Stability radii for real linear Hamiltonian systems with perturbed dissipation

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Abstract

We study linear dissipative Hamiltonian (DH) systems with real constant coefficients that arise in energy based modeling of dynamical systems. In this paper we analyze when such a system is on the boundary of the region of asymptotic stability, i.e., when it has purely imaginary eigenvalues, or how much the dissipation term has to be perturbed to be on this boundary. For unstructured systems the explicit construction of the *real distance to instability (real stability radius)* has been a challenging problem. In this paper, we analyze this real distance under different structured perturbations to the dissipation term that preserve the DH structure and we derive explicit formulas for this distance in terms of low rank perturbations. We also show (via numerical examples) that under real structured perturbations to the dissipation the asymptotical stability of a DH system is much more robust than for unstructured perturbations.

Dissipative Hamiltonian system, port-Hamiltonian system, real distance to instability, real structured distance to instability, restricted real distance to instability.
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1   Introduction

In this paper we study linear time-invariant systems with real coefficients. When a physical system is energy preserving, then a mathematical model should reflect this property and this is characterized by the property of the system being *Hamiltonian*. If energy dissipates from the system then it becomes a *dissipative Hamiltonian (DH) system*, which in the linear time-invariant case can be expressed as

$$\dot{x} = (J - R)\,Qx, \tag{1.1}$$

where the function $x \mapsto x^T Q x$, with $Q = Q^T \in \mathbb{R}^{n,n}$ positive definite, describes the energy of the system, $J = -J^T \in \mathbb{R}^{n,n}$ is the *structure matrix* that describes the energy flux among energy storage elements, and $R \in \mathbb{R}^{n,n}$ with $R = R^T \geq 0$ is the *dissipation matrix* that describes energy dissipation in the system.

Dissipative Hamiltonian systems are special cases of port-Hamiltonian systems, which recently have received at lot attention in energy based modeling, see, e.g., [3, 8, 19, 21, 22,

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26, 25, 27, 28, 29]. An important property of DH systems is that they are stable, which in the time-invariant case can be characterized by the property that all eigenvalues of $A = (J - R)Q$ are contained in the closed left half complex plane and all eigenvalues on the imaginary axis are semisimple, see, e.g. [20] for a simple proof which immediately carries over to the real case discussed here.

For general unstructured systems $\dot{x} = Ax$, if A has purely imaginary eigenvalues, then arbitrarily small perturbations (arising e.g. from linearization errors, discretization errors, or data uncertainties) may move eigenvalues into the right half plane and thus make the system unstable. So stability of the system can only be guaranteed when the system has a reasonable *distance to instability*, see [2, 5, 11, 12, 14, 31], and the discussion below.

When the coefficients of the system are real and it is clear that the perturbations are real as well, then the complex distance to instability is not the right measure, since typically the distance to instability under real perturbations is larger than the complex distance. A formula for the real distance under general perturbations was derived in the ground-breaking paper [23]. The computation of this *real distance to instability (or real stability radius)* is an important topic in many applications, see e.g. [10, 17, 18, 24, 30, 32].

In this paper, we study the distance to instability under real (structure preserving) perturbations for DH systems and we restrict ourselves to the case that only the dissipation matrix R is perturbed, because this is the part of the model that is usually most uncertain, due to the fact that modeling damping or friction is extremely difficult, see [10] and Example 1.1 below. Furthermore, the analysis of the perturbations in the matrices Q, J is not completely clear at this stage.

Since DH systems are stable, it is clear that perturbations that preserve the DH structure also preserve the stability. However, DH systems may not be *asymptotically stable*, since they may have purely imaginary eigenvalues, which happens e.g. when the dissipation matrix R vanishes. So in the case of DH systems, we discuss when the system is *robustly asymptotically stable*, i.e. small real and structure-preserving perturbations Δ_R to the coefficient matrix R keep the system asymptotically stable, and we determine the smallest perturbations that move the system to the boundary of the set of asymptotically stable systems. Motivated from an application in the area of disk brake squeal, we consider restricted structure-preserving perturbations of the form $\Delta_R = B\Delta B^T$, where $B \in \mathbb{R}^{n,r}$ is a so-called *restriction matrix* of full column rank. This restriction allows the consideration of perturbations that only affect selected parts of the matrix R. (We mention that perturbations of the form $B\Delta B^T$ should be called *structured perturbations* by the convention following [13], but we prefer the term *restricted* here, because of the danger of confusion with the term *structure-preserving* for perturbations that preserve the DH structure.)

Example 1.1 The finite element analysis of disk brake squeal [10] leads to large scale differential equations of the form

$$M\ddot{q} + (D + G)\dot{q} + (K + N)q = f,$$

where $M = M^T > 0$ is the mass matrix, $D = D^T \geq 0$ models material and friction induced damping, $G = -G^T$ models gyroscopic effects, $K = K^T > 0$ models the stiffness, and N is a nonsymmetric matrix modeling circulatory effects. Here $\dot{q}$ denotes the derivative with respect to time. An appropriate first order formulation is associated with the linear system

$\dot{z} = (J - R)Qz$, where

$$J := \begin{bmatrix} G & K + \frac{1}{2}N \\ -(K + \frac{1}{2}N^H) & 0 \end{bmatrix}, \ R := \begin{bmatrix} D & \frac{1}{2}N \\ \frac{1}{2}N^H & 0 \end{bmatrix}, \ Q := \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}^{-1}. \tag{1.2}$$

This system is in general not a DH system, since for $N \neq 0$ the matrix R is indefinite. But since brake squeal is associated with the eigenvalues in the right half plane, it is an important question for which perturbations the system is stable. This instability can be associated with the restricted indefinite perturbation matrix

$$R + \Delta R := \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}N \\ \frac{1}{2}N^H & 0 \end{bmatrix},$$

and this matrix underlies further restrictions, since only a small part of the finite elements is associated with the brake pad and thus the perturbations in N are restricted to the coefficients associated with these finite elements.

In the following $\|\cdot\|$ denotes the spectral norm of a vector or a matrix while $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. By $\Lambda(A)$ we denote the spectrum of a matrix $A \in \mathbb{R}^{n,n}$, where $\mathbb{R}^{n,r}$ is the set of real $n \times r$ matrices, with the special case $\mathbb{R}^n = \mathbb{R}^{n,1}$. For $A = A^T \in \mathbb{R}^{n,n}$, we use the notation $A \geq 0$ and $A \leq 0$ if A is positive or negative semidefinite, respectively, and $A > 0$ if A is positive definite. The Moore-Penrose of a matrix $\mathbb{R}^{n,r}$ is denoted by $A^\dagger$, see e.g. [9].

The paper is organized as follows. In Section 2 we study mapping theorems that are needed to construct the minimal perturbations. The stability radii for DH systems under perturbations in the dissipation matrix is studied in Section 3 and the results are illustrated with numerical examples in Section 4.

2 Mapping Theorems

As in the complex case that was treated in [20], the main tool in the computation of stability radii for real DH systems will be minimal norm solutions to structured mapping problems. To construct such mappings in the case of real matrices, and to deal with pairs of complex conjugate eigenvalues and their eigenvectors, we will need general mapping theorems for maps that map several real vectors $x_1, \ldots, x_m$ to vectors $y_1, \ldots, y_m$. We will first study the complex case and then use these results to construct corresponding mappings in the real case.

For given $X \in \mathbb{C}^{m,p}$, $Y \in \mathbb{C}^{n,p}$, $Z \in \mathbb{C}^{n,k}$, and $W \in \mathbb{C}^{m,k}$, the general *mapping problem*, see [16], asks under which conditions on X, Y, Z, and W the set

$$\mathcal{S} := \left\{ A \in \mathbb{C}^{n,m} \mid AX = Y, \ A^H Z = W \right\}$$

is non-empty, and if it is, then to construct the minimal spectral or Frobenius norm elements from $\mathcal{S}$. To solve this mapping problem, we will need the following well-known result.

Theorem 2.1 [4] *Let A, B, C be given complex matrices of appropriate dimensions. Then for any positive number μ satisfying*

$$\mu \geq \max \left(\left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|, \left\| \begin{bmatrix} A & C \end{bmatrix} \right\| \right), \tag{2.1}$$

there exists a complex matrix D of appropriate dimensions such that

$$\left\|\begin{bmatrix} A & C \\ B & D \end{bmatrix}\right\| \leq \mu. \tag{2.2}$$

Furthermore, any matrix D satisfying (2.2) is of the form

$$D = -KA^H L + \mu(I - KK^H)^{1/2} Z (I - LL^H)^{1/2},$$

where $K^H = (\mu^2 I - A^H A)^{-1/2} B^H$, $L = (\mu^2 I - AA^H)^{-1/2} C$, and Z is an arbitrary contraction, i.e., $\|Z\| \leq 1$.

Using Theorem 2.1 we obtain the following solution to the general mapping problem.

Theorem 2.2 *Let $X \in \mathbb{C}^{m,p}$, $Y \in \mathbb{C}^{n,p}$, $Z \in \mathbb{C}^{n,k}$, and $W \in \mathbb{C}^{m,k}$. Then the set*

$$\mathcal{S} := \left\{ A \in \mathbb{C}^{n,m} \,\middle|\, AX = Y, \ A^H Z = W \right\},$$

is non-empty if and only if $X^H W = Y^H Z$, $Y X^\dagger X = Y$, and $W Z^\dagger Z = W$. If this is the case, then

$$\mathcal{S} = \left\{ Y X^\dagger + (W Z^\dagger)^H - (W Z^\dagger)^H X X^\dagger + (I - ZZ^\dagger) R (I - XX^\dagger) \,\middle|\, R \in \mathbb{C}^{n,m} \right\}. \tag{2.3}$$

Furthermore, we have the following minimal-norm solutions to the mapping problem $AX = Y$ and $A^H Z = W$:

1) *The matrix*

$$G := Y X^\dagger + (W Z^\dagger)^H - (W Z^\dagger)^H X X^\dagger \tag{2.4}$$

   is the unique matrix from $\mathcal{S}$ of minimal Frobenius norm

$$\|G\|_F = \sqrt{\|Y X^\dagger\|_F^2 + \|W Z^\dagger\|_F^2 - \operatorname{trace}((W Z^\dagger)(W Z^\dagger)^H X X^\dagger)} = \inf_{A \in \mathcal{S}} \|A\|_F.$$

2) *The minimal spectral norm of elements from $\mathcal{S}$ is given by*

$$\mu := \max \left\{ \|Y X^\dagger\|, \|W Z^\dagger\| \right\} = \inf_{A \in \mathcal{S}} \|A\|. \tag{2.5}$$

   Moreover, assume that $\operatorname{rank}(X) = r_1$ and $\operatorname{rank}(Z) = r_2$. If $X = U\Sigma V^H$ and $Z = \widehat{U}\widehat{\Sigma}\widehat{V}^H$ are singular value decompositions of X and Z, respectively, where $U = [U_1, U_2]$ with $U_1 \in \mathbb{C}^{m,r_1}$ and $\widehat{U} = [\widehat{U}_1, \widehat{U}_2]$ with $\widehat{U}_1 \in \mathbb{C}^{n,r_2}$, then the infimum in (2.5) is attained by the matrix

$$F := Y X^\dagger + (W Z^\dagger)^H - (W Z^\dagger)^H X X^\dagger + (I - ZZ^\dagger) \widehat{U}_2 C U_2^H (I - XX^\dagger), \tag{2.6}$$

   where

$$\begin{aligned}
C &= -K(U_1^H (Y X^\dagger) \widehat{U}_1) L + \mu (I - KK^H)^{\frac{1}{2}} P (I - L^H L)^{\frac{1}{2}}, \\
K &= \left[(\mu^2 I - U_1^H (Y X^\dagger)^H ZZ^\dagger (Y X^\dagger) U_1)^{-\frac{1}{2}} (\widehat{U}_2^H Y X^\dagger U_1) \right]^H, \\
L &= \left(\mu^2 I - \widehat{U}_1^H (Y X^\dagger)(Y X^\dagger)^H \widehat{U}_1 \right)^{-\frac{1}{2}} \left(\widehat{U}_1^H (W Z^\dagger)^H U_2 \right),
\end{aligned}$$

   and P is an arbitrary contraction.

4

Proof. Suppose that $\mathcal{S}$ is non-empty. Then there exists $A \in \mathbb{C}^{n,m}$ such that $AX = Y$ and $A^H Z = W$. This implies that $X^H W = Y^H Z$. Using the properties of the Moore-Penrose pseudoinverse, we have that $YX^\dagger X = AXX^\dagger X = AX = Y$ and $WZ^\dagger Z = A^H ZZ^\dagger Z = A^H Z = W$.

