# The canonical generalized Polar decomposition

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# Polar decompositions

**Reminder**: **Polar decomposition** of $A \in \mathbb{F}^{m\times n}$, $m \geq n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$:

$$A = U \cdot H$$

1) $U \in \mathbb{F}^{m\times n}$ has orthonormal columns;

2) $H \in \mathbb{F}^{n\times n}$ is selfadjoint $(H^* = H)$ and positive semidefinite;

- decomposition always exists;
- $H = (A^*A)^{1/2}$ is uniquely determined;
- $U$ is uniquely determined iff $\operatorname{rank}(A) = n$.

$(A^*A)^{1/2}$ denotes the unique positive semidefinite square root of $A^*A$.

# Canonical polar decompositions

**Aim**: make also $U$ being unique by relaxing the condition of having orthonormal columns.

**Definition**: $U \in \mathbb{F}^{m\times n}$, $m \geq n$, is a **partial isometry** if $\|Ux\|_2 = \|x\|_2$ for all $x \in \operatorname{range} U^*$.

- $U$ is a partial isometry $\iff UU^*U = U \iff U^+ = U^*$.
- $U^+$ denotes the Moore-Penrose generalized inverse of $U$, i.e., the unique matrix $U^+ \in \mathbb{F}^{n\times m}$ satisfying

$$\begin{array}{ll} (1)\;\; UU^+U = U, & (2)\;\; U^+UU^+ = U^+, \\ (3)\;\; (UU^+)^* = UU^+, & (4)\;\; (U^+U)^* = U^+U. \end{array}$$

# Canonical polar decompositions

**Canonical polar decomposition** of $A \in \mathbb{F}^{m\times n}$, $m \geq n$:

$$A = U \cdot H$$

1) $U \in \mathbb{F}^{m\times n}$ is a partial isometry;

2) $H \in \mathbb{F}^{n\times n}$ is selfadjoint $(H^* = H)$ and positive semidefinite;

3) $\mathrm{range}\,(U^*) = \mathrm{range}\,(H)$;

- decomposition always exists;
- $H = (A^*A)^{1/2}$ is uniquely determined;
- $U$ is uniquely determined by $\mathrm{range}\,(U^*) = \mathrm{range}\,(H)$.

# Canonical polar decompositions

**Example**: Let $A \in \mathbb{F}^{m\times n}$, $m \geq n$, and let

$$A = P \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} Q^*$$

be an SVD of $A$, where $\Sigma_r \in \mathbb{F}^{r\times r}$ is invertible.

1) All polar decompositions $A = UH$ of $A$ have the forms

$$H = (A^*A)^{1/2} \quad \text{and} \quad U = P \begin{bmatrix} I_r & 0 \\ 0 & W \end{bmatrix} Q^* \in \mathbb{F}^{m\times n},$$

where $W$ has orthonormal columns.

2) The canonical polar decomposition $A = UH$ of $A$ is given by

$$H = (A^*A)^{1/2} \quad \text{and} \quad U = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^* \in \mathbb{F}^{m\times n}.$$

# Generalized polar decompositions

**Aim**: Generalize the concept of polar decomposition to spaces with a bilinear/sesquilinear form.

- $M \in \mathbb{F}^{n\times n}$ invertible defines a nondegenerate bilinear ($\mathbb{F} = \mathbb{R}$), resp. sesquilinear ($\mathbb{F} = \mathbb{C}$) form (**scalar product**) on $\mathbb{F}^n$:

$$\langle x, y\rangle_M := x^* M y \quad \text{for all } x, y \in \mathbb{F}^n;$$

- We say that $\langle \cdot, \cdot\rangle_M$ is **orthosymmetric**, i.e.,
  - $M^T = \pm M$ if $\mathbb{F} = \mathbb{R}$;
  - $M^* = \alpha M$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$ if $\mathbb{F} = \mathbb{C}$.

# Generalized polar decompositions

The **adjoint** of $A \in \mathbb{F}^{n\times n}$ with respect to $\langle\cdot,\cdot\rangle_M$ is the unique matrix $A^\star \in \mathbb{F}^{n\times n}$ satisfying

$$\langle x, Ay\rangle_M = \langle A^\star x, y\rangle_M \quad \text{for all } x, y \in \mathbb{F}^n$$

If $\langle\cdot,\cdot\rangle_M$ is orthosymmetric, we have $(A^\star)^\star = A$ for all $A \in \mathbb{F}^{n\times n}$.

