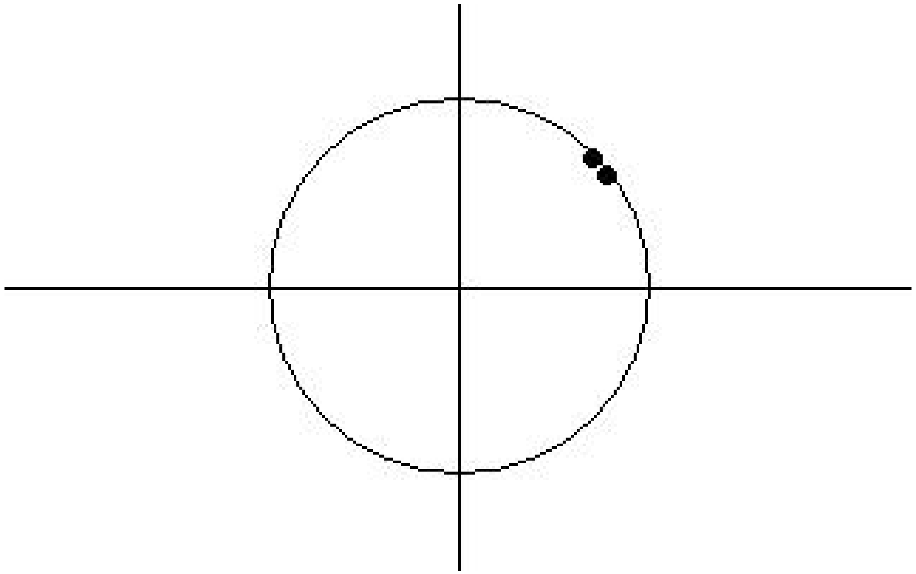

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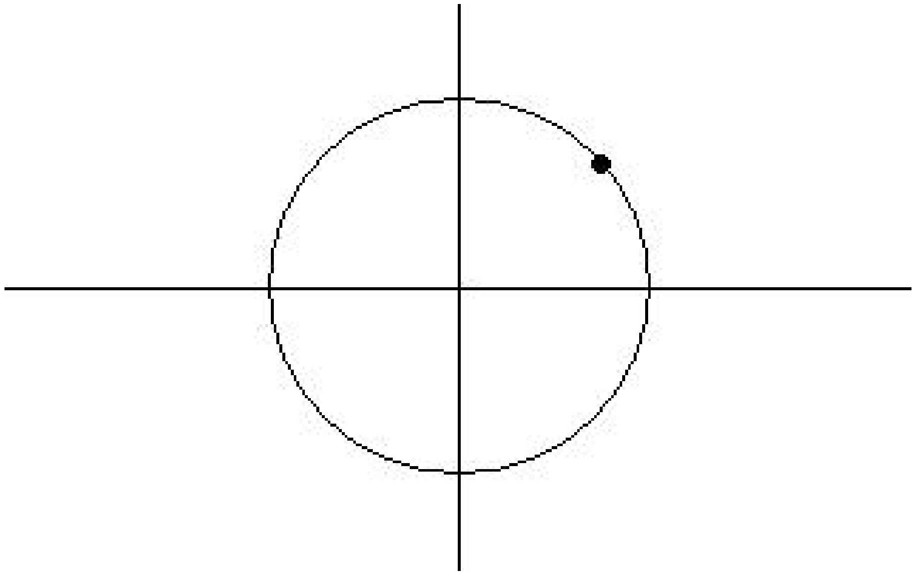

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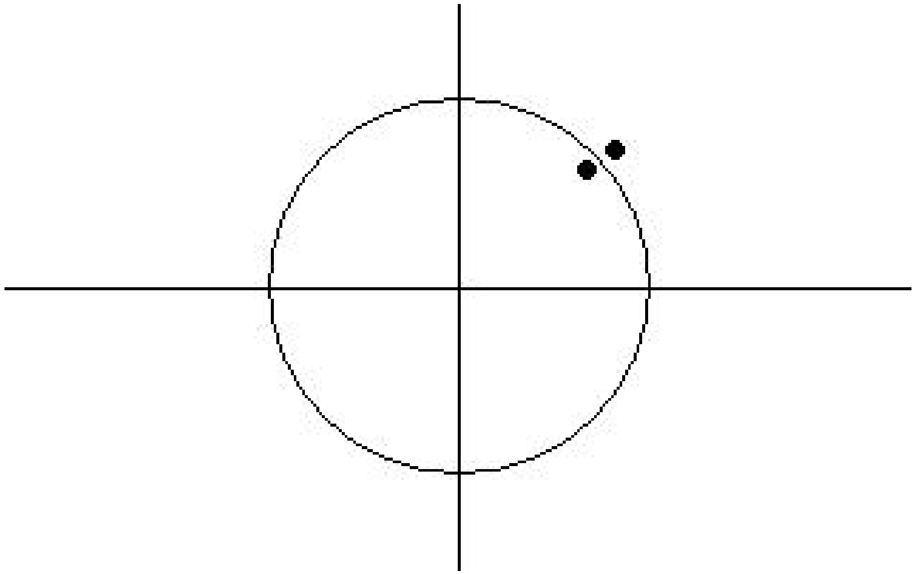

