Semidefinite invariant subspaces: degenerate inner products

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Abstract

The paper concerns several classes of matrices in possibly degenerate indefinite inner products, such as expansive, dissipative, normal and plus matrices. The main results concern existence of invariant maximal semidefinite subspaces for matrices in these classes.

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1 Introduction

The theory and applications of semidefinite invariant subspaces for certain classes of operators in indefinite inner product spaces, both finite and infinite dimensional, is well-developed by now (see, e.g., the monographs [1], [2], [4], [8], [14], [17]). However, most results in this area are available under the additional hypothesis that the indefinite inner product is regular, i.e., the only vector orthogonal to the whole space is the zero vector. At the same time, nonregular, or degenerate, indefinite inner products do appear in various applications (see [3], [16], [21]). Because of this, and of general mathematical interest, it is worthwhile to develop a more general theory of classes of operators and semidefinite invariant subspaces in indefinite inner products that does not presuppose regularity. Some work in this direction already exists (see [22]). In the present paper, we continue work in this direction, with emphasis on semidefinite invariant subspaces.

We confine ourselves to finite dimensional spaces, as proofs of several of our main results depend on finite dimensionality, although many statements in Section 2 can be extended to some infinite dimensional indefinite inner product spaces.

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Throughout the paper, we consider the vector space $\mathbb{F}^n$, where $\mathbb{F}$ is the real field or the complex field. We fix the indefinite inner product $[\cdot, \cdot]$ determined by a not necessarily invertible Hermitian (or symmetric in the real case) $n \times n$ matrix H via the formula

$$[x, y] = \langle Hx, y\rangle, \qquad x, y \in \mathbb{F}^n.$$

(Here, $\langle\cdot,\cdot\rangle$ denotes the standard inner product.) By $i_+(H)$ (respectively, $i_-(H), i_0(H)$) we denote the number of positive (respectively, negative, zero) eigenvalues (counted with multiplicities) of H.

The rest of the paper is organized as follows. Section 2 is preliminary and reviews the theory of semidefinite subspaces. Many results there may be well known but not easily found in the literature. In Sections 3 to 6, we then discuss the existence of maximal semidefinite subspaces that are invariant for matrices from various classes with respect to the indefinite inner product, namely:

a) expansive matrices A: $[Ax, Ax] \geq [x, x]$, $x \in \mathbb{F}^n$;

b) plus-matrices A: $[Ax, Ax] \geq 0$ for all $x \in \mathbb{F}^n$ such that $[x, x] \geq 0$;

c) dissipative matrices A: $\mathrm{Im}[Ax, x] \geq 0$, $x \in \mathbb{F}^n$ (in the complex case);

d) normal matrices A: $A^*HA = HAH^\dagger A^*H$, where $H^\dagger$ denotes the Moore-Penrose generalized inverse of H.

We note that in each case it is easy to check that if A is from one of the classes of matrices in a)–d) with respect to the inner product induced by H, then $P^{-1}AP$ is from the corresponding class with respect to the inner product induced by P^*HP, provided that P is nonsingular.

We start investigating expansive matrices in Section 3. The key result is that for an H-expansive matrix $A \in \mathbb{F}^{n\times n}$ and an A-invariant H-nonnegative subspace $\mathcal{M}_0$ there exists an A-invariant H-nonnegative subspace $\mathcal{M} \supseteq \mathcal{M}_0$ such that $\dim \mathcal{M} = i_+(H)+i_0(H)$. This result is the basis for analogous results for plus-matrices, described in Section 4, and for dissipative matrices, described in Section 5. In Section 6 we discuss H-normal matrices. We first prove that in a nondegenerate indefinite inner product space any H-normal matrix has an invariant maximal H-nonnegative subspace. Our proof is constructive, and in that way it differs from the one given in [2]. The second main result of the section concerns spaces with a degenerate indefinite inner product. We show that if N is H-normal, then there is an H-nonnegative N-invariant subspace $\mathcal{M}$ with $\dim \mathcal{M} = i_+(H) + i_0(H)$.

We shall use the following notations in the sequel: $\mathbb{N}$ is the set of positive numbers; $e_k = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{F}^n$ denotes the k^{th} standard unit vector (with 1 in the k^{th} position) - the dimension n is understood from context; $\mathrm{Span}\{x_1, \ldots, x_k\}$ is the subspace spanned by the vectors $x_1, \ldots, x_k$; $\mathrm{Im}\ X$ is the column space of a matrix X; for a complex number z, $\mathrm{Im}\ z$ is the imaginary part of z; Z_n is the $n \times n$ matrix with ones on the upper right - lower left diagonal and zeros elsewhere; $\mathcal{J}_n(\lambda)$ denotes a Jordan block of size n associated with the eigenvalue λ; I_p and 0_p stand for the $p \times p$ identity and the $p \times p$ zero matrix (if p is clear from context, it will often be omitted); $A_1 \oplus \cdots \oplus A_k$ is a block diagonal matrix with diagonal blocks $A_1, \ldots, A_k$ (in that order); A^T is the transpose of a matrix A; $A \geq 0$ stands for positive semidefiniteness of matrix a A; and $\|X\|$ denotes the operator norm (largest singular value) of a matrix X.

2 Semidefinite subspaces

In this section we develop a general framework to handle various types of semidefinite subspaces. The class of *H-nonnegative* subspaces is defined by

$$\mathcal{S}_{\geq 0}(H) = \Big\{ V : V \subseteq \mathbb{F}^n \text{ subspace such that } [x,x] \geq \ 0 \text{ for all } x \in V \setminus \{0\} \Big\}. \tag{2.1}$$

Analogously, we define the classes $\mathcal{S}_{>0}(H)$, $\mathcal{S}_{\leq 0}(H)$, $\mathcal{S}_{<0}(H)$, $\mathcal{S}_{=0}(H)$, of *H-positive*, *H-nonpositive*, *H-negative*, *H-neutral* subspaces, respectively, by replacing the symbol $\geq$ in (2.1) with $>, \leq, <, =$, respectively.

Clearly, if a subspace $V \subseteq \mathbb{F}^n$ belongs to one of these classes, then so does every subspace of V. Observe that by default (non-existence of nonzero vectors) the zero subspace is simultaneously H-positive, H-negative, and H-neutral.

A subspace $\mathcal{M} \in \mathcal{S}_{\geq 0}(H)$ is called *maximal H-nonnegative* if there is no larger subspace in the set $\mathcal{S}_{\geq 0}(H)$. Analogously, maximal H-positive, H-nonpositive, H-negative, and H-neutral subspaces are defined.

We note that the classes $S_\eta(H)$, where $\eta \in \{\geq 0, > 0, \leq 0, < 0, = 0\}$, are naturally transformed under congruence $H \mapsto S^*HS$, where $S \in \mathbb{F}^{n\times n}$ is invertible. Namely,

$$\mathcal{M} \in \mathcal{S}_\eta(H) \text{ if and only if } S^{-1}\mathcal{M} \stackrel{\text{def}}{=} \{S^{-1}x |\, x \in \mathcal{M}\} \in \mathcal{S}_\eta(S^*HS). \tag{2.2}$$

As a consequence, we obtain the following useful lemma (whose elementary proof is omitted).