Conversely, suppose that $X^H W = Y^H Z$, $YX^\dagger X = Y$, and $WZ^\dagger Z = W$. Then $\mathcal{S}$ is non-empty, since for any $C \in \mathbb{C}^{n,m}$ we have

$$A = YX^\dagger + (WZ^\dagger)^H - (WZ^\dagger)^H XX^\dagger + (I - ZZ^\dagger)C(I - XX^\dagger) \in \mathcal{S}.$$

In particular, this proves the inclusion "$\supseteq$" in (2.3). To prove the other direction "$\subseteq$", let $X = U\Sigma V^H$ and $Z = \widehat{U}\widehat{\Sigma}\widehat{V}^H$ be singular value decompositions of X and Z, respectively, partitioned as in 2), with $\Sigma_1 \in \mathbb{R}^{r_1,r_1}$ and $\widehat{\Sigma}_1 \in \mathbb{R}^{r_2,r_2}$ being the leading principle submatrices of Σ and $\widehat{\Sigma}$, respectively. Let $A \in \mathcal{S}$ and set

$$\widehat{A} := \widehat{U}^H A U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.7}$$

where $A_{11} \in \mathbb{C}^{r_1,r_2}$, $A_{12} \in \mathbb{C}^{r_1,m-r_2}$, $A_{21} \in \mathbb{C}^{n-r_1,r_2}$ and $A_{22} \in \mathbb{C}^{n-r_1,m-r_2}$. Then $\|A\|_F = \|\widehat{A}\|_F$ and $\|A\| = \|\widehat{A}\|$. Multiplying $\widehat{U}\widehat{A}U^H X = AX = Y$ by $\widehat{U}^H$ from the left, we obtain

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 V_1^H \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} X = \begin{bmatrix} \widehat{U}_1^H \\ \widehat{U}_2^H \end{bmatrix} Y = \begin{bmatrix} \widehat{U}_1^H Y \\ \widehat{U}_2^H Y \end{bmatrix},$$

which implies that

$$A_{11} = \widehat{U}_1^H Y V_1 \Sigma_1^{-1} \quad \text{and} \quad A_{21} = \widehat{U}_2^H Y V_1 \Sigma_1^{-1}. \tag{2.8}$$

Analogously, multiplying $U\widehat{A}^H \widehat{U}^H Z = A^H Z = W$ from the left by U^H, we obtain that

$$\begin{bmatrix} A_{11}^H & A_{21}^H \\ A_{12}^H & A_{22}^H \end{bmatrix} \begin{bmatrix} \widehat{\Sigma}_1 \widehat{V}_1^H \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11}^H & A_{21}^H \\ A_{12}^H & A_{22}^H \end{bmatrix} \begin{bmatrix} \widehat{U}_1^H \\ \widehat{U}_2^H \end{bmatrix} Z = \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} W = \begin{bmatrix} U_1^H W \\ U_2^H W \end{bmatrix}.$$

This implies that

$$A_{11} = (\widehat{\Sigma}_1^{-1})^H \widehat{V}_1^H W^H U_1 \quad \text{and} \quad A_{12} = (\widehat{\Sigma}_1^{-1})^H \widehat{V}_1^H W^H U_2. \tag{2.9}$$

Equating the two expressions for A_{11} from (2.8) and (2.9), we obtain

$$\widehat{U}_1^H Y V_1 \Sigma_1^\dagger = (\widehat{\Sigma}_1^\dagger)^H \widehat{V}_1^H W^H U_1$$

which shows that $X^H W = Y^H Z$. Furthermore, we get

$$\widehat{A} = \begin{bmatrix} \widehat{U}_1^H Y V_1 \Sigma_1^{-1} & (\widehat{\Sigma}_1^{-1})^H \widehat{V}_1^H S^H U_2 \\ \widehat{U}_2^H Y V_1 \Sigma_1^{-1} & A_{22} \end{bmatrix} = \begin{bmatrix} \widehat{U}_1^H Y X^\dagger U_1 & \widehat{U}_1^H (WZ^\dagger)^H U_2 \\ \widehat{U}_2^H Y X^\dagger U_1 & A_{22} \end{bmatrix}. \tag{2.10}$$

Thus, using $X^\dagger U_1 U_1^H = X^\dagger$, $Z^\dagger \widehat{U}_1 \widehat{U}_1^H = Z^\dagger$, and $U_1 U_1^H = XX^\dagger$, we obtain

$$
\begin{aligned}
A &= \widehat{U}\widehat{A}U^H = \widehat{U}_1 \widehat{U}_1^H Y X^\dagger U_1 U_1^H + \widehat{U}_2 \widehat{U}_2^H Y X^\dagger U_1 U_1^H + \widehat{U}_1 \widehat{U}_1^H (WZ^\dagger)^H U_2 U_2^H + \widehat{U}_2 A_{22} U_2^H \\
&= \widehat{U}_1 \widehat{U}_1^H Y X^\dagger U_1 U_1^H + (I - \widehat{U}_1 \widehat{U}_1^H) Y X^\dagger U_1 U_1^H + \widehat{U}_1 \widehat{U}_1^H (SZ^\dagger)^H (I - U_1 U_1^H) + \widehat{U}_2 A_{22} U_2^H \\
&= YX^\dagger + (WZ^\dagger)^H - (WZ^\dagger)^H XX^\dagger + (I - ZZ^\dagger)\widehat{U}_2 A_{22} U_2^H (I - XX^\dagger). \tag{2.11}
\end{aligned}
$$

1) In view of (2.10) and using that $X^\dagger U_2 = 0$ and $\widehat{U}_2^H (Z^\dagger)^H = 0$, we obtain that

$$
\begin{aligned}
\|A\|_F^2 = \|\widehat{A}\|_F^2 &= \left\| \begin{bmatrix} \widehat{U}_1^H Y X^\dagger U_1 \\ \widehat{U}_2^H Y X^\dagger U_1 \end{bmatrix} \right\|_F^2 + \|\widehat{U}_1^H (WZ^\dagger)^H U_2\|_F^2 + \|A_{22}\|_F^2 \\
&= \|\widehat{U}^H Y X^\dagger U\|_F^2 + \|\widehat{U}(WZ^\dagger)^H U\|_F^2 - \|\widehat{U}_1^H (WZ^\dagger)^H U_1\|_F^2 + \|A_{22}\|_F^2 \\
&= \|Y X^\dagger\|_F^2 + \|W Z^\dagger\|_F^2 - \operatorname{trace}\left((WZ^\dagger)(WZ^\dagger)^H X X^\dagger \right) + \|A_{22}\|_F^2.
\end{aligned}
$$

Thus, by setting $A_{22} = 0$ in (2.11), we obtain a unique element of $\mathcal{S}$ which minimizes the Frobenius norm giving

$$
\inf_{A \in \mathcal{S}} \|A\|_F = \sqrt{\|Y X^\dagger\|_F^2 + \|W Z^\dagger\|_F^2 - \operatorname{trace}\left((WZ^\dagger)(WZ^\dagger)^H X X^\dagger \right)}.
$$

2) By definition of μ, we have

$$
\begin{aligned}
\mu &= \max\left\{ \|Y X^\dagger\|, \|W Z^\dagger\| \right\} \\
&= \max\left\{ \left\| \begin{bmatrix} \widehat{U}_1^H Y X^\dagger U_1 \\ \widehat{U}_2^H Y X^\dagger U_1 \end{bmatrix} \right\|, \left\| \begin{bmatrix} \widehat{U}_1^H Y X^\dagger U_1 & \widehat{U}_1^H (WZ^\dagger)^H U_2 \end{bmatrix} \right\| \right\}.
\end{aligned}
$$

Then it follows that for any $A \in \mathcal{S}$, we have $\|A\| = \|\widehat{A}\| \geq \mu$ with $\widehat{A}$ as in (2.7). By Theorem 2.1 there exists matrices $A \in \mathcal{S}$ with $\|A\| \leq \mu$, i.e., $\inf_{A \in \mathcal{S}} \|A\| = \mu$. Furthermore, by Theorem 2.1 this infimum is attained with a matrix A as in (2.11), where

$$
\begin{aligned}
A_{22} &= -K \left(U_1^H (YX^\dagger)^H \widehat{U}_1 \right) L + \mu (I - KK^H)^{\frac{1}{2}} P (I - L^H L)^{\frac{1}{2}}, \\
K &= \left[\left(\mu^2 I - U_1^H (YX^\dagger)^H ZZ^\dagger (YX^\dagger) U_1 \right)^{-\frac{1}{2}} (\widehat{U}_2^H Y X^\dagger U_1) \right]^H, \\
L &= \left(\mu^2 I - \widehat{U}_1^H (YX^\dagger)(YX^\dagger)^H \widehat{U}_1 \right)^{-\frac{1}{2}} \left(\widehat{U}_1^H (WZ^\dagger)^H U_2 \right),
\end{aligned}
$$

and P is an arbitrary contraction. Hence the assertion follows by setting $C = A_{22}$. $\quad\square$

The special case $p = k = 1$ of Theorem 2.2 was originally obtained in [15, Theorem 2]. The next result characterizes the Hermitian positive semidefinite matrices that map a given $X \in \mathbb{C}^{n,m}$ to a given $Y \in \mathbb{C}^{n,m}$, and we include the solutions that are minimal with respect to the spectral or Frobenius norm. This generalizes [20, Theorem 2.3] which only covers the case $m = 1$.

Theorem 2.3 *Let $X, Y \in \mathbb{C}^{n,m}$ be such that $\operatorname{rank}(Y) = m$. Define*

$$
\mathcal{S} := \left\{ H \in \mathbb{C}^{n,n} \,\middle|\, H^H = H \geq 0,\ HX = Y \right\}.
$$

Then there exists $H = H^H \geq 0$ such that $HX = Y$ if and only if $X^H Y = Y^H X$ and $X^H Y > 0$. If this holds, then

$$
\widetilde{H} := Y (Y^H X)^{-1} Y^H \tag{2.12}
$$

is well-defined and Hermitian positive-semidefinite, and

$$
\mathcal{S} = \left\{ \widetilde{H} + \left(I_n - XX^\dagger \right) K^H K \left(I_n - XX^\dagger \right) \,\middle|\, K \in \mathbb{C}^{n,n} \right\}. \tag{2.13}
$$

Furthermore, we have the following minimal norm solutions to the mapping problem $HX = Y$:

6

1) *The matrix $\widetilde{H}$ from (2.12) is the unique matrix from $\mathcal{S}$ with minimal Frobenius norm*

$$\min\left\{\,\|H\|_F \mid H \in \mathcal{S}\,\right\} = \|Y(X^H Y)^{-1} Y^H\|_F.$$

2) *The minimal spectral norm of elements from $\mathcal{S}$ is given by*

$$\min\left\{\,\|H\| \mid H \in \mathcal{S}\,\right\} = \|Y(X^H Y)^{-1} Y^H\|$$

and the minimum is attained for the matrix $\widetilde{H}$ from (2.12).

Proof. If $H \in \mathcal{S}$, then $X^H Y = X^H H X = (HX)^H X = Y^H X$ and $X^H Y = X^H H X \geq 0$, since $H^H = H \geq 0$. If $X^H H X$ were singular, then there would exist a vector $v \in \mathbb{C}^m \setminus \{0\}$ such that $v^H X^H H X v = 0$ and, hence, $Yv = HXv = 0$ (as $H \geq 0$) in contradiction to the assumption that Y is of full rank. Thus, $X^H Y > 0$.

Conversely, let $X^H Y = Y^H X > 0$ (which implies that also $(X^H Y)^{-1} > 0$). Then $\widetilde{H}$ in (2.12) is well-defined. Clearly $\widetilde{H} X = Y$ and $\widetilde{H} = Y(Y^H X)^{-1} Y^H \geq 0$, which implies that $\widetilde{H} \in \mathcal{S}$. Let

$$H = \widetilde{H} + (I - XX^\dagger) K^H K (I - XX^\dagger) \in \mathbb{C}^{n,n}$$

be as in (2.13) for some $K \in \mathbb{C}^{n,n}$. Then clearly $H^H = H$, $HX = Y$ and also $H \geq 0$, as it is the sum of two positive semidefinite matrices. This proves the inclusion "$\supseteq$" in (2.13). For the converse inclusion, let $H \in \mathcal{S}$. Since $H \geq 0$, we have that $H = A^H A$ for some $A \in \mathbb{C}^{n,n}$. Therefore, $HX = Y$ implies that $(A^H A)X = Y$, and setting $Z = AX$, we have $AX = Z$ and $A^H Z = Y$. Since $\operatorname{rank}(Y) = m$, we necessarily also have that $\operatorname{rank}(Z) = m$ and $\operatorname{rank}(X) = m$. Therefore, by Theorem 2.2, A can be written as

$$A = ZX^\dagger + (YZ^\dagger)^H - (YZ^\dagger)^H XX^\dagger + (I - ZZ^\dagger)C(I - XX^\dagger) \tag{2.14}$$

for some $C \in \mathbb{C}^{n,n}$. Note that

$$(YZ^\dagger)^H XX^\dagger = (Z^\dagger)^H Y^H XX^\dagger = (Z^\dagger)^H Z^H ZX^\dagger = (ZZ^\dagger)^H ZX^\dagger = ZX^\dagger, \tag{2.15}$$

since $(ZZ^\dagger)^H = ZZ^\dagger$ and $Y^H X = Z^H Z$. By inserting (2.15) in (2.14), we obtain that

$$A = (YZ^\dagger)^H + (I - ZZ^\dagger)C(I - XX^\dagger),$$

and thus,

$$\begin{aligned}
H &= A^H A = (YZ^\dagger)(YZ^\dagger)^H + (I - XX^\dagger)C^H(I - ZZ^\dagger)(YZ^\dagger)^H \\
&\quad + (YZ^\dagger)(I - ZZ^\dagger)C(I - XX^\dagger) + (I - XX^\dagger)C^H(I - ZZ^\dagger)(I - ZZ^\dagger)C(I - XX^\dagger) \\
&= (YZ^\dagger)(YZ^\dagger)^H + (I - XX^\dagger)C^H(I - ZZ^\dagger)(I - ZZ^\dagger)C(I - XX^\dagger), \tag{2.16}
\end{aligned}$$

where the last equality follows since

$$(YZ^\dagger)(I - ZZ^\dagger) = Y(Z^\dagger - Z^\dagger ZZ^\dagger) = Y(Z^\dagger - Z^\dagger) = 0.$$

Setting $K = (I - ZZ^\dagger)C$ in (2.16) and using

$$Z^\dagger(Z^\dagger)^H = (Z^H Z)^{-1} Z^H\big((Z^H Z)^{-1} Z^H\big)^H = (Z^H Z)^{-1} = (Y^H X)^{-1}$$

proves the inclusion "$\subseteq$" in (2.13).

