Symmetric matrices with respect to sesquilinear forms

Christian Mehl[†] Leiba Rodman[‡]

Abstract

Simple forms are obtained for matrices that are symmetric with respect to degenerate sesquilinear forms on finite dimensional complex linear spaces of column vectors. Symmetric matrices and the sesquilinear forms are then representable as block diagonals having simple forms as the diagonal blocks. The notion of indecomposability for symmetric matrices is studied. An example shows that, in contrast with the non-degenerate sesquilinear forms, an indecomposable symmetric matrix with respect to a degenerate sesquilinear form may have arbitrarily many Jordan blocks. All indecomposable symmetric matrices are characterized in two situations: when the sesquilinear form has only one degree of degeneracy, and when the form is semidefinite.

Key Words. Degenerate inner products, sesquilinear forms, selfadjoint matrices.

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1 Introduction

Let $\mathcal{V}$ be a finite dimensional complex vector space, and let $[x,y]$, $x,y \in \mathcal{V}$, be a sesquilinear (linear in the first argument x, conjugate linear in the second argument y) form on $\mathcal{V}$. A linear transformation $A : \mathcal{V} \to \mathcal{V}$ will be called *symmetric* with respect to $[\cdot,\cdot]$ if the equality $[Ax,y] = [x,Ay]$ holds for every $x,y \in \mathcal{V}$. In this paper we study simple and indecomposable forms of such linear transformations.

It will be convenient to work with matrices. Thus, we identify $\mathcal{V}$ with $\mathbb{C}^n$, the complex linear space of column vectors having n components. Every sesquilinear form on $\mathcal{V}$ is given by the formula

$$[x,y] = \langle Hx, y\rangle, \qquad x,y \in \mathbb{C}, \tag{1}$$

where H is a uniquely determined Hermitian $n \times n$ matrix, and $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product in $\mathbb{C}$. Represent a linear transformation A on $\mathbb{C}$ as an $n \times n$ matrix, also denoted A, with respect to the standard orthonormal basis (made up of the

[†]Fakultät II; Institut für Mathematik, TU Berlin, D-10623 Berlin, Germany.

[‡]College of William and Mary, Department of Mathematics, P.O.Box 8795, Williamsburg, VA 23187-8795. *The research of this author is partially supported by* NSF Grant DMS-9988579.

columns of the $n \times n$ identity matrix). We then obtain that A is symmetric with respect to (1) if and only if the equality $HA = A^*H$ holds; in this case we say that A is *H-selfadjoint*.

Canonical forms of H-selfadjoint matrices are well known for the case when H is invertible, can be found in many sources (see, for example, [4]), and are widely used in applications. In contrast, the case when H is singular is not well studied (some works here, primarily concerning infinite dimensional degenerate Pontryagin spaces, include [11], [6] (and references there), [1], [8], and parts of the book [2]), although it does appear in applications [9]. Some results obtained in the theory of nondegenerate indefinite inner products can be extended to the singular case (no restriction on $H = H^*$) without difficulties, for example:

Theorem 1 *Let A be H-selfadjoint, and let $\mathcal{M} \subseteq \mathbb{C}^n$ be a subspace that is simultaneously A-invariant and H-nonnegative (or H-nonpositive). Then there exists a subspace $\mathcal{N} \subseteq \mathbb{C}^n$ that is simultaneously A-invariant and H-nonnegative (or H-nonpositive), and such that $\mathcal{N} \supseteq \mathcal{M}$ and $\dim \mathcal{N} = \nu_+(H) + \nu_0(H)$ (or $\dim \mathcal{N} = \nu_-(H) + \nu_0(H)$).*

Here $\nu_+(H)$, $\nu_0(H)$, and $\nu_-(H)$ are the numbers of positive, zero, and negative eigenvalues of H, respectively, counted with multiplicities. A subspace $\mathcal{M} \subseteq \mathbb{C}^n$ is called *H-nonnegative* (resp., *H-nonpositive*) if $\langle Hx, x\rangle \geq 0 \quad$ (resp., $\quad \langle Hx, x\rangle \leq 0$) holds for every $x \in \mathcal{M}$. It is well known that the dimension of a maximal (in the sense of set-theoretic containment) H-nonnegative (resp., H-nonpositive) subspace is equal to $\nu_+(H) + \nu_0(H)$ (resp. $\nu_-(H) + \nu_0(H)$).

The proof of Theorem 1 is easily obtained from the case when H is nonsingular (in this case the result is known as Pontryagin's theorem), using the fact that $\operatorname{Ker} H$ is A-invariant, for an H-selfadjoint matrix A.

Notwithstanding Theorem 1, and some other results that are known for the case of nonsingular H and can be extended without difficulty for singular H, it is of interest to develop an independent theory of simple, canonical, and indecomposable forms of H-selfadjoint matrices. In this paper we present some results in this direction.

Besides the introduction, the paper consists of five sections. The main result of Section 2 gives simple forms of H-selfadjoint matrices, so that every H-selfadjoint matrix can be reduced, together with the sesquilinear form defined by H, to a direct sum of these forms. Sections 3 to 5 are devoted to the notion of indecomposability. An example in Section 3 shows that, generally speaking, an H-selfadjoint H-indecomposable matrix can have arbitrarily many Jordan blocks in its Jordan canonical form. Indecomposable H-selfadjoint matrices are fully described in the case that the kernel of H is one-dimensional (Section 4) and in the case that H is semidefinite (Section 5). Finally, H-unitary matrices and their connections to H-skewadjoints and the indecomposability properties are briefly described in Section 6.

Throughout the paper, H denotes a (possibly singular) Hermitian $n \times n$ complex matrix. Furthermore, we use the following notation: $\mathbb{N} = \{1, 2, \cdots\}$; $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$; $\mathbb{R}$ is the field of real numbers; I_p is the $p \times p$ identity matrix; $\mathcal{J}_p(\lambda)$ is the $p \times p$ upper triangular Jordan block with eigenvalue λ; Z_p is the $p \times p$ matrix with ones on the southwest-northeast diagonal and zeros elsewhere, i.e.,

$$Z_p = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}_{p\times p}.$$

$X_1 \oplus \cdots \oplus X_k$ stands for the block diagonal matrix with the diagonal blocks $X_1, \ldots, X_k$ (in that order).

2 Simple form

We first recall the canonical form for H-selfadjoint matrices in the case that H is invertible.

Proposition 2 *Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$, such that*

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_k \qquad \text{and} \qquad P^*HP = H_1 \oplus \cdots \oplus H_k, \tag{2}$$

where A_j, H_j are of the same size and each pair (A_j, H_j) has one and only one of the following forms:

1. *blocks associated with real eigenvalues:*

$$A_j = \mathcal{J}_p(\lambda) \quad \text{and} \quad H_j = \varepsilon Z_p, \tag{3}$$

where $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, and $\varepsilon \in \{1, -1\}$;

2. *blocks associated with a pair of nonreal conjugate eigenvalues:*

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\overline{\lambda}) \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \tag{4}$$

where $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $p \in \mathbb{N}$.