Lemma 2.1 *Let $\mathcal{M}$ be H-positive, respectively, H-negative, with dimension d. Then there exists a nonsingular matrix $S \in \mathbb{F}^{n\times n}$ such that $S^{-1}\mathcal{M} = \mathrm{Span}\{e_1, \ldots, e_d\}$ and $S^*HS = H_{11} \oplus H_{22}$, where $H_{11} \in \mathbb{F}^{d\times d}$ is positive definite, respectively, negative definite.*

Proposition 2.2 *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. Then*
(a) *$\mathcal{M} \in \mathcal{S}_{\geq 0}(H)$ is maximal H-nonnegative if and only if* $\dim \mathcal{M} = i_+(H) + i_0(H)$*;*
(b) *$\mathcal{M} \in \mathcal{S}_{>0}(H)$ is maximal H-positive if and only if* $\dim \mathcal{M} = i_+(H)$*;*
(c) *$\mathcal{M} \in \mathcal{S}_{\leq 0}(H)$ is maximal H-nonpositive if and only if* $\dim \mathcal{M} = i_-(H) + i_0(H)$*;*
(d) *$\mathcal{M} \in \mathcal{S}_{<0}(H)$ is maximal H-negative if and only if* $\dim \mathcal{M} = i_-(H)$*;*
(e) *$\mathcal{M} \in \mathcal{S}_{=0}(H)$ is maximal H-neutral if and only if* $\dim \mathcal{M} = \min(i_+(H), i_-(H)) + i_0(H)$*.*

The "if" parts of Proposition 2.2 follow from the interlacing inequalities for eigenvalues of compressions of Hermitian matrices to a subspace (see, e.g., [13, Theorem 4.3.8] or [18, Chapter 8]). We omit the proofs of "only if" parts, noting only that Lemma 2.1 may be used for the proof of "only if" in (b), and the "only if" of (a) can be obtained from (b) by considering $H + \epsilon I$, where $\epsilon > 0$, and letting $\epsilon \longrightarrow 0$.

In the following, we will discuss the existence of maximal nonnegative subspaces that contain a given nonnegative subspace. Therefore, a description of nonnegative and maximal nonnegative subspaces in terms of the range of certain matrices is needed. Note that, setting $i_+(H) = p$, $i_-(H) = q$, and $i_0(H) = \nu$, we can assume that H has the form

$$H = I_p \oplus -I_q \oplus 0_\nu. \tag{2.3}$$

Indeed, by applying a congruence $H \mapsto S^*HS$ for a suitable invertible matrix S, and simultaneously transforming $\mathcal{M}_0 \mapsto S^{-1}\mathcal{M}_0$, the form (2.3) can always be achieved. Then, H-nonnegative subspaces can be conveniently described (see, e.g., [4] for the case of invertible H):

Lemma 2.3 *Let H be given by (2.3). Then a subspace $\mathcal{M} \subseteq \mathbb{F}^n$ of dimension $d > 0$ is H-nonnegative if and only if $\mathcal{M}$ has the form*

$$\mathcal{M} = \operatorname{Im} \begin{bmatrix} P & 0 \\ K & 0 \\ Y & X \end{bmatrix}, \tag{2.4}$$

where $P \in \mathbb{F}^{d\times r}$, $K \in \mathbb{F}^{q\times r}$, and $\begin{bmatrix} Y & X \end{bmatrix} \in \mathbb{F}^{\nu\times d}$ satisfy $P^*P = I_r$, $\|K\| \leq 1$, $X^*X = I_{d-r}$, and $X^*Y = 0$ for some r with $0 \leq r \leq d$.

Proof. It is easy to see that every subspace of the form (2.4) is H-nonnegative.

Conversely, let $\mathcal{M} \in \mathcal{S}_{\geq 0}(H)$, $\dim \mathcal{M} = d$. Let $f_1, \dots, f_d$ be a basis of $\mathcal{M}$, and partition

$$f_j = \begin{bmatrix} f_{1j} \\ f_{2j} \\ f_{3j} \end{bmatrix}, \quad f_{1j} \in \mathbb{F}^p,\ f_{2j} \in \mathbb{F}^q,\ f_{3j} \in \mathbb{F}^\nu.$$

We note that the vectors $[f_{1j}, f_{3j}]^T$, $j = 1, \dots, d$, are linearly independent; otherwise, $\mathcal{M}$ would contain a nonzero vector of the form $\left[0, f^T, 0\right]^T$, $f \in \mathbb{F}^q$, a contradiction with $\mathcal{M}$ being H-nonnegative.

Let r be the dimension of the span of the vectors $f_{1j}, j = 1, \dots, d$. Without loss of generality, we may assume that $f_{11}, \dots, f_{1r}$ are linearly independent. By subtracting from some of the f_j, $j > r$, linear combinations of $f_1, \dots, f_r$, we may moreover assume without loss of generality that $f_{1j} = 0$ for $j > r$. It then follows that also $f_{2j} = 0$ for $j > r$, as otherwise $\mathcal{M}$ contains H-negative vectors.

Put $F = [f_{11} \cdots f_{1r}]$. Let $P = FT$, where $T \in \mathbb{F}^{r\times r}$ is chosen so that $P^*P = I_e$. As F^*F is invertible such a choice of T is possible. Now we take $K = [f_{21} \cdots f_{2r}]\,T$.

Define $X_1 = [f_{3,r+1} \cdots f_{3d}]$ and $Y = [f_{31} \cdots f_{3r}]\,T$. Then

$$\mathcal{M} = \mathrm{Im}\,[f_1 \cdots f_d] = \mathrm{Im} \begin{bmatrix} P & 0 \\ K & 0 \\ Y & X_1 \end{bmatrix}.$$

It follows that rank $X_1 = d - r$. Then there is a matrix S such that $X_1 = XS$ with $X^*X = I_{d-r}$. So

$$\mathcal{M} = \mathrm{Im} \begin{bmatrix} P & 0 \\ K & 0 \\ Y & X \end{bmatrix}.$$

Next, we show that we can take Y in such a way that $X^*Y = 0$. Indeed, the latter condition simply means that Im X and Im Y are orthogonal. This can be achieved by observing that adding to Y a matrix of the form XW, where W is arbitrary, does not change $\mathcal{M}$. Then $X^*Y + X^*XW = X^*Y + W = 0$ if we take $W = -X^*Y$. Finally, a straightforward calculation shows that $\|K\| \leq 1$. $\square$

Note that the representation (2.4) for a given $\mathcal{M}$ is not unique. We have the following result.

Lemma 2.4 *If*

$$\mathrm{Im} \begin{bmatrix} P_1 & 0 \\ K_1 & 0 \\ Y_1 & X_1 \end{bmatrix} = \mathrm{Im} \begin{bmatrix} P_2 & 0 \\ K_2 & 0 \\ Y_2 & X_2 \end{bmatrix}, \tag{2.5}$$

where P_1 and P_2 are $p \times r_1$ and $p \times r_2$ matrices such that $P_1^*P_1 = I$ and $P_2^*P_2 = I$, and $X_1^*X_1 = I$, $X_2^*X_2 = I$, $X_i^*Y_i = 0$ for $i = 1, 2$, then $r_1 = r_2$ and $P_1 = P_2W$, $Y_1 = Y_2W$, $K_1 = K_2W$ for some unitary $d \times d$ matrix W, and $X_1 = X_2V$ for some unitary V. Such unitary matrices W and V are unique.

Proof. We have

$$\begin{bmatrix} P_1 & 0 \\ K_1 & 0 \\ Y_1 & X_1 \end{bmatrix} = \begin{bmatrix} P_2 & 0 \\ K_2 & 0 \\ Y_2 & X_2 \end{bmatrix} Q, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}, \quad Q_1 \in \mathbb{F}^{r_2 \times r_1}, \tag{2.6}$$

for some invertible matrix Q which is easily seen to be unique. Conditions $P_1^*P_1 = I$ and $P_2^*P_2 = I$ imply $Q_1^*Q_1 = I$; in particular, $r_2 \geq r_1$. Reversing the roles of the matrices in the left and the right hand sides of (2.6), we obtain also $r_1 \geq r_2$. Thus, $r_1 = r_2$, and Q_1 is unitary. Next, we obtain from (2.6):

$$Q^*(Z_2 \oplus I)Q = Z_1 \oplus I,$$

where $Z_j = I + K_j^*K_j + Y_j^*Y_j$, $j = 1, 2$. In particular,

$$Q_1^*Z_2Q_1 = Z_1 - Q_3^*Q_3, \tag{2.7}$$

and since upon subtracting a positive semidefinite matrix from a positive definite matrix the eigenvalues can only decrease, we obtain that $\lambda_k(Z_2) \le \lambda_k(Z_1)$, $k = 1, 2 \ldots, r_1$, where $\lambda_k(X)$ stand for the eigenvalues of a positive definite matrix X, in the non-increasing order. Reversing the roles of Z_1 and Z_2 we obtain also $\lambda_k(Z_1) \le \lambda_k(Z_2)$, and hence Z_1 and Z_2 have the same eigenvalues. Now (2.7) implies $Q_3 = 0$. Analogously $Q_2 = 0$. Now from (2.6) we have $X_1 = X_2 Q_4$, and (as for Q_1) we show that Q_4 is unitary. □

In particular Lemma 2.4 applies to an H-nonnegative subspace of the form (2.5). However, for the validity of the lemma, it is not necessary that $\|K_1\| \le 1$, $\|K_2\| \le 1$.

