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PARAMETER-DEPENDENT RANK-ONE PERTURBATIONS OF SINGULAR HERMITIAN OR SYMMETRIC PENCILS*

C. MEHL†, V. MEHRMANN†, and M. WOJTYLAK‡

Abstract. Structure-preserving generic low-rank perturbations are studied for classes of structured matrix pencils, including real symmetric, complex symmetric, and complex Hermitian pencils. For singular pencils it is analyzed which characteristic quantities stay invariant in the perturbed canonical form, and it is shown that the regular part of a structured matrix pencil is not affected by generic perturbations of rank one. When the rank-one perturbations involve a scaling parameter, the behavior of the canonical forms dependent on this parameter is analyzed as well.

Key words. Hermitian pencils, symmetric pencils, singular pencils, structured Kronecker canonical form, generic low-rank perturbations

AMS subject classifications. 15A18, 15A21, 15A22, 65F15, 65F22

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1. Introduction.

In this paper we study low-rank perturbations in the coefficients of linear differential-algebraic equations (DAEs) of the form

$$E\dot{x} + Ax = f, \tag{1.1}$$

which typically arise as linearizations around stationary solutions of general nonlinear DAEs of the form $F(t, x, \dot{x}) = 0$; see, e.g., [3]. The analysis of the solution behavior of (1.1) can be characterized via the *Kronecker canonical form* of the matrix pencil $\lambda E + A$; see [2, 9, 14]. It is well known that small perturbations can drastically change the canonical form and hence also the solution behavior of (1.1). This is particularly unfortunate if perturbations make the pencil $\lambda E + A$ singular, because then the perturbed system may not be (uniquely) solvable any more. A major motivation for our work comes from structured pencils arising in stability analysis of DAEs.

Example 1.1. In the finite element analysis of disc brake squeal [10, 20], large scale second order differential equations arise that have the form

$$M\ddot{q} + (D + G)\dot{q} + (K + N)q = f,$$

where $M = M^\top > 0$ is the mass matrix, $D = D^\top \geq 0$ models material and friction induced damping, $G = -G^\top$ models gyroscopic effects, $K = K^\top > 0$ models the stiffness, and $N = -N^\top$ is a nonsymmetric matrix modeling circulatory effects. (Here $>$ ($\geq$) denotes positive (semi)definiteness of a matrix). An appropriate first order formulation is associated with the linear pencil

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†Institut für Mathematik, MA 4-5, TU Berlin, D-10623 Berlin, Germany (mehl@math.tu-berlin.de, mehrmann@math.tu-berlin.de).
‡Instytut Matematyki, Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, 30-348 Kraków, Poland (michal.wojtylak@uj.edu.pl).

$$\lambda E+A+L:=\lambda\begin{bmatrix} M & 0\\ 0 & K\end{bmatrix}+\begin{bmatrix} D & \frac{1}{2}N\\ \frac{1}{2}N & 0\end{bmatrix}+\begin{bmatrix} G & K+\frac{1}{2}N\\ -(K+\frac{1}{2}N) & 0\end{bmatrix},$$

where E is real symmetric, L is skew-symmetric, and A is symmetric.

The classical modal truncation approach [4] used in commercial finite element packages computes the eigenvalues closest to 0 and the associated eigenvectors of the symmetric eigenvalue problem $\lambda E+A$, and projects the full problem into the subspace spanned by these eigenvectors. In the analysis presented in [10], it was noticed that the matrix pencil $\lambda E+A$ was close to a singular pencil, and this effect was traced back to the introduction of a small number of stiff springs instead of rigid connections. A similar behavior was observed in [13]. These low-rank perturbations in the modeling process lead to pencils that are close to being singular. This creates large difficulties in the numerical methods, because these pencils numerically behave as if they were singular pencils. Our analysis is motivated by the desire to understand the effect of low-rank perturbations in these situations.

The smallest perturbation (in some norm) that makes a pencil singular is called the *distance to singularity* and it is a long time open problem [1] to determine this distance. Some progress in the solution of this problem has been made recently in [18] for structured pencils and the case that the perturbations are restricted to be of rank one, and for general pencils using the integration of ordinary differential equations in [11].

In numerical analysis, e.g., in the solution of ill-conditioned linear systems, it is often better to consider a problem as being singular and to treat it as such, e.g., in the case of linear systems, as a least squares problem. With such a strategy in mind, to understand the behavior of the system and its associated pencil in the neighborhood of a singularity, we start from a singular pencil $\lambda E+A$ with exactly one singular block in its canonical form, so that generically a rank-one perturbation will make that pencil regular. In that situation, we investigate the dependence on the scalar parameter τ of the canonical form for the perturbations

$$\lambda E+A+\tau(\lambda e+a)Z, \tag{1.2}$$

where Z is a rank-one matrix and a, e are fixed scalars. We consider the following classes of structured pencils and their structure-preserving rank-one perturbations:

- Hermitian pencils: $A, E\in\mathbb{C}^{n\times n}$ and $A^*=A$, $E^*=E$, $Z=uu^*$, $u\in\mathbb{C}^n$, $a, e\in\mathbb{R}$;
- real symmetric pencils: $A, E\in\mathbb{R}^{n\times n}$ and $A=A^\top$, $E=E^\top$, $Z=uu^\top$, $u\in\mathbb{R}^n$, $a, e\in\mathbb{R}$;
- complex symmetric pencils: $A, E\in\mathbb{C}^{n\times n}$ and $A=A^\top$, $E=E^\top$, $Z=uu^\top$, $u\in\mathbb{C}^n$, $a, e\in\mathbb{C}$,

where $^\top$ denotes the transpose and * the conjugate transpose. Other important structures include real or complex $\top$-alternating pencils $\lambda E+A$, where $A=A^\top$ and $E=-E^\top$, or where $A=-A^\top$ and $E=E^\top$. We will not consider $\top$-alternating pencils in the main part of this paper, but for the sake of future reference, some preliminary results are formulated in a very general fashion so that they also cover $\top$-alternating pencils.

As particular rank-one perturbations may have very specialized effects, we will mainly consider *structure-preserving generic* rank-one perturbations, which is understood in the following sense. For $\mathbb{F}\in\{\mathbb{C},\mathbb{R}\}$, a set $\mathcal{A}\subseteq\mathbb{F}^n$ is called *algebraic*, if it is a set of common zeros of finitely many polynomials $p_1,\dots,p_k$ in n variables. An algebraic set $\mathcal{A}$ is called *proper* if $\mathcal{A}\neq\mathbb{F}^n$. A set $\Omega\subseteq\mathbb{F}^n$ is called *generic* if its complement $\mathbb{F}^n\setminus\Omega$ is contained in a proper algebraic set. Throughout the paper, we will be interested in statements "*for generic $u \in \mathbb{F}^n$ property X is satisfied*," which by definition means "*there exists a generic set $\Omega \subseteq \mathbb{F}^n$ such that for all $u \in \Omega$ property X is satisfied*."

For pencils without additional symmetry structure, the effect of generic low-rank perturbations has been studied in great detail in [5, 6, 7, 8]. We mention that this problem is essentially different from the one studied here, because generic rank-one perturbations that do not preserve one of the structures mentioned above do not have the form $(\lambda e + a)Z$ with Z being a rank-one matrix, but they consist of a pencil having precisely two singular blocks of size 1×2 and 1×0, or 2×1 and 0×1 in its Kronecker canonical form; see [6, Theorem 3.2] and also the discussion in the introduction of [7]. However, it follows from the results on canonical forms mentioned in section 2 that these types of rank-one perturbations can never be structure preserving for the structures considered in this paper.

It is known that for a perturbed singular pencil as in (1.2), the eigenvalues and their algebraic multiplicities are generically constant in the parameter $\tau \neq 0$; see [18]. Surprisingly, this need not be the case for the corresponding partial multiplicities which may depend on τ, as the following example shows.

Example 1.2. Consider the real symmetric singular pencil

$$P(\lambda) = \lambda \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] + \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Letting $u = e_1 + e_4 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix}^\top$ we obtain that

$$P_\tau(\lambda) := P(\lambda) + \tau u u^* = \begin{bmatrix} \tau & 0 & 0 & 1+\tau & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \\ 1+\tau & \lambda & 0 & \tau & 0 \\ 0 & 1 & \lambda & 0 & 0 \end{bmatrix}.$$

For $\tau \neq 0$ and

$$S_\tau^\top = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{1+\tau}{\tau} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we then obtain

$$S_\tau^\top P_\tau(\lambda) S_\tau = \left[\begin{array}{c|cccc} \tau & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & \lambda & -\frac{1}{\tau}(1+2\tau) & 0 \\ 0 & \lambda & 1 & 0 & 0 \end{array}\right].$$

Thus, for all $\tau \neq 0$, the pencil $P_\tau(\lambda)$ is regular and has the eigenvalue infinity with algebraic multiplicity one and the eigenvalue zero with algebraic multiplicity four. Note that for $\tau \neq -1/2$ the pencil has a block of size four corresponding to the eigenvalue zero in the structured Kronecker canonical form (see (2.1) below), while for $\tau = -1/2$ there are two blocks of size two.

Although having an eigenvalue of multiplicity four is a nongeneric property, we will show that such change in the canonical form for a specific value of the parameter τ is generic in the classes of real or complex symmetric pencils. More precisely, we will show that if a singular pencil becomes regular after a rank-one perturbation, then its regular part generically is not affected by the perturbation. Furthermore, there appear new eigenvalues whose location is independent of the norm of the perturbation as their algebraic multiplicities are constant in τ, but their partial multiplicities will only be constant in τ except for a specific value τ_0. In the case of real symmetric pencils, this discontinuity of the canonical form is accompanied by a switch of one sign from the so-called *sign characteristic* of the pencil.

For the sake of simplification of proofs in the remainder of this paper, we will use Möbius transformations to reduce the case of perturbations of the form (1.2) to the special case $e = 0$ which has the effect that just one of the coefficient matrices of the pencil is perturbed. Although the effect of Möbius transformations on most invariants of matrix pencils or matrix polynomials is well understood (see, e.g., [16]), it seems that the change of the sign characteristic of real eigenvalues of Hermitian or real symmetric pencils has not yet been considered in the literature. Therefore, we will fill this gap and investigate how the sign characteristic of a Hermitian or real symmetric pencil is related to the one that is obtained under a Möbius transformation that keeps the Hermitian or real symmetric structure of the pencil invariant.

The remainder of the paper is organized as follows. In section 2 we present structured canonical forms for Hermitian, real symmetric, and complex symmetric pencils. Section 3 discusses Möbius transformations of structured matrix pencils with special emphasis on their effect on the sign characteristic. Section 4 contains some general perturbation results that are applicable to several kinds of structured matrix pencils and the important result that the regular part of a structured matrix pencil is not affected by structure-preserving generic perturbations of rank one. In section 5, we investigate Hermitian and real or complex symmetric singular pencils that become regular after a structure-preserving rank-one perturbation. In particular, the behavior of their canonical forms in dependence of a scaling parameter in the rank-one perturbation is analyzed.

2. Canonical forms of pencils.

In this section, we recall basic decompositions for all three classes of structured pencils considered in this paper. We start with the canonical form for Hermitian pencils; see, e.g., [15, Theorem 4.1], [21, Theorem 1].