Moreover, the form $(P^{-1}AP, P^*HP)$ of (A, H) is uniquely determined up to the permutation of blocks and called the canonical form of (A, H).

Proof. See [4], for example. $\square$

We also use a slightly different form of the blocks of type (4). Namely, multiplying the matrices in (4) from both sides by $I_p \oplus Z_p$, one finds that (4) takes the form

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\lambda)^* \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \tag{5}$$

To state the main result, it is convenient to use the following notion. A matrix $X \in \mathbb{C}^{q\times p}$ will be called *special* if only the left-most column (if $q \leq p$) or only the bottom row (if $q > p$) of X may have nonzero entries.

Theorem 3 *Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^{-1}AP = A_{11} \oplus \cdots \oplus A_{kk} \quad \text{and} \quad P^*HP = H_{11} \oplus \cdots \oplus H_{kk}, \tag{6}$$

where, for each j, the blocks A_{jj} and H_{jj} have the same sizes and are of one of the following three types:

type 1:

$$A_{jj} = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

where $A_1, H_1 \in \mathbb{C}^{p\times p}$, $A_2 \in \mathbb{C}^{q\times p}$, $A_3 \in \mathbb{C}^{q\times q}$ for some $p \in \mathbb{N}, q \in \mathbb{Z}_+$. Moreover, $\sigma(A_{jj}) = \{\lambda\} \subset \mathbb{R}$, the matrix H_1 is nonsingular, (A_1, H_1) is in canonical form (2), A_3 is in Jordan canonical form. Furthermore, upon partitioning

$$A_1 = \mathcal{J}_{p_1}(\lambda) \oplus \cdots \oplus \mathcal{J}_{p_r}(\lambda), \quad A_3 = \mathcal{J}_{q_1}(\lambda) \oplus \cdots \oplus \mathcal{J}_{q_s}(\lambda), \quad A_2 = [A_{2,\alpha,\beta}]_{\alpha=1,\beta=1}^{s,r}, \tag{8}$$

where $A_{2,\alpha,\beta}$ is of size $q_\alpha \times p_\beta$, the matrices $A_{2,\alpha,\beta}$ are special;

type 2:

$$A_{jj} = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{9}$$

where $A_1, H_1 \in \mathbb{C}^{p\times p}$, $A_2 \in \mathbb{C}^{q\times p}$, $A_3 \in \mathbb{C}^{q\times q}$ for some $p \in \mathbb{N}$, $q \in \mathbb{Z}_+$. Moreover, $\sigma(A_{jj}) = \{\lambda, \bar{\lambda}\} \subset \mathbb{C} \setminus \mathbb{R}$, the matrix H_1 is nonsingular, (A_1, H_1) is in canonical form (2), and A_3 is in Jordan canonical form. Furthermore, upon partitioning

$$A_1 = \mathcal{J}_{p_1}(\lambda) \oplus \mathcal{J}_{p_2}(\bar{\lambda}) \cdots \oplus \mathcal{J}_{p_{2r-1}}(\lambda) \oplus \mathcal{J}_{p_{2r}}(\bar{\lambda}), \quad (\ p_{2j-1} = p_{2j}\ ),$$

$$A_3 = \mathcal{J}_{q_1}(\mu_1) \oplus \cdots \oplus \mathcal{J}_{q_s}(\mu_s), \quad (\ \mu_j \in \{\lambda, \bar{\lambda}\}\ ), \quad A_2 = [A_{2,\alpha,\beta}]_{\alpha=1,\beta=1}^{s,2r},$$

where $A_{2,\alpha,\beta}$ is of size $q_\alpha \times p_\beta$, the matrix $A_{2,\alpha,\beta}$ is zero if $\mu_\alpha \neq \lambda$ (in case β is odd) or if $\mu_\alpha \neq \bar{\lambda}$ (in case β is even); otherwise, $A_{2,\alpha,\beta}$ is special;

type 3:

$$A_{jj} = \mathcal{J}_p(\lambda) \quad \text{and} \quad H_{jj} = 0 \in \mathbb{C}^{p\times p}, \tag{10}$$

for some $p \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.

The following two propositions will be used in the proof of Theorem 3. Their proofs are straightforward, and are therefore omitted.

Proposition 4 *Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint, such that*

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix},$$

where H is partitioned corresponding to A. If A_{11}^* and A_{22} have no common eigenvalues, then $H_{12} = 0$.

Proposition 5 *Let be given two Jordan blocks $\mathcal{J}_p(\lambda)$ and $\mathcal{J}_q(\lambda)$ having the same eigenvalue. Then for every $A \in \mathbb{C}^{q\times p}$ there exist $X \in \mathbb{C}^{q\times p}$ such that the matrix $X\mathcal{J}_p(\lambda) - \mathcal{J}_q(\lambda)X + A$ is special.*

We now proceed with the proof of Theorem 3.

Proof. By Sylvester's Law of Inertia, we may assume that $H = \tilde{H} \oplus 0$, where $\tilde{H}$ is a nonsingular diagonal matrix. Partitioning A correspondingly

$$A = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix},$$

we obtain from the identity $A^*H = HA$ that $\tilde{A}_2 = 0$ and $\tilde{A}_1^*\tilde{H} = \tilde{H}\tilde{A}_1$, i.e., $\tilde{A}_1$ is $\tilde{H}_1$-selfadjoint. Thus, applying a similarity transformation with a block diagonal matrix, we may furthermore assume that $(\tilde{A}_1, \tilde{H})$ is in canonical form (2) and that $\tilde{A}_4$ is in Jordan canonical form. Next, let us moreover assume that $\tilde{A}_1$ and $\tilde{A}_4$ have the block diagonal forms

$$\tilde{A}_1 = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix} \quad \text{and} \quad \tilde{A}_4 = \begin{bmatrix} \tilde{A}_{33} & 0 \\ 0 & \tilde{A}_{44} \end{bmatrix},$$

where either $\sigma(\tilde{A}_{11}) = \{\lambda\}$ for some $\lambda \in \mathbb{R}$ or $\sigma(\tilde{A}_{11}) = \{\lambda, \bar{\lambda}\}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and, furthermore, $\sigma(\tilde{A}_{11}) \cap \sigma(\tilde{A}_{22}) = \emptyset$, $\sigma(\tilde{A}_{11}) \cap \sigma(\tilde{A}_{44}) = \emptyset$, and $\sigma(\tilde{A}_{33}) \subseteq \sigma(\tilde{A}_{11})$. Since $\tilde{A}_1$ is $\tilde{H}$-selfadjoint, it follows from Proposition 4 that A and H take the form

$$A = \begin{bmatrix} \tilde{A}_{11} & 0 & 0 & 0 \\ 0 & \tilde{A}_{22} & 0 & 0 \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & 0 \\ \tilde{A}_{41} & \tilde{A}_{42} & 0 & \tilde{A}_{44} \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} \tilde{H}_{11} & 0 & 0 & 0 \\ 0 & \tilde{H}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\tilde{A}_{jj}$ and $\tilde{H}_{jj}$ have the same size. Since the spectra of $\tilde{A}_{11}$ and $\tilde{A}_{44}$ are disjoint, there exists a unique solution X of the equation $\tilde{A}_{44}X - X\tilde{A}_{11} = -\tilde{A}_{41}$, see, e.g., §3.4.1 in [3]. Setting