1) If $H \in \mathcal{S}$, then $X^H Y = Y^H X > 0$ and we obtain

$$H = Y(Y^H X)^{-1} Y^H + (I - XX^\dagger) K^H K (I - XX^\dagger), \tag{2.17}$$

for some $K \in \mathbb{C}^{n,n}$. By using the formula $\|BB^H + DD^H\|_F^2 = \|BB^H\|_F^2 + 2\|DB\|_F^2 + \|D^H D\|_F^2$ for $B = Y(Y^H X)^{-1/2}$ and $D = K(I - XX^\dagger)$, we obtain

$$\|H\|_F^2 = \|Y(Y^H X)^{-1} Y^H\|_F^2 + 2\|K(I - XX^\dagger)Y(Y^H X)^{-\frac{1}{2}}\|_F^2 + \|(I - XX^\dagger) K^H K (I - XX^\dagger)\|_F^2.$$

Hence, setting $K = 0$, we obtain $\widetilde{H} = Y(Y^H X)^{-1} Y^H$ as the unique minimal Frobenius norm solution.

2) Let $H \in \mathcal{S}$ be of the form (2.17) for some $K \in \mathbb{C}^{n,n}$. Since $Y(Y^H X)^{-1} Y^H \geq 0$ and $(I - XX^\dagger) K^H K (I - XX^\dagger) \geq 0$, we have

$$\|Y(Y^H X)^{-1} Y^H\| \leq \|Y(Y^H X)^{-1} Y^H + (I - XX^\dagger) K^H K (I - XX^\dagger)\|.$$

This implies that

$$\|Y(Y^H X)^{-1} Y^H\| \leq \inf_{K \in \mathbb{C}^{n,n}} \|Y(Y^H X)^{-1} Y^H + (I - XX^\dagger) K^H K (I - XX^\dagger)\| = \inf_{H \in \mathcal{S}} \|H\|. \tag{2.18}$$

One possible choice for obtaining equality in (2.18) is $K = 0$ which gives $\widetilde{H} = Y(Y^H X)^{-1} Y^H \in \mathcal{S}$ and $\|\widetilde{H}\| = \|Y(Y^H X)^{-1} Y^H\| = \min_{H \in \mathcal{S}} \|H\|$. $\square$

Having presented the complex versions of the mapping theorems, we now adapt these for the case of real perturbations. Here, "*real*" refers to the mappings, but not to the vectors that are mapped, because we need to apply the mapping theorems to eigenvectors which may be complex even if the matrix under consideration is real.

Remark 2.4 If $X \in \mathbb{C}^{m,p}$, $Y \in \mathbb{C}^{n,p}$, $Z \in \mathbb{C}^{n,k}$, $W \in \mathbb{C}^{m,k}$ are such that $\operatorname{rank}([X \ \bar{X}]) = 2p$ and $\operatorname{rank}([Z \ \bar{Z}]) = 2k$ (or, equivalently, $\operatorname{rank}([\operatorname{Re} X \ \operatorname{Im} X]) = 2p$ and $\operatorname{rank}([\operatorname{Re} Z \ \operatorname{Im} Z]) = 2k$), then minimal norm solutions to the mapping problem $AX = Y$ and $A^H Z = W$ with $A \in \mathbb{R}^{n,m}$ can easily be obtained from Theorem 2.2. This follows from the observation that with $AX = Y$ and $A^H Z = W$ we also have $A\bar{X} = \bar{Y}$ and $A^H \bar{Z} = \bar{W}$ and thus

$$A[\operatorname{Re} X \ \operatorname{Im} X] = [\operatorname{Re} Y \ \operatorname{Im} Y] \quad \text{and} \quad A^H[\operatorname{Re} Z \ \operatorname{Im} Z] = [\operatorname{Re} W \ \operatorname{Im} W].$$

We can then apply Theorem 2.2 to the real matrices $\mathcal{X} = [\operatorname{Re} X \ \operatorname{Im} X]$, $\mathcal{Y} = [\operatorname{Re} Y \ \operatorname{Im} Y]$, $\mathcal{Z} = [\operatorname{Re} Z \ \operatorname{Im} Z]$, and $\mathcal{W} = [\operatorname{Re} W \ \operatorname{Im} W]$. Indeed, whenever there exists a complex matrix $A \in \mathbb{C}^{n,m}$ satisfying $A\mathcal{X} = \mathcal{Y}$ and $A^H \mathcal{Z} = \mathcal{W}$, then there also exists a real one, because it is easily checked that the minimal norm solutions in Theorem 2.2 are real. (Here, we assume that for the case of the spectral norm the real singular value decompositions of $\mathcal{X}$ and $\mathcal{Z}$ are taken, and the contraction P is also chosen to be real.)

A similar observation holds for solution of the real version of the Hermitian positive semidefinite mapping problem in Theorem 2.3.

Since the real version of Theorem 2.2 (and similarly of Theorem 2.3) is straightforward in view of Remark 2.4, we refrain from an explicit statement. The situation, however, changes considerably if the assumptions $\operatorname{rank}([\operatorname{Re} X \ \operatorname{Im} X]) = 2p$ and $\operatorname{rank}([\operatorname{Re} Z \ \operatorname{Im} Z]) = 2k$ as in

Remark 2.4 are dropped. In this case, it seems that a full characterization of real solutions to the mapping problems is highly challenging and very complicated. Therefore, we only consider the generalization of Theorem 2.3 to real mappings for the special case $m = 1$ which is in fact the case needed for the computation of the stability radii. We obtain the following two results.

Theorem 2.5 *Let $x, y \in \mathbb{C}^n$ be such that* $\mathrm{rank}([y \ \bar{y}]) = 2$*. Then the set*

$$\mathcal{S} := \left\{ H \ \middle| \ H \in \mathbb{R}^{n,n}, \ H^T = H \geq 0, \ Hx = y \right\}$$

is non-empty if and only if $x^H y > |x^T y|$ (which includes the condition $x^H y \in \mathbb{R}$). In this case let

$$X := [\mathrm{Re}\, x \ \ \mathrm{Im}\, x] \quad and \quad Y := [\mathrm{Re}\, y \ \ \mathrm{Im}\, y].$$

Then the matrix

$$\widetilde{H} := Y \left(Y^H X \right)^{-1} Y^H \tag{2.19}$$

is well defined and real symmetric positive semidefinite, and

$$\mathcal{S} = \left\{ \widetilde{H} + (I - XX^\dagger) K (I - XX^\dagger) \ \middle| \ K \in \mathbb{R}^{n,n}, \ K^T = K \geq 0 \right\}. \tag{2.20}$$

1) *The minimal Frobenius norm of an element in $\mathcal{S}$ is given by*

$$\min \left\{ \|H\|_F \ \middle| \ H \in \mathcal{S} \right\} = \left\| Y \left(Y^T X \right)^{-1} Y^T \right\|_F$$

   and this minimal norm is uniquely attained by the matrix $\widetilde{H}$ in (2.19).

2) *The minimal spectral norm of an element in $\mathcal{S}$ is given by*

$$\min \left\{ \|H\| \ \middle| \ H \in \mathcal{S} \right\} = \left\| Y \left(Y^T X \right)^{-1} Y^T \right\|$$

   and the matrix $\widetilde{H}$ in (2.19) is a matrix that attains this minimum.

Proof. Let $H \in \mathcal{S}$, i.e. $H^T = H \geq 0$ and $Hx = y$. Since H is real, we also have $HX = Y$. Thus, by Theorem 2.3, X and Y satisfy $X^H Y = Y^H X$ and $Y^H X > 0$. The first condition is equivalent to $(\mathrm{Re}\, x)^T \mathrm{Im}\, y = (\mathrm{Re}\, y)^T \mathrm{Im}\, x$ which in turn is equivalent to $x^H y \in \mathbb{R}$ and the second condition is easily seen to be equivalent to $x^H y > |x^T y|$. Conversely, if $x^H y = y^H x$ and $x^H y > |x^T y|$, then $Y^T X > 0$ and hence

$$\widetilde{H} := Y \left(Y^T X \right)^{-1} Y^T$$

is positive semidefinite. Moreover, we obviously have $\widetilde{H} X = Y$ and thus $\widetilde{H} x = y$. This implies that $\widetilde{H} \in \mathcal{S}$.

The inclusion "$\supseteq$" in (2.20) is straightforward. For the other inclusion let $H \in \mathcal{S}$, i.e., $H^T = H \geq 0$ and $Hx = y$, and thus $HX = Y$. Then, by Theorem 2.3, there exist $L \in \mathbb{C}^{n,n}$ such that

$$\begin{aligned} H &= \widetilde{H} + (I - XX^\dagger) L^H L (I - XX^\dagger) \\ &= \widetilde{H} + (I - XX^\dagger) \left(\mathrm{Re}(L^H) \mathrm{Re}(L) + \mathrm{Im}(L)^H \mathrm{Im}(L) \right) (I - XX^\dagger), \end{aligned}$$

9

where for the last identity we made use of the fact that H is real. Thus, by setting $K = (\operatorname{Re}(L^H)\operatorname{Re}(L) + \operatorname{Im}(L^H)\operatorname{Im}(L))$, we get the inclusion "$\subseteq$" in (2.20). The norm minimality in 1) and 2) follows immediately from Theorem 2.3, because any real map $H \in S$ also satisfies $HX = Y$. □

Theorem 2.5 does not consider the case that y and $\bar{y}$ are linearly dependent. In that case, we obtain the following result.

Theorem 2.6 *Let $x, y \in \mathbb{C}^n$, $y \neq 0$ be such that $\operatorname{rank}[y\ \bar{y}] = 1$. Then the set*

$$\mathcal{S} = \left\{ H \,\middle|\, H \in \mathbb{R}^{n,n},\ H^T = H \geq 0,\ Hx = y \right\}$$

is non-empty if and only if $x^H y > 0$. In that case

$$\widetilde{H} := \frac{yy^H}{x^H y} \tag{2.21}$$

is well-defined and real symmetric positive semidefinited. Furthermore, we have:

1) *The minimal Frobenius norm of an element in $\mathcal{S}$ is given by*

$$\min \left\{ \|H\|_F \,\middle|\, H \in \mathcal{S} \right\} = \frac{\|y\|^2}{x^H y}$$

   and this minimal norm is uniquely attained by the matrix $\widetilde{H}$ in (2.21).

2) *The minimal spectral norm of an element in $\mathcal{S}$ is given by*

$$\min \left\{ \|H\| \,\middle|\, H \in \mathcal{S} \right\} = \frac{\|y\|^2}{x^H y}$$

   and the matrix $\widetilde{H}$ in (2.19) is a matrix that attains this minimum.

Proof. If $H \in \mathcal{S}$, i.e., $H^T = H \geq 0$ and $Hx = y$, then by Theorem 2.3 (for the case $m = 1$), we have that $x^H y > 0$. Conversely, assume that x and y satisfy $x^H y > 0$. Since y and $\bar{y}$ are linearly dependent, there exists a unimodular $\alpha \in \mathbb{C}$ such that αy ist real. But then also $yy^H = (\alpha y)(\alpha y)^H$ is real and hence the matrix $\widetilde{H}$ in (2.21) is well defined and real. By Theorem 2.3, it is the unique element from $\mathcal{S}$ with minimal Frobenius norm and also an element from $\mathcal{S}$ of minimal spectral norm. □

Remark 2.7 Note that results similar to Theorem 2.5 and Theorem 2.6 can also be obtained for real negative semidefinite maps. Indeed, for $x, y \in \mathbb{C}^n$ such that $\operatorname{rank}[y\ \bar{y}] = 2$, there exist a real negative semidefinite matrix $H \in \mathbb{R}^{n,n}$ such that $Hx = y$ if and only if $x^H y = y^H x$ and $-x^H y > |x^T y|$. Furthermore, it follows immediately from Theorem 2.5 by replacing y with $-y$ and H with $-H$ that a minimal solution in spectral and Frobenius norm is given by $\widetilde{H} = [\operatorname{Re} y\ \operatorname{Im} y]\left([\operatorname{Re} y\ \operatorname{Im} y]^H[\operatorname{Re} x\ \operatorname{Im} x]\right)^{-1}[\operatorname{Re} y\ \operatorname{Im} y]^H$. An analogous argument holds for Theorem 2.6. Therefore, we will refer to Theorem 2.5 and Theorem 2.6 also in the case that we are seeking solutions for the negative semidefinite mapping problem.