THEOREM 2.1 (Hermitian canonical form). *Let $A, E \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Then there exists an invertible matrix $S \in \mathbb{C}^{n\times n}$ such that the pencil $S^*(\lambda E + A)S$ is block-diagonal with diagonal blocks of one of the following forms:*

(i) *blocks corresponding to a pair of conjugate complex eigenvalues $\gamma, \overline{\gamma}$, where $\gamma \in \mathbb{C}^+ := \{z \in \mathbb{C} \,|\, \operatorname{Im} z > 0\}$:*

$$\mathcal{J}_{k,k,\gamma}(\lambda) := \begin{bmatrix} 0 & \mathcal{J}^1_{k,\gamma}(\lambda) \\ \mathcal{J}^1_{k,\bar{\gamma}}(\lambda) & 0 \end{bmatrix} \in \mathbb{C}^{2k\times 2k}, \tag{2.1}$$

where $\mathcal{J}^1_{k,\gamma}(\lambda)$ and $\mathcal{J}^1_{k,\bar{\gamma}}(\lambda)$ are defined as in (2.2);

(ii) *blocks corresponding to a real eigenvalue $\gamma \in \mathbb{R}$:*

$$\mathcal{J}^s_{k,\gamma}(\lambda) := s \begin{bmatrix} & & & \lambda-\gamma \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ \lambda-\gamma & 1 & & \end{bmatrix} \in \mathbb{R}^{k\times k}, \;\; s \in \{-1, 1\}\,; \tag{2.2}$$

(iii) *blocks corresponding to the eigenvalue infinity:*

$$\mathcal{J}^s_{k,\infty}(\lambda) := s \begin{bmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \lambda \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & \lambda & & \end{bmatrix} \in \mathbb{R}^{k\times k},\ s \in \{-1,1\}\,; \tag{2.3}$$

(iv) *singular blocks:*

$$\mathcal{L}_{2k+1}(\lambda) := \begin{bmatrix} 0 & \mathcal{G}_k(\lambda) \\ \mathcal{G}_k^\top(\lambda) & 0 \end{bmatrix} \in \mathbb{R}^{(2k+1)\times(2k+1)}, \tag{2.4}$$

where

$$\mathcal{G}_k(\lambda) = \begin{bmatrix} 1 & & \\ \lambda & \ddots & \\ & \ddots & 1 \\ & & \lambda \end{bmatrix} \in \mathbb{R}^{(k+1)\times k}. \tag{2.5}$$

The parameters $\gamma \in \mathbb{C}$, $s \in \{-1,1\}$, and $k \in \mathbb{N}$ (for blocks of types (i)–(iii) *we have $k \geq 1$) depend on the particular block and hence may be different in different blocks. Moreover, the canonical form is unique up to permutation of diagonal blocks.*

A Hermitian pencil is *singular* if and only if it contains blocks of the form (2.4) and *infinity* is an eigenvalue if and only if it contains a block of the form (2.3). The collection of the signs s appearing in the blocks associated with a fixed real eigenvalue or the eigenvalue infinity, respectively, is called the *sign characteristic* of the corresponding eigenvalue.

Note that the canonical form as we have presented it here is consistent with interpreting a Hermitian pencil as a degree one matrix polynomial $\lambda E + A$. In the literature, Hermitian pencils are also written in the form $\lambda E - A$ and then, instead of $\mathcal{J}^s_{k,\infty}(\lambda)$, a block of the form

$$\widetilde{\mathcal{J}}^{\tilde{s}}_{k,\infty}(\lambda) := \tilde{s} \begin{bmatrix} & & & -1 \\ & & \cdot^{\cdot^{\cdot}} & \lambda \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ -1 & \lambda & & \end{bmatrix} \in \mathbb{R}^{k\times k},\ \tilde{s} \in \{-1,1\}\,,$$

is occurring in the canonical form. This has an effect on the definition of the sign characteristic at infinity via the canonical form, because $\mathcal{J}^s_{k,\infty}(\lambda)$ and $\widetilde{\mathcal{J}}^{\tilde{s}}_{k,\infty}(\lambda)$ are congruent if k is odd, but if k is even, then $\mathcal{J}^s_{k,\infty}(\lambda)$ and $\widetilde{\mathcal{J}}^{-\tilde{s}}_{k,\infty}(\lambda)$ are congruent; see also [19] for a detailed discussion of this issue.

Next, we recall a corresponding theorem for the case of real symmetric pencils; see [22, Theorem 2]. Note that most of the blocks in the canonical form in Theorem 2.1 are already real. Only for blocks of the form (2.1) is a different representation needed.

THEOREM 2.2 (real symmetric canonical form). *Let $A, E \in \mathbb{R}^{n\times n}$ be symmetric matrices. Then there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that the pencil $S^\top(\lambda E + A)S$ is block-diagonal with diagonal blocks of one of the following forms:*

(i) *blocks corresponding to a pair of conjugate complex eigenvalues* $\alpha \pm i\beta$, *where* $\alpha, \beta \in \mathbb{R}$ *and* $\beta > 0$:

$$\mathcal{J}_{k,k,\alpha,\beta}(\lambda) := \begin{bmatrix} & & & \lambda R_2 - Z_{\alpha,\beta} \\ & & \iddots & R_2 \\ & \iddots & \iddots & \\ \lambda R_2 - Z_{\alpha,\beta} & R_2 & & \end{bmatrix} \in \mathbb{R}^{2k \times 2k}, \tag{2.6}$$

where

$$Z_{\alpha,\beta} := \begin{bmatrix} -\beta & \alpha \\ \alpha & \beta \end{bmatrix} \in \mathbb{R}^{2,2} \quad \text{and} \quad R_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \tag{2.7}$$

(ii) *blocks corresponding to a real eigenvalue* $\gamma \in \mathbb{R}$: $\mathcal{J}^s_{k,\gamma}(\lambda)$, *where* $\mathcal{J}^s_{k,\gamma}(\lambda)$ *is as in* (2.2) *and* $s \in \{-1, 1\}$;

(iii) *blocks corresponding to the eigenvalue infinity:* $\mathcal{J}^s_{k,\infty}(\lambda)$, *where* $\mathcal{J}^s_{k,\infty}(\lambda)$ *is as in* (2.3) *and* $s \in \{-1, 1\}$;

(iv) *singular blocks:* $\mathcal{L}_{2k+1}(\lambda)$, *where* $\mathcal{L}_{2k+1}(\lambda)$ *is as in* (2.4).

The parameters $\alpha, \beta, \gamma \in \mathbb{R}$, $s \in \{-1, 1\}$, *and* $k \geq 0$ *depend on the particular block and, hence, may be different in different blocks. Moreover, the canonical form is unique up to permutation of diagonal blocks.*

The third class considered in the paper are complex symmetric matrices. Here, we have the following canonical form; see [22, Theorem 1].

THEOREM 2.3 (complex symmetric canonical form). *Let* $A, E \in \mathbb{C}^{n \times n}$ *be symmetric matrices. Then there exists an invertible matrix* $S \in \mathbb{C}^{n \times n}$ *such that the pencil* $S^\top(\lambda E + A)S$ *is block-diagonal with diagonal blocks of one of the following forms:*

(i) *blocks corresponding to a complex or real eigenvalue* $\gamma \in \mathbb{C}$:

$$\mathcal{J}_{k,\gamma}(\lambda) := \begin{bmatrix} & & & \lambda - \gamma \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ \lambda - \gamma & 1 & & \end{bmatrix} \in \mathbb{C}^{k \times k}; \tag{2.8}$$

(ii) *blocks corresponding to the eigenvalue infinity:* $\mathcal{J}_{k,\infty}(\lambda) := \mathcal{J}^1_{k,\infty}(\lambda)$, *where* $\mathcal{J}^1_{k,\infty}(\lambda)$ *is as in* (2.3);

(iii) *singular blocks:* $\mathcal{L}_{2k+1}(\lambda)$, *where* $\mathcal{L}_{2k+1}(\lambda)$ *is as in* (2.4).

The parameters $\gamma \in \mathbb{C}$ *and* $k \geq 0$ *depend on the particular block and hence may be different in different blocks. Moreover, the canonical form is unique up to permutation of diagonal blocks.*

If we want to determine the sign characteristic of Hermitian or real symmetric matrices in the following, we will do that by computing the *inertia index* of particular Hermitian matrices. Recall that the inertia index $\operatorname{ind}(H) = (n_+, n_-, n_0)$ of a Hermitian or real symmetric matrix H consists of the numbers n_+ of positive eigenvalues, n_- of negative eigenvalues, and n_0 of zero eigenvalues. We will then frequently make use of the following lemma which is straightforward to prove.

LEMMA 2.4. *Let the* $n \times n$ *Hermitian matrix*

$$H = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} 0 & & a_{1m} \\ & \iddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{bmatrix} \in \mathbb{C}^{m \times m}$$

be such that A is nonsingular, where $m = 2k+1$ is odd. Then the inertia index of H is given by

$$\operatorname{ind}(H) = \begin{cases} (k+1, k, n-m) & \text{if } a_{k+1,k+1} > 0, \\ (k, k+1, n-m) & \text{if } a_{k+1,k+1} < 0. \end{cases}$$

Having presented the structured canonical forms, in the next section we will discuss how these structures change under Möbius transformations and then in the remainder of the paper under structure-preserving generic low-rank perturbations.

3. Möbius transformations. For $b, c \in \mathbb{C} \setminus \{(0,0)\}$, we define the Möbius transformation of a complex number γ (including ∞) as $M_{b,c}(\gamma) = \frac{b\gamma + c}{-c\gamma + b}$. (Here and throughout the remainder of the paper, we interpret a fraction $\frac{d}{0}$ with $d \in \mathbb{C} \setminus \{0\}$ as ∞.) Following [16], we extend this definition onto linear pencils as

$$M_{b,c}(\lambda E + A) = \lambda(cA + bE) + (bA - cE).$$

Note that if $P(\lambda)$ is a Hermitian (real symmetric) pencil, then for $b, c \in \mathbb{R}$, $M_{b,c}P(\lambda)$ is Hermitian (resp., symmetric) as well. If $P(\lambda)$ is complex symmetric, then $M_{b,c}P(\lambda)$ is complex symmetric for $b, c \in \mathbb{C}$. By [16, Propositions 3.23 and 3.16] Möbius transformations preserve congruence and the block-diagonal structure of matrix pencils. Thus, we may assume without loss of generality that the pencil under consideration is already in one of the canonical forms from section 2, and it is sufficient to investigate the effect of the Möbius transformation on each diagonal block independently. It was shown in [16] that the eigenvalues of $M_{b,c}P(\lambda)$ are the Möbius transforms of the eigenvalues of $P(\lambda)$ and the Kronecker structure is preserved. In particular, Theorem 5.3 of [16] implies the following two lemmas displaying the effect of Möbius transformations on the blocks in the structured canonical forms.

LEMMA 3.1. *For $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $\big(b, c \in \mathbb{C}^2 \setminus \{(0,0)\}\big)$, and $k \geq 1$ the Hermitian canonical form and the real symmetric canonical form (resp; the complex symmetric canonical form) of $M_{b,c}\mathcal{L}_{2k+1}(\lambda)$ is equal to $\mathcal{L}_{2k+1}(\lambda)$.*

LEMMA 3.2. *For $b, c, \gamma \in \mathbb{C}$, $(b,c) \neq (0,0)$, and $k \geq 1$ the complex symmetric canonical form of $M_{b,c}\mathcal{J}_{k,\gamma}(\lambda)$ is equal to $\mathcal{J}_{k, M_{b,c}(\gamma)}(\lambda)$.*

These two lemmas completely determine the action of $M_{b,c}(\gamma)$ on canonical forms of complex symmetric pencils. The corresponding result for Hermitian and real symmetric pencils, however, is not as immediate, because the sign characteristic of real eigenvalues is involved.