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ X & 0 & 0 & I \end{bmatrix},$$

we obtain that

$$T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & 0 & 0 & 0 \\ 0 & \tilde{A}_{22} & 0 & 0 \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & 0 \\ 0 & \tilde{A}_{42} & 0 & \tilde{A}_{44} \end{bmatrix} \quad \text{and} \quad T^*HT = H = \begin{bmatrix} \tilde{H}_{11} & 0 & 0 & 0 \\ 0 & \tilde{H}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In an analogous way, we can zero out $\tilde{A}_{32}$. Since the spectrum of $\tilde{A}_{11}$ is either $\{\lambda\}$ for some $\lambda \in \mathbb{R}$, or $\{\lambda, \bar{\lambda}\}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we can split off blocks of the forms (7) and (9) by applying a row and column permutation. We then repeat the procedure for $\tilde{A}_{22}$ and $\tilde{A}_{44}$. If the size of $\tilde{A}_{22}$ and $\tilde{H}_{22}$ is zero, then we are left with the block $\tilde{A}_{44}$ and a corresponding zero block in H. Decomposing $\tilde{A}_{44}$ according to its Jordan structure, we obtain blocks of the form (10).

It remains to show that the matrices A_2 in (7) and in (9) can be made to satisfy the required properties. We show this for A_2 as in (7); the proof for A_2 as in (9) is analogous. Using Proposition 5, for every α ($\alpha = 1, \ldots, s$) and every β ($\beta = 1, \ldots, r$) find $X_{\alpha,\beta}$ such that the matrix

$$-X_{\alpha,\beta}\mathcal{J}_{p_\beta}(\lambda) + \mathcal{J}_{q_\alpha}(\lambda)X_{\alpha,\beta} - A_{2,\alpha,\beta}$$

is special. Then let $X = [X_{\alpha,\beta}]_{\alpha=1,\beta=1}^{s,r}$ and $T = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$. One checks that

$$T^* \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T^{-1} \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} T = \begin{bmatrix} A_1 & 0 \\ \widetilde{A}_2 & A_3 \end{bmatrix},$$

where upon partitioning $\widetilde{A}_2 = [\widetilde{A}_{2,\alpha,\beta}]_{\alpha=1,\beta=1}^{s,r}$ conformably with (8), we have that each $\widetilde{A}_{2,\alpha,\beta}$ is special. □

3 Indecomposability

Theorem 3 is a first step towards a canonical form of H-selfadjoint matrices for the case that H is possibly singular. However, the simple form in Theorem 3 does not display the Jordan structure of the H-selfadjoint matrix A. In this section, we discuss the classification of H-selfadjoint matrices. A key term in this discussion is H-decomposability.

A matrix $X \in \mathbb{C}^{n\times n}$ is called *H-decomposable* if there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P^{-1}XP = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \quad \text{and} \quad P^*HP = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix},$$

where X_j and H_j have the same nonzero size. If X is not H-decomposable, then it is called *H-indecomposable*. (We will sometimes use the term "(in)decomposable" instead of "H-(in)decomposable" when it is clear which H we are talking about.)

Clearly, any matrix X can be decomposed into a direct sum of indecomposables, i.e., there exists a nonsingular matrix P, such that $P^{-1}XP = X_1 \oplus \cdots \oplus X_k$ and $P^*HP = H_1 \oplus \cdots \oplus H_k$, where each X_j is H_j-indecomposable. Note that, in contrast to the decomposition (2) in the case H invertible, the blocks of the decomposition (6) in Theorem 3 need not be indecomposable.

The following lemma is well known.

Lemma 6 *Let*

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix}, \quad A_j = \underbrace{\mathcal{J}_{p_j}(\lambda) \oplus \cdots \oplus \mathcal{J}_{p_j}(\lambda)}_{m_j \text{ blocks}}$$

$j = 1, \ldots, k$, where $\lambda \in \mathbb{C}$, $m_j \in \mathbb{N}$, and $p_1 > \ldots > p_k$, i.e., A is a matrix in Jordan canonical form such that all Jordan blocks of same size p_j are collected in a larger block A_j. Furthermore, let

$$P = \begin{bmatrix} P_{11} & \ldots & P_{1k} \\ \vdots & \ddots & \vdots \\ P_{k1} & \ldots & P_{kk} \end{bmatrix}$$

be partitioned conformably with A and let A and P commute. If P is nonsingular, then so are the diagonal blocks $P_{11}, \ldots, P_{kk}$.

Proof. See, e.g., [10], Lemma 10. □

By [3], a matrix P that commutes with a matrix $A = \mathcal{J}_{p_1} \oplus \cdots \oplus \mathcal{J}_{p_k}$, $p_1 \geq \cdots \geq p_k$, has the form

$$P = \begin{bmatrix} P_{11} & \ldots & P_{1k} \\ \vdots & \ddots & \vdots \\ P_{k1} & \ldots & P_{kk} \end{bmatrix},$$

where

$$P_{jl} = \begin{bmatrix} T_{jl} \\ 0 \end{bmatrix} \quad \text{for } j \geq l, \qquad P_{jl} = \begin{bmatrix} 0 & T_{jl} \end{bmatrix} \quad \text{for } j < l,$$

and the T_{jl} are upper triangular $p_l \times p_l$ or, respectively, $p_j \times p_j$ Toeplitz matrices. We will call matrices of the form P_{jl} *rectangular upper triangular Toeplitz matrices.* The following lemmas yield information about the rank of the sum and the product of two rectangular upper triangular Toeplitz matrices.

Lemma 7 *Let $M, N \in \mathbb{C}^{p \times l}$ be rectangular upper triangular Toeplitz matrices. Then $M + N$ is a rectangular upper triangular Toeplitz matrix and if $\operatorname{rank}(M) > \operatorname{rank}(N)$, then $\operatorname{rank}(M + N) = \operatorname{rank}(M)$.*

The proof is straightforward.