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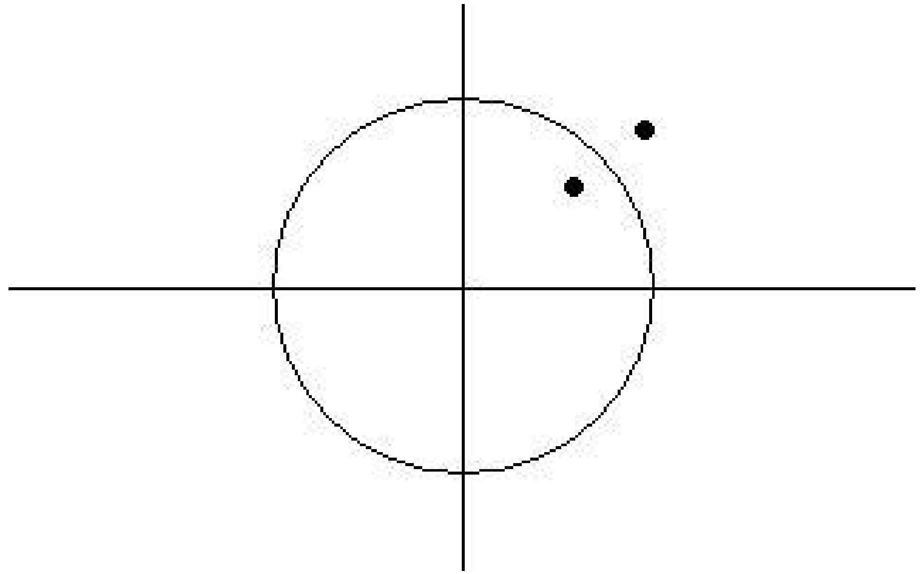

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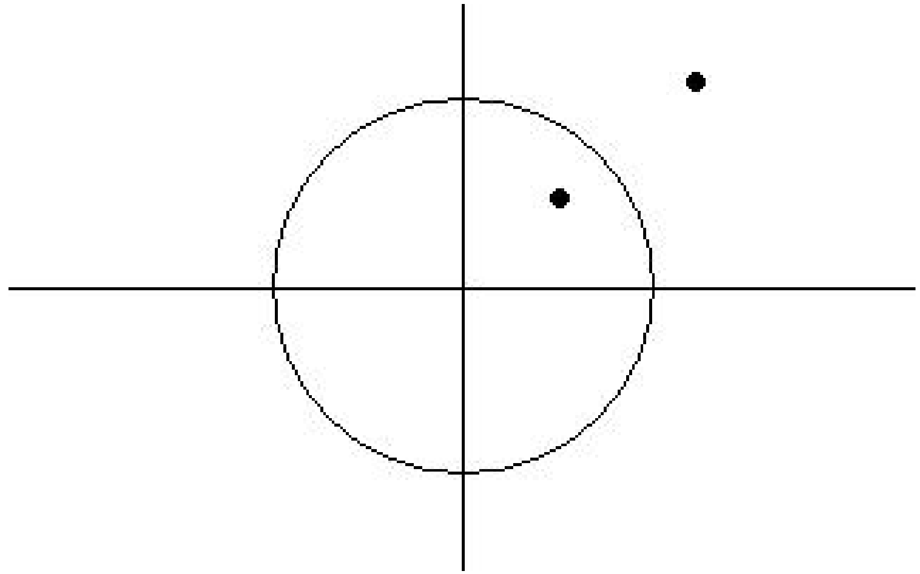

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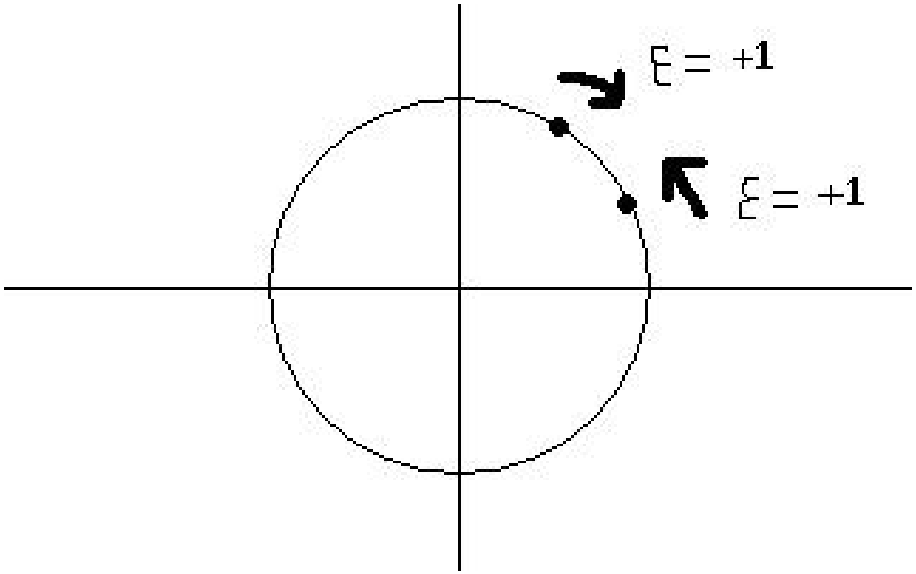