THEOREM 3.3. *Let $P(\lambda)$ be a Hermitian pencil having the canonical form*

$$\left(\bigoplus_{i=1}^{m} \mathcal{J}_{k_i, k_i, \gamma_i}(\lambda) \right) \oplus \left(\bigoplus_{i=m+1}^{l} \mathcal{J}_{k_i, \gamma_i}^{s_i}(\lambda) \right) \oplus S(\lambda)$$

from Theorem 2.1*, with $k_i \in \mathbb{N} \setminus \{0\}$ for $i = 1, \ldots, l$, $\gamma_i \in \mathbb{C}^+$ for $i = 1, \ldots, m$, $\gamma_i \in \mathbb{R} \cup \{\infty\}$, and $s_i \in \{-1, 1\}$ for $i = m+1, \ldots, l$, and where $S(\lambda)$ contains all the singular blocks from the canonical form of $P(\lambda)$. Furthermore, let $b, c \in \mathbb{R}$ with $b^2 + c^2 > 0$. Then the canonical form (as a Hermitian pencil) of the Möbius transformation $M_{b,c}\big(P(\lambda)\big)$ is given by*

$$\left(\bigoplus_{i=1}^{m} \mathcal{J}_{k_i, k_i, M_{b,c}(\gamma_i)}(\lambda) \right) \oplus \left(\bigoplus_{i=m+1}^{l} \mathcal{J}_{k_i, M_{b,c}(\gamma_i)}^{\tilde{s}_i}(\lambda) \right) \oplus S(\lambda),$$

where $M_{b,c}(\gamma_i) \in \mathbb{C}^+$ *for* $i = 1, \dots, m$, *and for* $i = m+1, \dots, l$ *we have* $M_{b,c}(\gamma_i) \in \mathbb{R} \cup \{\infty\}$ *as well as*

$$\tilde{s}_i = s_i \cdot \begin{cases} (-1)^n \operatorname{sgn}(c) & \text{for} \quad M_{b,c}(\gamma_i) = \infty, \\ (-1)^{n+1} \operatorname{sgn}(c) & \text{for} \quad \gamma_i = \infty, \\ \operatorname{sgn}(cM_{b,c}(\gamma_i) + b) & \text{for} \quad \gamma_i, M_{b,c}(\gamma_i) \in \mathbb{R},\ n \text{ even}, \\ \operatorname{sgn}(c\gamma_i + b) & \text{for} \quad \gamma_i, M_{b,c}(\gamma_i) \in \mathbb{R},\ n \text{ odd}, \end{cases}$$

if $c \neq 0$, *and* $\tilde{s}_i = \operatorname{sgn}(b) \cdot s_i$ *if* $c = 0$.

Proof. By the discussion at the beginning of this section it is sufficient to consider the case that $P(\lambda)$ is already in canonical form and consists of exactly one block as in (i)–(iv) of Theorem 2.1. The case that $P(\lambda)$ is a singular block is covered by Lemma 3.1. Next, let $P(\lambda)$ be a block as in (i) of Theorem 2.1. As $b^2 + c^2 > 0$ it follows that the scalar Möbius transformation maps the upper, respectively, lower, complex half-plane onto itself. The assertion now follows directly from Lemma 3.2.

Thus, it remains to investigate the case that $P(\lambda)$ is a block of type (ii) or (iii) as in Theorem 2.1, i.e., $P(\lambda) = \mathcal{J}^s_{k,\gamma}(\lambda)$, where $k \in \mathbb{N}$, $\gamma \in \mathbb{R} \cup \{\infty\}$, and $s \in \{-1, 1\}$. Without loss of generality we may assume that $s = 1$. Note that for $\gamma \in \mathbb{R}$, by definition, we have

$$M_{b,c}\mathcal{J}^1_{k,\gamma}(\lambda) = \begin{bmatrix} & & & \lambda(b-c\gamma) - b\gamma - c \\ & & \iddots & \lambda c + b \\ & \iddots & \iddots & \\ \lambda(b-c\gamma) - b\gamma - c & \lambda c + b & & \end{bmatrix}.$$

If $c = 0$, then $M_{b,c} = b \cdot Id$ and the statement is obvious, so we assume $c \neq 0$ in the following and distinguish three cases.

Case (1): If $\gamma \in \mathbb{R}$ and $b - c\gamma = 0$, i.e., $M_{b,c}(\gamma) = \infty$, then

$$M_{b,c}\mathcal{J}^1_{k,\gamma}(\lambda) = \lambda c \begin{bmatrix} & & & 0 \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix} + c \begin{bmatrix} & & & -1-\gamma^2 \\ & & \iddots & \gamma \\ & \iddots & \iddots & \\ -1-\gamma^2 & \gamma & & \end{bmatrix}.$$

If n is even, then by Lemma 2.4 the inertia index of the leading coefficient $M_{b,c}\mathcal{J}^1_{k,\gamma}(\infty)$ of the pencil $M_{b,c}\mathcal{J}^1_{k,\gamma}(\lambda)$ equals

$$\operatorname{ind}\big(M_{b,c}\mathcal{J}^1_{k,\gamma}(\infty)\big) = \begin{cases} \big(\frac{n}{2}, \frac{n}{2} - 1, 1\big) & \text{for } c > 0, \\ \big(\frac{n}{2} - 1, \frac{n}{2}, 1\big) & \text{for } c < 0, \end{cases} \tag{3.1}$$

while in general

$$\operatorname{ind}\big(\mathcal{J}^t_{k,\infty}(\infty)\big) = \begin{cases} \big(\frac{n}{2}, \frac{n}{2} - 1, 1\big) & \text{for } t = 1, \\ \big(\frac{n}{2} - 1, \frac{n}{2}, 1\big) & \text{for } t = -1. \end{cases} \tag{3.2}$$

Comparing (3.1) and (3.2), and using the fact that a congruence transformation keeps the inertia indices invariant, we see that $\tilde{s} = \operatorname{sgn}(c)s$. Similarly for n odd, we compare the inertia indices of the trailing coefficient matrices

$$\operatorname{ind}\big(M_{b,c}\mathcal{J}^1_{k,\gamma}(0)\big) = \begin{cases} \big(\frac{n-1}{2}, \frac{n+1}{2}, 0\big) & \text{for } c > 0, \\ \big(\frac{n+1}{2}, \frac{n-1}{2}, 0\big) & \text{for } c < 0, \end{cases}$$

and

$$\operatorname{ind}\big(\mathcal{J}^1_{k,\infty}(0)\big) = \begin{cases} \big(\frac{n+1}{2}, \frac{n-1}{2}, 0\big) & \text{for } t = 1, \\ \big(\frac{n-1}{2}, \frac{n+1}{2}, 0\big) & \text{for } t = -1, \end{cases}$$

and obtain $\tilde{s} = -\operatorname{sgn}(c)s$.

Case (2): The case $\gamma = \infty$ follows from Case (1) by considering the inverse transformation $M_{\tilde{b},\tilde{c}} = M^{-1}_{b,c}$ with $\operatorname{sgn}(\tilde{c}) = -\operatorname{sgn}(c)$.

Case (3): In the case $\gamma \in \mathbb{R}$ and $b - c\gamma \neq 0$, assume first that n is even. Evaluating $M_{b,c}\mathcal{J}^1_{k,\gamma}(\lambda)$ at $\widetilde{\gamma} = M_{b,c}(\gamma) = \frac{b\gamma+c}{b-c\gamma}$ we get

$$M_{b,c}\mathcal{J}^1_{k,\gamma}(\widetilde{\gamma}) = \begin{bmatrix} & & & 0 \\ & & \iddots & \widetilde{\gamma}c + b \\ & \iddots & \iddots & \\ 0 & \widetilde{\gamma}c + b & & \end{bmatrix} \quad \text{and} \quad \widetilde{\gamma}c + b = \frac{b^2 + c^2}{b - c\gamma} \neq 0$$

and, hence, again applying Lemma 2.4, that

$$\operatorname{ind}\big(M_{b,c}\mathcal{J}^1_{k,\gamma}(\widetilde{\gamma})\big) = \begin{cases} \big(\frac{n}{2}, \frac{n}{2} - 1, 1\big) & \text{for } \widetilde{\gamma}c + b > 0, \\ \big(\frac{n}{2} - 1, \frac{n}{2}, 1\big) & \text{for } \widetilde{\gamma}c + b < 0. \end{cases}$$

Comparing this with

$$\operatorname{ind}\big(\mathcal{J}^1_{k,\tilde{\gamma}}(\tilde{\gamma})\big) = \begin{cases} \big(\frac{n}{2}, \frac{n}{2} - 1, 1\big) & \text{for } t = 1, \\ \big(\frac{n}{2} - 1, \frac{n}{2}, 1\big) & \text{for } t = -1, \end{cases}$$

the assertion follows. If n is odd, then evaluating the pencils at $\zeta = -\frac{b}{c}$ we get

$$M_{b,c}\mathcal{J}^1_{k,\gamma}(\zeta) = \begin{bmatrix} & & -\frac{b^2+c^2}{c} \\ & \iddots & \\ -\frac{b^2+c^2}{c} & & \end{bmatrix},$$

$$\mathcal{J}^t_{k,\tilde{\gamma}}(\zeta) = t\begin{bmatrix} & & & -\frac{b}{c} - \gamma \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ -\frac{b}{c} - \gamma & 1 & & \end{bmatrix}.$$

Hence, the inertia indices to compare are

$$\operatorname{ind}\big(M_{b,c}\mathcal{J}^1_{k,\gamma}(\zeta)\big) = \begin{cases} \big(\frac{n-1}{2}, \frac{n+1}{2}, 0\big) & \text{for } c > 0, \\ \big(\frac{n+1}{2}, \frac{n-1}{2}, 0\big) & \text{for } c < 0, \end{cases}$$

and

$$\operatorname{ind}\big(\mathcal{J}^t_{k,\tilde{\gamma}}(\zeta)\big) = \begin{cases} \big(\frac{n-1}{2}, \frac{n+1}{2}, 0\big) & \text{for } t(\frac{b}{c} + \gamma) > 0, \\ \big(\frac{n+1}{2}, \frac{n-1}{2}, 1\big) & \text{for } t(\frac{b}{c} + \gamma) < 0, \end{cases}$$

and thus, $\tilde{s} = s \cdot \operatorname{sgn}(b + c\gamma)$. ∎

Remark 3.4. There is a corresponding result for real symmetric pencils that follows directly from the relation between the Hermitian canonical form and the real symmetric canonical form of a real symmetric pencil. Since this is straightforward, we do not state this result here, but only mention that for $\alpha, \beta, b, c \in \mathbb{R}$ with $b^2 + c^2 > 0$, $\beta > 0$, and $k \geq 1$, the real symmetric canonical form of $M_{b,c}\mathcal{J}_{k,k,\alpha,\beta}(\lambda)$ is given by $\mathcal{J}_{k,k,\tilde{\alpha},\tilde{\beta}}(\lambda)$, where $\tilde{\alpha} + i\tilde{\beta} = M_{b,c}(\alpha + i\beta)$.