Lemma 8 *Let $M \in \mathbb{C}^{p \times l}$ and $N \in \mathbb{C}^{l \times k}$ be rectangular upper triangular Toeplitz matrices. Then MN is a rectangular upper triangular Toeplitz matrix and*

$$\operatorname{rank}(MN) = \max\Big(0,\ \operatorname{rank}(M) + \operatorname{rank}(N) - l\Big).$$

Proof. Let $\operatorname{rank}(M) = r_m$ and $\operatorname{rank}(N) = r_n$. Clearly, MN is again a rectangular upper triangular Toeplitz matrix. If $r_n + r_m \leq l$, then it is easily verified that $MN = 0$. Thus, assume that $r_n + r_m > l$. Since M and N are rectangular upper triangular Toeplitz matrices, we can write them in the form

$$M = \begin{array}{c} \\ r_n + r_m - l \\ l - r_n \\ p - r_m \end{array} \!\!\begin{array}{c} \begin{array}{ccc} l - r_m & r_n + r_m - l & l - r_n \end{array} \\ \begin{bmatrix} 0 & M_1 & M_3 \\ 0 & 0 & M_2 \\ 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$
N = \begin{array}{c} \\ l-r_m \\ r_n+r_m-l \\ l-r_n \end{array}
\begin{array}{c} \begin{array}{ccc} k-r_n & l-r_m & r_n+r_m-l \end{array} \\
\begin{bmatrix} 0 & N_1 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}
$$

for some matrices M_j, N_j of appropriate dimensions. In particular, M_1 and N_3 are nonsingular. This implies

$$
MN = \begin{array}{c} \\ r_n+r_m-l \\ l-r_n \\ p-r_m \end{array}
\begin{array}{c} \begin{array}{ccc} k-r_n & l-r_m & r_n+r_m-l \end{array} \\
\begin{bmatrix} 0 & 0 & M_1N_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.
$$

Thus, $\mathrm{rank}(MN) = \mathrm{rank}(M) + \mathrm{rank}(N) - l$. □

It seems that the problem of finding a complete classification of indecomposable H-selfadjoint matrices for the general case of singular H is intractable. For example, the number of Jordan blocks of an indecomposable H-selfadjoint matrix associated with the same eigenvalue may be greater than one, in contrast to the nondegenerate case. In fact, this number may be arbitrarily large, as the following example shows.

Example 9 Let λ be a real number and

$$
A_p = \begin{bmatrix} \mathcal{J}_{2p-1}(\lambda) & & 0 \\ & \ddots & \\ 0 & & \mathcal{J}_1(\lambda) \end{bmatrix}, \quad
H_p = \begin{bmatrix} H_{pp} & H_{p,p-1} & \dots & H_{p1} \\ H_{p,p-1}^* & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{p1}^* & 0 & \dots & 0 \end{bmatrix},
$$

where

$$
H_{pk} = \begin{array}{c} \\ 2(p-k) \\ 2k-1 \end{array}
\begin{array}{c} 2k-1 \\ \begin{bmatrix} 0 \\ Z_{2k-1}\mathcal{J}_{2k-1}(0)^{k-1} \end{bmatrix} \end{array}
$$

for $k = p, \dots, 1$, with the understanding that $\mathcal{J}_1(0)^0 = 1$. For example, we have

$$
A_2 = \left[\begin{array}{ccc|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \lambda \end{array}\right], \quad
H_2 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right]
$$

$$
A_3 = \left[\begin{array}{ccccc|ccc|c}
\lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda
\end{array}\right], \quad
H_3 = \left[\begin{array}{ccccc|ccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]
$$

One verifies easily that A_p is H_p-selfadjoint. Next, we show that, in general, A_p is H_p-indecomposable. To see this, let us assume that A_p is H_p-decomposable. Then there exists a nonsingular matrix P with $P^{-1}A_pP = A_p$ such that if

$$P^*H_pP = \begin{bmatrix} \tilde{H}_{pp} & \dots & \tilde{H}_{p1} \\ \vdots & \ddots & \vdots \\ \tilde{H}_{p1}^* & \dots & \tilde{H}_{11} \end{bmatrix}, \tag{11}$$

is partitioned conformably with A_p, then some blocks among $\tilde{H}_{pj}$, $j = p-1, \dots, 1$, are zero. (This can be seen as follows: the decomposability of A_p implies the existence of a nonsingular matrix Q such that

$$Q^{-1}A_pQ = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix} \quad \text{and} \quad Q^*H_pQ = \begin{bmatrix} \tilde{H}_1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix}.$$

Without loss of generality, we may assume that $\tilde{A}_1$ and $\tilde{A}_2$ are in Jordan canonical form. Then a block row and column permutation yields the desired result.)

We now show that for any nonsingular P with $A_pP = PA_p$ all blocks $\tilde{H}_{pj}$, $j = p-1, \dots, 1$, in (11) are nonzero in contradiction to the H_p-decomposability of A_p. By Lemma 6, a matrix P that commutes with A_p has the form

$$P = \begin{bmatrix} P_{pp} & \dots & P_{p1} \\ \vdots & \ddots & \vdots \\ P_{1p} & \dots & P_{11} \end{bmatrix},$$

where each P_{jl} is an rectangular upper triangular Toeplitz matrix. Since P is nonsingular, so are $P_{11}, \dots, P_{pp}$ by Lemma 6. Thus, we have $\operatorname{rank}(P_{jj}) = 2j-1$ for $j = 1, \dots, p$. Moreover, we have

$$\operatorname{rank}(P_{jl}) \leq \min(2j-1,\ 2l-1) \quad \text{and} \quad \operatorname{rank}(H_{pj}) = j, \quad j = 1, \dots, p.$$

In particular, $\operatorname{rank}(P_{pk}) \leq 2k-1$. The block $\tilde{H}_{pk}$ in (11) is given by

$$P_{pp}^*H_{pp}P_{pk} + P_{p-1,p}^*H_{p,p-1}^*P_{pk} + \dots + P_{1p}^*H_{p1}^*P_{pk} + P_{pp}^*H_{p,p-1}P_{p-1,k} + \dots + P_{pp}^*H_{p1}P_{1k}. \tag{12}$$

Note that $Z_{2p-1}H_{pj}$ and $Z_{2j-1}H_{pj}^*$ are rectangular upper triangular Toeplitz matrices. Thus, applying Lemma 8 to $Z_{2j-1}H_{pj}^*P_{pk}$, we obtain for $k < p$ that

$$\operatorname{rank}(H_{pj}^*P_{pk}) = \operatorname{rank}(Z_{2j-1}H_{pj}^*P_{pk}) \leq \max\Big(0,\ j+2k-1-(2p-1)\Big) \leq \max(0,\ 2k-p) < k,$$

since $j \leq p$. Furthermore applying Lemma 8 to $Z_{2p-1}H_{pj}P_{jk}$ and recalling the nonsingularity of P_{pp}, we obtain for $j \neq k$ that

$$\begin{aligned} \operatorname{rank}(P_{pp}^*H_{pj}P_{jk}) &= \operatorname{rank}(Z_{2p-1}H_{pj}P_{jk}) \leq \max\Big(0,\ j + \min(2j-1, 2k-1) - (2j-1)\Big) \\ &= \max\Big(0,\ \min(2j, 2k) - j\Big) = \left\{ \begin{array}{ll} j & \text{for } j < k \\ \max(0, 2k-j) & \text{for } j > k \end{array} \right\} < k \end{aligned}$$

On the other hand, we have $\mathrm{rank}(P_{pp}^* H_{pk} P_{kk}) = k$, i.e., only one summand in the sum (12) has rank k, all the others have smaller ranks. Thus, applying Lemma 7 to $Z_{2p-1}\tilde{H}_{pk}$, it follows that $\mathrm{rank}(\tilde{H}_{pk}) = k$, i.e., H_{pk} is nonzero for $k = 1, \ldots, p-1$. Hence, A_p is H_p-indecomposable.