- let S have two close **unimodular eigenvalues** with **equal signs**;
- if S is perturbed and the two eigenvalues meet, they *cannot* form a Jordan block, and they *must* remain on the unit circle;

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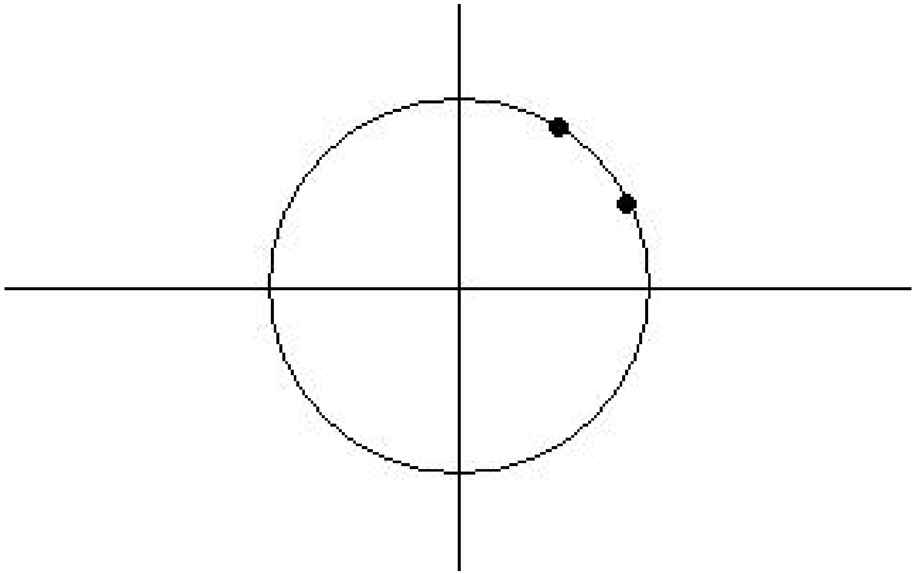

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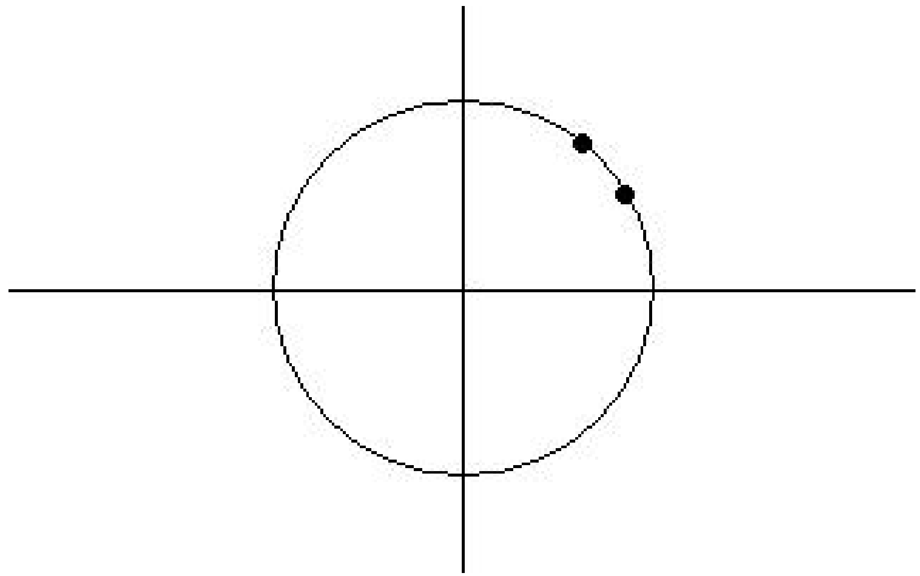