The presented results on the transformation of the canonical forms under Möbius transformations will be used in the following sections to simplify the proofs of the perturbation results.

4. Invariance of the regular part under structure-preserving rank-one perturbations. In this section we show that the regular part of a pencil of one of the structures mentioned in the introduction stays intact under generic structure-preserving rank-one perturbations. We present Theorems 4.1 and 4.2 below in a very general setting. Note that they both cover not only the three main classes considered in the paper, i.e., Hermitian pencils, real symmetric pencils, and complex symmetric pencils, but also other structures such as real or complex $\top$-alternating pencils.

THEOREM 4.1. *Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, let $\star \in \{*, \top\}$, and let $A, E \in \mathbb{F}^{n\times n}$ be such that $A^\star = \delta_A A$ and $E^\star = \delta_E E$ with $\delta_A, \delta_E \in \{+1, -1\}$. Then there exists a nonsingular matrix $U \in \mathbb{F}^{n\times n}$ such that*

$$\lambda U^\star E U + U^\star A U = \begin{bmatrix} S(\lambda) & 0 & 0 \\ 0 & R_f(\lambda) & 0 \\ 0 & 0 & R_i(\lambda) \end{bmatrix},$$

where $R_f(\lambda) = \lambda E_f + A_f$ with E_f nonsingular (this part contains the finite eigenvalues of the pencil $\lambda E + A$), $R_i(\lambda) = \lambda E_i + A_i$ with A_i nonsingular, and E_i being nilpotent (this part contains the infinite eigenvalues of $\lambda E + A$), and

$$S(\lambda) = \operatorname{diag}\big(\mathcal{L}^{\delta_A,\delta_E}_{2k_1+1}(\lambda), \dots, \mathcal{L}^{\delta_A,\delta_E}_{2k_l+1}(\lambda)\big)$$

with $k_1, \dots, k_l \in \mathbb{N}$ and

$$\begin{aligned} \mathcal{L}^{\delta_A,\delta_E}_{2k_j+1}(\lambda) :&= \begin{bmatrix} 0 & \mathcal{G}_{k_j}(\lambda) \\ \delta_A \mathcal{G}^\top_{k_j}(\delta_A\delta_E\lambda) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & G^{\mathrm{u}}_{k_j} + \lambda G^{\mathrm{d}}_{k_j} \\ \delta_A (G^{\mathrm{u}}_{k_j})^\top + \lambda \delta_E (G^{\mathrm{d}}_{k_j})^\top & 0 \end{bmatrix}, \end{aligned}$$

where $\mathcal{G}_{k_j}(\lambda)$ is as in (2.5) *and*

$$G^{\mathrm{u}}_{k_j} =: \begin{bmatrix} I_{k_j} \\ 0 \end{bmatrix}, \quad G^{\mathrm{d}}_{k_j} := \begin{bmatrix} 0 \\ I_{k_j} \end{bmatrix} \tag{4.1}$$

for $j = 1, \dots, l$. In particular, $S(\lambda)$ is uniquely determined up to a permutation of the l singular blocks on its block-diagonal.

Proof. The proof follows immediately, by inspection, from the canonical form of pairs of Hermitian matrices in [21] or of real or complex pairs of matrices that are either symmetric or skew symmetric given in [22]. ∎

We call $R(\lambda) = \operatorname{diag}\big(R_f(\lambda), R_i(\lambda)\big)$ the *regular part* and $S(\lambda)$ the *singular part* of the pencil $\lambda E + A$, respectively. (Note, however, that both parts are only unique up to appropriate congruence transformations.)

In the following, we will present a general result about the effect of generic structure-preserving rank-one perturbations on the regular part of a structured pencil $\lambda E + A$. Again, we present this theorem in a very general setting by simultaneously considering the real and complex case and, in the complex case, symmetry structures with respect to both the transpose and the conjugate transpose. Instead of expressing the next theorem in terms of rank-one perturbations in the form

$$\lambda(E + euu^\star) + (A + auu^\star), \tag{4.2}$$

we rather consider the unperturbed pencil and the vector u separately, thus interpreting (4.2) as a pair $\big(P(\lambda),u\big)$, consisting of a structured pencil $P(\lambda)=\lambda E+A$ with $A,E\in\mathbb{F}^{n\times n}$ and a "perturbation vector" $u\in\mathbb{F}^n$. A congruence transformation on the pencil (4.2) will then take the form $\big(P(\lambda),u\big)\mapsto\big(U^\star P(\lambda)U,U^\star u\big)$, where $U\in\mathbb{F}^{n\times n}$ is nonsingular.

THEOREM 4.2. *Let $\mathbb{F}\in\{\mathbb{C},\mathbb{R}\}$, let $\star\in\{*,\top\}$, and let $P(\lambda)=\lambda E+A$ with $A,E\in\mathbb{F}^{n\times n}$ be such that $A^\star=\delta_A A$ and $E^\star=\delta_E E$ with $\delta_A,\delta_E\in\{+1,-1\}$. Furthermore, let $P(\lambda)$ have at least one singular block $\mathcal{L}_{2m+1}^{\delta_A,\delta_E}(\lambda)$, $m\geq 0$, in its canonical form. Then for a generic $u\in\mathbb{F}^n$, there exists a nonsingular matrix $U\in\mathbb{F}^{n\times n}$ (depending on u) such that*

$$U^\star P(\lambda)U=\begin{array}{c c} & \begin{array}{cc} n_s & n_r \end{array} \\ \begin{array}{c} n_s \\ n_r \end{array} & \begin{bmatrix} S(\lambda) & 0 \\ 0 & R(\lambda)\end{bmatrix}\end{array},\quad U^\star u=\begin{array}{c} n_s \\ n_r \end{array}\begin{bmatrix} u_s \\ 0\end{bmatrix}, \tag{4.3}$$

where $R(\lambda)$ and $S(\lambda)$ are the regular and singular parts of the pencil, respectively.

Proof. Without loss of generality we may assume that $P(\lambda)$ has the form discussed in Theorem 4.1. Furthermore, we may assume that it has one singular block $\mathcal{L}_{2m+1}^{\delta_A,\delta_E}(\lambda)$ only. Similarly, we can consider $R_f(\lambda)$ and $R_i(\lambda)$ from Theorem 4.1 separately, so we may assume without loss of generality that $R(\lambda)=\lambda R_E+R_A$, where $R_A,R_E\in\mathbb{F}^{k\times k}$ and where either R_A or R_E is invertible. By interchanging the roles of A and E, if necessary, we may further assume that it is R_E which is invertible. Note that interchanging A and E may change the actual structure of the pencil, e.g., in the case of an alternating pencil from even to odd, but this is not of importance. Indeed, if (4.3) is shown to hold for the pencil $\lambda A+E$, then it obviously also holds for the pencil $\lambda E+A$.

Let $u=[u_1^\top,u_2^\top,u_3^\top]^\top\in\mathbb{F}^n$, where $u_1\in\mathbb{F}^{m+1}$, $u_2\in\mathbb{F}^m$, $u_3\in\mathbb{F}^k$, and let $D\in\mathbb{F}^{k\times(m+1)}$ be a matrix satisfying

$$u_3=-Du_1 \tag{4.4}$$

with entries still to be determined. Then with

$$\mathcal{D}:=\begin{bmatrix} I_{m+1} & 0 & 0 \\ 0 & I_m & 0 \\ D & 0 & I_k\end{bmatrix}^\star,$$

we obtain that $\mathcal{D}^\star u=[u_1^\top,u_2^\top,0]^\top$, and we have that $\widetilde{A}:=\mathcal{D}^\star A\mathcal{D}$ and $\widetilde{E}:=\mathcal{D}^\star E\mathcal{D}$ take the forms

$$\widetilde{A}=\begin{bmatrix} 0 & G^{\mathrm{u}} & 0 \\ \delta_A(G^{\mathrm{u}})^\top & 0 & \delta_A(G^{\mathrm{u}})^\top D^\star \\ 0 & DG^{\mathrm{u}} & R_A\end{bmatrix},\quad \widetilde{E}=\begin{bmatrix} 0 & G^{\mathrm{d}} & 0 \\ \delta_E(G^{\mathrm{d}})^\top & 0 & \delta_E(G^{\mathrm{d}})^\top D^\star \\ 0 & DG^{\mathrm{d}} & R_E\end{bmatrix},$$

respectively, where $G^{\mathrm{u}}:=G_m^{\mathrm{u}}$ and $G^{\mathrm{d}}:=G_m^{\mathrm{d}}$ are as in (4.1). Thus, the transformed vector $\mathcal{D}^\star u$ has the desired form, but we have introduced unwanted nonzero entries in the $(2,3)$ and $(3,2)$ blocks of $\widetilde{A}$ and $\widetilde{E}$. We eliminate these blocks with the help of a transformation of the form

$$\mathcal{X}:=\begin{bmatrix} I_m & 0 & 0 \\ 0 & I_{m-1} & X \\ 0 & 0 & I_k\end{bmatrix}^\star,$$

where $X \in \mathbb{F}^{(m-1)\times k}$ still has to be determined. Note that $\mathcal{X}^\star\mathcal{D}^\star u = \mathcal{D}^\star u$ and

$$\begin{aligned}\widehat{A} &:= \mathcal{X}^\star \widetilde{A}\mathcal{X}\\ &= \begin{bmatrix} 0 & G^{\mathrm{u}} & 0 \\ \delta_A (G^{\mathrm{u}})^\top & XDG^{\mathrm{u}} + \delta_A (G^{\mathrm{u}})^\top D^* X^\star + XR_A X^\star & \delta_A (G^{\mathrm{u}})^\top D^\star + XR_A \\ 0 & DG^{\mathrm{u}} + R_A X^\star & R_A \end{bmatrix},\end{aligned}$$

as well as

$$\begin{aligned}\widehat{E} &:= \mathcal{X}^\star \widetilde{E}\mathcal{X}\\ &= \begin{bmatrix} 0 & G^{\mathrm{d}} & 0 \\ \delta_E (G^{\mathrm{d}})^\top & XDG^{\mathrm{d}} + \delta_E (G^{\mathrm{d}})^\top D^\star X^\star + XR_E X^\star & \delta_E (G^{\mathrm{d}})^\top D^\star + XR_E \\ 0 & DG^{\mathrm{d}} + R_E X^\star & R_E \end{bmatrix}.\end{aligned}$$

Suppose that X and D can be chosen such that

$$DG^{\mathrm{u}} + R_A X^\star = 0 \quad \text{and} \quad DG^{\mathrm{d}} + R_E X^\star = 0 \tag{4.5}$$

(we will show below that this is indeed possible). Then the $(3, 2)$ block-entries of $\widehat{A}$ and $\widehat{E}$ are zero and by (skew) symmetry this also holds for the $(2, 3)$ block-entry. In that case, we can define the transformation

$$\mathcal{Y} = Y \oplus I_k,$$

where Y is chosen in such a way that the block upper 2×2 block of $\mathcal{Y}^\star(\lambda\widehat{E}+\widehat{A})\mathcal{Y}$ is in the reduced form of Theorem 4.1. Then, due to (4.5), the pencil $\mathcal{Y}^\star\mathcal{X}^\star\mathcal{D}^\star P(\lambda)\mathcal{D}\mathcal{X}\mathcal{Y}$ is in the corresponding form of Theorem 4.1 as well. Hence,

$$Y^\star\left(\begin{bmatrix} 0 & G^{\mathrm{u}} \\ (G^{\mathrm{u}})^\top & * \end{bmatrix} + \lambda \begin{bmatrix} 0 & G^{\mathrm{d}} \\ (G^{\mathrm{d}})^\top & * \end{bmatrix}\right) Y = \begin{bmatrix} 0 & G^{\mathrm{u}} \\ (G^{\mathrm{u}})^\top & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & G^{\mathrm{d}} \\ (G^{\mathrm{d}})^\top & 0 \end{bmatrix},$$

because $\mathcal{Y}^\star(\lambda\widehat{E} + \widehat{A})\mathcal{Y}$ and $P(\lambda)$ are congruent and thus $\mathcal{Y}^\star(\lambda\widehat{E} + \widehat{A})\mathcal{Y}$ must contain exactly one singular block $\mathcal{L}^{\delta_A,\delta_E}_{2m+1}(\lambda)$. Thus, choosing $U = \mathcal{D}\mathcal{X}\mathcal{Y}$ finishes the proof under the assumptions (4.4) and (4.5).