4 Indecomposable H-selfadjoints: rank$(H) = n - 1$

Example 9 shows that, in contrast to the nondegenerate case, the Jordan structure of an indecomposable block may be very complicated, i.e., we may have more than one Jordan block (in fact: arbitrarily many ones) associated with the same eigenvalue. From this point of view, it seems that the problem of finding a complete classification of indecomposable H-selfadjoint matrices is intractable. Therefore, we restrict ourselves to special cases in which the matrix H that induces the degenerate inner product has some additional properties, and classify indecomposable H-selfadjoint matrices in these cases. In this section we assume that the $n \times n$ Hermitian matrix H has rank $n-1$, and in the next section it will be assumed that H is semidefinite.

Theorem 10 *Let H have rank $n-1$ and let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint and indecomposable. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that $\tilde{A} = P^{-1}AP$ and $\tilde{H} = P^*HP$ are of one of the following types:*

type (1):

$$\tilde{A} = \mathcal{J}_n(\lambda) \quad \text{and} \quad \tilde{H} = \varepsilon Z_n \mathcal{J}_n(0) = \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & Z_{n-1} \end{bmatrix},$$

where $\lambda \in \mathbb{R}$, $n > 1$, and $\varepsilon \in \{-1, +1\}$;

type (2):

$$\tilde{A} = \begin{bmatrix} \mathcal{J}_{p+1}(\lambda) & 0 \\ 0 & \mathcal{J}_p(\bar{\lambda}) \end{bmatrix} \quad \text{and} \quad \tilde{H} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & Z_p \\ \hline 0 & Z_p & 0 \end{array}\right]$$

where $\lambda \in \mathbb{C}$ and $n = 2p+1 > 1$;

type (3):

$$\tilde{A} = \begin{bmatrix} \lambda \end{bmatrix} \quad \text{and} \quad \tilde{H} = \begin{bmatrix} 0 \end{bmatrix},$$

where $\lambda \in \mathbb{C}$ and $n = 1$.

Proof. By Theorem 3, we may assume that A and H are of one of the forms (7), (9), or (10). If A and H are of the form (10), i.e., $H = 0$, then H can have rank $n-1$ only if $n = 1$ which gives us the blocks of type (3) that are clearly indecomposable. For the remainder of the proof, it is sufficient to consider the following two cases:

Case (1): A and H are in the form (7). Since H has rank $n-1$, we find that according to Theorem 3, A and H have the forms

$$A = \begin{bmatrix} A_1 & 0 \\ a^* & \lambda \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $A_1 = \mathcal{J}_{p_1}(\lambda) \oplus \cdots \oplus \mathcal{J}_{p_k}(\lambda)$, $H_1 = \varepsilon_1 Z_{p_1} \oplus \cdots \oplus \varepsilon_k Z_{p_k}$ for some $\varepsilon_1, \ldots, \varepsilon_k \in \{+1, -1\}$, $\lambda \in \mathbb{R}$, and $p_1 + \cdots + p_k = n - 1$. Moreover, a^* consists of k blocks that are special, i.e.,

$$a^* = [\underbrace{a_1\ 0\ \ldots\ 0}_{p_1 \text{ entries}}\ \ldots\ \underbrace{a_k\ 0\ \ldots\ 0}_{p_k \text{ entries}}]$$

Note that we must have $a_1, \ldots, a_k \neq 0$. Otherwise, one can easily see that A is decomposable.

Next, we will show that $k = 1$ or ($k = 2$ and $p_1 = p_2$, $\varepsilon_1 = -\varepsilon_2$, $|a_1| = |a_2|$).

Thus, assume that $k > 1$. We may furthermore assume without loss of generality that $p_1 \leq \cdots \leq p_k$. Setting

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U_1 = \left[\begin{array}{c|c} I_{p_1} & \begin{bmatrix} 0 & \alpha I_{p_1} \end{bmatrix} \\ \hline \begin{bmatrix} \beta I_{p_1} \\ 0 \end{bmatrix} & I_{p_2} \end{array}\right] \oplus I_{p_3} \oplus \cdots \oplus I_{p_k},$$

where $\beta = -\frac{a_1}{a_2}$ and $\alpha = -\varepsilon_1\varepsilon_2\beta^*$, we find that A_1 and U_1 commute. Moreover, U is easily seen to be invertible if $p_1 < p_2$. In the case $p_1 = p_2$, we have

$$U_1 = \begin{bmatrix} I_{p_1} & \alpha I_{p_1} \\ \beta I_{p_1} & I_{p_1} \end{bmatrix} \oplus I_{p_3} \oplus \cdots \oplus I_{p_k},$$

which implies

$$\det U_1 = (1 - \alpha\beta)^{p_1} = \left(1 + \varepsilon_1\varepsilon_2 \frac{|a_1|^2}{|a_2|^2}\right)^{p_1}.$$

Thus, U is invertible unless $\varepsilon_1\varepsilon_2 = -1$ and $|a_1| = |a_2|$. This implies

$$U^{-1}AU = \begin{bmatrix} A_1 & 0 \\ a^*U_1 & \lambda \end{bmatrix}, \quad a^*U_1 = [\underbrace{a_1 + \beta a_2\ 0\ \ldots\ 0}_{p_1 \text{ entries}}\ \underbrace{a_2\ 0\ \ldots\ 0}_{p_2 \text{ entries}}\ \ldots\ \underbrace{a_k\ 0\ \ldots\ 0}_{p_k \text{ entries}}].$$