**Matrices with symmetries w.r.t. $\langle\cdot,\cdot\rangle_M$:**

| | adjoint | |
|---|---|---|
| $H$ **$M$-selfadjoint** | $H^\star = H$ | $H^* M = MH$ |
| $S$ **$M$-skew-adjoint** | $S^\star = -S$ | $S^* M = -MS$ |
| $U$ **$M$-unitary** | $U^\star = U^{-1}$ | $U^* MU = M$ |

# Generalized polar decompositions

**Generalized polar decomposition** of $A \in \mathbb{F}^{n\times n}$ with $\sigma(A) \subseteq \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

$$A = W \cdot S$$

1) $W \in \mathbb{F}^{n\times n}$ is $M$-unitary;

2) $S \in \mathbb{F}^{n\times n}$ is $M$-selfadjoint and $\sigma(S) \subseteq \mathbb{C}_+$;

- decomposition exists if $(A^\star)^\star = A$ (guaranteed in the orthosymmetric case);
- $S$ uniquely determined by spectral properties;
- notation:
  $\mathbb{C}_+$: open right half plane
  $\mathbb{R}_{<0}$: open negative real axis

# Generalized polar decompositions

**Generalized polar decomposition** of $A \in \mathbb{F}^{n \times n}$ with $\sigma(A) \subseteq \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

$$A = W \cdot S$$

1) $W \in \mathbb{F}^{n \times n}$ is $M$-unitary;

2) $S \in \mathbb{F}^{n \times n}$ is $M$-selfadjoint and $\sigma(S) \subseteq \mathbb{C}_+$;

**Some history**:

- Higham, Mackey, Mackey, Tisseur 2005: definition as above;

- Bolshakov, van der Mee, Ran, Reichstein, Rodman 1997:
  Hermitian scalar products, no assumptions on spectrum of $S$;

# Generalized polar decompositions

**Generalized polar decomposition** of $A \in \mathbb{F}^{n \times n}$ with $\sigma(A) \subseteq \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

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1) $W \in \mathbb{F}^{n \times n}$ is $M$-unitary;

2) $S \in \mathbb{F}^{n \times n}$ is $M$-selfadjoint and $\sigma(S) \subseteq \mathbb{C}_+$;

**Question**: Can we extend the concept to more general matrices?
(A singular, A rectangular?)

# Generalized polar decompositions

**Generalized polar decomposition** of $A \in \mathbb{F}^{n \times n}$ with $\sigma(A) \subseteq \mathbb{C} \setminus \mathbb{R}_{<0}$

$$A = W \cdot S$$

1) $W \in \mathbb{F}^{n \times n}$ is $M$-unitary;

2) $S \in \mathbb{F}^{n \times n}$ is $M$-selfadjoint and $\sigma(S) \subseteq \mathbb{C}_+ \cup \{0\}$;

**Problems**:

1) the generalized polar decomposition need not exist;

2) if it exists, even the selfadjoint factor need not be unique;

# Problems

**Problem 1**: generalized polar decompositions need not exist:
If $A = WS$ were a generalized polar decomposition then

$$A^\star A = S^\star W^\star WS = SW^{-1}WS = S^2$$

**Example**:

$$A = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad A^\star = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$$A^\star A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \stackrel{?}{=} S^2$$

$A^{[*]}A$ does not have a square root let alone an $M$-selfadjoint one.

# Problems

**Problem 2**: generalized polar decompositions need not be unique (not even the selfadjoint factor):

**Example**:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

These are two different generalized polar decompositions with two distinct selfadjoint polar factors.

# Problems

**Theorem**: Let $\langle \cdot, \cdot \rangle_M$ be orthosymmetric. Then $A \in \mathbb{F}^{n \times n}$ has a generalized polar decomposition $A = WS$ with a unique selfadjoint polar factor if and only if

1) $A^\star A$ has no negative real eigenvalues;
2) if zero is an eigenvalue of $A^\star A$, then it is semisimple;
3) $\mathrm{Ker}\ A^\star A = \mathrm{Ker}\ A$.