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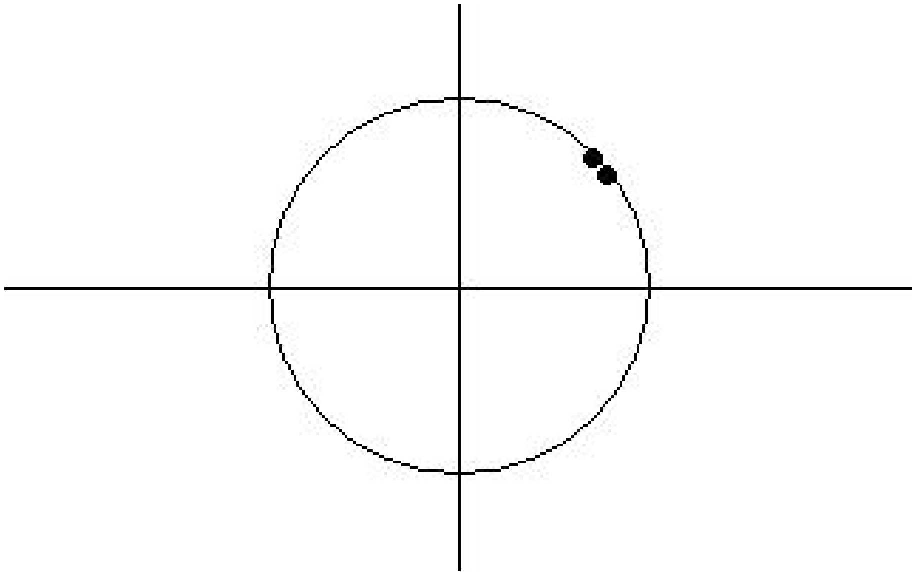

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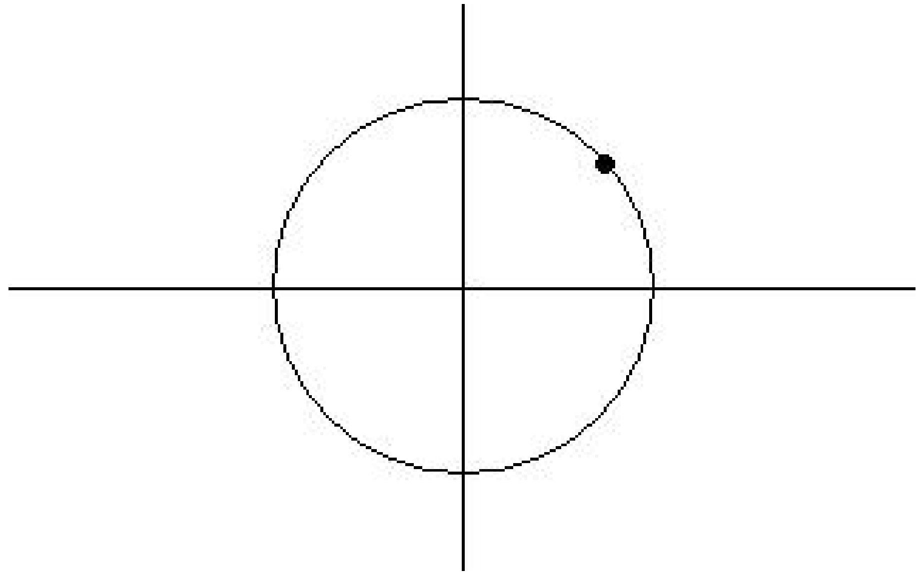

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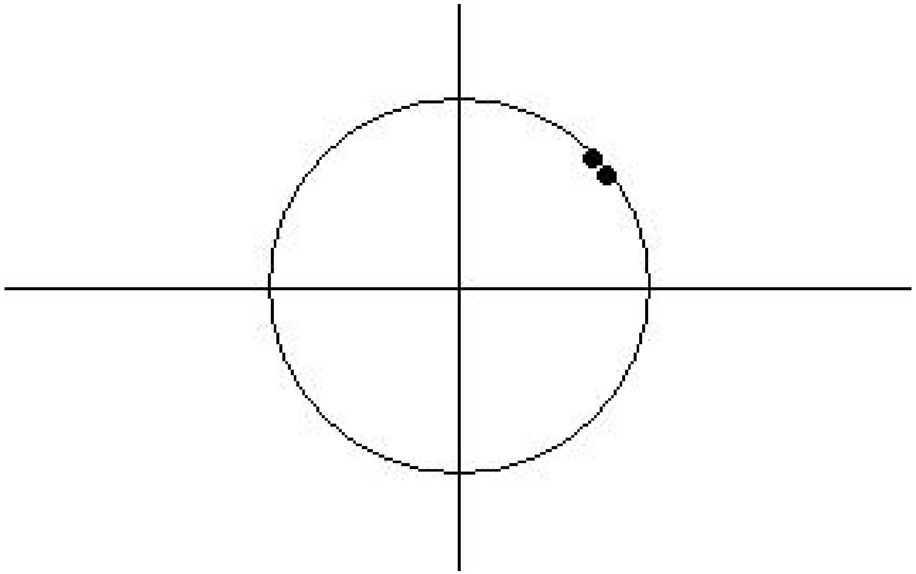