The first entry of a^*U_1 is zero. On the other hand, U^*HU is still block diagonal. To see this, it is sufficient to consider the uppermost $(p_1 + p_2) \times (p_1 + p_2)$ principal submatrix of H: if

$$\begin{aligned} &\begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{12}^* & \tilde{H}_{22} \end{bmatrix} \\ := &\left[\begin{array}{c|c} I_{p_1} & \begin{bmatrix} \beta^* I_{p_1} & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 \\ \alpha^* I_{p_1} \end{bmatrix} & I_{p_2} \end{array}\right] \begin{bmatrix} \varepsilon_1 Z_{p_1} & 0 \\ 0 & \varepsilon_2 Z_{p_2} \end{bmatrix} \left[\begin{array}{c|c} I_{p_1} & \begin{bmatrix} 0 & \alpha I_{p_1} \end{bmatrix} \\ \hline \begin{bmatrix} \beta I_{p_1} \\ 0 \end{bmatrix} & I_{p_2} \end{array}\right], \end{aligned}$$

then

$$\begin{aligned} \tilde{H}_{12} &= \varepsilon_1 Z_{p_1} \begin{bmatrix} 0 & \alpha I_{p_1} \end{bmatrix} + \varepsilon_2 \begin{bmatrix} \beta^* I_{p_1} & 0 \end{bmatrix} \begin{bmatrix} 0 & Z_{p_1} \\ Z_{p_2 - p_1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \varepsilon_1 \alpha Z_{p_1} \end{bmatrix} + \begin{bmatrix} 0 & \varepsilon_2 \beta^* Z_{p_1} \end{bmatrix} = 0, \end{aligned}$$

because of $\varepsilon_2\beta^* = -\varepsilon_1\alpha$. But then, since the $(1,1)$-element of a^* is zero, it follows that A is decomposable. Thus $k = 1$ or ($p_1 = p_2$, $\varepsilon_2 = -\varepsilon_1$, and $|a_2| = |a_1|$). It remains to show $k = 2$ for the latter case. But if $k > 2$ and $p_3 > p_1 = p_2$, we can use a similar argument as above to zero out a_1. And if $p_3 = p_1$, then $\varepsilon_3 = \varepsilon_1$ or $\varepsilon_3 = \varepsilon_2$, and we can zero out a_1 or a_2, respectively, by the procedure described above. Again, it follows that A is decomposable and thus, we must have $k = 2$.

We now consider the cases $k = 1$ and $k = 2$ separately.

Case (1a): $k = 1$. Then we may assume that A and H have the forms

$$A = \begin{bmatrix} \mathcal{J}_{n-1}(\lambda) & 0 \\ a_1 e_1^* & \lambda \end{bmatrix}, \quad H = \begin{bmatrix} \varepsilon Z_{n-1} & 0 \\ 0 & 0 \end{bmatrix},$$

where $a_1 \neq 0$, $\varepsilon = \pm 1$, and e_1 denotes the first $(n-1)$-dimensional unit vector. Clearly, we may also assume that $a_1 = 1$. Otherwise, we apply an H-unitary similarity transformation on A with the matrix $S = I_{n_1} \oplus \begin{bmatrix} a_1 \end{bmatrix}$. So, assuming that $a_1 = 1$ and setting

$$P = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix},$$

we obtain that

$$P^{-1}AP = \begin{bmatrix} \lambda & e_1^* \\ 0 & \mathcal{J}_{n-1}(\lambda) \end{bmatrix} = \mathcal{J}_n(\lambda) \quad \text{and} \quad P^*HP = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon Z_{n-1} \end{bmatrix} = \varepsilon Z_n \mathcal{J}_n(0),$$

which gives us the blocks of type (1) of the theorem. Since A has only one Jordan block, it is clearly indecomposable.

Case (1b): $k = 2$. In this case, we may assume that A and H have the forms

$$A = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & \mathcal{J}_p(\lambda) & 0 \\ a_1 e_1^* & a_2 e_1^* & \lambda \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} -Z_p & 0 & 0 \\ 0 & Z_p & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $|a_1| = |a_2| \neq 0$, $2p + 1 = n$, and e_1 denotes the first p-dimensional unit vector. Analogously to Case (1a), we can apply an H-unitary similarity transformation with

$$S = I_p \oplus \left(a_2^{-1} a_1\right) I_p \oplus \begin{bmatrix} \sqrt{2} a_1 \end{bmatrix}$$

on A to set a_1 and a_2 equal to $\frac{1}{\sqrt{2}}$. Then setting

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & I_p & -I_p \\ 0 & I_p & I_p \\ \sqrt{2} & 0 & 0 \end{bmatrix},$$

we obtain that

$$P^{-1}AP = \begin{bmatrix} \lambda & e_1^* & 0 \\ 0 & \mathcal{J}_p(\lambda) & 0 \\ 0 & 0 & \mathcal{J}_p(\lambda) \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{p+1}(\lambda) & 0 \\ 0 & \mathcal{J}_p(\lambda) \end{bmatrix}, \quad P^*HP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & Z_p \\ 0 & Z_p & 0 \end{bmatrix},$$

which gives us the blocks of type (2) of the theorem for the case $\lambda \in \mathbb{R}$. It remains to show that a block of type (2) is indecomposable. Indeed, let $\tilde{A} = P^{-1}AP$ and $\tilde{H} = P^*HP$. Analogously to the argument used in Example 9, it suffices to show that for any nonsingular matrix Q satisfying $Q\tilde{A} = \tilde{A}Q$ and

$$Q^*\tilde{H}Q = \begin{bmatrix} \tilde{H}_1 & \tilde{H}_2 \\ \tilde{H}_2^* & \tilde{H}_3 \end{bmatrix}$$

we have that $\tilde{H}_2 \neq 0$. By Lemma 6, such Q has the form

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix},$$

where Q_1 and Q_4 are nonsingular $(p+1) \times (p+1)$ and $p \times p$ upper triangular Toeplitz matrices, respectively, and Q_2 and Q_3 are rectangular upper triangular Toeplitz matrices with rank smaller than or equal to p. We obtain that $\tilde{H}_2$ has the form

$$\tilde{H}_2 = Q_3^* \begin{bmatrix} 0 & Z_p \end{bmatrix} Q_2 + Q_1^* \begin{bmatrix} 0 \\ Z_p \end{bmatrix} Q_4.$$

Applying Lemma 8 to the rectangular upper triangular Toeplitz matrix $Z_p \begin{bmatrix} 0 & Z_p \end{bmatrix} Q_2$, we obtain that

$$\operatorname{rank}\left(Q_3^* \begin{bmatrix} 0 & Z_p \end{bmatrix} Q_2\right) \leq \operatorname{rank}\left(\begin{bmatrix} 0 & Z_p \end{bmatrix} Q_2\right) \leq p + p - (p+1) = p - 1 < p.$$

On the other hand, we have that

$$\operatorname{rank}\left(Q_1^* \begin{bmatrix} 0 \\ Z_p \end{bmatrix} Q_4\right) = p.$$

Thus, $\operatorname{rank}(\tilde{H}_2) = p$, by Lemma 7 applied to $Z_{p+1}\tilde{H}_2$. This implies the indecomposability of blocks of type (2).