Corollary 2.5 *Every maximal H-nonnegative subspace $\mathcal{M}$ can be uniquely written in the form*

$$\mathcal{M} = \operatorname{Im} \begin{bmatrix} I_p & 0 \\ K & 0 \\ 0 & I_\nu \end{bmatrix}, \tag{2.8}$$

where $K \in \mathbb{F}^{q \times p}$ satisfies $\|K\| \le 1$. Conversely, every subspace of the form (2.8) is maximal H-nonnegative.

Indeed, the uniqueness of (2.8) follows from Lemma 2.4. Existence of (2.8) follows from Lemma 2.3 in which P is unitary, because in view of Proposition 2.2(a) the dimension of $\mathcal{M}$ is equal to $p + \nu$.

Next, we express containment of an H-nonnegative subspace in a maximal such subspace, in terms of the representation (2.4).

Lemma 2.6 *Let*

$$\mathcal{M}_0 = \operatorname{Im} \begin{bmatrix} P_0 & 0 \\ K_0 & 0 \\ Y_0 & X_0 \end{bmatrix} \in \mathcal{S}_{\ge 0}(H), \quad \mathcal{M} = \operatorname{Im} \begin{bmatrix} I_p & 0 \\ K & 0 \\ 0 & I_\nu \end{bmatrix} \in \mathcal{S}_{\ge 0}(H),$$

where $P_0 \in \mathbb{F}^{p \times r}$, $K_0 \in \mathbb{F}^{q \times r}$, $K \in \mathbb{F}^{q \times p}$ are such that $P_0^* P_0 = I$ and $\|K_0\| \le 1$, $\|K\| \le 1$, and $X_0^* X_0 = I$, $X_0^* Y_0 = 0$. (In particular, $\mathcal{M}$ is maximal H-nonnegative). Then $\mathcal{M}_0 \subseteq \mathcal{M}$ if and only if $K_0 = K P_0$.

Proof. If $K_0 = K P_0$, then obviously $\begin{bmatrix} P_0 \\ K_0 \end{bmatrix} = \begin{bmatrix} I \\ K \end{bmatrix} P_0$, and therefore $\mathcal{M}_0 \subseteq \mathcal{M}$. Conversely, if $\mathcal{M}_0 \subseteq \mathcal{M}$, then there exist matrices B_{11}, B_{12}, B_{21}, B_{22} such that

$$\begin{bmatrix} P_0 & 0 \\ K_0 & 0 \\ Y_0 & X_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ K & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

It is immediate from this equality that in fact $B_{12} = 0$, $B_{22} = X_0$, $B_{21} = Y_0$ and $B_{11} = P_0$. It then follows that $K_0 = K B_{11} = K P_0$. □

We will need also the following description of H-neutral subspaces:

Lemma 2.7 *Let H be invertible and let $\mathcal{M}$ be H-neutral with dimension d. Then $2d \leq n$ and there exists a nonsingular $P \in \mathbb{F}^{n\times n}$ such that $P^{-1}\mathcal{M} = \mathrm{Span}\{e_1, \dots, e_d\}$ and*

$$P^*HP = \begin{bmatrix} 0 & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & H_{33} \end{bmatrix}.$$

Proof. The result follows from the properties of skewly-linked neutral subspaces (see, for example, [14, Lemma 3.1], or [7]). For completeness, we offer an independent proof. By equation (2.2), we may assume that $\mathcal{M} = \mathrm{Span}\{e_1, \dots, e_d\}$. Since H is nonsingular, we obtain from Proposition 2.2 that $2d \leq n$. Partition

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^* & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix}.$$

Then $H_{11} = 0$ and hence, $\begin{bmatrix} H_{12} & H_{13} \end{bmatrix}$ is of full row rank. But then, there exists matrices $Q_1 \in \mathbb{F}^{d\times d}$ and $Q_2 \in \mathbb{F}^{(n-2d)\times(n-2d)}$ such that $Q_1 \begin{bmatrix} H_{12} & H_{13} \end{bmatrix} Q_2 = \begin{bmatrix} I_d & 0 \end{bmatrix}$. Setting $P_1 = Q_1 \oplus Q_2$, we obtain

$$P_1^*HP_1 = \begin{bmatrix} 0 & I_d & 0 \\ I_d & \hat{H}_{22} & \hat{H}_{23} \\ 0 & \hat{H}_{23}^* & \hat{H}_{33}^* \end{bmatrix}, \quad P_1^{-1}\mathcal{M} = \mathcal{M}.$$

Setting furthermore

$$P = P_1 \begin{bmatrix} I & -\frac{1}{2}\hat{H}_{22} & -\hat{H}_{23} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

we obtain that $P^{-1}\mathcal{M} = \mathcal{M}$ and P^*HP has the desired form. $\square$

3 Expansive matrices

In this section we focus on H-expansive matrices. Recall that a matrix $A \in \mathbb{F}^{n\times n}$ is called H-*expansive* if $[Ax, Ax] \geq [x, x]$ for all $x \in \mathbb{F}^n$, or, equivalently, if $A^*HA - H \geq 0$. It turns out that the kernel $\mathrm{Ker}\, A := \{x \in \mathbb{F}^n \,:\, Ax = 0\}$ of an H-expansive matrix A is H-negative (as long as H is invertible). To prove this, we need the following auxiliary result.

Lemma 3.1 *Let $A \in \mathbb{F}^{n\times n}$ be H-expansive. Then there exists a nonsingular matrix $S \in \mathbb{F}^{n\times n}$ such that*

$$S^*HS = H_1 \oplus (-I_{p_2}) \oplus 0_{p_3}, \quad S^*A^*HAS = M_1 \oplus 0_{p_2} \oplus 0_{p_3}, \tag{3.1}$$

where $M_1, H_1 \in \mathbb{F}^{p_1\times p_1}$ are nonsingular and $p_1, p_2, p_3 \in \mathbb{N} \cup \{0\}$.

Proof. By the well-known canonical forms under congruence for pairs of Hermitian matrices or pairs of symmetric matrices in the real case (see, e.g., [24], [25]), we may assume that H and A^*HA have the forms

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_m, \quad A^*HA = M_1 \oplus M_2 \oplus \cdots \oplus M_m,$$

where M_j and H_j have the same size, H_1 and M_1 are nonsingular, and M_j and H_j, $j > 1$, are blocks of one of the following types:

type 1: $H_j = \varepsilon Z_p$ and $M_j = \varepsilon Z_p \mathcal{J}_p(0)$ for some $p \in \mathbb{N}$, $\varepsilon = \pm 1$;

type 2: $p \in \mathbb{N} \cup \{0\}$,

$$H_j = \begin{bmatrix} 0 & 0 & I_p \\ 0 & 0 & 0 \\ I_p & 0 & 0 \end{bmatrix} \in \mathbb{F}^{(2p+1)\times(2p+1)}, \quad M_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_p \\ 0 & I_p & 0 \end{bmatrix} \in \mathbb{F}^{(2p+1)\times(2p+1)},$$

type 3: $H_j = \varepsilon Z_p \mathcal{J}_p(0)$ and $M_j = \varepsilon Z_p$ for some $p \in \mathbb{N}$, $\varepsilon = \pm 1$.