The minimal norm solutions for the real symmetric mapping problem with respect to both the spectral norm and Frobenius norm are well known, see [1, Theorem 2.2.3]. We do not restate this result in its full generality, but in terms of the following two theorems that are formulated in such a way that they allow a direct application in the remainder of this paper.

Theorem 2.8 *Let $x, y \in \mathbb{C}^n \setminus \{0\}$ be such that $\mathrm{rank}([x \; \bar{x}]) = 2$. Then*

$$\mathcal{S} := \{H \in \mathbb{R}^{n,n} | \; H^T = H, Hx = y\}$$

is nonempty if and only if $x^H y = y^H x$. Furthermore, define

$$X := [\mathrm{Re}\,x \;\; \mathrm{Im}\,x], \quad Y := [\mathrm{Re}\,y \;\; \mathrm{Im}\,y], \quad \widetilde{H} := YX^\dagger + (YX^\dagger)^T - (XX^\dagger)^T YX^\dagger. \quad (2.22)$$

1) *The minimal Frobenius norm of an element in $\mathcal{S}$ is given by*

$$\min_{H \in \mathcal{S}} \|H\|_F = \|\widetilde{H}\|_F = \sqrt{2\|YX^\dagger\|_F^2 - \mathrm{trace}\,(YX^\dagger(YX^\dagger)^T XX^\dagger)}$$

   and the minimum is uniquely attained by $\widetilde{H}$ in (2.22).

2) *To characterize the minimal spectral norm, consider the singular value decomposition $X = U\Sigma V^T$ and let $U = [U_1 \; U_2]$ where $U_1 \in \mathbb{R}^{n,2}$. Then*

$$\min_{H \in \mathcal{S}} \|H\| = \|YX^\dagger\|,$$

   and the minimum is attained by

$$\widehat{H} = \widetilde{H} - (I_n - XX^\dagger)KU_1^T YX^\dagger U_1 K^T (I_n - XX^\dagger), \quad (2.23)$$

   where $K = YX^\dagger U_1(\mu^2 I_2 - U_1^T YX^\dagger YX^\dagger U_1)^{-1/2}$ and $\mu := \|YX^\dagger\|$.

Proof. Observe that for $H^T = H \in \mathbb{R}^{n,n}$ the identity $Hx = y$ is equivalent to $HX = Y$. Thus, the result follows immediately from [1, Theorem 2.2.3] applied to the mapping problem $HX = Y$. $\square$

Theorem 2.9 *Let $x, y \in \mathbb{C}^n \setminus \{0\}$ be such that $\mathrm{rank}([x \; \bar{x}]) = 1$. Then*

$$\mathcal{S} := \{H \in \mathbb{R}^{n,n} | \; H^T = H, Hx = y\}$$

is nonempty if and only if $x^H y = y^H x$. Furthermore, we have:

1) *The minimal Frobenius norm of an element in $\mathcal{S}$ is given by*

$$\min_{H \in \mathcal{S}} \|H\|_F = \frac{\|y\|}{\|x\|}$$

   and the minimum is uniquely attained by the real matrix

$$\widetilde{H} := \frac{yx^H}{\|x\|^2} + \frac{xy^H}{\|x\|^2} - \frac{(x^H y)xx^H}{\|x\|^4}. \quad (2.24)$$

   (If x and y are linearly dependent, then $\widetilde{H} = \frac{yx^H}{x^H x}$.)

2) *The minimal spectral norm of an element in $\mathcal{S}$ is given by*

$$\min_{H \in \mathcal{S}} \|H\| = \frac{\|y\|}{\|x\|},$$

   and the minimum is attained by the real matrix

$$\widehat{H} := \frac{\|y\|}{\|x\|} \begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|} \end{bmatrix} \begin{bmatrix} \frac{y^H x}{\|x\|\|y\|} & 1 \\ 1 & \frac{x^H y}{\|x\|\|y\|} \end{bmatrix}^{-1} \begin{bmatrix} \frac{y}{\|y\|} & \frac{x}{\|x\|} \end{bmatrix}^H \quad (2.25)$$

   if x and y are linearly independent and for $\widehat{H} := \frac{yx^H}{x^H x}$ otherwise.

Proof. By [1, Theorem 2.2.3] (see also [20, Theorem 2.1] and [16]) the matrices $\widetilde{H}$ and $\widehat{H}$ are the minimal Frobenius resp. spectral norm solutions to the complex Hermitian mapping problem $Hx = y$. Thus, it only remains to show that $\widetilde{H}$ and $\widehat{H}$ are real. Since x and $\bar{x}$ are linearly dependent, there exists a unimodular $\alpha \in \mathbb{C}$ such that αx is real. But then also $\alpha y = H(\alpha x)$ and thus $xx^H = (\alpha x)(\alpha x)^H$ and $yx^H = (\alpha y)(\alpha x)^H$ are real which implies the realness of $\widetilde{H}$. Analogously, $\widehat{H}$ can be shown to be real. $\square$

Obviously, the minimal Frobenius or spectral norm solutions from Theorem 2.9 have either rank one or two. The following lemma characterizes the rank of the minimal Frobenius or spectral norm solutions from Theorem 2.8 as well as the number of their negative and their positive eigenvalues, respectively.

Lemma 2.10 *Let $x, y \in \mathbb{C}^n \setminus \{0\}$ be such that $x^H y = y^H x$ and $\mathrm{rank}([x\ \bar{x}]) = 2$. If $\widetilde{H}$ and $\widehat{H}$ are defined as in (2.22) and (2.23), respectively, then $\mathrm{rank}(\widetilde{H}), \mathrm{rank}(\widehat{H}) \leq 4$ and both $\widetilde{H}$ and $\widehat{H}$ have at most two negative eigenvalues and at most two positive eigenvalues.*

Proof. Recall from Theorem 2.8 that $X = [\mathrm{Re}\,x\ \mathrm{Im}\,x]$, $Y = [\mathrm{Re}\,y\ \mathrm{Im}\,y]$ and consider the singular value decomposition

$$X = U\Sigma V^T = U \left[\begin{array}{c} \widetilde{\Sigma} \\ 0 \end{array} \right] V^T$$

with $U = [U_1\ U_2] \in \mathbb{R}^{n,n}$, $U_1 \in \mathbb{R}^{n,2}$, $\widetilde{\Sigma} \in \mathbb{R}^{2,2}$, and $V \in \mathbb{R}^{2,2}$. If we set

$$Y = U \left[\begin{array}{c} Y_1 \\ Y_2 \end{array} \right] V^T,$$

where $Y_1 \in \mathbb{R}^{2,2}$ and $Y_2 \in \mathbb{R}^{n-2,2}$, then

$$
\begin{aligned}
\widetilde{H} &= YX^\dagger + (YX^\dagger)^T - (XX^\dagger)^T YX^\dagger \\
&= U \left[\begin{array}{cc} Y_1\widetilde{\Sigma}^{-1} & 0 \\ Y_2\widetilde{\Sigma}^{-1} & 0 \end{array} \right] U^T + U \left[\begin{array}{cc} \widetilde{\Sigma}^{-1}Y_1^T & \widetilde{\Sigma}^{-1}Y_2^H \\ 0 & 0 \end{array} \right] U^T - U \left[\begin{array}{cc} Y_1\widetilde{\Sigma}^{-1} & 0 \\ 0 & 0 \end{array} \right] U^T \\
&= U \left[\begin{array}{cc} \widetilde{\Sigma}^{-1}Y_1^T & \widetilde{\Sigma}^{-1}Y_2^T \\ Y_2\widetilde{\Sigma}^{-1} & 0 \end{array} \right] U^T.
\end{aligned}
\tag{2.26}
$$

Thus, $\widetilde{H}$ is of rank at most four, and also $\widetilde{H}$ has a $n-2$ dimensional *neutral subspace*, i.e., a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ satisfying $z^T \widetilde{H} \widetilde{z} = 0$ for all $z, \widetilde{z} \in \mathcal{V}$. This means that the restriction of $\widetilde{H}$ to its range still has a neutral subspace of dimension at least

$$
\max\{0, \mathrm{rank}(\widetilde{H}) - 2\} = \left\{ \begin{array}{ll} 2 & \text{if } \mathrm{rank}(\widetilde{H}) = 4, \\ 1 & \text{if } \mathrm{rank}(\widetilde{H}) = 3, \\ 0 & \text{if } \mathrm{rank}(\widetilde{H}) \leq 2. \end{array} \right.
$$

Thus, it follows by applying [7, Theorem 2.3.4] that $\widetilde{H}$ has at most two negative and at most two positive eigenvalues. On the other hand, we have

$$\widehat{H} = \widetilde{H} - (I_n - XX^\dagger)KU_1^T YX^\dagger U_1 K^T(I_n - XX^\dagger),\tag{2.27}$$

where $K = YX^\dagger U_1 W$, $W := (\mu^2 I_2 - U_1^T YX^\dagger YX^\dagger U_1)^{-1/2}$ and $\mu := \|YX^\dagger\|$. Then we obtain

$$K = YX^\dagger U_1 W = U \begin{bmatrix} Y_1\widetilde{\Sigma}^{-1} & 0 \\ Y_2\widetilde{\Sigma}^{-1} & 0 \end{bmatrix} U^T U_1 W = U \begin{bmatrix} Y_1\widetilde{\Sigma}^{-1} \\ Y_2\widetilde{\Sigma}^{-1} \end{bmatrix} W.$$

Also,

$$\begin{aligned} \mathcal{H} &:= (I_n - XX^\dagger) K U_1^T YX^\dagger U_1 K^T (I_n - XX^\dagger) \\ &= U \begin{bmatrix} 0 \\ Y_2\widetilde{\Sigma}^{-1} \end{bmatrix} W U_1^T U \begin{bmatrix} Y_1\widetilde{\Sigma}^{-1} & 0 \\ Y_2\widetilde{\Sigma}^{-1} & 0 \end{bmatrix} U^T U_1 W^T \begin{bmatrix} 0 & \widetilde{\Sigma}^{-1} Y_2^T \end{bmatrix} U^T \\ &= U \begin{bmatrix} 0 & 0 \\ 0 & Y_2\widetilde{\Sigma}^{-1} W Y_1 \widetilde{\Sigma}^{-1} W^T \widetilde{\Sigma}^{-1} Y_2^T \end{bmatrix} U^T. \end{aligned} \tag{2.28}$$

Inserting (2.26) and (2.28) into (2.27), we get

$$\begin{aligned} \widehat{H} &= \widetilde{H} - \mathcal{H} = U \begin{bmatrix} \widetilde{\Sigma}^{-1} Y_1^T & \widetilde{\Sigma}^{-1} Y_2^T \\ Y_2\widetilde{\Sigma}^{-1} & -Y_2\widetilde{\Sigma}^{-1} W Y_1 \widetilde{\Sigma}^{-1} W^T \widetilde{\Sigma}^{-1} Y_2^T \end{bmatrix} U^T \\ &= U \begin{bmatrix} Z & L^T \\ L & -LWZ^T W^T L^T \end{bmatrix} U^T, \end{aligned}$$

where $Z := \widetilde{\Sigma}^{-1} Y_1^T$ and $L := Y_2\widetilde{\Sigma}^{-1}$. This implies that

$$U^T \widehat{H} U = \begin{bmatrix} Z & L^T \\ L & -LWZ^T W^T L^T \end{bmatrix}$$

and Z are real symmetric. Let $\widehat{U} \in \mathbb{R}^{n-2,n-2}$ be invertible such that

$$\widehat{U} L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{12} & l_{22} & 0 & \dots & 0 \end{bmatrix}^T = \begin{bmatrix} \widehat{L} \\ 0 \end{bmatrix}$$

is in row echelon form, where $\widehat{L} = \begin{bmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{bmatrix}$. Then

$$\begin{bmatrix} I_2 & 0 \\ 0 & \widehat{U} \end{bmatrix} \begin{bmatrix} Z & L^T \\ L & -LWZW^T L^T \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ 0 & \widehat{U}^T \end{bmatrix} = \begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where

$$Z_{11} = \begin{bmatrix} Z & \widehat{L}^T \\ \widehat{L} & -\widehat{L}WZW^T \widehat{L}^T \end{bmatrix} \in \mathbb{R}^{4,4}.$$

This implies that $\mathrm{rank}(\widehat{H}) = \mathrm{rank}(Z_{11}) \le 4$. Furthermore, by Sylvester's law of inertia, $\widehat{H}$ and Z_{11} have the same number of negative eigenvalues. Thus, the assertion follows if we show that Z_{11} has at most two negative eigenvalues. For this, first suppose that $\widehat{L}$ is singular, i.e., $l_{22} = 0$, which would imply that $\mathrm{rank}(Z_{11}) \le 3$. If $\mathrm{rank}(Z_{11}) < 3$, then clearly Z_{11} can have at most two negative eigenvalues and at most two positive eigenvalues. If $\mathrm{rank}(Z_{11}) = 3$, then we have that Z_{11} is indefinite. Indeed, if Z is indefinite then so is Z_{11}. If Z is positive (negative) semidefinite then $-\widehat{L}WZW^T \widehat{L}^T$ is negative (positive) semidefinite. In this case Z_{11}

13

has two real symmetric matrices of opposite definiteness as block matrices on the diagonal. This shows that Z_{11} is indefinite and thus can have at most two negative eigenvalues and at most two positive eigenvalues.