Case (2): A and H are in the form (9). Using formulas (5), after reordering some blocks, we may assume that

$$A = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1^* & 0 & 0 \\ A_2 & 0 & A_4 & 0 \\ 0 & A_3 & 0 & A_5 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $A_1 \in \mathbb{C}^{p\times p}$, $\sigma(A_1) = \{\lambda\}$, $\sigma(A_4) \subseteq \{\lambda\}$, $\sigma(A_5) \subseteq \{\bar{\lambda}\}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since H has rank $n-1$, one of the matrices A_4 and A_5 must have size one and the other one must have size zero. Without loss of generality, we may assume that A_5 has size zero. We are then left with the situation when

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_1^* & 0 \\ a^* & 0 & \lambda \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $A_1 = \mathcal{J}_{p_1}(\lambda) \oplus \cdots \oplus \mathcal{J}_{p_k}(\lambda)$ for $p_1 \leq \cdots \leq p_k$, $p = p_1 + \cdots + p_k$, and $a \in \mathbb{C}^p$. Moreover, a^* consists of k special blocks, i.e.,

$$a^* = [\underbrace{a_1\, 0 \ldots 0}_{p_1 \text{ entries}}\ \ldots\ \underbrace{a_k\, 0 \ldots 0}_{p_k \text{ entries}}],$$

where $a_1, \ldots, a_k \neq 0$. (Otherwise, A would be decomposable). Assume $k > 1$. Then

$$U_1 = \begin{bmatrix} I_{p_1} & 0 \\ \begin{bmatrix} \beta I_{p_1} \\ 0 \end{bmatrix} & I_{p_2} \end{bmatrix} \oplus I_{p_3} \oplus \cdots \oplus I_{p_k}, \quad \beta = -\frac{a_1}{a_2},$$

is invertible, even in the case $p_1 = p_2$, and commutes with A_1. Moreover, for $U := U_1 \oplus (U_1^*)^{-1} \oplus \begin{bmatrix} 1 \end{bmatrix}$ we have

$$U^{-1}AU = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_1^* & 0 \\ a^*U_1 & 0 & \lambda \end{bmatrix}, \quad U^*HU = \begin{bmatrix} 0 & I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$a^*U_1 = [\underbrace{a_1 + \beta a_2\, 0 \ldots 0}_{p_1 \text{ entries}}\ \underbrace{a_2\, 0 \ldots 0}_{p_2 \text{ entries}}\ \ldots\ \underbrace{a_k\, 0 \ldots 0}_{p_k \text{ entries}}].$$

Once again, the first entry of a^*U_1 is zero which implies the decomposability of A. Thus, $k = 1$ and after having applied a scaling transformation as in Case (1), we may assume that A and H have the forms

$$A = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & \mathcal{J}_p(\lambda)^* & 0 \\ e_1^* & 0 & \lambda \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $2p + 1 = n$ and e_1 denotes the first p-dimensional unit column vector. It is now easy to check that these blocks can be transformed into the blocks of type (2) of the theorem for the case $\lambda \in \mathbb{C} \setminus \mathbb{R}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ I_p & 0 & 0 \\ 0 & Z_p & 0 \end{bmatrix} \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & \mathcal{J}_p(\lambda)^* & 0 \\ e_1^* & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & I_p & 0 \\ 0 & 0 & Z_p \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{p+1}(\lambda) & 0 \\ 0 & \mathcal{J}_p(\bar{\lambda}) \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ I_p & 0 & 0 \\ 0 & Z_p & 0 \end{bmatrix} \begin{bmatrix} 0 & I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I_p & 0 \\ 0 & 0 & Z_p \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & Z_p \\ 0 & Z_p & 0 \end{bmatrix}.$$

The indecomposability of these blocks is clear for $\bar{\lambda} \neq \lambda$. □

We emphasize that the case $\lambda \in \mathbb{R}$ is possible in type (2) of Theorem 10. Thus, we see that if $\operatorname{rank}(H) = n - 1$, then an indecomposable H-selfadjoint matrix may have two, but not more than two, Jordan blocks associated with the same eigenvalue. As we know from Example 9, this number may be arbitrarily large in the general case. This gives rise to the following problem.

Problem 11 *Find the maximal possible number of Jordan blocks associated with the same eigenvalue of an indecomposable H-selfadjoint matrix, depending on the rank of H.*

Theorem 10 also shows that if $\operatorname{rank}(H) = n-1$ and if A is an indecomposable H-selfadjoint matrix having more than one Jordan block associated with the same eigenvalue, then the difference of their sizes is at most one. This difference, however, may be arbitrarily large even in the case that $\operatorname{rank}(H) = n-2$ as the following example shows.

Example 12 Let $p \geq 2$, let e_p denote the pth p-dimensional unit column vector, and let

$$A = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} Z_p\mathcal{J}_p(0) & e_p \\ e_p^* & 0 \end{bmatrix}.$$

Then the H-selfadjoint matrix A is H-indecomposable. Indeed, let P be nonsingular such that $P^{-1}AP = A$ and

$$P^*HP = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}. \tag{13}$$

A is only H-decomposable if there exists a matrix P as above such that $H_{12} = 0$ in (13). However, the nonsingularity and the identity $P^{-1}AP = A$ imply that P has the form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$

where P_1 and P_4 are nonsingular and $P_2 = p_2e_1$, $P_3 = p_3e_p^*$ for some $p_2, p_3 \in \mathbb{C}$. Here, e_1 denotes the first p-dimensional unit vector. But then we obtain

$$H_{12} = P_1^*Z_p\mathcal{J}_p(0)P_2 + P_3^*e_p^*P_2 + P_1^*e_pP_4 = P_1^*e_pP_4,$$

because of $\mathcal{J}_p(0)e_1 = 0$ and $e_p^*e_1 = 0$. Thus, H_{12} in (13) is nonzero for all possible choices of P. This implies that A is indecomposable. Clearly, we have $\operatorname{rank}(H) = p-1 = (\operatorname{size} H)-2$ and the difference of sizes of Jordan blocks of A is $p-1$.

5 Indecomposable H-selfadjoints: Semidefinite H

In this section, we classify indecomposable H-selfadjoint matrices for the case that H is semidefinite. Of course, it is sufficient to consider the case that H is positive semidefinite. Otherwise, we can replace H by $-H$.

Theorem 13 *Let H be positive semidefinite and let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint and indecomposable. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that $\tilde{A} = P^{-1}AP$ and $\tilde{H} = P^*HP$ are of one of the following forms:*

type (1): $\tilde{A} = \mathcal{J}_p(\lambda)$, $\tilde{H} = Z_p\mathcal{J}_p(0)^{p-1}$, where $p \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and it is understood that $J_1(0)^0 = 1$;

type (2): $\tilde{A} = \mathcal{J}_p(\lambda)$, $\tilde{H} = 0$, where $p \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.