It remains to show that X and D can be chosen such that both (4.4) and (4.5) are satisfied. Since we have assumed that R_E is invertible, the second equation in (4.5) can be solved for $X^\star$ which gives $X^\star = -R_E^{-1} DG^{\mathrm{d}}$. Inserting this into the first equation in (4.5), we obtain

$$DG^{\mathrm{u}} = -R_A X^\star = R_A R_E^{-1} DG^{\mathrm{d}}$$

or, equivalently,

$$R_E^{-1} DG^{\mathrm{u}} = R_E^{-1} R_A R_E^{-1} DG^{\mathrm{d}}. \tag{4.6}$$

Note that the matrix on the left-hand side of (4.6) consists of the first $m-1$ columns of $R_E^{-1}D$, while the matrix on the right-hand side of (4.6) consists of the last $m-1$ columns of $R_E^{-1}R_A R_E^{-1}D$. Setting $R_E^{-1}D = [d_1, \ldots, d_m]$, we thus obtain

$$R_E^{-1}D = \begin{bmatrix} (R_E^{-1}R_A)^{m-1} d_m & \ldots & R_E^{-1}R_A d_m & d_m \end{bmatrix}, \tag{4.7}$$

i.e., the matrix is uniquely determined by its last column d_m. Conversely, for every choice of d_m the matrix D defined by (4.7) satisfies (4.5). Thus, we aim to choose d_m in such a way that (4.4) is satisfied, which translates into the condition

$$R_E^{-1} u_3 = -R_E^{-1} D u_1.$$

Note that

$$-R_E^{-1}Du_1 = -Md_m,$$

where

$$M = \sum_{j=0}^{m-1} u_{1j}(R_E^{-1}R_A)^j,$$

and the u_{1j}, $j = 0, \ldots, m-1$, denote the components of u_1. Thus, if M is invertible then the choices $d_m = -M^{-1}R_E^{-1}u_3$,

$$D = R_E \left[\begin{array}{cccc} (R_E^{-1}R_A)^{m-1}d_m & \ldots & R_E^{-1}R_A d_m & d_m \end{array} \right],$$

and

$$X^\star = -R_E^{-1}DG^{\mathrm{d}}$$

satisfy both equations (4.4) and (4.5). Note that $\det M$ is a polynomial in the entries of u_1. Furthermore, for $u_{10} = 1$ and $u_{1j} = 0$, $j > 0$, we have $M = (R_E^{-1}R_1)^0 = I$ and, hence, $\det M$ is a nonzero polynomial in the entries of u_1. This shows that for a generic vector u the matrix M is invertible, which finishes the proof. ∎

As a direct corollary we obtain an invariance result for the regular part under structure-preserving generic rank-one perturbations.

COROLLARY 4.3. *Let* $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$, *let* $\star \in \{*, \top\}$, *and let* $P(\lambda) = \lambda E + A$ *with* $A, E \in \mathbb{F}^{n \times n}$ *be such that* $A^\star = \delta_A A$ *and* $E^\star = \delta_E E$ *with* $\delta_A, \delta_E \in \{+1, -1\}$. *Furthermore, let* $P(\lambda)$ *have at least one singular block* $\mathcal{L}_{2m+1}^{\delta_A,\delta_E}(\lambda)$, $m \geq 0$, *in its canonical form. Then for generic* $u \in \mathbb{F}^n$ *and for all* $(e, a) \in \mathbb{F}^2$, *the regular part of* $\lambda E + A$ *is contained in the regular part of the pencil* $\lambda(E + euu^\star) + (A + auu^\star)$ *in the following sense: There exist nonsingular matrices* $U_1, U_2 \in \mathbb{F}^{n \times n}$ *such that*

$$U_1^\star(\lambda E + A)U_1 = \left[\begin{array}{cc} R_1(\lambda) & 0 \\ 0 & S_1(\lambda) \end{array} \right],$$

$$U_2^\star(\lambda E + A + (\lambda e + a)uu^\star)U_2 = \left[\begin{array}{cc} R_2(\lambda) & 0 \\ 0 & S_2(\lambda) \end{array} \right],$$

and

$$R_2(\lambda) = \left[\begin{array}{cc} R_1(\lambda) & 0 \\ 0 & \widetilde{R}(\lambda) \end{array} \right]$$

for some $\widetilde{R}(\lambda)$, *where* $R_1(\lambda)$ *and* $R_2(\lambda)$ *are, respectively, the regular parts of* $\lambda E + A$ *and* $\lambda(E + euu^\star) + (A + auu^\star)$.

Note that for the sake of structure preservation, we may have to restrict the sets of possible values for a and e in Corollary 4.3 depending on the considered field and the corresponding involution $\star$, in order to guarantee that the perturbed matrix pencil is from the same set of structured pencils as the unperturbed one. For example, if $\mathbb{F} = \mathbb{C}$, $A^* = A$, and $E^* = E$, i.e., the pencil $\lambda E + A$ is Hermitian, then we must have $a, e \in \mathbb{R}$ to ensure that also $\lambda(E + euu^*) + (A + auu^*)$ is Hermitian. If, on the other hand, we have, e.g., $\mathbb{F} = \mathbb{R}$, $A^\top = -A$, and $E^\top = E$, i.e., the pencil $\lambda E + A$ is real $\top$-odd, then we must require that $a = 0$ and $e \in \mathbb{R}$. However, note that the statement of Corollary 4.3 remains true even in the case that a and e have been chosen such that the perturbed pencil does no longer have the same structure as the unperturbed pencil.

5. Perturbation of a single singular block. For the remainder of the paper, we will focus on Hermitian, complex symmetric, and real symmetric pencils. We will investigate what happens to a matrix pencil under a generic structure-preserving rank-one perturbation when the given pencil has exactly one singular block in the structured canonical form. As we have seen in the previous section that generically the regular part of the given singular pencil will not be affected by a structure-preserving rank-one perturbation, we can focus our attention on the singular part of the pencil only.

We start the analysis with the following lemma on the roots of a certain polynomial. Here and in the following, we use the notation $\bar{q}$ for the polynomial whose coefficients are the complex conjugates of the coefficients of the polynomial q, i.e., if $q(\lambda) = a_k\lambda^k + \cdots + a_1\lambda + a_0$, then

$$\bar{q}(\lambda) := \bar{a}_k\lambda^k + \cdots + \bar{a}_1\lambda + \bar{a}_0.$$

Lemma 5.1. *Let $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Then, for a generic vector $u \in \mathbb{F}^{k+1}$ the polynomial*

$$q(\lambda) = \lambda^k - \frac{u_2}{u_1}\lambda^{k-1} + \cdots + (-1)^k \frac{u_{k+1}}{u_1}\lambda^0 \tag{5.1}$$

has simple complex roots only. Furthermore, if $\mathbb{F} = \mathbb{C}$ and if $\mathbb{C}$ is interpreted as $\mathbb{R}^2$ then for generic $(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R}^{2(k+1)}$ and for each nonreal root λ_0, the conjugate $\bar{\lambda}_0$ is not a root of $q(\lambda)$. If $\mathbb{F} = \mathbb{R}$, then for all $u \in \mathbb{R}^k$ and for each nonreal root λ_0, the conjugate $\bar{\lambda}_0$ is a root of $q(\lambda)$.

Proof. For the proof let us recall that two polynomials p_1, p_2 have a common root if and only if their Sylvester resultant matrix $S(p_1, p_2)$ is singular; see, e.g., [12, Theorem 5.7].

Under the generic assumption that $u_1 \neq 0$, the roots of $q(\lambda)$ coincide with the roots of $u_1 q_1(\lambda) = u_1\lambda^k - u_2\lambda^{k-1} + \cdots + (-1)^k u_{k+1}\lambda^0$. To prove the first assertion, note that $\det S\big(u_1 q_1(\lambda), u_1 q_1'(\lambda)\big)$ is a nonzero polynomial in $u_1, \ldots, u_k$.

To see the "furthermore" part, note that $\big|\det S\big(u_1 q_1(\lambda), \bar{u}_1 \bar{q}_1(\lambda)\big)\big|^2$ is a real nonzero polynomial in the entries of $(\operatorname{Re} u, \operatorname{Im} u)$ in the case $\mathbb{F} = \mathbb{C}$. The proof for the case $\mathbb{F} = \mathbb{R}$ is elementary. ∎

Remark 5.2. Observe that, with respect to q, the set $\mathbb{R}^k$ can be divided into two disjoint subsets. The first set contains all vectors $u \in \mathbb{R}^k$ defined by the property that the polynomial q made of the components of u as in (5.1) has a real root, while the second subset consists of all $u \in \mathbb{R}^k$ such that q has only complex roots. Note that if k is even, then both sets have a nonempty interior while if k is odd, then the latter set is empty.