If $\widehat{L}$ is invertible, then

$$\left[\begin{array}{cc} I_2 & 0 \\ 0 & W^{-1}\widehat{L}^{-1} \end{array}\right] \left[\begin{array}{cc} Z & \widehat{L}^T \\ \widehat{L} & -\widehat{L}WZW^T\widehat{L}^T \end{array}\right] \left[\begin{array}{cc} I_2 & 0 \\ 0 & \widehat{L}^{-T}W^{-T} \end{array}\right] = \left[\begin{array}{cc} Z & W^{-T} \\ W^{-1} & -Z \end{array}\right]$$

is a real symmetric Hamiltonian matrix. Therefore, by using the Hamiltonian spectral symmetry with respect to the imaginary axis, it follows that Z_{11} has two positive and two negative eigenvalues. □

In the next section we will discuss real stability radii under restricted perturbations, where the restrictions will be expressed with the help of a *restriction matrix* $B \in \mathbb{R}^{n,r}$. To deal with those, we will need the following lemmas.

Lemma 2.11 ([20]) *Let $B \in \mathbb{R}^{n,r}$ with $\operatorname{rank}(B) = r$, let $y \in \mathbb{C}^r \setminus \{0\}$, and let $z \in \mathbb{C}^n \setminus \{0\}$. Then for all $A \in \mathbb{R}^{r,r}$ we have $BAy = z$ if and only if $Ay = B^\dagger z$ and $BB^\dagger z = z$.*

Lemma 2.12 *Let $B \in \mathbb{R}^{n,r}$ with $\operatorname{rank}(B) = r$, let $y \in \mathbb{C}^r \setminus \{0\}$ and $z \in \mathbb{C}^n \setminus \{0\}$.*

1) *If $\operatorname{rank}([z\ \bar{z}]) = 1$, then there exists a positive semidefinite $A = A^T \in \mathbb{R}^{r,r}$ satisfying $BAy = z$ if and only if $BB^\dagger z = z$ and $y^H B^\dagger z > 0$.*

2) *If $\operatorname{rank}([z\ \bar{z}]) = 2$. then there exists a positive semidefinite $A = A^T \in \mathbb{R}^{r,r}$ satisfying $BAy = z$ if and only if $BB^\dagger z = z$ and $y^H B^\dagger z > |y^T B^\dagger z|$.*

Proof. Let $A \in \mathbb{R}^{r,r}$, then by Lemma 2.11 we have that $BAy = z$ if and only if

$$BB^\dagger z = z \quad \text{and} \quad Ay = B^\dagger z. \tag{2.29}$$

If $\operatorname{rank}([z\ \bar{z}]) = 1$, then by Theorem 2.6 the identity (2.29) is equivalent to $BB^\dagger z = z$ and $y^H B^\dagger z > 0$. If $\operatorname{rank}([z\ \bar{z}]) = 2$ then (2.29) is equivalent to

$$BB^\dagger[z\ \bar{z}] = [z\ \bar{z}] \quad \text{and} \quad A[y\ \bar{y}] = [B^\dagger z\ B^\dagger \bar{z}], \tag{2.30}$$

because A is real. Note that $\operatorname{rank}([B^\dagger z\ B^\dagger \bar{z}]) = 2$, because otherwise there would exist $\alpha \in \mathbb{C}^2 \setminus \{0\}$ such that $[B^\dagger z\ B^\dagger \bar{z}]\alpha = 0$. But this implies that $[z\ \bar{z}]\alpha = [BB^\dagger z\ BB^\dagger \bar{z}]\alpha = 0$ in contradiction to the fact that $\operatorname{rank}([z\ \bar{z}]) = 2$. Thus, by Theorem 2.5, there exists $0 \leq A \in \mathbb{R}^{r,r}$ satisfying (2.30) if and only if $BB^\dagger z = z$ and $y^H B^\dagger z > |y^T B^\dagger z|$. □

The following is a version of Lemma 2.12 without semidefiniteness.

Lemma 2.13 *Let $B \in \mathbb{R}^{n,r}$ with $\operatorname{rank}(B) = r$, let $y \in \mathbb{C}^r \setminus \{0\}$ and $z \in \mathbb{C}^n \setminus \{0\}$. Then there exist a real symmetric matrix $A \in \mathbb{R}^{r,r}$ satisfying $BAy = z$ if and only if $BB^\dagger z = z$ and $y^H B^\dagger z \in \mathbb{R}$.*

Proof. The proof is analogous to that of Lemma 2.12, using Theorem 2.8 and Theorem 2.9 instead of Theorem 2.5 and Theorem 2.6. □

In this section we have presented several mapping theorems, in particular for the real case. These will be used in the next section to determine formulas for the real stability radii of DH systems.

3 Real stability radii for DH systems

Consider a real linear time-invariant dissipative Hamiltonian (DH) system

$$\dot{x} = (J - R)Qx, \tag{3.1}$$

where $J, R, Q \in \mathbb{R}^{n,n}$ are such that $J^T = -J$, $R^T = R \geq 0$ and $Q^T = Q > 0$. For perturbations in the dissipation matrix we define the following real distances to instability.

Definition 3.1 *Consider a real DH system of the form (3.1) and let $B \in \mathbb{R}^{n,r}$ and $C \in \mathbb{R}^{q,n}$ be given restrictions matrices. Then for $p \in \{2, F\}$ we define the following stability radii.*

1) *The* stability radius $r_{\mathbb{R},p}(R; B, C)$ *of system (3.1) with respect to real general perturbations to R under the restriction (B, C) is defined by*

$$r_{\mathbb{R},p}(R; B, C) := \inf \left\{ \|\Delta\|_p \,\Big|\, \Delta \in \mathbb{R}^{r,q},\ \Lambda\big((J - R)Q - (B\Delta C)Q\big) \cap i\mathbb{R} \neq \emptyset \right\}.$$

2) *The* stability radius $r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B)$ *of system (3.1) with respect to real structure-preservig semidefinite perturbations from the set*

$$\mathcal{S}_d(R, B) := \left\{ \Delta \in \mathbb{R}^{r,r} \,\big|\, \Delta^T = \Delta \leq 0 \ \text{ and } \ (R + B\Delta B^T) \geq 0 \right\} \tag{3.2}$$

under the restriction (B, B^T) is defined by

$$r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) := \inf \left\{ \|\Delta\|_p \,\Big|\, \Delta \in \mathcal{S}_d(R, B),\ \Lambda\big((J - R)Q - (B\Delta B^T)Q\big) \cap i\mathbb{R} \neq \emptyset \right\}.$$

3) *The* stability radius $r_{\mathbb{R},p}^{\mathcal{S}_i}(R; B)$ *of system (3.1) with respect to structure-preserving indefinite perturbations from the set*

$$\mathcal{S}_i(R, B) := \left\{ \Delta \in \mathbb{R}^{r,r} \,\big|\, \Delta^T = \Delta \ \text{ and } \ (R + B\Delta B^T) \geq 0 \right\} \tag{3.3}$$

under the restriction (B, B^T) is defined by

$$r_{\mathbb{R},p}^{\mathcal{S}_i}(R; B) := \inf \left\{ \|\Delta\|_p \,\Big|\, \Delta \in \mathcal{S}_i(R, B),\ \Lambda\big((J - R)Q - (B\Delta B^T)Q\big) \cap i\mathbb{R} \neq \emptyset \right\}.$$

The formula for the stability radius $r_{\mathbb{R},2}(R; B, C)$ is a direct consequence of the following well-known result.

Theorem 3.2 ([23]) *For a given $M \in \mathbb{C}^{p,m}$, define*

$$\mu_{\mathbb{R}}(M) := \left(\inf \left\{ \|\Delta\| \,\big|\, \Delta \in \mathbb{R}^{m,p},\ \det(I_m - \Delta M) = 0 \right\} \right)^{-1}.$$

Then

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma \in (0,1]} \sigma_2 \left(\begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \right),$$

where $\sigma_2(A)$ is the second largest singular value of a matrix A. Furthermore, an optimal Δ that attains the value of $\mu_{\mathbb{R}}(M)$ can be chosen of rank at most two.

Applying this theorem to DH systems we obtain the following corollary.

Corollary 3.3 *Consider an asymptotically stable DH system of the form (3.1) and let $B \in \mathbb{R}^{n,r}$ and $C \in \mathbb{R}^{q,n}$ be given restriction matrices. Then*

$$r_{\mathbb{R},2}(R; B, C) = \inf_{\omega \in \mathbb{R}} \left(\inf_{\gamma \in (0,1]} \sigma_2 \left(\begin{bmatrix} \operatorname{Re} M(\omega) & -\gamma \operatorname{Im} M(\omega) \\ \gamma^{-1} \operatorname{Im} M(\omega) & \operatorname{Re} M(\omega) \end{bmatrix} \right) \right)^{-1} \tag{3.4}$$

and

$$r_{\mathbb{R},2}(R; B, C) \leq r_{\mathbb{R},F}(R; B, C) \leq \sqrt{2}\, r_{\mathbb{R},2}(R; B, C), \tag{3.5}$$

where $M(\omega) := CQ((J - R)Q - i\omega I_n)^{-1}B$.

Proof. By definition, we have

$$\begin{aligned}
r_{\mathbb{R},2}(R; B, C) &= \inf \left\{ \|\Delta\| \,\Big|\, \Delta \in \mathbb{R}^{r,q},\ \Lambda\big((J - R)Q - (B\Delta C)Q\big) \cap i\mathbb{R} \neq \emptyset \right\} \\
&= \inf \left\{ \|\Delta\| \,\Big|\, \Delta \in \mathbb{R}^{r,q},\ \omega \in \mathbb{R},\ \det\big(i\omega I_n - (J - R)Q + B\Delta CQ\big) = 0 \right\} \\
&= \inf \left\{ \|\Delta\| \,\Big|\, \Delta \in \mathbb{R}^{r,q},\ \omega \in \mathbb{R},\ \det\big(I_n - B\Delta CQ((J - R)Q - i\omega I_n)^{-1}\big) = 0 \right\},
\end{aligned}$$

where the last equality follows, since $(J - R)Q$ is asymptotically stable so that the inverse of $(J - R)Q - i\omega I_n$ exists for all $\omega \in \mathbb{R}$. Thus we have

$$\begin{aligned}
r_{\mathbb{R},2}(R; B, C) &= \inf \left\{ \|\Delta\| \,\Big|\, \Delta \in \mathbb{R}^{r,q},\ \omega \in \mathbb{R},\ \det\big(I_n - \Delta CQ((J - R)Q - i\omega I_n)^{-1}B\big) = 0 \right\} \\
&= \inf_{\omega \in \mathbb{R}} \left(\mu_{\mathbb{R}}\big(M(\omega)\big) \right)^{-1},
\end{aligned}$$

where $M(\omega) := CQ((J - R)Q - i\omega I_n)^{-1}B$. Therefore, (3.4) follows from Theorem 3.2, and if $\widehat{\Delta}$ is of rank at most two such that $\|\widehat{\Delta}\| = r_{\mathbb{R},2}(R; B, C)$, then (3.5) follows by using the definition of $r_{\mathbb{R},F}(R; B, C)$ and by the fact that

$$\|\widehat{\Delta}\| = r_{\mathbb{R},2}(R; B, C) \leq r_{\mathbb{R},F}(R; B, C) \leq \|\widehat{\Delta}\|_F \leq \sqrt{\operatorname{rank}(\Delta)}\, \|\widehat{\Delta}\|_2 = \sqrt{2}\, r_{\mathbb{R},2}(R; B, C). \qquad \square$$

To derive the real structured stability radii, we follow the strategy in [20] for the complex case, to reformulate the problem of computing $r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B)$ or $r_{\mathbb{R},p}^{\mathcal{S}_i}(R; B)$ in terms of real structured mapping problems. The following lemma of [20] (expressed here for real matrices) gives a characterization when a DH system of the form (3.1) has eigenvalues on the imaginary axis.