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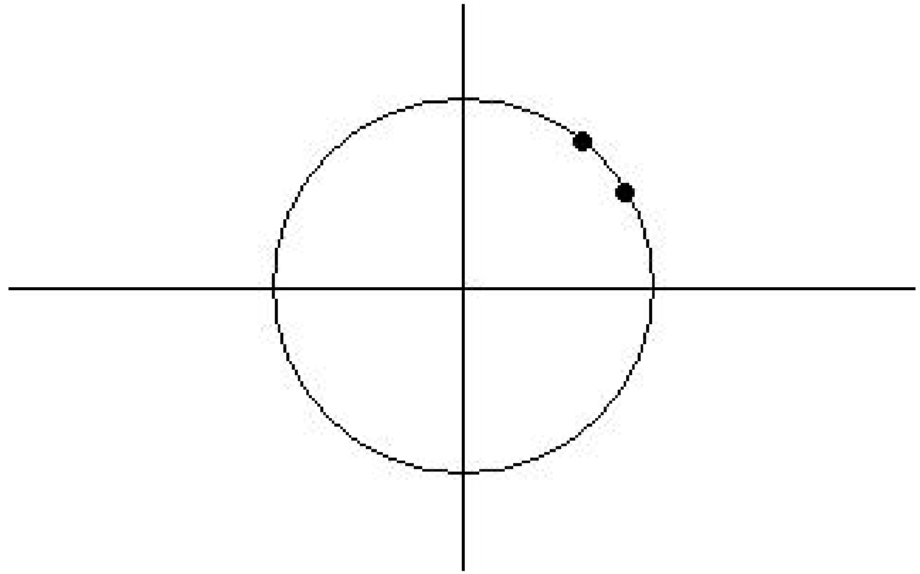

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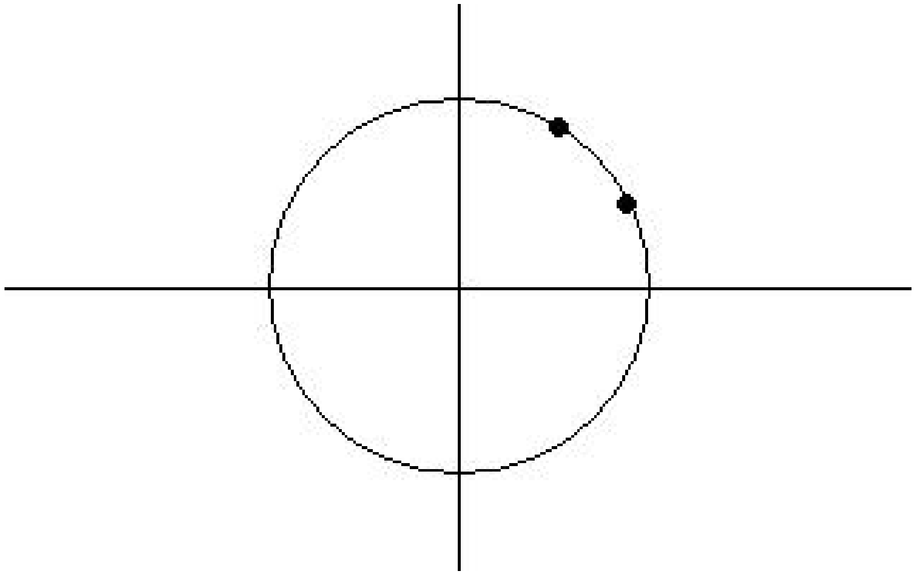

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Canonical forms

Conclusions

- canonical forms are more complicated than in the definite case;
- sign characteristic is crucial for deeper understanding of structure-preserving algorithms, e.g.,
  - theory of structured perturbations;
  - existence of Schur-like forms
  - existence of Lagrangian subspaces (important in the solution of control problems and Riccati equations)

2) Normal matrices

Normal matrices

Definite Linear Algebra:

- A matrix $X \in \mathbb{C}^{n\times n}$ is called **normal** if $XX^* = X^*X$;
- normal matrices generalize Hermitian, skew-Hermitian, and unitary matrices;
- normal matrices have "nice properties", because they are unitarily diagonalizable;
- they are "**good guys**".

Normal matrices

Indefinite Linear Algebra:

- Let $H \in \mathbb{C}^{n \times n}$ be Hermitian and invertible;
- A matrix $X \in \mathbb{C}^{n \times n}$ is called **H-normal** if $X^{[*]}X = XX^{[*]}$;
- H-normal matrices generalize H-selfadjoint, H-skewadjoint, and H-unitary matrices;
- Are H-normal matrices "**good guys**", too?

Classification of normal matrices

H-Indecomposability:

$A \in \mathbb{C}^{n\times n}$ is called **H-decomposable**, if there exists $P \in \mathbb{C}^{n\times n}$ invertible such that

$$P^{-1}AP = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad P^*HP = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad A_j, H_j \in \mathbb{F}^{n_j\times n_j}, n_j > 0$$

Otherwise A is called **H-indecomposable**.

Clear: Any $A \in \mathbb{C}^{n\times n}$ can be decomposed as

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_k, \quad P^*HP = H_1 \oplus \cdots \oplus H_k,$$

where each A_j is H_j-indecomposable.