Clearly, since A is H-expansive, each block $M_j - H_j$ is positive semidefinite. It is easy to check that this is possible if and only if $p = 1$ and $\varepsilon = -1$ if H_j and M_j are of type 1, $p = 1$ and $\varepsilon = 1$ if H_j and M_j are of type 3, and $p = 0$ if H_j and M_j are of type 2. But then, after eventually permuting some blocks, H and A^*HA have the forms

$$H = H_1 \oplus 0_{p_2} \oplus (-I_{p_3}) \oplus 0_{p_4}, \quad A^*HA = M_1 \oplus 0_{p_2} \oplus 0_{p_3} \oplus I_{p_4}.$$

Note that $M_1 - H_1$ is still positive semidefinite, i.e., the number of positive eigenvalues of M_1 is larger or equal to the number of positive eigenvalues of H_1 (see, e.g., 7.7.4 in [13]). From the well-known fact that the number of positive (negative) eigenvalues of A^*HA is always less or equal to the number of positive (negative, respectively) eigenvalues of H (see, e.g., 4.5.11 in [13]), it follows that blocks of type 3 cannot occur and hence, A^*HA and H have forms as in (3.1). $\square$

Corollary 3.2 *Let H be invertible and let $A \in \mathbb{F}^{n\times n}$ be H-expansive. Then* Ker A *is H-negative.*

Proof. If A is nonsingular, Ker A is H-negative by definition. Otherwise, let $y \in \operatorname{Ker} A$. Then $y \in \operatorname{Ker} A^*HA$. Since H is invertible, it follows immediately from Lemma 3.1 and equation (2.2) that $y^*Hy < 0$. $\square$

Proposition 3.3 *Let $A \in \mathbb{F}^{n\times n}$ be H-expansive. Then* Ker H *is A-invariant.*

Proof. Applying a transformation of the form $(A, H) \mapsto (S^{-1}AS, S^*HS)$, we may assume that H and A^*HA have the forms as in (3.1). Applying one more transformation on M_1 and H_1, we may furthermore assume that

$$H = I_{p_1} \oplus -I_{p_2} \oplus -I_{p_3} \oplus 0, \quad A^*HA = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} \oplus 0,$$

where $p_1, p_2, p_3 \in \mathbb{N} \cup \{0\}$, $M_{11} \in \mathbb{F}^{p_1 \times p_1}$, $M_{12} \in \mathbb{F}^{p_1 \times p_2}$, $M_{22} \in \mathbb{F}^{p_2 \times p_2}$. Let A be partitioned conformably:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

With $A^*HA - H$ also $M_{11} - I_{p_1} = A_{11}^* A_{11} - A_{21}^* A_{21} - A_{31}^* A_{31} - I_{p_1}$ must be positive semidefinite. This is possible only if $A_{11}^* A_{11}$ is positive definite, i.e., only if A_{11} is nonsingular. Next, we show $A_{14} = 0$. To see this, assume that A_{14} is not zero. Then there exists a matrix $P \in \mathbb{F}^{p_4 \times p_1}$, $p_4 = n - p_1 - p_2 - p_3$, such that $A_{11} - PA_{14}$ is singular. (For example, if $v \in \mathbb{F}^{p_4}$ is such that $A_{14}v \neq 0$ choose P such that $PA_{14}v = A_{11}v$.) Applying the transformation $(A, H) \mapsto (\mathcal{P}^{-1}A\mathcal{P}, \mathcal{P}^* H \mathcal{P})$ with

$$\mathcal{P} = \begin{bmatrix} I_{p_1} & 0 & 0 & 0 \\ 0 & I_{p_2} & 0 & 0 \\ 0 & 0 & I_{p_3} & 0 \\ P & 0 & 0 & I_{p_4} \end{bmatrix},$$

we find that the $(1,1)$-block of $\mathcal{P}^{-1}A\mathcal{P}$ is $A_{11} - PA_{14}$, whereas $\mathcal{P}^* H \mathcal{P} = H$ and $\mathcal{P}^* A^* H A \mathcal{P} = A^* H A$. This contradicts the fact just mentioned that the $(1,1)$-block of $\mathcal{P}^{-1}A\mathcal{P}$ must be nonsingular. But $A_{14} = 0$ implies that the $(4,4)$-block of A^*HA has the form $-A_{24}^* A_{24} - A_{34}^* A_{34} = 0$. This is possible only if $A_{24} = 0$ and $A_{34} = 0$. Hence, Ker H is A-invariant. $\square$

The key result in this section is the following theorem.

Theorem 3.4 *Let $A \in \mathbb{F}^{n \times n}$ be H-expansive and let $\mathcal{M}_0 \subseteq \mathbb{F}^n$ be an A-invariant subspace which is H-nonnegative. Then there exists an H-nonnegative A-invariant subspace $\mathcal{M} \supseteq \mathcal{M}_0$ such that* $\dim \mathcal{M} = i_+(H) + i_0(H)$.

Proof. For the case when H is invertible, the proof follows by a well-known argument that originated with M. G. Kreĭn [15] using the Schauder's fixed point theorem [11, Section 106] and the representation of $\mathcal{M}_0$ according to Lemma 2.3:

$$\mathcal{M}_0 = \operatorname{Im} \begin{bmatrix} P_0 \\ K_0 \end{bmatrix},$$

where $P_0^* P_0 = I$, $\|K_0\| \leq 1$; and see [2, Section 3.3] for an application of a fixed point theorem in a more general Krein space context.

Thus, consider the case that H is singular. Without loss of generality, we may assume that H has the form (2.3), i.e., $H = H_1 \oplus 0$, where $H_1 = I_p \oplus (-I_q)$. Then Proposition 3.3 implies that A takes the form

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{F}^{(p+q)\times(p+q)}$ is easily seen to be H_1-expansive. Represent $\mathcal{M}_0$ according to Lemma 2.3:

$$\mathcal{M}_0 = \operatorname{Im} \begin{bmatrix} Q_0 & 0 \\ Y_0 & X_0 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} P_0 \\ K_0 \end{bmatrix},$$

where $P_0^* P_0 = I_p$, $\|K_0\| \le 1$, $X_0^* X_0 = I$, and $X_0^* Y_0 = 0$. Then $\widetilde{\mathcal{M}_0} = \operatorname{Im} Q_0$ is H_1-nonnegative and A_{11}-invariant. By the part already proved, there exists an $i_+(H)$-dimensional, H_1-nonnegative, and A_{11}-invariant subspace $\widetilde{\mathcal{M}}$ that contains $\mathcal{M}_0$. Let $\widetilde{\mathcal{M}} = \operatorname{Im} Q$ for some matrix Q of appropriately chosen dimension, i.e., $Q_0 = QW_0$ for some matrix W_0. Then $\mathcal{M} := \operatorname{Im}(Q \oplus I_\nu)$ is H-nonnegative and A-invariant. Furthermore, it contains $\mathcal{M}_0$, and has dimension $i_+(H) + i_0(H)$. □

Obviously, every H-isometric matrix is H-expansive (recall that a matrix A is called *H-isometric* if $[Ax, Ax] = [x, x]$ for every $x \in \mathbb{F}^n$), Thus, as an important corollary of Theorem 3.4, we obtain:

Theorem 3.5 *Let $A \in \mathbb{F}^{n\times n}$ be an H-isometric, and let $\mathcal{M}_0 \subseteq \mathbb{F}^n$ be an A-invariant H-nonnegative, respectively, H-nonpositive, subspace. Then there exists an A-invariant H-nonnegative, respectively, H-nonpositive, subspace $\mathcal{M}$ such that $\mathcal{M} \supseteq \mathcal{M}_0$ and* $\dim \mathcal{M} = i_+(H)$*, respectively,* $\dim \mathcal{M} = i_-(H)$*.*

The part of Theorem 3.5 concerning H-nonpositive subspaces follows by noticing that A is also expansive with respect to $-H$, and applying Theorem 3.4 with H replaced by $-H$.

4 Plus-matrices

Recall that a matrix $A \in \mathbb{F}^{n\times n}$ is called a *plus-matrix* if $[Ax, Ax] \ge 0$ for every $x \in \mathbb{F}^n$ such that $[x, x] \ge 0$. The following lemma is well known in the complex case (and can be proved by using the Toeplitz-Hausdorff theorem on convexity of the numerical range [1], and see [6] for a proof in the real case).