Lemma 3.4 [20, Lemma 3.1] *Let $J, R, Q \in \mathbb{R}^{n,n}$ be such that $J^T = -J$, $R^T = R \geq 0$ and $Q^T = Q > 0$. Then $(J - R)Q$ has an eigenvalue on the imaginary axis if and only if $RQx = 0$ for some eigenvector x of JQ. Furthermore, all purely imaginary eigenvalues of $(J - R)Q$ are semisimple.*

After these preliminaries we obtain the following formulas for the stability radii.

3.1 The stability radius $r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B)$

To derive formulas for the stability radius under real structure-preserving restricted and semidefinite perturbations we need the following two lemmas.

Lemma 3.5 *Let $H^T = H \in \mathbb{R}^{n,n}$ be positive semidefinite. Then $x^H H x \geq |x^T H x|$ for all $x \in \mathbb{C}^n$, and equality holds if and only if Hx and $H\bar{x}$ are linearly dependent.*

Proof. Let $S \in \mathbb{R}^{n,n}$ be a symmetric positive semidefinite square root of H, i.e. $S^2 = H$. Then using the Cauchy-Schwarz inequality we obtain that

$$\left|x^T H x\right| = \left|\langle S\bar{x}, Sx\rangle\right| \leq \|S\bar{x}\| \cdot \|Sx\| = \sqrt{\bar{x}^H H \bar{x}} \cdot \sqrt{x^H H x} = x^H H x,$$

because $\bar{x}^H H \bar{x} = \overline{x^H H x} = x^H H x$ as H is real. In particular equality holds if and only if Sx and $S\bar{x}$ are linearly dependent which is easily seen to be equivalent to the linear dependence of Hx and $H\bar{x}$. $\quad\square$

Lemma 3.6 *Let $R, W \in \mathbb{R}^{n,n}$ be such that $R^T = R \geq 0$ and $W^T = W > 0$. If $x \in \mathbb{C}^n$ is such that $\text{rank}([x\ \bar{x}]) = 2$ and $x^H W^T R W x > |x^T W^T R W x|$, set*

$$\Delta_R := -RW[\text{Re}\,x\ \ \text{Im}\,x]\left([\text{Re}\,x\ \ \text{Im}\,x]^H WRW[\text{Re}\,x\ \ \text{Im}\,x]\right)^{-1}[\text{Re}\,x\ \ \text{Im}\,x]^H WR.$$

Then $R + \Delta_R$ is symmetric positive semidefinite.

Proof. Since R and W are real symmetric and $x^H W^T R W x > |x^T W^T R W x|$, the matrix $[\text{Re}\,x\ \ \text{Im}\,x]^H WRW[\text{Re}\,x\ \ \text{Im}\,x]$ is real symmetric and positive definite and, therefore, Δ_R is well defined and we have $\Delta_R = \Delta_R^T \leq 0$. We prove that $R + \Delta_R \geq 0$ by showing that all its eigenvalues are nonnegative. Since W is nonsingular, we have that $Wx \neq 0$. Also Δ_R is a real matrix of rank two satisfying $\Delta_R W[\text{Re}\,x\ \ \text{Im}\,x] = -RW[\text{Re}\,x\ \ \text{Im}\,x]$. This implies that

$$(R + \Delta_R)W[\text{Re}\,x\ \ \text{Im}\,x] = 0. \tag{3.6}$$

Since $\text{rank}[\text{Re}\,x\ \ \text{Im}\,x] = \text{rank}[x\ \bar{x}] = 2$ and since W is nonsingular, we have that $W\,\text{Re}\,x$ and $W\,\text{Im}\,x$ are linearly independent eigenvectors of $R + \Delta_R$ corresponding to the eigenvalue zero.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of R and let $\eta_1, \ldots, \eta_n$ be the eigenvalues of $R + \Delta_R$, where both lists are arranged in nondecreasing order, i.e.,

$$0 \leq \lambda_1 \leq \cdots \leq \lambda_n \quad \text{and} \quad \eta_1 \leq \cdots \leq \eta_n.$$

Since Δ_R is of rank two, by the Cauchy interlacing theorem [6],

$$\lambda_k \leq \eta_{k+2} \quad \text{and} \quad \eta_k \leq \lambda_{k+2} \quad \text{for } k = 1, \ldots, n-2. \tag{3.7}$$

This implies that $0 \leq \eta_3 \leq \cdots \leq \eta_n$, and thus the assertion follows once we show that $\eta_1 = 0$ and $\eta_2 = 0$. If R is positive definite, then $\lambda_1, \ldots, \lambda_n$ satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n$ and, therefore, $0 < \eta_3 \leq \cdots \leq \eta_n$. Therefore we must have $\eta_1 = 0$ and $\eta_2 = 0$ by (3.6).

If R is positive semidefinite but singular, then let k be the dimension of the kernel of R. We then have $k < n$, because $R \neq 0$. Letting ℓ be the dimension of kernel of $R + \Delta_R$, then using (3.7) we have that

$$k - 2 \leq \ell \leq k + 2,$$

and we have $\eta_1 = 0$ and $\eta_2 = 0$ if we show that $\ell = k + 2$. Since W is nonsingular, the kernels of R and RW have the same dimension k. Let $x_1, \ldots, x_k$ be linearly independent eigenvectors of RW associated with the eigenvalue zero, i.e., we have $RWx_i = 0$ for all $i = 1, \ldots, k$. Then $\Delta_R W x_i = 0$ for all $i = 1, \ldots, k$, and hence $(R + \Delta_R)W x_i = 0$ for all $i = 1, \ldots, k$. The linear independence of $x_1, \ldots, x_k$ together with the nonsingularity of W implies that $Wx_1, \ldots, Wx_k$ are linearly independent. By (3.6) we have $(R + \Delta_R)W\,\text{Re}\,x = 0$ and $(R + \Delta_R)W\,\text{Im}\,x = 0$.

Moreover, the vectors $W \operatorname{Re} x, W \operatorname{Im} x, W x_1, \ldots, W x_k$ are linearly independent. Indeed, let $\alpha, \beta, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$ be such that

$$\alpha W \operatorname{Re} x + \beta W \operatorname{Im} x + \alpha_1 W x_1 + \cdots + \alpha_k W x_k = 0.$$

Then we have $R(\alpha W \operatorname{Re} x + \beta W \operatorname{Im} x) = 0$ as $R W x_i = 0$ for all $i = 1, \ldots, k$. This implies that $\alpha = 0$ and $\beta = 0$, because $R W \operatorname{Re} x$ and $R W \operatorname{Im} x$ are linearly independent. The linear independence of $W x_1, \ldots, W x_k$ then implies that $\alpha_i = 0$ for all $i = 1, \ldots, k$, and hence $W \operatorname{Re} x, W \operatorname{Im} x, W x_1, \ldots, W x_k$ are linearly independent eigenvectors of $R + \Delta_R$ corresponding to the eigenvalue zero. Thus, the dimension of the kernel of $R + \Delta_R$ is at least $k + 2$ and hence we must have $\eta_1 = 0$ and $\eta_2 = 0$. $\quad\square$

Using these lemmas, we obtain a formula for the structured real stability radius of DH systems.

Theorem 3.7 *Consider an asymptotically stable DH system of the form (3.1). Let $B \in \mathbb{R}^{n,r}$ with $\operatorname{rank}(B) = r$, and let $p \in \{2, F\}$. Furthermore, for $j \in \{1, 2\}$ let Ω_j denote the set of all eigenvectors x of JQ such that $(I_n - BB^\dagger)RQx = 0$ and $\operatorname{rank}\left(\begin{bmatrix} RQx & RQ\bar{x} \end{bmatrix}\right) = j$, and let $\Omega := \Omega_1 \cup \Omega_2$. Then $r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B)$ is finite if and only if Ω is non-empty. If this is the case, then*

$$r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) = \min \left\{ \inf_{x \in \Omega_1} \left\| \frac{(B^\dagger RQx)(B^\dagger RQx)^H}{x^H QRQx} \right\|_p, \ \inf_{x \in \Omega_2} \| Y(Y^H X)^{-1} Y^H \|_p \right\}, \tag{3.8}$$

where $X = B^T Q[\operatorname{Re} x \ \operatorname{Im} x]$ and $Y = B^\dagger RQ[\operatorname{Re} x \ \operatorname{Im} x]$.

Proof. By definition, we have

$$r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) := \inf \left\{ \| \Delta \|_p \ \middle| \ \Delta \in \mathcal{S}_d(R, B), \ \Lambda\left((J - R)Q - (B\Delta B^T)Q\right) \cap i\mathbb{R} \neq \emptyset \right\},$$

where $\mathcal{S}_d(R, B) := \left\{ \Delta \in \mathbb{R}^{r,r} \,\middle|\, \Delta^T = \Delta \leq 0 \ \text{and} \ (R + B\Delta B^T) \geq 0 \right\}$. By using Lemma 3.4, we obtain that

$$r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) = \inf \left\{ \| \Delta \|_p \ \middle| \ \Delta \in \mathcal{S}_d(R, B), \ (R + B\Delta B^T)Qx = 0 \text{ for some eigenvector } x \text{ of } JQ \right\}$$

$$= \inf \left\{ \| \Delta \|_p \ \middle| \ \Delta \in \mathcal{S}_d(R, B), \ B\Delta B^T Qx = -RQx \text{ for some eigenvector } x \text{ of } JQ \right\}$$

$$= \inf \left\{ \| \Delta \|_p \ \middle| \ \Delta \in \mathcal{S}_d(R, B), \ \Delta B^T Qx = -B^\dagger RQx \text{ for some } x \in \Omega \right\}, \tag{3.9}$$

since by Lemma 2.11 we have $B\Delta B^T Qx = -RQx$ if and only if $\Delta B^T Qx = -B^\dagger RQx$ and $BB^\dagger RQx = RQx$, and thus $x \in \Omega$. From (3.9) and $\mathcal{S}_d(R, B) \subseteq \{\Delta \in \mathbb{R}^{r,r} \,|\, \Delta^T = \Delta \leq 0\}$, we obtain that

$$r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) \geq \inf \left\{ \| \Delta \|_p \ \middle| \ \Delta = \Delta^T \in \mathbb{R}^{r,r}, \ \Delta \leq 0, \ \Delta B^T Qx = -B^\dagger RQx \text{ for some } x \in \Omega \right\} =: \varrho. \tag{3.10}$$

The infimum on the right hand side of (3.10) is finite if and only if Ω is non-empty. The same will also hold for $r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B)$ if we show equality in (3.10). To this end we will use the abbreviations

$$\varrho_j := \inf \left\{ \| \Delta \|_p \,|\, \Delta = \Delta^T \in \mathbb{R}^{r,r}, \ \Delta \leq 0, \ \Delta B^T Qx = -B^\dagger RQx \text{ for some } x \in \Omega_j \right\}$$

for $j \in \{1, 2\}$, i.e., we have $\varrho = \min\{\varrho_1, \varrho_2\}$, and we consider two cases.

Case (1): $\varrho = \varrho_1$. If $x \in \Omega_1$, then $BB^\dagger RQx = RQx$, and RQx and $RQ\bar{x}$ are linearly dependent. But then also $B^\dagger RQx$ and $B^\dagger RQ\bar{x}$ are linearly dependent and hence, by Lemma 2.12 there exists $\Delta \in \mathbb{R}^{r,r}$ such that $\Delta \leq 0$ and $\Delta B^T Qx = -B^\dagger RQx$ (and thus $(-\Delta) \geq 0$ and $(-\Delta)B^T Qx = B^\dagger RQx$) if and only if $x^H QRQx > 0$. This condition is satisfied for all $x \in \Omega_1$. Indeed, since R is positive semidefinite, we find that $x^H QRQx \geq 0$, and $x^H QRQx = 0$ would imply $RQx = 0$ and thus $(J-R)Qx = JQx$ which means that x is an eigenvector of $(J-R)Qx$ associated with an eigenvalue on the imaginary axis in contradiction to the assumption that $(J - R)Q$ only has eigenvalues in the open left half plane.

Using minimal norm mappings from Theorem 2.6 we thus have

$$
\begin{aligned}
r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) \;\; &\geq \;\; \varrho_1 = \inf\left\{\|\Delta\|_p \mid \Delta = \Delta^T \in \mathbb{R}^{r,r}, \; \Delta \leq 0, \; \Delta B^T Qx = -B^\dagger RQx, \; x \in \Omega_1 \right\} \\
&= \;\; \inf_{x \in \Omega_1}\left\{\left\|\frac{(B^\dagger RQx)(B^\dagger RQx)^H}{x^H QBB^\dagger RQx}\right\|\right\} = \inf_{x \in \Omega_1}\left\{\left\|\frac{(B^\dagger RQx)(B^\dagger RQx)^H}{x^H QRQx}\right\|\right\}. \;\; (3.11)
\end{aligned}
$$

As the expression in (3.11) is invariant under scaling of x, it is sufficient to take the infimum over all $x \in \Omega_1$ of norm one. Then a compactness argument shows that the infimum is actually attained for some $\widetilde{x} \in \Omega_1$ and by [20, Theorem 4.2], the matrix

$$
\widehat{\Delta} := \frac{(B^\dagger RQ\widetilde{x})(B^\dagger RQ\widetilde{x})^H}{\widetilde{x}^H QRQ\widetilde{x}}
$$

is the unique (resp. a) complex matrix of minimal Frobenius (resp. spectral) norm satisfying $\widehat{\Delta}^T = \widehat{\Delta} \leq 0$ and $\widehat{\Delta}B^T Qx = -B^\dagger RQx$. Also, by Theorem 2.6 the matrix $\widehat{\Delta}$ is real and by [20, Lemma 4.1] we have $R + B\widehat{\Delta}B^T \geq 0$. Thus $\widehat{\Delta} \in \mathcal{S}_d(R, B)$. But this means that $r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) = \varrho_1 = \varrho$.