5.1. Hermitian case. In this section, we will assume that the given singular pencil is Hermitian.

Proposition 5.3. *Let $u \in \mathbb{C}^{2k+1}$ with $u_1 \neq 0$. If $k \geq 1$, then for $\tau \in \mathbb{R}$ we have*

$$\det\big(\mathcal{L}_{2k+1}(\lambda) + \tau uu^*\big) = \tau \cdot (q\bar{q})(\lambda),$$

where

$$q(\lambda) = \lambda^k - \frac{u_2}{u_1}\lambda^{k-1} + \cdots + (-1)^k \frac{u_{k+1}}{u_1}\lambda^0.$$

Furthermore, for $\tau \in \mathbb{R} \setminus \{0\}$, infinity is an eigenvalue of $\mathcal{L}_{2k+1}(\lambda) + \tau uu^*$ with the corresponding single block $\mathcal{J}^s_{1,\infty}(\lambda)$, where $s = \operatorname{sgn}\tau$. Also, if $k = 0$ then

$$\det\big(\mathcal{L}_{2k+1}(\lambda) + \tau uu^*\big) = \tau|u_1|^2.$$

Proof. Consider the matrix

$$\mathcal{S} := \begin{bmatrix} \frac{1}{u_1} & & & \\ -\frac{u_2}{u_1} & 1 & & \\ \vdots & & \ddots & \\ -\frac{u_{2k+1}}{u_1} & & & 1 \end{bmatrix}^*$$

such that $\mathcal{S}^* u = e_1$. Then

$$\mathcal{S}^*\big(\mathcal{L}_{2k+1}(\lambda) + \tau u u^*\big)\mathcal{S} =$$

$$\left[\begin{array}{ccccc|cccc} \tau & & & & & \frac{1}{u_1} & & & \\ & & & & & \lambda - \frac{u_2}{u_1} & 1 & & \\ & & & & & -\frac{u_3}{u_1} & \lambda & \ddots & \\ & & & & & \vdots & & \ddots & 1 \\ & & & & & -\frac{u_{k+1}}{u_1} & & & \lambda \\ \hline \frac{1}{\bar{u}_1} & \lambda - \frac{\bar{u}_2}{\bar{u}_1} & -\frac{\bar{u}_3}{\bar{u}_1} & \cdots & -\frac{\bar{u}_{k+1}}{\bar{u}_1} & -2\mathrm{Re}\,\frac{u_{k+2}}{u_1} & -\frac{\bar{u}_{k+3}}{\bar{u}_1} & \cdots & -\frac{\bar{u}_{2k+1}}{\bar{u}_1} \\ & 1 & \lambda & & & -\frac{u_{k+3}}{u_1} & & & \\ & & \ddots & \ddots & & \vdots & & & \\ & & & 1 & \lambda & -\frac{u_{2k+1}}{u_1} & & & \end{array}\right]$$

with the nonindicated entries being zeros. Eliminating the entries in the positions $(1, k+2)$ and $(k+2, 1)$ by adding a multiple of the first row and column, respectively, we obtain a congruent pencil

$$[\tau] \oplus \left[\begin{array}{cccc|cccc} & & & & \lambda - \frac{u_2}{u_1} & 1 & & \\ & & & & -\frac{u_3}{u_1} & \lambda & \ddots & \\ & & & & \vdots & & \ddots & 1 \\ & & & & -\frac{u_{k+1}}{u_1} & & & \lambda \\ \hline \lambda - \frac{\bar{u}_2}{\bar{u}_1} & -\frac{\bar{u}_3}{\bar{u}_1} & \cdots & -\frac{\bar{u}_k}{\bar{u}_1} & f(u) & -\frac{\bar{u}_{k+3}}{\bar{u}_1} & \cdots & -\frac{\bar{u}_{2k+1}}{\bar{u}_1} \\ 1 & \lambda & & & -\frac{u_{k+3}}{u_1} & & & \\ & \ddots & \ddots & & \vdots & & & \\ & & 1 & \lambda & -\frac{u_{2k+1}}{u_1} & & & \end{array}\right], \tag{5.2}$$

where

$$f(u) = -2\mathrm{Re}\,\frac{u_{k+2}}{u_1} - \frac{1}{\tau |u_1|^2}.$$

As the determinants of the upper-right and lower-left block are equal to $q(\lambda)$ and $\bar{q}(\lambda)$, respectively, the proof is finished. ◻

COROLLARY 5.4. *For every monic complex polynomial q of degree $k \geq 1$ there exists a vector $u \in \mathbb{C}^{2k+1}$ with $u_1 \neq 0$ such that the characteristic polynomial of the pencil $\mathcal{L}_{2k+1}(\lambda) + uu^*$ equals $(q\bar{q})(\lambda)$.*

We now present the main theorem for generic Hermitian rank-one perturbations of Hermitian pencils having exactly one singular block.

THEOREM 5.5. *Let $A, E \in \mathbb{C}^{n\times n}$ be Hermitian and such that the pencil $\lambda E+A$ has precisely one singular block in its canonical form, say $\mathcal{L}_{2k+1}(\lambda)$, $k \geq 0$. Furthermore, let $(a,e) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then for $u \in \mathbb{C}^n$, with $(\operatorname{Re} u, \operatorname{Im} u) \in \mathbb{R}^{2n}$ being generic, the canonical form of $\lambda E + A + \tau(\lambda e + a)uu^*$ is constant for $\tau \in \mathbb{R}\setminus\{0\}$ and equals*

$$\mathcal{J}^s_{1,-\frac{a}{e}}(\lambda) \oplus \left(\bigoplus_{j=1}^{k} \mathcal{J}_{1,1,\lambda_j}(\lambda)\right) \oplus R_f(\lambda) \oplus R_i(\lambda), \tag{5.3}$$

where $\lambda_1, \dots \lambda_k \in \mathbb{C}^+$ are mutually distinct eigenvalues, and $R_f(\lambda) \oplus R_i(\lambda)$ is the (canonical form of the) regular part of $\lambda E + A$, and where

$$s = \begin{cases} \operatorname{sgn}\tau \cdot \operatorname{sgn} e & \text{if } e \neq 0, \\ \operatorname{sgn}\tau \cdot \operatorname{sgn} a & \text{if } e = 0. \end{cases}$$

In the case $k = 0$, the form (5.3) reduces to $\mathcal{J}^s_{1,-\frac{a}{e}}(\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda)$.

Proof. First we consider the case $a = 1$, $e = 0$. By Theorem 4.2 we may assume without loss of generality that we have $\lambda E + A = \mathcal{L}_{2k+1}(\lambda)$. By Proposition 5.3, the statement is trivial for the case $k = 0$ and for $k > 0$ the finite eigenvalues of $\lambda E+A+\tau uu^*$ are the roots of $q(\lambda)$ and $\bar{q}(\lambda)$. By Lemma 5.1 all roots of the polynomial q are generically simple and different from their conjugates, which determines the canonical form to be as in (5.3).

To prove the general case we use Möbius transformations. Let $b, c \in \mathbb{R}$ be such that

$$M_{b,c}(a + \lambda e)uu^* = uu^*,$$

i.e., $M_{b,c} = M^{-1}_{a,e}$. Then we obtain

$$M_{b,c}\big(\lambda E + A + \tau(\lambda e + a)uu^*\big) = M_{b,c}(\lambda E + A) + \tau uu^*.$$

By Theorem 3.3 the pencil $M_{b,c}(\lambda E + A)$ has precisely one singular block $\mathcal{L}_{2k+1}(\lambda)$ in its canonical form. Hence, by the first part of the proof, the canonical form of $M_{b,c}(\lambda E + A) + \tau uu^*$ equals

$$\mathcal{J}^{\tilde{s}}_{1,\infty}(\lambda) \oplus \bigoplus_{j=1}^{k} \mathcal{J}_{1,1,\mu_j}(\lambda) \oplus M_{b,c}\big(R_f(\lambda) \oplus R_i(\lambda)\big)$$

with $\tilde{s} = \operatorname{sgn}\tau$ and some $\mu_1, \dots, \mu_k \in \mathbb{C}^+$. Applying the transformation $M_{a,e}$ we get by Theorem 3.3 that the canonical form of $\lambda E + A + \tau(a + \lambda e)uu^*$ equals

$$\mathcal{J}^{s}_{1,\lambda_0}(\lambda) \oplus \bigoplus_{j=1}^{k} \mathcal{J}_{1,1,\lambda_j}(\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda)$$

with $\lambda_0 = M_{a,e}(\infty) = -\frac{a}{e}$, $\lambda_j = M_{a,e}(\mu_j) \in \mathbb{C}^+$ for $j = 1, \dots, k$, and $s = \operatorname{sgn}\tau \cdot \operatorname{sgn} e$ if $e \neq 0$, and $s = \operatorname{sgn}\tau \cdot \operatorname{sgn} a$ if $e = 0$. ∎

5.2. Real symmetric case. In this section, we assume that the given pencil $\lambda E + A$ is real symmetric. By taking the vector u in the statement of Proposition 5.3 to be real, we get the following result.

PROPOSITION 5.6. *Let $u \in \mathbb{R}^{2k+1}$ with $u_1 \neq 0$. If $k \geq 1$, then for $\tau \in \mathbb{R}$ we have*

$$\det\left(\mathcal{L}_{2k+1}(\lambda) + \tau u u^\top\right) = \tau q^2(\lambda),$$

where

$$q(\lambda) = \lambda^k - \frac{u_2}{u_1}\lambda^{k-1} + \cdots + (-1)^k \frac{u_{k+1}}{u_1}\lambda^0.$$

Furthermore, for $\tau \in \mathbb{R} \setminus \{0\}$, we have that ∞ is an eigenvalue of $\mathcal{L}_{2k+1}(\lambda) + \tau u u^\top$ with the corresponding single block $\mathcal{J}^s_{1,\infty}(\lambda)$, where $s = \operatorname{sgn}\tau$.

Moreover, if $k = 0$ then

$$\det\left(\mathcal{L}_{2k+1}(\lambda) + \tau u u^\top\right) = \tau u_1^2.$$

Note that by Proposition 5.6 every root of q is precisely a double eigenvalue of $\mathcal{L}_{2k+1}(\lambda) + uu^\top$ and, in addition for each nonreal root of q there is a corresponding complex conjugate root; see Lemma 5.1 and Remark 5.2.

We will study now the generic canonical structure corresponding to these real and nonreal double eigenvalues. For this we introduce the following lemma.

LEMMA 5.7. *Let $\lambda_0 \in \mathbb{R}$ be an eigenvalue of algebraic multiplicity two of the real regular symmetric matrix pencil*

$$P(\lambda) := \begin{bmatrix} 0 & \lambda C + B \\ \lambda C^\top + B^\top & \lambda F + D \end{bmatrix},$$

where $B, C, D, F \in \mathbb{R}^{n\times n}$ and $D = D^\top$, $F = F^\top$. Then exactly one of the following statements holds:

(i) *The geometric multiplicity of λ_0 is equal to one and the sign s of λ_0 in the sign characteristic of $P(\lambda)$ is given by*

$$s = \begin{cases} +1 & \text{if } \operatorname{ind}\big(P(\lambda_0)\big) = (n, n-1, 1), \\ -1 & \text{if } \operatorname{ind}\big(P(\lambda_0)\big) = (n-1, n, 1). \end{cases}$$

(ii) *The geometric multiplicity of λ_0 is equal to two and the signs in the sign characteristic of λ_0 are $+1$ and -1.*

Proof. For the first part of the proof, we will ignore that the pencil $P(\lambda)$ is real and consider it as a particular complex Hermitian pencil. Clearly, there exist unitary matrices $U, V \in \mathbb{C}^{n\times n}$ such that $U^*(\lambda C + B)V$ is in antitriangular form. (This follows by computing the generalized Schur decomposition of $\lambda C + B$ and then applying a row permutation.) Thus, $\operatorname{diag}(U^*, V^*)P(\lambda)\operatorname{diag}(U, V)$ is in antitriangular form as well. Then it follows from [17, Theorem 15 and Remark 16] that for each real eigenvalue γ of $P(\lambda)$ in the (complex) canonical form, the number of blocks associated with γ and having odd size is even, say $2m$, and exactly m of them have sign $+1$ in the sign characteristic (and the other m have the sign -1 in the sign characteristic). To be more precise, if

$$\mathcal{J}^{s_1}_{k_1,\gamma}(\lambda), \dots, \mathcal{J}^{s_{2m}}_{k_{2m},\gamma}(\lambda), \mathcal{J}^{s_{2m+1}}_{k_{2m+1},\gamma}(\lambda), \dots, \mathcal{J}^{s_{2m+l}}_{k_{2m+l},\gamma}(\lambda)$$

are the blocks associated with γ in the canonical form in Theorem 2.1, where the indices $k_1, \dots, k_{2m}$ are odd and $k_{2m+1}, \dots, k_{2m+l}$ are even, then exactly m of the signs $s_1, \ldots, s_{2m}$ are equal to $+1$ and exactly m of them are equal to -1. In the following we will call this condition the "*sign restriction for* γ."