Lemma 4.1 *If A is a plus-matrix, then there exists a nonnegative number k such that $[Ax, Ax] \ge k[x, x]$ for all $x \in \mathbb{F}^n$.*

Denote by $k(A)$ the smallest value of $k \ge 0$ for which Lemma 4.1 holds.

Theorem 4.2 *Let A be a plus matrix such that* $\operatorname{Im} A$ *is not H-nonnegative, and let $\mathcal{M}_0 \subseteq \mathbb{F}^n$ be an A-invariant subspace which is H-nonnegative. Then there exists H-nonnegative A-invariant subspace $\mathcal{M} \supseteq \mathcal{M}_0$ such that dim $\mathcal{M} = i_+(H) + i_0(H)$.*

Proof. By Lemma 4.1, we have $[Ax, Ax] \ge k(A)[x, x]$ for all $x \in \mathbb{F}^n$. Since $\operatorname{Im} A$ is not H-nonnegative, it follows that $k(A) > 0$. Scaling A, if necessary, we can assume that $k(A) = 1$. Then A is H-expansive, and the result follows from Theorem 3.4. □

The hypothesis that $\operatorname{Im} A$ is not H-nonnegative is essential in Theorem 3.4. The following example illustrates this.

Example 4.3 Let p and q are real numbers such that $4p+q^2>0$. Furthermore, let

$$A = \begin{bmatrix} 0 & 1 & 0 & p \\ -1 & 0 & 1 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{M}_0 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Then, A is a plus matrix; in fact, $[Ax, Ax] = 0$ for every $x \in \mathbb{F}^4$. Moreover, $A\mathcal{M}_0 = \{0\}$ which implies, in particular, that $\mathcal{M}_0$ is A-invariant. The subspace $\mathcal{M}_0$ is H-positive. On the other hand, Ker $A = \text{Span}\{[1,0,1,0]^T, [0,-p,-q,1]^T\}$ is not H-nonnegative. If $F = \mathbb{R}$, then the only two-dimensional A-invariant subspaces are Ker A and Im A, and the latter does not contain $\mathcal{M}_0$. If $F = \mathbb{C}$, then, besides Ker A, there are two two-dimensional A-invariant subspaces that contain $\mathcal{M}_0$, namely, $\text{Span}\left\{[1\ 0\ 1\ 0]^T,\ \ [1\ \ \pm i\ 0\ 0]^T\right\}$. But

$$\begin{bmatrix} 1 & \mp i & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} H \begin{bmatrix} 1 & 1 \\ \pm i & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix},$$

so neither of these two subspaces is H-nonnegative. $\square$

Theorem 4.4 *Let $A \in \mathbb{F}^{n\times n}$ be a plus-matrix. Then there exists an H-nonnegative A-invariant subspace $\mathcal{M}$ such that* $\dim \mathcal{M} = i_+(H) + i_0(H)$.

Proof. By Lemma 4.1 we have $[Ax, Ax] \geq k(A)[x,x]$ for all $x \in \mathbb{F}^n$ and $k(A) \geq 0$. If $k(A) = 0$, then the range of A is an H-nonnegative subspace, and we are done. Indeed, any maximal H-nonnegative subspace $\mathcal{M}$ containing the range of A is A-invariant. Otherwise, apply the previous theorem with $\mathcal{M}_0 = \{0\}$. $\square$

5 Dissipative matrices

Let us consider the case $\mathbb{F} = \mathbb{C}$ first. A matrix $B \in \mathbb{C}^{n\times n}$ is called *H-dissipative* if $\text{Im}[Bx, x] \geq 0$ for every $x \in \mathbb{C}^n$. The dissipativity condition can be easily interpreted in terms of positive definiteness: A matrix B is H-dissipative if and only if

$$i(HB - B^*H) \leq 0. \tag{5.1}$$

Lemma 5.1 (a) *Let A be H-expansive, and let $w, \eta \in \mathbb{C}$ be such that w has positive imaginary part, $|\eta| = 1$, and η is not an eigenvalue of A. Then the matrix*

$$B = (wA - \overline{w}\eta I)(A - \eta I)^{-1} \tag{5.2}$$

is H-dissipative.

(b) *Let B be H-dissipative, and let $w, \eta \in \mathbb{C}$ be such that $|\eta| = 1$, w has positive imaginary part, and is not an eigenvalue of B. Then the matrix*

$$A = \eta(B - \overline{w}I)(B - wI)^{-1} \tag{5.3}$$

is H-expansive.

The proof is obtained by elementary algebraic manipulations and therefore is omitted.

Theorem 5.2 ($\mathbb{F} = \mathbb{C}$) *Let $B \in \mathbb{C}^{n\times n}$ be H-dissipative, and let $\mathcal{M}_0 \subseteq \mathbb{C}^n$ be a B-invariant H-nonnegative, respectively, H-nonpositive subspace. Then there exists a B-invariant maximal H-nonnegative, respectively, maximal H-nonpositive subspace $\mathcal{M}$ such that $\mathcal{M} \supseteq \mathcal{M}_0$.*

Proof. Assume $\mathcal{M}_0$ is H-nonnegative. Let A be given by (5.3). Then A is H-expansive. Note also that $\mathcal{M}_0$ is A-invariant, because A is a function of B. By Theorem 3.4, there exists an A-invariant subspace $\mathcal{M}$ which is maximal H-nonnegative and contains $\mathcal{M}_0$. Since B is a function of A (given by formula (5.2)) $\mathcal{M}$ is also B-invariant. This proves the part of Theorem 5.2 for H-nonnegative subspaces. If $\mathcal{M}_0$ is H-nonpositive, apply the already proved part of Theorem 5.2 to the $(-H)$-dissipative matrix $-B$. $\square$

Recall that a matrix A is called H-skewadjoint if $HA = -A^*H$. Since every H-selfadjoint matrix is obviously H-dissipative and since every H-skewadjoint matrix is just an H-selfadjoint matrix multiplied with the imaginary unit, we immediately obtain the following corollary:

Corollary 5.3 ($\mathbb{F} = \mathbb{C}$) *Let $B \in \mathbb{C}^{n\times n}$ be H-selfadjoint or H-skewadjoint, and let $\mathcal{M}_0 \subseteq \mathbb{C}^n$ be a B-invariant H-nonnegative, respectively, H-nonpositive subspace. Then there exists a B-invariant maximal H-nonnegative, respectively, maximal H-nonpositive subspace $\mathcal{M}$ such that $\mathcal{M} \supseteq \mathcal{M}_0$.*

Theorem 5.2 can also be extended to the real case. But first, we have to modify the definition for H-dissipative matrices, since (5.1) for real matrices B and H implies that B is H-selfadjoint, and not every real H-selfadjoint matrix has an invariant subspace that is maximal H-nonnegative, as easy examples show. Thus, for the case $\mathbb{F} = \mathbb{R}$, let us call $C \in \mathbb{R}^{n\times n}$ *real H-dissipative* if $HC + C^T H \leq 0$, i.e., if $-iC$ is H-dissipative. (Note that this corresponds exactly to the definition of dissipative matrices (or operators) in the sense of [23].) Clearly, in the complex case the theory of real H-dissipative matrices is essentially the same as the theory of H-dissipative matrices, but in the real case, the two definitions lead to fundamentally different classes of matrices.