Proof. Again, we may assume that A and H are of one of the types listed in Theorem 3. Since H is positive semidefinite, only blocks of the form (7) or (10) may occur. Since an indecomposable block of the form (10) is necessarily of the form of type (2), it is sufficient to consider the following situation:

$$A = \begin{bmatrix} \lambda I_k & 0 \\ A_2 & A_3 \end{bmatrix}, \quad H = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} \mathcal{J}_{p_1}(\lambda) & & 0 \\ & \ddots & \\ 0 & & \mathcal{J}_{p_m}(\lambda) \end{bmatrix},$$

where $\lambda \in \mathbb{R}$, and $p_1 \geq p_2 \geq \cdots \geq p_m$. Moreover, A_2 consists of mk subblocks that are special, i.e.,

$$A_2 = \begin{bmatrix} \tilde{a}_{11} & \dots & \tilde{a}_{1k} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \dots & \tilde{a}_{mk} \end{bmatrix}, \quad \text{where } \tilde{a}_{ij} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{ij} \end{bmatrix} \in \mathbb{C}^{p_i}.$$

Choosing a unitary $T_1 \in \mathbb{C}^{k \times k}$ and setting $T = T_1 \oplus I_{n-k}$, we obtain furthermore that

$$T^{-1}AT = \begin{bmatrix} \lambda I_k & 0 \\ A_2 T_1 & A_3 \end{bmatrix} \quad \text{and} \quad T^* H T = H.$$

Hence, choosing T_1 appropriately otherwise, we may moreover assume that $\tilde{a}_{12} = \cdots = \tilde{a}_{1k} = 0$. If a_{11} is zero, then it is easy to see that A is decomposable. (This follows by applying an appropriate permutation of rows and columns.) Thus, a_{11} is nonzero. Then setting

$$\mathcal{Q}^{-1} = I_k \oplus Q, \quad \text{where } Q = \begin{bmatrix} \frac{1}{a_{11}} I_{p_1} & & & 0 \\ \begin{bmatrix} 0 & -\frac{a_{21}}{a_{11}} I_{p_2} \end{bmatrix} & I_{p_2} & & \\ \vdots & & \ddots & \\ \begin{bmatrix} 0 & -\frac{a_{m1}}{a_{11}} I_{p_m} \end{bmatrix} & & & I_{p_m} \end{bmatrix},$$

we obtain that Q commutes with A_3 and that

$$\mathcal{Q}^{-1} A \mathcal{Q} = \begin{bmatrix} A_1 & 0 \\ Q A_2 & A_3 \end{bmatrix}, \quad \mathcal{Q}^* H \mathcal{Q} = H.$$

Thus, we can zero out $a_{21}, \dots, a_{m1}$. But this means that A is decomposable, unless $k = m = 1$. Hence, we finally may assume that A and H have the forms

$$A = \left[\begin{array}{c|c} \lambda & 0 \\ \hline e_p & \mathcal{J}_p(\lambda) \end{array}\right] \quad \text{and} \quad H = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] \in \mathbb{C}^{(p+1)\times(p+1)},$$

where $p = p_1$ and e_p denotes the pth p-dimensional unit vector. Up to a row and column permutation, these blocks are of type (1). $\square$

6 H-skewadjoint and H-unitary matrices

Clearly, the results of Sections 2 and 3 can be applied to H-skewadjoint matrices S, i.e., such that $HS = -S^*H$, as well, noting that a matrix S is H-skewadjoint if and only if iS is H-selfadjoint. Thus, solving the problem of classifying all indecomposable H-selfadjoint matrices also solves the same problem for the class of indecomposable H-skewadjoint matrices.

A matrix U is called *H-unitary* if $U^*HU = H$. It is easy to see that the set of all H-unitary matrices forms a semigroup, and the set of all nonsingular H-unitary matrices is a group (with respect to the standard matrix multiplication).

In this section, we show that also the problem of classifying indecomposable H-unitary matrices can be traced back to the analogous problem for H-selfadjoints (or, more precisely, for H-skewadjoints). If H is singular, then there exist also singular H-unitary matrices. The following proposition shows that it is sufficient to consider nonsingular H-unitary matrices only.

Proposition 14 *Let $U \in \mathbb{C}^{n\times n}$ be H-unitary. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^{-1}UP = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \quad \textit{and} \quad P^*HP = \begin{bmatrix} H_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where U_1 is nonsingular, U_2 is nilpotent, and H_1 (possibly singular) has the same size as U_1.

Proof. Let $P \in \mathbb{C}^{n\times n}$ be nonsingular and such that

$$P^{-1}UP = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \quad \text{and} \quad P^*HP = \begin{bmatrix} H_1 & H_2 \\ H_2^* & H_3 \end{bmatrix},$$

where U_1 is nonsingular and U_2 is nilpotent. Then the fact that $(U^*)^m HU^m = H$ implies that

$$H_2 = (U_1^*)^m H_2 U_2^m \quad \text{and} \quad H_3 = (U_2^*)^m H_3 U_2^m$$

for all $m \in \mathbb{N}$. Since U_2 is nilpotent, it follows that $H_2 = 0$ and $H_3 = 0$. $\square$

Thus, a singular H-unitary matrix decomposes into a nonsingular and a singular part, the classification of singular H-unitary matrices being trivial. Hence, it is sufficient to consider the Lie group of nonsingular H-unitary matrices only. It is easily seen that the exponential map *exp* maps the Lie algebra of H-skewadjoints into the Lie group of invertible H-unitary matrices. We show next that this map is onto, i.e., any nonsingular H-unitary matrix has an H-skewadjoint logarithm.

Proposition 15 *The group of nonsingular H-unitary matrices coincides with the set of exponentials of H-skewadjoint matrices.*

Proof. Let U be a nonsingular H-unitary matrix, and let Γ be a suitable simple closed rectifiable contour in the complex plane such that the eigenvalues of U are inside Γ, for example,

$$\Gamma = \{re^{i\theta} : \theta_1 \leq \theta \leq \theta_2\} \cup \{\frac{1}{r}e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\} \cup \{xe^{i\theta_1} : r \leq x \leq \frac{1}{r}\} \cup \{xe^{i\theta_2} : r \leq x \leq \frac{1}{r}\}$$

for suitable $r < 1$ and θ_1, θ_2 ($\theta_1 < \theta_2 < \theta_1 + 2\pi$). Define

$$S := \log(U) := \frac{1}{2\pi i} \int\limits_{\Gamma} \log(z)(zI - U)^{-1} dz, \tag{14}$$

where log is the branch of the logarithmic function such that $\log(z)$ is real for $z > 0$. Then $-S^* = \log((U^*)^{-1})$, see Theorem 6.4.20 in [5]. Approximating the integral in (14) by Riemann sums

$$\sum_{j=0}^{N-1} (z_{j+1} - z_j) \log(z_j)(z_j I - U)^{-1},$$

where $z_0, \ldots, z_{N-1}$ are consecutive partition points on Γ in the counterclockwise direction and $z_N = z_0$, and using the easily verified property

$$(zI - (U^*)^{-1})^{-1} H = H(zI - U)^{-1},$$

we find that $-S^* H = \log((U^*)^{-1})H = H \log(U) = HS$. Thus, S is H-skewadjoint. □

Clearly, U is indecomposable if and only if $\log(U)$ is indecomposable. This follows from the fact that both maps *log* and *exp* preserve block diagonal structures of matrices. Thus, we have reduced the problem of classifying indecomposable H-unitary matrices to the corresponding problem for H-skewadjoint matrices.

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