Classification of H-normal matrices

Example: nilpotent indecomposable matrices

- canonical form for H-selfadjoint matrices:

$$X = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}_{n\times n}, \quad H = \varepsilon \begin{bmatrix} 0 & & 1 \\ & .\cdot^{\cdot} & \\ 1 & & 0 \end{bmatrix}_{n\times n}, \quad \varepsilon = \pm 1;$$

- canonical form for H-normal matrices, when H has two negative eigenvalues: 17 different types of blocks, e.g.,

$$X = \begin{bmatrix} 0 & 1 & ir & isz \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \varepsilon \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad |z| = 1,\ r, s \in \mathbb{R}, \varepsilon = \pm 1$$

Classification of H-normal matrices

- Gohberg/Reichstein 1990: The problem of classifying H-normal matrices is as hard as the problem of classifying a pair of commuting matrices under simultaneous similarity.

- Gohberg/Reichstein 1990: Complete classification when H has one negative eigenvalue.

- Holtz/Strauss 1996: Complete classification when H has two negative eigenvalues.

Conclusion: The class of H-normal matrices is **too large**!

Classes of H-normal matrices

Question: Is there a "better" definition for H-normality?

Conditions equivalent to normality (in the case $H = I$):

- Grone/Johnson/Sa/Wolkowicz (1987): 69 conditions
- Elsner/Ikramov (1998): 20 conditions
- all together: conditions (1) – (89)
- **Best candidate**:

  (17) There exists a polynomial p such that $X^{[*]} = p(X)$.

Polynomially normal matrices

Definition: A matrix $X \in \mathbb{F}^{n\times n}$ is called **polynomially H-normal** if there exists a polynomial $p \in \mathbb{F}[t]$ such that $X^{[*]} = p(X)$.

Properties:

- p is unique if it is of minimal degree and monic.
- X is polynomially H-normal $\Rightarrow$ X is H-normal
  $\not\Leftarrow$

Examples:

- H-selfadjoint matrices are polynomially H-normal with $p(t) = t$;
- H-skew-adjoint matrices are polynomially H-normal with $p(t) = -t$;
- H-unitary matrices are polynomially H-normal ($U^{-1} = p(U)$).

Polynomially normal matrices

- M. 2006: Canonical forms for real and complex polynomially H-normal matrices.

- **Spectral symmetries**:

	y^*Hx	y^THx	y^THx
	$\mathbb{F}=\mathbb{C}$	$\mathbb{F}=\mathbb{C}$	$\mathbb{F}=\mathbb{R}$
H-selfadjoints	$\lambda, \overline{\lambda}$	λ	$\lambda, \overline{\lambda}$
H-skew-adjoints	$\lambda, -\overline{\lambda}$	$\lambda, -\lambda$	$\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}$
H-unitaries	$\lambda, \overline{\lambda}^{-1}$	λ, λ^{-1}	$\lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1}$
polynomially H-normals	$\lambda, \overline{p(\lambda)}$	$\lambda, p(\lambda)$	$\lambda, p(\lambda), \overline{\lambda}, \overline{p(\lambda)}$

Normal matrices

Conclusions

- The class of H-normal matrices is too large.
  H-normal matrices are "**bad guys**";

- Polynomially H-normals are "**nicer guys**";

- Canonical forms for polynomially H-normals generalize H-selfadjoints, H-skew-adjoints, H-unitaries;

- unifying theory (e.g. existence of semidefinite invariant subspaces).

3) Polar decompositions

H-polar decompositions

Definite Linear Algebra: Let $A \in \mathbb{C}^{n \times n}$. Then there exist a unitary matrix $U \in \mathbb{C}^{n \times n}$ and a Hermitian positive semidefinite matrix $H \in \mathbb{C}^{n \times n}$ such that

$$A = UH.$$

Indefinite Linear Algebra: **H-polar decomposition** of a matrix $X \in \mathbb{C}^{n \times n}$:

$$X = UA, \quad U \text{ is } H\text{-unitary, } A \text{ is } H\text{-selfadjoint}$$

Note: Sometimes, additional assumptions on A are imposed, e.g.

- $HA \geq 0$ (Bolshakov, van der Mee, Ran, Reichstein, Rodman, 1996);
- $\sigma(A) \subseteq \mathbb{C}_+$ (Higham, Mackey, Mackey, Tisseur, 2004).

How to construct H-polar decompositions

Observations for polar decompositions $X = UA$:

- $X^{[*]} = A^{[*]}U^{[*]} = AU^{-1}$
- $X^{[*]}X = AU^{-1}UA = A^2$
- $\mathrm{Ker}\ X = \mathrm{Ker}\ A$.

Construction of H-polar decompositions:

i) compute H-selfadjoint square root A of $X^{[*]}X$ s.t. $\mathrm{Ker}\ X = \mathrm{Ker}\ A$;

ii) compute H-unitary U such that $X = UA$

An example

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad X^{[*]} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

i) computation of H-selfadjoint factor A:

$$X^{[*]}X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{take, e.g., } A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

ii) computation of H-unitary factor U:

$$U = XA^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note: $HA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is NOT positive semidefinite, but $\sigma(A) \subseteq \mathbb{C}_+$.