If we evaluate the pencil $\mathcal{J}^{s_j}_{k_j,\gamma}(\lambda)$ at $\lambda = \mu \in \mathbb{R} \setminus \{\gamma\}$, then by Lemma 2.4 the resulting matrix $\mathcal{J}^{s_j}_{k_j,\gamma}(\mu)$ will have the inertia index

$$\begin{cases} \left(\frac{k_j}{2}, \frac{k_j}{2}, 0\right) & \text{if } k_j \text{ is even,} \\ \left(\frac{k_j+1}{2}, \frac{k_j-1}{2}, 0\right) & \text{if } k_j \text{ is odd and } s_j(\gamma - \mu) > 0, \\ \left(\frac{k_j-1}{2}, \frac{k_j+1}{2}, 0\right) & \text{if } k_j \text{ is odd and } s_j(\gamma - \mu) < 0. \end{cases}$$

Thus, it follows that the sign restriction for γ has the effect that the inertia index of the matrix

$$\mathcal{J}_\gamma(\mu) := \bigoplus_{j=1}^{2m+l} \mathcal{J}^{s_j}_{k_j,\gamma}(\mu) \tag{5.4}$$

is equal to $\left(\frac{a_\gamma}{2}, \frac{a_\gamma}{2}, 0\right)$ for all $\mu \in \mathbb{C} \setminus \{\gamma\}$, where a_γ denotes the algebraic multiplicity of the eigenvalue γ. A similar observation shows that we also have

$$\operatorname{ind}\bigl(\mathcal{J}_\infty(\mu)\bigr) = \left(\tfrac{a_\infty}{2}, \tfrac{a_\infty}{2}, 0\right),$$

where a_∞ is the algebraic multiplicity of infinity as the eigenvalue of $P(\lambda)$, and where $\mathcal{J}_\infty(\mu)$ is defined analogously to (5.4). Thus, if $\mathcal{J}_\infty(\lambda)$ is the matrix pencil consisting of all blocks associated with the eigenvalue infinity of the pencil $P(\lambda)$, then $\mathcal{J}_\infty(\mu)$ is the matrix obtained by evaluating $\mathcal{J}_\infty(\lambda)$ at $\lambda = \mu$.

For the remainder of the proof, let us consider the real canonical form $S^\top P(\lambda) S$ of $P(\lambda)$ as in Theorem 2.2. Although we have been arguing via the complex canonical form so far, note that the blocks associated with real eigenvalues or with the eigenvalue infinity coincide in the real and complex canonical form. Thus, by our considerations above and observing that each block of the form $\mathcal{J}_{k,k,\alpha,\beta}(\lambda)$ as in (2.6) has inertia index $(k, k, 0)$ for all $\lambda \in \mathbb{R}$, we see that the matrix $P(\lambda_0)$ obtained by evaluating the pencil $P(\lambda)$ at $\lambda = \lambda_0$ has the inertia index

$$\operatorname{ind}\bigl(P(\lambda_0)\bigr) = (n-1, n-1, 0) + \operatorname{ind}\bigl(\mathcal{J}_{\lambda_0}(\lambda_0)\bigr).$$

Here the sum of inertia indices is taken componentwise. By assumption, the algebraic multiplicity of λ_0 as an eigenvalue of $P(\lambda)$ is equal to two and thus its geometric multiplicity is either one or two. If it is one, then

$$\mathcal{J}_{\lambda_0}(\lambda) = \mathcal{J}^{s}_{2,\lambda_0}(\lambda) = s \begin{bmatrix} 0 & \lambda - \lambda_0 \\ \lambda - \lambda_0 & 1 \end{bmatrix}$$

for some sign $s \in \{+1, -1\}$ and evaluating this at $\lambda = \lambda_0$ it immediately follows that

$$\operatorname{ind}\bigl(\mathcal{J}_{\lambda_0}(\lambda_0)\bigr) = \left(\tfrac{1+s}{2}, \tfrac{1-s}{2}, 1\right) = \begin{cases} (1,0,1) & \text{if } s = 1, \\ (0,1,1) & \text{if } s = -1 \end{cases}$$

which proves (i).

If, on the other hand, the geometric multiplicity of λ_0 is two, then we have $\mathcal{J}_{\lambda_0}(\lambda) = \mathcal{J}^{s_1}_{1,\lambda_0}(\lambda) \oplus \mathcal{J}^{s_2}_{1,\lambda_0}(\lambda)$ for some signs $s_1, s_2 \in \{+1, -1\}$. Since we have two blocks of odd size one in this case, the sign restriction for λ_0 requires that the signs s_1, s_2 are opposite which proves (ii). ∎

Using these lemmas we obtain the following theorem.

THEOREM 5.8. *Let $A, E \in \mathbb{R}^{n\times n}$ be symmetric and such that the pencil $\lambda E+A$ has precisely one singular block in its canonical form, say $\mathcal{L}_{2k+1}(\lambda), k \geq 0$. Furthermore, let $(a,e) \in \mathbb{R}^2 \setminus \{(0,0\}$. Then, for generic $u \in \mathbb{R}^n$, the canonical form of the pencil $\lambda E + A + \tau(\lambda e + a)uu^\top$ is*

$$\mathcal{J}^s_{1,-\frac{a}{e}}(\lambda) \oplus \bigoplus_{j=1}^{k'} \mathcal{J}_{2,2,\alpha_j,\beta_j}(\lambda) \oplus \bigoplus_{j=1}^{k''} \mathcal{S}_{2,\lambda_j}(\tau,\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda),$$

where

(1) $s = \operatorname{sgn}\tau \cdot \operatorname{sgn} e$;
(2) $4k' + 2k'' = 2k$;
(3) $\alpha_j + i\beta_j$, $j = 1,\dots,k'$ *are mutually distinct nonreal eigenvalues with* $\alpha_j, \beta_j \in \mathbb{R}$ *and* $\beta_j > 0$;
(4) λ_j, $j = 1,\dots,k''$ *are mutually distinct real eigenvalues different from* $-\frac{a}{e}$ *and of algebraic multiplicity two. The corresponding blocks* $\mathcal{S}_{2,\lambda_j}(\tau,\lambda)$ *have the form*

$$\mathcal{S}_{2,\lambda_j}(\tau,\lambda) = \begin{cases} \mathcal{J}^{s_j}_{2,\lambda_j}(\lambda) & \text{if } \tau^{-1} < \tau_j, \\ \mathcal{J}^{1}_{1,\lambda_j}(\lambda) \oplus \mathcal{J}^{-1}_{1,\lambda_j}(\lambda) & \text{if } \tau^{-1} = \tau_j, \\ \mathcal{J}^{-s_j}_{2,\lambda_j}(\lambda) & \text{if } \tau^{-1} > \tau_j, \end{cases}$$

where $\tau_1,\dots,\tau_{k''} \in \mathbb{R}$ *depend on* a, e, u *and where*

$$s_j = \begin{cases} \operatorname{sgn}\big(e\frac{e\lambda_j+a}{e-a\lambda_j} + a\big) & \text{if } n \text{ is even}, \\ \operatorname{sgn}(e\lambda_j + a) & \text{if } n \text{ is odd} \end{cases}$$

for $j = 1,\dots,k''$ *when* $e \neq 0$, *and* $s_1 = \cdots = s_{k''} = \operatorname{sgn} a$ *when* $e = 0$;
(5) $R_f(\lambda) \oplus R_i(\lambda)$ *is (the canonical form of) the regular part of* $\lambda E + A$.

In the case $k = 0$, *the canonical form reduces to* $\mathcal{J}^s_{1,-\frac{a}{e}}(\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda)$.

Proof. Consider first the case $a = 1$, $e = 0$. By Theorem 4.2 we may assume without loss of generality that

$$\lambda E + A = \mathcal{L}_{2k+1}(\lambda).$$

The case $k = 0$ is trivial, so we may assume $k > 0$ and we will continue the congruence transformations from the proof of Proposition 5.3 starting from (5.2). Note that all transformations appearing therein were real for $u \in \mathbb{R}^{2k+1}$. We may simply assume first that $u_1 = 1$, which gives that the second diagonal block of the pencil in (5.2) has the form

$$R(\lambda,\tau) := \begin{bmatrix} 0 & \lambda I - A_1 \\ \lambda I - A_1^\top & B(\tau) \end{bmatrix}$$

$$:= \left[\begin{array}{cccc|cccc} & & & & \lambda - u_2 & 1 & & \\ & & & & -u_3 & \lambda & \ddots & \\ & & & & \vdots & & \ddots & 1 \\ & & & & -u_{k+1} & & & \lambda \\ \hline \lambda - u_2 & -u_3 & \cdots & -u_{k+1} & f(\tau,u) & -u_{k+3} & \dots & -u_{2k+1} \\ 1 & \lambda & & & -u_{k+3} & & & \\ & \ddots & \ddots & & \vdots & & & \\ & & 1 & \lambda & -u_{2k+1} & & & \end{array}\right],$$

where

$$f(\tau, u) = -2u_{k+2} - \frac{1}{\tau}$$

is the only entry depending on τ.

For the case $k = 1$, we make the genericity assumption that $u_3 \neq 0$ and then

$$\begin{bmatrix} 0 & \lambda I - A_1 \\ \lambda I - A_1^\top & B(\tau) \end{bmatrix} = \begin{bmatrix} 0 & \lambda - u_2 \\ \lambda - u_2 & -2u_3 - \tau^{-1} \end{bmatrix},$$

which has the canonical form

$$\begin{cases} \mathcal{J}^1_{2,u_2}(\lambda) & \text{for } \tau^{-1} < -2u_3, \\ \mathcal{J}^1_{1,u_2}(\lambda) \oplus \mathcal{J}^{-1}_{1,u_2}(\lambda) & \text{for } \tau^{-1} = -2u_3, \\ \mathcal{J}^{-1}_{2,u_2}(\lambda) & \text{for } \tau^{-1} > -2u_3 \end{cases}$$

by Lemma 5.7. Now let us deal with the case $k \geq 2$. Let λ_0 be an eigenvalue of A_1. (For the moment, we do not fix this eigenvalue to be real or nonreal, but if it is nonreal, then we may assume that $\operatorname{Im} \lambda_0 > 0$ by otherwise switching to its conjugate which must also be an eigenvalue of $P(\lambda)$.) Clearly, λ_0 is then also an eigenvalue of $A_1^\top$. Furthermore, we make the genericity assumption that q has simple roots only (see Lemma 5.1) and we add the genericity assumption that $u_{k+1} \neq 0$. Hence, λ_0 is a simple, nonzero eigenvalue of both A_1 and $A_1^\top$. Let us further transform the matrix

$$R(\lambda_0, \tau) = \begin{bmatrix} 0 & \lambda_0 I - A_1 \\ \lambda_0 I - A_1^\top & B(\tau) \end{bmatrix}. \tag{5.5}$$

As λ_0 is an eigenvalue of A_1, there exists a nonsingular matrix $T_1 \in \mathbb{C}^{k\times k}$ such that

$$(\lambda_0 I - A_1)T_1 = \begin{bmatrix} 0 & 1 & & \\ 0 & \lambda_0 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & & & \lambda_0 \end{bmatrix} =: A_2.$$

Indeed, let

$$T_1 = \begin{bmatrix} 1 & & & \\ x_3 & 1 & & \\ \vdots & & \ddots & \\ x_{k+1} & & & 1 \end{bmatrix}$$

with

$$\begin{aligned} x_{k+1} &= \frac{u_{k+1}}{\lambda_0}, \\ x_{j-1} &= \frac{u_{j-1}}{\lambda_0} - \frac{x_j}{\lambda_0}, \quad j = k+1, \dots, 4, \end{aligned}$$

so that the entries $2, \dots, k$ of the first column of $(\lambda_0 I - A_1)T_1$ are eliminated. Furthermore,

$$x_3 = \frac{u_3}{\lambda_0} - \frac{u_4}{\lambda_0^2} + \cdots + (-1)^k \frac{u_{k+1}}{\lambda_0^{k-1}},$$

and hence, with $q(\lambda)$ being the characteristic polynomial of the matrix A_1, we have

$$\lambda_0 - u_2 + x_3 = \lambda_0^{-k+1} q(\lambda_0) = 0,$$

which gives a zero in the $(1,1)$-entry of $(\lambda_0 I - A_1)T$.