Theorem 5.4 ($\mathbb{F} = \mathbb{R}$) *Let $C \in \mathbb{R}^{n\times n}$ be real H-dissipative, and let $\mathcal{M}_0 \subseteq \mathbb{R}^n$ be a C-invariant H-nonnegative, respectively, H-nonpositive subspace. Then there exists a C-invariant maximal H-nonnegative, respectively, maximal H-nonpositive subspace $\mathcal{M}$ such that $\mathcal{M} \supseteq \mathcal{M}_0$.*

Proof. Let $\varrho < 0$ such that ϱ is neither an eigenvalue of C, nor of $-C$. Then the real matrix $A = (C + \varrho I)(C - \varrho I)^{-1}$ is H-expansive by a simple computation. The rest of the proof proceeds analogously to the proof of Theorem 5.2. □

Since any H-skewadjoint matrix is real H-dissipative, we immediately obtain the following corollary:

Corollary 5.5 ($\mathbb{F} = \mathbb{R}$) *Let $C \in \mathbb{R}^{n\times n}$ be H-skewadjoint, and let $\mathcal{M}_0 \subseteq \mathbb{R}^n$ be a C-invariant H-nonnegative (respectively, H-nonpositive) subspace. Then there exists a C-invariant maximal H-nonnegative (respectively, maximal H-nonpositive) subspace $\mathcal{M}$ such that $\mathcal{M} \supseteq \mathcal{M}_0$.*

6 Normal matrices

First, let H be invertible. Then for $M \in \mathbb{F}^{n\times n}$, the matrix $M^{[*]_H} := H^{-1}M^*H$, or $M^{[*]}$ if there is no risk of confusion, is called the H*-adjoint* of M. This is the unique matrix satisfying

$$[Mx, y] = [x, M^{[*]}y] \quad \text{for all } x, y \in \mathbb{F}^n. \tag{6.1}$$

A matrix $N \in \mathbb{F}^{n\times n}$ is called H*-normal* if and only if N and $N^{[*]}$ commute, i.e., if and only if $H^{-1}N^*HN = NH^{-1}N^*H$. If H is singular, then for $M \in \mathbb{F}^{n\times n}$ an "H-adjoint", i.e., a matrix $M^{[*]}$ satisfying (6.1) need not exist, and if it exists, it need not be unique. Therefore, we have to modify the definition of H-normal matrices in this case. Noting that H-normality is equivalent to the identity $N^*HN = HNH^{-1}N^*H$, we say that a matrix $N \in \mathbb{F}^{n\times n}$ is H*-normal* if and only if

$$N^*HN = HNH^{\dagger}N^*H,$$

where $H^{\dagger}$ denotes the Moore-Penrose generalized inverse of H. (This definition has been introduced in [19] and used subsequnetly also in [5].) In the case $\mathbb{F} = \mathbb{C}$ and H invertible, it is well known that there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P^{-1}NP = N_1 \oplus \cdots \oplus N_m, \quad P^*HP = H_1 \oplus \cdots \oplus H_m, \tag{6.2}$$

where, for each j, N_j and H_j have the same size and N_j has at most two distinct eigenvalues, see [10], and [12] for a corresponding result in the real case. This is no longer true in the case that H is singular as the following example illustrates:

Example 6.1 Let

$$N = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad H^{\dagger} = \frac{1}{16}\begin{bmatrix} 3 & 4 & 5 \\ 4 & 0 & 12 \\ 5 & 12 & 3 \end{bmatrix}.$$

Then it is easy to check that A is H-normal, in fact, $N^*HN = HNH^{\dagger}N^*H = 0$. Suppose, a decomposition as in (6.2) with $m > 1$ exists. Note that the eigenvectors of N associated with the eigenvalues -1 and 3 are H-neutral and that the eigenvector associated with the eigenvalue 0 is H-negative. Then the fact that Ker H is not N-invariant and that $i_+(H) = i_-(H) = i_0(H) = 1$ implies that the only possible decomposition as in (6.2) with $m > 1$ is

$$P^{-1}NP = N_1 \oplus \left[\, 0 \,\right], \quad P^*HP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus \left[\, -1 \,\right]$$

for some nonsingular matrix $P \in \mathbb{F}^{3\times 3}$ (up to congruent forms for the blocks of P^*HP). This contradicts the fact that the subspace spanned by the eigenvectors of N associated with the eigenvalues -1 and 3 is H-neutral.

The question whether every H-normal matrix has an invariant subspace that is maximal H-nonnegative has a negative answer in the case $\mathbb{F} = \mathbb{R}$ as can be seen from the example

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where α, β are real and $\beta \neq 0$. The answer, however, is affirmative in the case $\mathbb{F} = \mathbb{C}$ (and invertible H), see [2, Corollary 3.4.12]. For the sake of completeness, we present an independent proof for this result.

Theorem 6.2 *Let H be invertible and let $N \in \mathbb{C}^{n\times n}$ be H-normal. Then there exists an H-nonnegative N-invariant subspace $\mathcal{M}$ such that* $\dim \mathcal{M} = i_+(H)$.

Proof. Let $N = A + S$ be the decomposition of N into its H-selfadjoint part A and its H-skewadjoint part S, i.e., $A = 1/2(N + N^{[*]})$ and $S = 1/2(N - N^{[*]})$. Then it is easy to check that N is H-normal if and only if A and S commute. We will now show by induction on n that A and S have a common invariant subspace that is maximal H-nonnegative. Clearly, this subspace is also N-invariant. The case $n = 1$ is clear, since either the zero space $\{0\}$ or the full space $\mathbb{C}$ is maximal H-nonnegative. Next, let $n > 1$. Since A and S commute, they have a common eigenvector $v \in \mathbb{C}^n$.

Case (1): Span$\{v\}$ is not H-neutral, i.e., Span$\{v\}$ is H-positive of H-negative. By Lemma 2.1, we may assume that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad H = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix},$$

where $A_{11}, S_{11}, H_{11} \in \mathbb{C}$, $H_{11} \neq 0$. Since A is H-selfadjoint and S is H-skewadjoint, i.e., $A^*H = HA$ and $-S^*H = HS$, we obtain that $A_{12} = S_{12} = 0$. Then the result follows by using the induction hypothesis on A_{11}, S_{11} and A_{22}, S_{22}.

Case (2): Span$\{v\}$ is H-neutral. By Lemma 2.7, we may assume that

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ 0 & S_{22} & S_{23} \\ 0 & S_{32} & S_{33} \end{bmatrix} H = \begin{bmatrix} 0 & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & H_{33} \end{bmatrix}.$$

Then the fact that A is H-selfadjoint and S is H-skewadjoint implies $A_{23} = S_{23} = 0$ as it can easily be seen from the identities $A^*H = HA$ and $-S^*H = HS$. Also A_{33} is H_{33}-selfadjoint, and S_{33} is H_{33}-skewadjoint. From $A_{23} = S_{23} = 0$ one easily sees by considering the $3,3$-block entry in $SA = AS$, that also $A_{33}S_{33} = S_{33}A_{33}$. Then, by the induction hypothesis, A_{33} and S_{33} have a common H-nonnegative invariant subspace $\widetilde{\mathcal{M}}$ of dimension $\tilde{d} = i_+(H_{33})$. Writing $\widetilde{\mathcal{M}} = \mathrm{Im}\, X$ for some matrix $X \in \mathbb{C}^{n\times\tilde{d}}$ of full column rank, we find that

$$\mathcal{M} = \mathrm{Im} \begin{bmatrix} I_d & 0 \\ 0 & 0 \\ 0 & X \end{bmatrix}$$

is both A- and S-invariant, H-nonnegative, and has dimension $i_+(H) = d + i_+(H_{33})$. $\square$

Next, we consider the question if every H-normal matrix $N \in \mathbb{C}^{n\times n}$ has an invariant subspace that is maximal H-nonnegative and that contains a given H-nonnegative N-invariant subspace $\mathcal{M}_0$. (Note that, in general, $\mathcal{M}_0$ is invariant neither for the H-selfadjoint part A, nor for the H-skewadjoint part S of N.) In its full generality, this question is an open problem, but for the special case that $\mathcal{M}_0$ is neutral, we can give an affirmative answer, even for a larger class than the class of normal matrices. For this, let us introduce the following notation. If $\mathcal{M} \subseteq \mathbb{C}^n$ is a subspace, then

$$\mathcal{M}^{[\perp]} := \{v \in \mathbb{C}^n : [v, w] = 0 \text{ for all } w \in \mathcal{M}\}.$$