Case (2): $\varrho = \varrho_2$. If $x \in \Omega_2$, then $BB^\dagger RQx = RQx$, and $\operatorname{Re} RQx$ and $\operatorname{Im} RQx$ are linearly independent. But then, also $\operatorname{Re} B^\dagger RQx$ and $\operatorname{Im} B^\dagger RQx$ are linearly independent, because otherwise, also $\operatorname{Re} RQx = \operatorname{Re} BB^\dagger RQx$ and $\operatorname{Im} RQ\bar{x} = \operatorname{Im} BB^\dagger RQ\bar{x}$ would be linearly dependent. Thus, by Lemma 2.12 there exists $\Delta \in \mathbb{R}^{r,r}$ such that $\Delta \leq 0$ and $\Delta B^T Qx = -B^\dagger RQx$ if and only if $x^H QRQx > |x^T QRQx|$. By Lemma 3.5 this condition is satisfied for all $x \in \Omega_2$, because R and thus QRQ is positive semidefinite.

Using minimal norm mappings from Theorem 2.5 we then obtain

$$
\begin{aligned}
&r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) \\
&\geq \;\; \varrho_2 = \inf\left\{\|\Delta\|_p \mid \Delta = \Delta^T \in \mathbb{R}^{r,r}, \; \Delta \leq 0, \; \Delta B^T Qx = -B^\dagger RQx, \; x \in \Omega_2 \right\} \qquad (3.12) \\
&= \;\; \inf\left\{\|Y(Y^H X)^{-1}Y^H\|_p \;\middle|\; X = B^T Q\,[\operatorname{Re} x \;\; \operatorname{Im} x], \; Y = B^\dagger RQ\,[\operatorname{Re} x \;\; \operatorname{Im} x], \; x \in \Omega_2 \right\} \\
&= \;\; \left\|\widetilde{Y}(\widetilde{Y}^H \widetilde{X})^{-1}\widetilde{Y}^H\right\|_p,
\end{aligned}
$$

for some $\tilde{x} \in \Omega_2$, where $\widetilde{X} = B^T Q\,[\operatorname{Re}\tilde{x} \;\; \operatorname{Im}\tilde{x}]$ and $\widetilde{Y} = B^\dagger RQ\,[\operatorname{Re}\tilde{x} \;\; \operatorname{Im}\tilde{x}]$. Indeed, the matrix $Y(Y^H X)^{-1}Y^H$ is invariant under scaling of x, so that it is sufficient to take the infimum over all $x \in \Omega_2$ having norm one. Then a compactness argument shows that the infimum is actually a minimum and attained for some $\widehat{x} \in \Omega_2$. Then setting

$$
\tilde{\Delta}_R := -\widetilde{Y}(\widetilde{Y}^H \widetilde{X})^{-1}\widetilde{Y}^H,
$$

we have equality in (3.12) if we show that $R + B\tilde{\Delta}_R B^T$ is positive semidefinite, because this would imply that $\tilde{\Delta}_R \in \mathcal{S}_d(R, B)$. But this follows from Lemma 3.6 by noting that $BB^\dagger RQ\,[\text{Re}\,\tilde{x}\ \text{Im}\,\tilde{x}] = RQ\,[\text{Re}\,\tilde{x}\ \text{Im}\,\tilde{x}]$. Indeed, by the definition of Ω_2, the vectors $\text{Re}\,RQ\tilde{x}$ and $\text{Im}\,RQ\tilde{x}$ are linearly independent, and

$$
\begin{aligned}
R + B\tilde{\Delta}_R B^T &= R - B\widetilde{Y}(\widetilde{Y}^H \widetilde{X})^{-1}\widetilde{Y}^H B^T \\
&= R - BB^\dagger RQ[\text{Re}\,x\ \text{Im}\,x]\big([\text{Re}\,x\ \text{Im}\,x]^H QRQ[\text{Re}\,x\ \text{Im}\,x]\big)^{-1}[\text{Re}\,x\ \text{Im}\,x]^H QR(B^\dagger)^T B^T \\
&= R - RQ[\text{Re}\,x\ \text{Im}\,x]\big([\text{Re}\,x\ \text{Im}\,x]^H QRQ[\text{Re}\,x\ \text{Im}\,x]\big)^{-1}[\text{Re}\,x\ \text{Im}\,x]^H QR.
\end{aligned}
$$

is positive semidefinite by Lemma 3.6. This proves that $r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B) = \varrho_2 = \varrho$. $\quad\square$

Remark 3.8 In the case that $R > 0$ and J are invertible, the set Ω_1 from Theorem 3.7 is empty and hence $\Omega = \Omega_2$, because if $R > 0$ and if x is an eigenvector of JQ with $\text{rank}\left(\begin{bmatrix} RQx & RQ\bar{x} \end{bmatrix}\right) = 1$ then, x is necessarily an eigenvector associated with the eigenvalue zero of JQ. Indeed, if RQx and $RQ\bar{x}$ are linearly dependent, then x and $\bar{x}$ are linearly independent, because RQ is nonsingular as $R > 0$ and $Q > 0$. This is only possible if x is associated with a real eigenvalue, and since the eigenvalues of JQ are on the imaginary axis, this eigenvalue must be zero.

Remark 3.9 We mention that in the case that R and J are invertible, $r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B)$ is also the minimal norm of a structure-preserving perturbation of rank at most two that moves an eigenvalue of $(J - R)Q$ to the imaginary axis. To be more precise, if

$$
\mathcal{S}_2(R, B) := \left\{ \Delta \in \mathbb{R}^{r,r} \,\middle|\, \Delta^T = \Delta, \quad \text{rank}\,\Delta \le 2, \quad \text{and} \quad (R + B\Delta B^T) \ge 0 \right\} \tag{3.13}
$$

and

$$
r_{\mathbb{R},p}^{\mathcal{S}_2}(R; B) := \inf \left\{ \|\Delta\|_p \,\middle|\, \Delta \in \mathcal{S}_2(R, B),\ \Lambda\big((J - R)Q - (B\Delta B^T)Q\big) \cap i\mathbb{R} \neq \emptyset \right\},
$$

then we have $r_{\mathbb{R},p}^{\mathcal{S}_2}(R; B) = r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B)$. Indeed, assume that $\Delta \in \mathcal{S}_2(R, B)$ is such that $(J - R)Q - (B\Delta B^T)Q$ has an eigenvalue on the imaginary axis. By Lemma 3.4 we then have $(R - B\Delta B^T)Qx = 0$ for some eigenvector x of JQ. Since R and J are invertible, it follows from Remark 3.8 that RQx and $RQ\bar{x}$ are linearly independent. Since Δ has rank at most two, it follows that the kernel of Δ has dimension at least $n - 2$. Thus, let $w_3, \ldots, w_n$ be linearly independent vectors from the kernel of Δ. Then $B^T Qx, B^T Q\bar{x}, w_3, \ldots, w_n$ is a basis of $\mathbb{C}^n$. Indeed, let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be such that

$$
\alpha_1 B^T Qx + \alpha_2 B^T Qx + \alpha_3 w_3 + \cdots + \alpha_n w_n = 0. \tag{3.14}
$$

Then multiplication with $B\Delta$ yields

$$
0 = \alpha_1 B\Delta B^T Qx + \alpha_2 B\Delta B^T Q\bar{x} = \alpha_1 RQx + \alpha_2 RQ\bar{x}
$$

and we obtain $\alpha_1 = \alpha_2 = 0$, because RQx and $RQ\bar{x}$ are linearly independent. But then (3.14) and the linear independence of $w_3, \ldots, w_n$ imply $\alpha_3 = \cdots = \alpha_n = 0$. Thus, setting $T := [B^T Qx, B^T Q\bar{x}, w_3, \ldots, w_n]$ we obtain that T is invertible and $T^H \Delta T = \text{diag}(D, 0)$, where

$$
D = \begin{bmatrix} x^H QB\Delta B^T Qx & \overline{x^T QB\Delta B^T Qx} \\ x^T QB\Delta B^T Qx & \overline{x^H QB\Delta B^T Qx} \end{bmatrix} = \begin{bmatrix} x^H QRQx & \overline{x^T QRQx} \\ x^T QRQx & \overline{x^H QRQx} \end{bmatrix}.
$$

20

Since by Lemma 3.5 we have $x^H QRQx > |x^T QRQx|$, it follows that D is positive definite which implies $\Delta \in \mathcal{S}_d(R; B)$ and hence $r_{\mathbb{R},p}^{\mathcal{S}_2}(R; B) \geq r_{\mathbb{R},p}^{\mathcal{S}_d}(R; B)$. The inequality "$\leq$" is trivial as minimal norm elements from $\mathcal{S}_d(R; B)$ have rank at most two.

In this subsection we have characterized the real structured restricted stability radius under positive semidefinite perturbations to the dissipation matrix R. In the next subsection we extend these results to indefinite perturbations that keep R semidefinite.

3.2 The stability radius $r_{\mathbb{R},p}^{\mathcal{S}_i}(R; B)$

If the perturbation matrices ΔR are allowed to be indefinite then the perturbation analysis is more complicated. We start with the following lemma.

Lemma 3.10 *Let* $0 < R = R^T$, $\Delta_R = \Delta_R^T \in \mathbb{R}^{n,n}$ *be such that* Δ_R *has at most two negative eigenvalues. If* $\dim (\mathrm{Ker}(R + \Delta_R)) = 2$, *then* $R + \Delta_R \geq 0$.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Δ_R. As Δ_R has at most two negative eigenvalues we may assume that $\lambda_3, \lambda_4, \ldots, \lambda_n \geq 0$ and we have the spectral decomposition

$$\Delta_R = \sum_{i=1}^n \lambda_i u_i u_i^H$$

with unit norm vectors $u_1, \ldots, u_n \in \mathbb{R}^n$. Since

$$\widetilde{R} = R + \sum_{i=3}^n \lambda_i u_i u_i^H > 0.$$

and $\lambda_1 u_1 u_1^H + \lambda_2 u_2 u_2^H$ is of rank two, we can apply the Cauchy interlacing theorem and obtain that $R + \Delta_R = \widetilde{R} + \lambda_1 u_1 u_1^H + \lambda_2 u_2 u_2^H$ has at least $n - 2$ positive eigenvalues. But then using the fact that $\dim(\mathrm{Ker}(R + \Delta_R)) = 2$ we get $R + \Delta_R \geq 0$.   □

Using this lemma, we obtain the following results on the stability radius $r_{\mathbb{R},p}^{\mathcal{S}_i}(R; B)$.

Theorem 3.11 *Consider an asymptotically stable DH system of the form* (3.1). *Let* $B \in \mathbb{R}^{n,r}$ *with* $\mathrm{rank}(B) = r$ *and let* $p \in \{2, F\}$. *Furthermore, for* $j \in \{1, 2\}$ *let* Ω_j *denote the set of all eigenvectors* x *of* JQ *such that* $BB^\dagger RQx = RQx$ *and* $\mathrm{rank}\left(\begin{bmatrix} B^T Qx & B^T Q\bar{x} \end{bmatrix} \right) = j$, *and let* $\Omega = \Omega_1 \cup \Omega_2$.

1) *If* $R > 0$, *then* $r_{\mathbb{R},p}^{\mathcal{S}_i}(R; B)$ *is finite if and only if* Ω *is non-empty. In that case we have*

$$r_{\mathbb{R},2}^{\mathcal{S}_i}(R; B) = \min\left\{ \inf_{x \in \Omega_1} \frac{\|(B^\dagger RQx)\|}{\|B^T Qx\|}, \ \inf_{x \in \Omega_2} \|YX^\dagger\| \right\} \tag{3.15}$$

and

$$r_{\mathbb{R},F}^{\mathcal{S}_i}(R; B) = \min\left\{ \inf_{x \in \Omega_1} \frac{\|(B^\dagger RQx)\|}{\|B^H Qx\|}, \ \inf_{x \in \Omega_2} \sqrt{\|YX^\dagger\|_F^2 - \mathrm{trace}\,(YX^\dagger (YX^\dagger)^H XX^\dagger)} \right\}, \tag{3.16}$$

where $X = [\mathrm{Re}\, B^T Qx \ \mathrm{Im}\, B^T Qx]$ *and* $Y = [\mathrm{Re}\, B^\dagger RQx \ \mathrm{Im}\, B^\dagger RQx]$ *for* $x \in \Omega_2$.