Another example

$$X = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad X^{[*]} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X^{[*]}X = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \stackrel{?}{=} A^2$$

- If A would be such an H-selfadjoint square root, then $\sigma(A) \subseteq \{-i, i\}$.
- The spectrum of H-selfadjoint matrices is symmetric w.r.t. real axis $\Rightarrow \ \sigma(A) = \{-i, i\}$.
- There is no H-selfadjoint square root for $X^{[*]}X$.

When do H-polar decompositions exist?

Question: When does X have an H-polar decomposition?

Theorem [Bolshakov, van der Mee, Ran, Reichstein, Rodman, 1996]: Let $X \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:

1) X has an H-polar decomposition.

2) $X^{[*]}X$ has an H-selfadjoint square root A satisfying $\mathrm{Ker}\ X = \mathrm{Ker}\ A$.

3) a) each Jordan block $\mathcal{J}_p(\lambda)$ associated with $\lambda < 0$ in the canonical form for $(X^{[*]}X, H)$ occurs an even number (say $2m$) of times such that there are exactly m blocks with sign $\varepsilon = +1$;

   b) several conditions on eigenvalue $\lambda = 0$ are satisfied.
   (one set of conditions comes from $X^{[*]}X = A^2$; a second set of conditions comes from $\mathrm{Ker}\ X = \mathrm{Ker}\ A$.)

Polar decompositions of normal matrices

Question: Let X be H-normal. Does X have an H-polar decomposition?

Answers:

- Bolshakov, van der Mee, Ran, Reichstein, Rodman, 1996: invertible H-normals have H-pd's;
- Bolshakov, van der Mee, Ran, Reichstein, Rodman, 1996: H-normals have H-pd's if H has at most one negative eigenvalue;
- Lins, Meade, M., Rodman, 2001: H-normals have H-pd's if H has at most two negative eigenvalues;

Polar decompositions of normal matrices

Theorem [M., Ran, Rodman, 2004] Let X be H-normal. Then X admits an H-polar decomposition.

Proof:

- induction on dim$(\mathrm{Ker}\ X)$;
- basic idea: construct an H-selfadjoint square root A of $X^{[*]}X$ satisfying $\mathrm{Ker}\ X = \mathrm{Ker}\ A$ from an H-polar decomposition of a smaller submatrix;

Corollary [conjectured by Kintzel, 2002] $X \in \mathbb{C}^{n\times n}$ admits an H-polar decomposition $X = UA$ if and only if $XX^{[*]}$ and $X^{[*]}X$ are H-unitarily similar.

Polar decompositions

Conclusions:

- theory on polar decompositions in **Indefinite Linear Algebra**;
- applications in linear optics and Procrustes problems;
- H-normal matrices are the prototypes of matrices allowing an H-polar decomposition, so they are **good guys** at the end;
- there has been progress in developing algorithms for computing H-polar decompositions (Kintzel, Higham/Mackey/Mackey/Tisseur, Higham/M./Tisseur).

4) Singular value decompositions

The Singular Value Decomposition

Definite Linear Algebra: Let $A \in \mathbb{C}^{m\times n}$. Then there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that

$$U^*AV = \left[\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} \sigma_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \sigma_r & 0 \\ \hline 0 & \dots & 0 & 0 \end{array}\right]$$

where $\sigma_1 \geq \cdots \geq \sigma_r > 0$. The parameters $\sigma_1, \ldots, \sigma_r$ are uniquely defined and the (nonzero) **singular values** of A.
Moreover,

$$AA^* = \left[\begin{array}{c|c} \Sigma^2 & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times m} \quad \text{and} \quad A^*A = \left[\begin{array}{c|c} \Sigma^2 & 0 \\ \hline 0 & 0 \end{array}\right]_{n\times n}$$

The Singular Value Decomposition

Aspects: of the singular value decomposition:

- allows computation of the polar decomposition;
- displays eigenvalues of the Hermitian matrices AA^* and A^*A;
- allows numerical computation of the rank of a matrix;
- allows construction of optimal low-rank approximations;
- useful tool in Numerical Linear Algebra;

Indefinite Singular Value Decompositions

Problem: Given $A \in \mathbb{C}^{m\times n}$, compute a canonical form that displays

- the Jordan canonical form of $A^{[*]}A$ and $AA^{[*]}$, where $A^{[*]} = H^{-1}A^*H$ is the adjoint with respect to a Hermitian sesquilinear form $[\cdot,\cdot] = (H\cdot,\cdot)$; ($A^{[*]}A$ and $AA^{[*]}$ are selfadjoint with respect to $[\cdot,\cdot]$);
- the Jordan canonical form of A^TA and AA^T; (these are complex symmetric matrices);
- the Jordan canonical form of $A^{[T]}A$ and $AA^{[T]}$, where $A^{[T]}$ is the adjoint with respect to a complex symmetric or complex skew-symmetric bilinear form $[\cdot,\cdot]$;

Indefinite Singular Value Decompositions

General formulation of the problem: let $A \in \mathbb{C}^{m\times n}$ and $\star \in \{*, T\}$;