Theorem 6.3 *Let H be invertible, let $X \in \mathbb{C}^{n\times n}$, and let $\mathcal{M}_0$ be an H-neutral X-invariant subspace. If $\mathcal{M}_0$ is $(XX^{[*]} - X^{[*]}X)$-invariant and if $\mathcal{M}_0^{[\perp]} \cap (H^{-1}\mathcal{M}_0)^{[\perp]}$ is $(XX^{[*]} - X^{[*]}X)$-neutral, then there exists an H-nonnegative X-invariant subspace $\mathcal{M}$ of dimension $i_+(H)$ such that $\mathcal{M}_0 \subseteq \mathcal{M} \subseteq \mathcal{M}_0^{[\perp]}$.*

Proof. If $d := \dim \mathcal{M}_0 = 0$, then $\mathcal{M}_0^{[\perp]} \cap (H^{-1}\mathcal{M}_0)^{[\perp]} = \mathbb{C}^n$. Since this space is $(XX^{[*]} - X^{[*]}X)$-neutral by assumption, it follows that $(XX^{[*]} - X^{[*]}X) = 0$, i.e., X is H-normal. Then the result follows from Theorem 6.2. If $n = 2d$, then $\mathcal{M}_0$ is already maximal H-nonnegative. The remainder of the proof now proceeds by induction on n. The case $n = 1$ is clear, since necessarily $\mathcal{M}_0 = \{0\}$. Thus, let $n > 1$. By the above, we may assume $0 < d < n/2$. By Lemma 2.7, we may moreover assume that $\mathcal{M}_0 = \mathrm{Span}\{e_1, \ldots, e_d\}$ and

$$P^*HP = \begin{bmatrix} 0 & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & H_{33} \end{bmatrix}.$$

Partitioning X conformably, we obtain that

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{bmatrix}, \quad X^{[*]} = \begin{bmatrix} X_{22}^* & X_{12}^* & X_{32}^*H_{33} \\ 0 & X_{11}^* & 0 \\ H_{33}^{-1}X_{23}^* & H_{33}^{-1}X_{13}^* & X_{33}^{[*]} \end{bmatrix}, \tag{6.3}$$

Note that

$$\mathcal{M}_0^{[\perp]} = \mathrm{Im} \begin{bmatrix} I_d & 0 \\ 0 & 0 \\ 0 & I_{n-2d} \end{bmatrix}, \quad H^{-1}\mathcal{M}_0^{[\perp]} = \mathrm{Im} \begin{bmatrix} 0 & 0 \\ I_d & 0 \\ 0 & I_{n-2d} \end{bmatrix}.$$

Thus, the fact that $\mathcal{M}_0$ is $(XX^{[*]} - X^{[*]}X)$-invariant and that $\mathcal{M}_0^{[\perp]} \cap (H^{-1}\mathcal{M}_0)^{[\perp]}$ is $(XX^{[*]} - X^{[*]}X)$-neutral implies that $XX^{[*]} - X^{[*]}X$ has the pattern

$$XX^{[*]} - X^{[*]}X = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & 0 \end{bmatrix}$$

if it is partitioned conformably with X. Computing the $(2,1)$-, $(3,1)$-, and $(3,3)$-block of $XX^{[*]} - X^{[*]}X$, we obtain the identities

$$X_{23}H_{33}^{-1}X_{23}^* = 0, \tag{6.4}$$

$$X_{33}H_{33}^{-1}X_{23}^* = H_{33}^{-1}X_{23}^*X_{11}, \tag{6.5}$$

$$X_{33}X_{33}^{[*]} - X_{33}^{[*]}X_{33} = H_{33}^{-1}X_{23}^*X_{13} + H_{33}^{-1}X_{13}^*X_{23}. \tag{6.6}$$

Let us consider the subspace $\widetilde{\mathcal{M}}_0 = \mathrm{Im}\ H_{33}^{-1}X_{23}^*$. Then $\widetilde{\mathcal{M}}_0$ is H_{33}-neutral (because of (6.4)), X_{33}-invariant (because of (6.5)), and $(X_{33}X_{33}^{[*]} - X_{33}^{[*]}X_{33})$-invariant, because by (6.6) we have that

$$(X_{33}X_{33}^{[*]} - X_{33}^{[*]}X_{33})H_{33}^{-1}X_{23}^* = H_{33}^{-1}X_{23}^*X_{13}H_{33}^{-1}X_{23}^* + \underbrace{H_{33}^{-1}X_{13}^*X_{23}H_{33}^{-1}X_{23}^*}_{=0 \text{ by } (6.4)}$$

Next, we show that $\widetilde{\mathcal{M}}_0^{[\perp]} \cap (H_{33}^{-1}\widetilde{\mathcal{M}}_0)^{[\perp]}$ is $(X_{33}X_{33}^{[*]} - X_{33}^{[*]}X_{33})$-neutral. For this, let B be a matrix such that $\widetilde{\mathcal{M}}_0^{[\perp]} \cap (H_{33}^{-1}\widetilde{\mathcal{M}}_0)^{[\perp]} = \mathrm{Im}\ B$. Then

$$B^*H_{33}^{-1}X_{23}^* = B^*H_{33}(H_{33}^{-1}H_{33}^{-1}X_{23}^*) = 0, \qquad \text{because } \mathrm{Im}\ B \subseteq (H_{33}^{-1}\widetilde{\mathcal{M}}_0)^{[\perp]}, \tag{6.7}$$

$$\text{and } X_{23}B = (H_{33}^{-1}X_{23}^*)^*H_{33}B = 0, \qquad \text{because } \mathrm{Im}\ B \subseteq \widetilde{\mathcal{M}}_0^{[\perp]}. \tag{6.8}$$

Using (6.6), we obtain

$$B^*\Big(X_{33}X_{33}^{[*]} - X_{33}^{[*]}X_{33}\Big)B = B^*H_{33}^{-1}X_{23}^*X_{13}B + B^*H_{33}^{-1}X_{13}^*X_{23}B = 0,$$

by (6.7) and (6.8), and thus, $\mathrm{Im}\ B$ is $(X_{33}X_{33}^{[*]} - X_{33}^{[*]}X_{33})$-neutral. By the induction hypothesis, there exists an H_{33}-nonnegative X_{33}-invariant subspace $\widetilde{\mathcal{M}}$ of dimension $i_+(H_{33})$ such that $\widetilde{\mathcal{M}}_0 \subseteq \widetilde{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}_0^{[\perp]}$. Let C be a matrix such that $\widetilde{\mathcal{M}} = \mathrm{Im}\ C$. Since $\widetilde{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}_0^{[\perp]}$, we obtain in particular that $X_{23}C = (H_{33}^{-1}X_{23}^*)^*H_{33}C = 0$, and since $\widetilde{\mathcal{M}}$

is X_{33}-invariant, there exists a matrix $Y \in \mathbb{C}^{(n-2d)\times(n-2d)}$ such that $X_{33}C = CY$. Now choose

$$\mathcal{M} = \operatorname{Im} \begin{bmatrix} I_d & 0 \\ 0 & 0 \\ 0 & C \end{bmatrix}.$$

Clearly, $\mathcal{M}$ is nonnegative with dimension $i_+(H) = d + i_+(H_{33})$ and $\mathcal{M}_0 \subseteq \mathcal{M} \subseteq \mathcal{M}_0^{[\perp]}$. Moreover, we obtain

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{bmatrix} \begin{bmatrix} I_d & 0 \\ 0 & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} I_d & 0 \\ 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} X_{11} & X_{13}C \\ 0 & Y \end{bmatrix},$$

i.e., $\mathcal{M}$ is X-invariant. This concludes the proof. $\square$

If $XX^{[*]} - X^{[*]}X = 0$, then clearly any subspace is $(XX^{[*]} - X^{[*]}X)$-invariant and $(XX^{[*]} - X^{[*]}X)$-neutral. Hence, we immediately obtain the following corollary.

Corollary 6.4 *Let H be invertible, let $N \in \mathbb{C}^{n\times n}$ be H-normal, and let $\mathcal{M}_0$ be an H-neutral N-invariant subspace. Then there exists an H-nonnegative N-invariant subspace $\mathcal{M}$ of dimension $i_+(H)$ such that $\mathcal{M}_0 \subseteq \mathcal{M} \subseteq \mathcal{M}_0^{[\perp]}$.*

Note that a direct proof of Corollary 6.4 cannot proceed completely analogously to the proof of Theorem 6.3, because X_{33} in (6.3) need not be H_{33}-normal.