**Question**: Can we make the generalized polar decomposition unique by relaxing the condition that $W$ be $M$-unitary?

# Canonical generalized polar decomposition

**Definition**: $W \in \mathbb{F}^{n\times n}$ is called a **partial $M$-isometry** if $WW^\star W = W$.

- $W$ partial isometry $\Rightarrow \langle Wx, Wy\rangle_M = \langle x, y\rangle_M$ for all $x, y \in \mathrm{range}\,(W^\star)$
- $\langle Wx, Wy\rangle_M = \langle x, y\rangle_M$ for all $x, y \in \mathrm{range}\,(W^\star) \not\Rightarrow W$ partial isometry

**Example**:

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow W^\star = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\langle Wx, Wy\rangle_M = \langle x, y\rangle_M = 0$ for all $x, y \in \mathrm{range}\,(W^\star) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$,
but $0 = WW^\star W \neq W$

## Canonical generalized polar decomposition

**Canonical generalized polar decomposition** of $A \in \mathbb{F}^{n\times n}$, where $\langle\cdot,\cdot\rangle_M$ is orthosymmetric:

$$A = W \cdot S$$

1) $W \in \mathbb{F}^{n\times n}$ is a partial $M$-isometry;

2) $S \in \mathbb{F}^{n\times n}$ is $M$-selfadjoint and $\sigma(S) \subseteq \mathbb{C}_+ \cup \{0\}$;

3) $\mathrm{range}\ (W^\star) = \mathrm{range}\ (S)$;

**Theorem**: Let $\langle\cdot,\cdot\rangle_M$ be orthosymmetric. Then $A \in \mathbb{F}^{n\times n}$ has a unique canonical generalized polar decomposition $A = WS$ if and only if

1) $A^\star A$ has no negative real eigenvalues;

2) if zero is an eigenvalue of $A^\star A$, then it is semisimple;

3) $\mathrm{Ker}\ A^\star A = \mathrm{Ker}\ A$.

# Computational considerations

**Theorem**: Let $A \in \mathbb{F}^{n \times n}$ have a unique canonical generalized polar decomposition $A = WS$. Let $g$ be any scalar function of the form $g(x) = xh(x^2)$ such that

1) the iteration $X_{k+1} = g(X_k)$ converges to $\mathrm{sign}(X_0)$ with order of convergence $p$ whenever $\mathrm{sign}(X_0)$ is defined,

2) $g(0) = 0$,

3) for sesquilinear forms, $g(X^\star) = g(X)^\star$ for all $X \in \mathbb{C}^{n \times n}$ in the domain of $g$.

Then the iteration

$$Y_{k+1} = Y_k h(Y_k^\star Y_k), \qquad Y_0 = A$$

converges to $W$ with order of convergence $p$.

# Computational considerations

**Example**: The iteration

$$S_{k+1} = 2S_k(I + S_k^2)^{-1}, \quad S_0 = S$$

which converges quadratically to $\mathrm{sign}(S)$;

leads to the iteration

$$X_{k+1} = 2X_k \left(I + X_k^\star X_k\right)^{-1}, \quad X_0 = A$$

which converges quadratically to $W$, when a unique canonical generalized polar decomposition $A = WS$ exists;

$S$ can be computed via $S = W^\star A$.

# Computational considerations

**Example**: $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$

The iteration

$$X_{k+1} = 2X_k \left(I + X_k^\star X_k\right)^{-1}, \quad X_0 = A$$

produces iterates with the following relative differences:

| iterate | $\|X_k - X_{k-1}\|_1 / \|X_k\|1$ |
|---|---|
| $X_1$ | 5.00e-1 |
| $X_2$ | 5.56e-2 |
| $X_3$ | 1.73e-3 |
| $X_4$ | 1.50e-6 |
| $X_5$ | 1.13e-12 |
| $X_6$ | 1.05e-16 |

# Conclusions

- concept of generalized polar decomposition can be generalized to the singular case;
- the concept of canonical polar decomposition can be generalized to orthosymmetric scalar products;
- the concept can even be generalized to rectangular matrices;
- the uniqueness of the decomposition allows to use well-known matrix iterations for the computation of the decomposition.

**Thank you for your attention!**