- allow two inner products given by $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$;
- compute a canonical form for the triple $(A, G, \hat{G})$ via

  $(A_{\mathsf{CF}}, G_{\mathsf{CF}}, \hat{G}_{\mathsf{CF}}) = (Y^\star AX, X^\star GX, Y^\star \hat{G} Y)$, where X, Y are nonsingular;
- let this form display the eigenvalues of
  - the matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^\star G^{-1} A$;
  - the matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^\star$;

This makes sense, because

$$Y^{-1}\hat{\mathcal{H}}Y = \hat{G}_{\mathsf{CF}}^{-1} A_{\mathsf{CF}}^\star G_{\mathsf{CF}}^{-1} A_{\mathsf{CF}} \quad \text{and} \quad X^{-1}\mathcal{H}X = G_{\mathsf{CF}}^{-1} A_{\mathsf{CF}} \hat{G}_{\mathsf{CF}}^{-1} A_{\mathsf{CF}}^\star$$

Indefinite Singular Value Decompositions

General formulation of the problem: let $A \in \mathbb{C}^{m\times n}$ and $\star \in \{*, T\}$;

- allow two inner products given by $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$;
- compute a canonical form for the triple $(A, G, \hat{G})$ via

  $(A_{\mathsf{CF}}, G_{\mathsf{CF}}, \hat{G}_{\mathsf{CF}}) = (Y^\star A X, X^\star G X, Y^\star \hat{G} Y)$, where X, Y are nonsingular;
- let this form display the eigenvalues of
  - the matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^\star G^{-1} A$;
  - the matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^\star$;

Then $G = H^{-1}$, $\hat{G} = H$, $\star = *$: $\rightsquigarrow$ forms for $\hat{\mathcal{H}} = A^{[*]}A$ and $\mathcal{H} = AA^{[*]}$;

Indefinite Singular Value Decompositions

General formulation of the problem: let $A \in \mathbb{C}^{m\times n}$ and $\star \in \{*, T\}$;

- allow two inner products given by $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$;
- compute a canonical form for the triple $(A, G, \hat{G})$ via

  $(A_{\mathsf{CF}}, G_{\mathsf{CF}}, \hat{G}_{\mathsf{CF}}) = (Y^\star AX, X^\star GX, Y^\star \hat{G} Y),$ where X, Y are nonsingular;
- let this form display the eigenvalues of
  - the matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^\star G^{-1} A$;
  - the matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^\star$;

Then $G = I$, $\hat{G} = I$, $\star = *$: $\rightsquigarrow$ SVD if we require $X^* G X = I$, $Y^* \hat{G} Y = I$;

Indefinite Singular Value Decompositions

General formulation of the problem: let $A \in \mathbb{C}^{m\times n}$ and $\star \in \{*, T\}$;

- allow two inner products given by $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$;
- compute a canonical form for the triple $(A, G, \hat{G})$ via

  $(A_{\mathsf{CF}}, G_{\mathsf{CF}}, \hat{G}_{\mathsf{CF}}) = (Y^\star A X, X^\star G X, Y^\star \hat{G} Y)$, where X, Y are nonsingular;
- let this form display the eigenvalues of
  - the matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^\star G^{-1} A$;
  - the matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^\star$;

Then $G = I$, $\hat{G} = I$, $\star = T$: $\rightsquigarrow$ forms for $\hat{\mathcal{H}} = A^T A$ and $\mathcal{H} = AA^T$.

Indefinite Singular Value Decompositions

Problem: $A \in \mathbb{C}^{n\times n}$ singular. Then $\mathcal{H}$ and $\hat{\mathcal{H}}$ may have different Jordan canonical forms.

Example

$$A = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad G = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad \hat{G} = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\mathcal{H} = \hat{G}^{-1}A^TG^{-1}A = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \hat{\mathcal{H}} = G^{-1}A\hat{G}^{-1}A^T = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

We have to allow rectangular blocks as “indefinite singular values”;

The singular values: $*$ and T case

Special case: $G = I_m$, $\hat{G} = I_n$. The singular values of $A \in \mathbb{C}^{m \times n}$ are:

***-case**: $\sigma_1, \ldots, \sigma_{\min(m,n)} \geq 0$ (related to the eigenvalues of A^*A and AA^*);

T-case: $\mathcal{J}_{\xi_1}(\mu_1)$, ..., $0_{m_0 \times n_0}$, $\mathcal{J}_{2p_1}(0)$, ..., $\begin{bmatrix} 0 \\ I_{q_1} \end{bmatrix}$, ... , $\begin{bmatrix} 0 & I_{r_1} \end{bmatrix}$, ...,
where $\arg(\mu_j) \in [0, \pi)$ and the "values" are related to the Jordan blocks of A^TA and AA^T.

Uniqueness: Singular values are unique both in the $*$-case and T-case!

Conclusions

- **Indefinite Linear Algebra** occurs frequently in applications.
- **Indefinite Linear Algebra** is challenging.
- There is much more to say... but not today!

Thank you for your attention!

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