Let us now consider the case that H is not necessarily invertible. As we can see from Example 6.1, the kernel of H need not be invariant for an H-normal matrix $N \in \mathbb{C}^{n\times n}$. Therefore, to generalize Theorem 6.2 to the case of singular H, we need some preparations. Let us start with a simple form for H-normal matrices.

Theorem 6.5 *Let $N \in \mathbb{C}^{n\times n}$ be H-normal. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^{-1}NP = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & B_2 \\ 0 & 0 & A_3 & 0 & B_3 \\ 0 & 0 & 0 & A_4 & 0 \\ C_1 & C_2 & C_3 & C_4 & D \end{bmatrix}, \quad P^*HP = \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (6.9)$$

where $A_1, H_1 \in \mathbb{C}^{n_1\times n_1}$, $A_2, H_2 \in \mathbb{C}^{n_2\times n_2}$, $A_3, A_4 \in \mathbb{C}^{m\times m}$, and the other blocks have corresponding sizes. Moreover, A_1, H_1, H_2, A_3 are nonsingular, A_2, A_4 are nilpotent, A_1 is H_1-normal, A_2 is H_2-normal, A_3 and A_4^* commute, and $A_2^*H_2B_2 = 0$, $B_2^*H_2B_2 = 0$, $A_4^*B_3 = 0$.

Proof. Without loss of generality, we may assume that

$$H = \begin{bmatrix} \tilde{H} & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $\tilde{H}$ is nonsingular and N is partitioned conformably with H. Since N is H-normal, we obtain that

$$\left[\begin{array}{cc} A^*\tilde{H}A & A^*\tilde{H}B \\ B^*\tilde{H}A & B^*\tilde{H}B \end{array}\right] = N^*HN = HNH^\dagger N^*H = \left[\begin{array}{cc} \tilde{H}A\tilde{H}^{-1}A^*\tilde{H} & 0 \\ 0 & 0 \end{array}\right]. \tag{6.10}$$

In particular, A is $\tilde{H}$-normal. Thus, since $\tilde{H}$ is invertible, there exists a nonsingular matrix Q such that

$$Q^{-1}AQ = A_{11} \oplus \cdots \oplus A_{kk}, \quad Q^*\tilde{H}Q = H_{11} \oplus \cdots \oplus H_{kk},$$

where, for each j, A_{jj} and H_{jj} have the same sizes, and either A_{jj} has only one eigenvalue, or A_{jj} has two distinct eigenvalues and

$$A_{jj} = \left[\begin{array}{cc} A_{jj1} & 0 \\ 0 & A_{jj2} \end{array}\right], \quad H_{jj} = \left[\begin{array}{cc} 0 & I_{p_j} \\ I_{p_j} & 0 \end{array}\right],$$

where both $A_{jj1} \in \mathbb{C}^{p_j \times p_j}$ and $A_{jj2} \in \mathbb{C}^{p_j \times p_j}$ have only one eigenvalue. (For a proof, see [20], for example.) If A_{jj} is singular and has two distinct eigenvalues, then one of the blocks A_{jj1}, A_{jj2} must be nilpotent. Clearly, we may assume that in this case always A_{jj2} is nilpotent, applying a permutation otherwise. Let us group together all nonsingular blocks, all singular blocks that have only one eigenvalue, and all singular blocks that have two distinct eigenvalues. Thus, after applying an appropriate block permutation, we may assume that N and H have the forms

$$N = \left[\begin{array}{ccccc} A_1 & 0 & 0 & 0 & B_1 \\ 0 & A_2 & 0 & 0 & B_2 \\ 0 & 0 & A_3 & 0 & B_3 \\ 0 & 0 & 0 & A_4 & B_4 \\ C_1 & C_2 & C_3 & C_4 & D \end{array}\right], \quad P^*HP = \left[\begin{array}{ccccc} H_1 & 0 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where $A_1, H_1 \in \mathbb{C}^{n_1 \times n_1}$ are nonsingular, $A_2, H_2 \in \mathbb{C}^{n_2 \times n_2}$, H_2 is nonsingular, A_2 is nilpotent, $A_3 \in \mathbb{C}^{m \times m}$ is nonsingular, and $A_4 \in \mathbb{C}^{m \times m}$ is nilpotent. The fact that A is $\tilde{H}$-normal implies furthermore that A_1 is H_1-normal, A_2 is H_2-normal, and $A_3A_4^* = A_4^*A_3$. Finally, equation (6.10) implies $B_1 = 0$, $B_4 = 0$, $A_2^*H_2B_2 = 0$, $B_2^*H_2B_2 = 0$, and $A_4^*B_3 = 0$. $\square$

We are now able to generalize Theorem 6.2 to the case that H is singular.

Theorem 6.6 *Let $N \in \mathbb{C}^{n \times n}$ be H-normal. Then there exists an H-nonnegative N-invariant subspace $\mathcal{M}$ such that* $\dim \mathcal{M} = i_+(H) + i_0(H)$.

Proof. Without loss of generality, we may assume that N and H are in the simple form (6.9). Using the same notation as in Theorem 6.5, let

$$\mathcal{M}_0 = \operatorname{Im}\left[\begin{array}{cccc} B_2 & A_2B_2 & \ldots & A_2^{n-1}B_2 \end{array}\right],$$

i.e., $\mathcal{M}_0$ is the controllable subspace of the pair (A, B) (for basic properties of controllable subspaces see, e.g., [9, Section 2.8] or [17, Chapter 4]). Then $\mathcal{M}_0$ is A_2-invariant and contains Im B_2, see, for example, [17, Proposition 4.1.2]. Next, we show that $\mathcal{M}_0$ is H_2-neutral. Therefore, it is sufficient to prove that

$$B_2^*(A_2^*)^i H_2 A_2^j B_2 = 0 \qquad \text{for all } i, j = 0, \dots, n-1. \tag{6.11}$$

Since by Theorem 6.5 we have $B_2^* H_2 B_2 = 0$ (which covers the case $i = j = 0$), equation (6.11) is guaranteed if $A_2^* H_2 A_2^j B_2 = 0$ for $j = 0, \dots, n-1$ which we will prove by induction on j. For $j = 0$, this follows directly from Theorem 6.5. If $j > 0$, then we have

$$A_2^* H_2 A_2^j B_2 = (A_2^* H_2 A_2) A_2^{j-1} B_2 = H_2 A_2 H_2^{-1} A_2^* H_2 A_2^{j-1} B_2 = 0,$$

because A_2 is H_2-normal and because of the induction hypothesis. Then, applying Corollary 6.4 on A_2 and $\mathcal{M}_0$, there exists an H_2-nonnegative A_2-invariant subspace $\mathcal{M}_2 \supseteq \mathcal{M}_0$ of dimension $i_+(H_2)$. Furthermore, there exists an H_1-nonnegative A_1-invariant subspace $\mathcal{M}_1$ of dimension $i_+(H_1)$ by Theorem 6.2. Let $\mathcal{M}_1 = \text{Im } X_1$ and $\mathcal{M}_2 = \text{Im } X_2$ for some matrices X_1, X_2 of appropriate dimensions and set[1]

$$\mathcal{M} = \text{Im} \begin{bmatrix} X_1 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_\nu \end{bmatrix} \subseteq \mathbb{C}^n,$$

where $\nu = n - n_1 - n_2 - 2m$. Then $\mathcal{M}$ is H-nonnegative and $\dim \mathcal{M} = i_+(H)$. Moreover, the fact that Im $B_2 \subseteq \mathcal{M}_0 \subseteq \mathcal{M}_2 = \text{Im } X_2$ implies that $\mathcal{M}$ is N-invariant. This concludes the proof. $\square$

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[1] Corrected formula. In the published version the I_m block appeared incorrectly in the $(4,3)$-position and not in the $(3,3)$-position.

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