Furthermore, we have

$$T_2 A_2 = \begin{bmatrix} 0 & 1 & & \\ 0 & & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} =: A_3, \quad \text{where } T_2 = \begin{bmatrix} 1 & & & \\ -\lambda_0 & \ddots & & \\ \vdots & \ddots & \ddots & \\ (-1)^k \lambda_0^k & \dots & -\lambda_0 & 1 \end{bmatrix}.$$

Setting $\mathcal{T} := \operatorname{diag}(T_2^\top, T_1)$, we obtain

$$\begin{aligned} \mathcal{T}^\top \begin{bmatrix} 0 & \lambda_0 I - A_1 \\ \lambda_0 I - A_1^\top & B(\tau) \end{bmatrix} \mathcal{T} &= \begin{bmatrix} 0 & T_2(\lambda_0 I - A_1)T_1 \\ T_1^\top(\lambda_0 I - A_1^\top)T_2^\top & T_1^\top B(\tau) T_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & A_3 \\ A_3^\top & T_1^\top B(\tau) T_1 \end{bmatrix}. \end{aligned}$$

Observe that

$$T_1^\top B(\tau) T_1 = \begin{bmatrix} g(\tau,u) & -u_{k+3} & \dots & -u_{2k+1} \\ -u_{k+3} & & & \\ \vdots & & & \\ -u_{2k+1} & & & \end{bmatrix}$$

with

$$g(\tau,u) = -2u_{k+2} - \frac{1}{\tau} - 2u_{k+3}x_3 - \cdots - 2u_{2k+1}x_{k+1}. \tag{5.6}$$

Let

$$\mathcal{S}^\top := \left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline 0 & \cdots & \cdots & 0 & 1 & & & \\ u_{k+3} & & & \vdots & & \ddots & & \\ & \ddots & & \vdots & & & \ddots & \\ & & u_{2k+1} & 0 & & & & 1 \end{array}\right],$$

then

$$\mathcal{S}^\top \mathcal{T}^\top \begin{bmatrix} 0 & \lambda_0 I - A_1 \\ \lambda_0 I - A_1^\top & B(\tau) \end{bmatrix} \mathcal{T}\mathcal{S} = \begin{bmatrix} & & & I_{k-1} \\ & 0 & 0 & \\ & 0 & g(\tau,u) & \\ I_{k-1} & & & \end{bmatrix}, \tag{5.7}$$

where the indicated zeros in the matrix on the right-hand side are scalar entries.

Consider first the case that λ_0 is nonreal. Recall that we work under the genericity assumption $u_1, u_{k+1} \neq 0$. Observe that $x_3, \dots, x_{k+1}$ depend polynomially on $u_1, \dots, u_{n+1}$ and that x_{k+1} is nonreal. Hence, for generic u the expression $g(1,u)$ is nonreal which implies that the expression $g(\tau,u)$ is nonzero for all $\tau \in \mathbb{R} \setminus \{0\}$. Thus, for generic u we have

$$\dim\ker R(\lambda_0,\tau) = 1, \quad \tau \in \mathbb{R} \setminus \{0\},$$

i.e., λ_0 is an eigenvalue of $\mathcal{L}(\lambda_0)+\tau uu^*$ of geometric multiplicity one for all $\tau \in \mathbb{R}\setminus\{0\}$. Hence, if we set $\lambda_0 = \alpha + i\beta$, then there is a block $\mathcal{J}_{2,2,\alpha,\beta}(\lambda)$ in the real canonical form of the pencil $\mathcal{L}(\lambda_0) + \tau uu^*$ for all $\tau \in \mathbb{R} \setminus \{0\}$.

Next, consider the case that λ_0 is real. Then for generic u we have that

$$\tau_0 := -2u_{k+2} - 2u_{k+3}x_3 - \cdots - 2u_{2k+1}x_{k+1} \neq 0. \tag{5.8}$$

Indeed, τ_0 depends polynomially on the entries of u and is nonzero as a polynomial, because for $u_{k+2} \neq 0$ and $u_j = 0$, $j = k+3, \ldots, 2k+1$, we have $\tau_0 \neq 0$. From (5.7) we see that for $\tau^{-1} \neq \tau_0$ there exists a Jordan chain of length two corresponding to the zero eigenvalue of the matrix $R(\lambda_0, \tau)$ and, thus, one block of size two $\mathcal{J}_{2,\lambda_0}^{s(\tau)}(\lambda)$ corresponding to λ_0 occurs in the canonical form of $R(\lambda, \tau)$. By Lemma 5.7 applied to the pencil $R(\lambda, \tau)$, in view of (5.7), we have that the sign $s(\tau)$ is equal to the sign of $g(\tau, u)$ which clearly is

$$s(\tau) = \begin{cases} +1 & \text{if } g(\tau, u) > 0, \quad \text{i.e., } \tau^{-1} < \tau_0, \\ -1 & \text{if } g(\tau, u) < 0, \quad \text{i.e., } \tau^{-1} > \tau_0. \end{cases}$$

Finally, $\dim\ker R(\lambda_0, \tau_0^{-1}) = 2$ and therefore, taking into account Lemma 5.7 again, there are two blocks $\mathcal{J}_{1,\lambda_0}^{1}(\lambda)$ and $\mathcal{J}_{1,\lambda_0}^{-1}(\lambda)$ in the canonical form of $P(\lambda)$. This finishes the case $a = 1$ and $e = 0$.

The case of general $a, e \in \mathbb{R}$ with $e \neq 0$ then follows from the case $a = 1$, $e = 0$ as in the corresponding part of the proof of Theorem 5.5, with an additional remark that by Theorem 3.3 we have $M_{a,e}\mathcal{J}_{2,\lambda_j}^{s_j}(\lambda) = \mathcal{J}_{2,M_{a,e}(\lambda_j)}^{\tilde{s}_j}(\lambda)$, where for $j = 1, \ldots, k''$ we have

$$\tilde{s}_j = s_j \cdot \begin{cases} \operatorname{sgn}\big(eM_{a,e}(\lambda_j) + a\big) & \text{if } n \text{ is even,} \\ \operatorname{sgn}(e\lambda_j + a) & \text{if } n \text{ is odd} \end{cases}$$

if $e \neq 0$, and $\tilde{s}_j = s_j \cdot \operatorname{sgn}(a)$ if $e = 0$. $\square$

5.3. Complex symmetric case. Finally, we consider a complex symmetric rank-one perturbation of a complex symmetric pencil, $\lambda E + A + \tau(\lambda e + a)uu^\top$, where $u \in \mathbb{C}^n$ is a generic vector and the parameter τ is complex. Similarly to Proposition 5.3 we obtain the following result.

PROPOSITION 5.9. *Let $u \in \mathbb{C}^{2k+1}$ with $u_1 \neq 0$. If $k \geq 1$, then for $\tau \in \mathbb{C}$ we have*

$$\det\left(\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top\right) = \tau q^2(\lambda),$$

where

$$q(\lambda) = \lambda^k - \frac{u_2}{u_1}\lambda^{k-1} + \cdots + (-1)^k \frac{u_{k+1}}{u_1}\lambda^0.$$

Furthermore, for $\tau \in \mathbb{C}\setminus\{0\}$ we have that ∞ is a simple eigenvalue of $\mathcal{L}_{2k+1}(\lambda)+\tau uu^\top$. Moreover, if $k = 0$ then

$$\det(\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top) = \tau u_1^2.$$

Proposition 5.9 states that each finite eigenvalue of $\mathcal{L}_{2k+1}(\lambda) + \tau uu^\top$ is a double eigenvalue. Now let us investigate the block structure for these double eigenvalues.

THEOREM 5.10. *Let $A, E \in \mathbb{C}^n$ be symmetric and such that the pencil $\lambda E + A$ has precisely one singular block in its canonical form, say $\mathcal{L}_{2k+1}(\lambda)$, $k \geq 0$. Furthermore,* *let* $(a,e) \in \mathbb{C}^2 \setminus \{(0,0\}$*. Then for generic* $u \in \mathbb{C}^n$ *the canonical form of the pencil* $\lambda E + A + \tau(\lambda e + a)uu^\top$ $(\tau \in \mathbb{C})$ *is*

$$\mathcal{J}_{1,-\frac{a}{e}}(\lambda) \oplus \bigoplus_{j=1}^{k} \mathcal{S}_{2,\lambda_j}(\tau,\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda),$$

where

(1) $\lambda_j \in \mathbb{C} \cup \{\infty\}$*,* $j = 1,\ldots,k'$*, are mutually distinct complex eigenvalues of algebraic multiplicity two and the corresponding blocks have the form*

$$\mathcal{S}_{2,\lambda_j}(\tau,\lambda) = \begin{cases} \mathcal{J}_{2,\lambda_j}(\lambda) & \text{if } \tau^{-1} \neq \tau_j, \\ \mathcal{J}_{1,\lambda_j}(\lambda) \oplus \mathcal{J}_{1,\lambda_j}(\lambda) & \text{if } \tau^{-1} = \tau_j, \end{cases}$$

where $\tau_1,\ldots,\tau_k$ *are some complex numbers depending on* u*;*

(2) $R_f(\lambda) \oplus R_i(\lambda)$ *is (the canonical form of) the regular part of* $\lambda E + A$*.*

In the case $k = 0$*, the canonical form reduces to* $\mathcal{J}_{1,-\frac{a}{e}}(\lambda) \oplus R_f(\lambda) \oplus R_i(\lambda)$*.*

Proof. First let us consider the case $a = 1$, $e = 0$. We repeat the congruence transformations from the proof of Theorem 5.8 to get (5.7) and (5.6), having in mind that now they are complex congruences. Note that the same argument as in the real case applies to prove that $\tau_0 \neq 0$, where τ_0 is defined as in (5.8). The remainder of the proof, together with the general case, follows the same lines as the proof of Theorem 5.8 with the simplification that now no sign characteristic is involved. ◻

Conclusions. We have analyzed the behavior of the structured Kronecker canonical form of singular structured matrix pencils under generic structure-preserving rank-one perturbations. In this case the regular Kronecker structure of the pencil is not affected by the perturbation and the behavior of the canonical form of the newly generated regular blocks can be characterized when the perturbations involve a scalar parameter.

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