21

2) *If $R \geq 0$ is singular and if $r^{\mathcal{S}_i}_{\mathbb{R},p}(R; B)$ is finite, then Ω is non-empty and we have*

$$r^{\mathcal{S}_i}_{\mathbb{R},2}(R; B) \geq \min \left\{ \inf_{x \in \Omega_1} \frac{\|(B^\dagger RQx)\|}{\|B^T Qx\|}, \inf_{x \in \Omega_2} \|YX^\dagger\| \right\} \tag{3.17}$$

and

$$r^{\mathcal{S}_i}_{\mathbb{R},F}(R; B) \geq \min \left\{ \inf_{x \in \Omega_1} \frac{\|(B^\dagger RQx)\|}{\|B^T Qx\|}, \inf_{x \in \Omega_2} \sqrt{\|YX^\dagger\|_F^2 - \operatorname{trace}\left(YX^\dagger (YX^\dagger)^H XX^\dagger\right)} \right\}, \tag{3.18}$$

where $X = [\operatorname{Re} B^T Qx \; \operatorname{Im} B^T Qx]$ and $Y = [\operatorname{Re} B^\dagger RQx \; \operatorname{Im} B^\dagger RQx]$ for $x \in \Omega_2$.

Proof. By definition

$$r^{\mathcal{S}_i}_{\mathbb{R},p}(R; B) = \inf \left\{ \|\Delta\|_p \; \Big| \; \Delta \in \mathcal{S}_i(R, B), \; \Lambda\left((J - R)Q - (B\Delta B^T)Q\right) \cap i\mathbb{R} \neq \emptyset \right\},$$

where $\mathcal{S}_i(R, B) := \left\{ \Delta = \Delta^T \in \mathbb{R}^{r,r} \; \big| \; (R + B\Delta B^T) \geq 0 \right\}$. Using Lemma 3.4 and Lemma 2.11 and following the lines of the proof of Theorem 3.7, we get

$$r^{\mathcal{S}_i}_{\mathbb{R},p}(R; B) = \inf \left\{ \|\Delta\|_p \; \big| \; \Delta \in \mathcal{S}_i(R, B), \; \Delta B^T Qx = -B^\dagger RQx \text{ for some} \right.$$
$$\left. \text{eigenvector } x \text{ of } JQ \text{ satisfying } BB^\dagger RQx = RQx \right\}.$$

Since all elements of $\mathcal{S}_i(R, B)$ are real symmetric, we obtain that

$$r^{\mathcal{S}_i}_{\mathbb{R},p}(R; B) \geq \inf \left\{ \|\Delta\|_p \; \big| \; \Delta = \Delta^T \in \mathbb{R}^{r,r}, \; \Delta B^T Qx = -B^\dagger RQx \text{ for some } x \in \Omega \right\} =: \varrho^{(p)}. \tag{3.19}$$

The infimum in the right hand side of (3.19) is finite if Ω is non-empty, as by Lemma 2.13 for $x \in \Omega$ there exist $\Delta = \Delta^T \in \mathbb{R}^{r,r}$ such that $\Delta B^T Qx = -B^\dagger RQx$ if and only if $x^H QBB^\dagger RQx \in \mathbb{R}$. This condition is satisfied because of the fact that $BB^\dagger RQx = RQx$ and R is real symmetric. If $r^{\mathcal{S}_i}_{\mathbb{R},p}(R; B)$ is finite, then Ω is non-empty because otherwise the right hand side of (3.19) would be infinite. To continue, we will use the abbreviations

$$\varrho^{(p)}_j := \inf \left\{ \|\Delta\|_p \; \big| \; \Delta = \Delta^T \in \mathbb{R}^{r,r}, \; \Delta B^T Qx = -B^\dagger RQx \text{ for some } x \in \Omega_j \right\}$$

for $j \in \{1, 2\}$, i.e., $\varrho^{(p)} = \min\{\varrho^{(p)}_1, \varrho^{(p)}_2\}$, and we consider two cases.

Case (1): $\varrho^{(p)} = \varrho^{(p)}_1$. If $x \in \Omega_1$, then $BB^\dagger RQx = RQx$ and $B^T Qx$ and $B^T Q\bar{x}$ are linearly dependent. Then using mappings of minimal spectral resp. Frobenius norms (and again using compactness arguments), we obtain from Theorem 2.9 that

$$\begin{aligned} r^{\mathcal{S}_i}_{\mathbb{R},p}(R; B) &\geq \varrho^{(p)}_1 = \inf \left\{ \|\Delta\|_p \; \big| \; \Delta = \Delta^T \in \mathbb{R}^{r,r}, \; \Delta B^T Qx = -B^\dagger RQx, \; x \in \Omega_1 \right\} \\ &= \inf_{x \in \Omega_1} \left\{ \frac{\|B^\dagger RQx\|_p}{\|B^T Qx\|_p} \right\} = \frac{\|B^\dagger RQ\widehat{x}\|_p}{\|B^T Q\widehat{x}\|_p} \end{aligned}$$

for some $\tilde{x} := \widehat{x} \in \Omega_1$. This proves 2) and " $\geq$ " in (3.15) and (3.16) in the case $\varrho^{(p)} = \varrho^{(p)}_1$.

Case (2): $\varrho^{(p)} = \varrho^{(p)}_2$. If $x \in \Omega_2$, then $BB^\dagger RQx = RQx$ and $B^T Qx$ and $B^T Q\bar{x}$ are linearly independent. Using mappings of minimal spectral resp. Frobenius norms (and once more

using compactness arguments), we obtain from Theorem 2.8 that

$$
\begin{aligned}
r^{\mathcal{S}_i}_{\mathbb{R},2}(R;B) \; \geq \; \varrho_2^{(2)} &= \inf\left\{ \|\Delta\| \;\Big|\; \Delta = \Delta^T \in \mathbb{R}^{r,r}, \; \Delta B^T Q x = -B^\dagger R Q x, \; x \in \Omega_2 \right\} \\
&= \inf\left\{ \|Y X^\dagger\| \;\Big|\; X = [\operatorname{Re} B^T Q x \;\; \operatorname{Im} B^T Q x], \; Y = [\operatorname{Re} B^\dagger R Q x \;\; \operatorname{Im} B^\dagger R Q x], \; x \in \Omega_2 \right\} \\
&= \|\widehat{Y} \widehat{X}^\dagger\|,
\end{aligned}
$$

for some $\widehat{x} \in \Omega_2$, where $\widehat{X} = [\operatorname{Re} B^T Q \widehat{x} \;\; \operatorname{Im} B^T Q \widehat{x}]$ and $\widehat{Y} = [\operatorname{Re} B^\dagger R Q \widehat{x} \;\; \operatorname{Im} B^\dagger R Q \widehat{x}]$, and

$$
\begin{aligned}
\left(r^{\mathcal{S}_i}_{\mathbb{R},F}(R;B) \right)^2 \; \geq \; \left(\varrho_2^{(F)} \right)^2 &= \inf\left\{ \|\Delta\|_F^2 \;\Big|\; \Delta = \Delta^T \in \mathbb{R}^{r,r}, \; \Delta B^T Q x = -B^\dagger R Q x, \; x \in \Omega_2 \right\} \\
&= \inf\left\{ 2\|Y X^\dagger\|_F^2 - \operatorname{trace}\left(Y X^\dagger (Y X^\dagger)^H X X^\dagger \right) \;\Big|\; X = [\operatorname{Re} B^T Q x \;\; \operatorname{Im} B^T Q x], \right. \\
&\qquad\qquad \left. Y = [\operatorname{Re} B^\dagger R Q x \;\; \operatorname{Im} B^\dagger R Q x], \; x \in \Omega_2 \right\} \\
&= 2\|\widetilde{Y} \widetilde{X}^\dagger\|_F^2 - \operatorname{trace}\left(\widetilde{Y} \widetilde{X}^\dagger (\widetilde{Y} \widetilde{X}^\dagger)^H \widetilde{X} \widetilde{X}^\dagger \right)
\end{aligned}
$$

for some $\tilde{x} \in \Omega_2$, where $\widetilde{X} = [\operatorname{Re} B^T Q \tilde{x} \;\; \operatorname{Im} B^T Q \tilde{x}]$ and $\widetilde{Y} = [\operatorname{Re} B^\dagger R Q \tilde{x} \;\; \operatorname{Im} B^\dagger R Q \tilde{x}]$. This proves 2) and "$\geq$" in (3.15) and (3.16) in the case $\varrho^{(p)} = \varrho_2^{(p)}$.

In both cases (1) and (2), it remains to show that equality holds in (3.15) and (3.16) when $R > 0$. This would also prove that in the case $R > 0$ the non-emptyness of Ω implies the finiteness of $r^{\mathcal{S}_i}_{\mathbb{R},F}(R;B)$.

Thus, assume that $R > 0$ and let $\widehat{\Delta} = \widehat{\Delta}^T \in \mathbb{R}^{r,r}$ and $\widetilde{\Delta} = \widetilde{\Delta}^T \in \mathbb{R}^{r,r}$ be such that they satisfy

$$
\widehat{\Delta} B^T Q \widehat{x} = -B^\dagger R Q \widehat{x} \quad \text{and} \quad \widetilde{\Delta} B^T Q \tilde{x} = -B^\dagger R Q \tilde{x}, \tag{3.20}
$$

and such that they are mappings of minimal spectral or Frobenius norm, respectively, as in Theorem 2.8 or Theorem 2.9, respectively. The proof is complete if we show that $(R + B\widehat{\Delta} B^T) \geq 0$ and $(R + B\widetilde{\Delta} B^T) \geq 0$, because this would imply that $\widehat{\Delta}$ and $\widetilde{\Delta}$ belong to the set $\mathcal{S}_i(R,B)$. In the case $\varrho^{(p)} = \varrho_1^{(p)}$ this follows exactly as in the proof of [20, Theorem 4.5] which is the corresponding result to Theorem 3.11 in the complex case. (In fact, in this case $\widetilde{\Delta}$ and $\widehat{\Delta}$ coincide with the corresponding complex mappings of minimal norm.)

In the case $\varrho^{(p)} = \varrho_2^{(p)}$, we obtain from Lemma 2.10, that the matrices $\widehat{\Delta}$ and $\widetilde{\Delta}$ are of rank at most four with at most two negative eigenvalues. This implies that also the matrices $B\widehat{\Delta} B^T$ and $B\widetilde{\Delta} B^T$ individually have at most two negative eigenvalues. Indeed, let $B_1 \in \mathbb{R}^{n,n-r}$ be such that $[B \; B_1] \in \mathbb{R}^{n,n}$ is invertible then we can write

$$
B\widehat{\Delta} B^T = [B \; B_1] \begin{bmatrix} \widehat{\Delta} & 0 \\ 0 & 0 \end{bmatrix} [B \; B_1]^H
$$

and by Sylvester's law of inertia also $B\widehat{\Delta} B^T$ has at most two negative eigenvalues. A similar argument proves the assertion for $B\widetilde{\Delta} B^T$.

Furthermore, using (3.20), we obtain

$$
(R + B\widehat{\Delta} B^T) Q \widehat{x} = R Q \widehat{x} - B B^\dagger R Q \widehat{x} = R Q \widehat{x} - R Q \widehat{x} = 0
$$

and

$$
(R + B\widetilde{\Delta} B^T) Q \tilde{x} = R Q \tilde{x} - B B^\dagger R Q \tilde{x} = R Q \tilde{x} - R Q \tilde{x} = 0,
$$

since $\widehat{x}, \tilde{x} \in \Omega$, i.e., $BB^{\dagger}RQ\widehat{x} = RQ\widehat{x}$ and $BB^{\dagger}RQ\tilde{x} = RQ\tilde{x}$. Also rank$[\widehat{x} \ \bar{\widehat{x}}] = 2$ and rank$[\tilde{x} \ \bar{\tilde{x}}] = 2$, respectively, imply that

$$\dim\left(\mathrm{Ker}(R + B\widehat{\Delta}B^{T})\right) = 2 \quad \text{and} \quad \dim\left(\mathrm{Ker}(R + B\widetilde{\Delta}B^{T})\right) = 2.$$

Thus, Lemma 3.10 implies that $(R + B\widehat{\Delta}B^{T}) \geq 0$ and $(R + B\widetilde{\Delta}B^{T}) \geq 0$. $\quad\square$

Remark 3.12 It follows from the proof of Theorem 3.11 that in the case $R \geq 0$, the inequalities in (3.17) and (3.18) are actually equalities if the minimal norm mappings $\widehat{\Delta}$ and $\widetilde{\Delta}$ from (3.20) satisfy $(R + B\widehat{\Delta}B^{T}) \geq 0$ and $(R + B\widetilde{\Delta}B^{T}) \geq 0$, respectively. In our numerical experiments this was always the case, so that we conjecture that in general equality holds in (3.17) and (3.18).

4 Numerical experiments

In this section, we present some numerical experiments to illustrate that the real structured stability radii are indeed larger than the real unstructured ones. To compute the distances, in all cases we used the function `fminsearch` in Matlab Version No. 7.8.0 (R2009a) to solve the associated optimization problems.

We computed the real stability radii $r_{\mathbb{R},2}(R; B, B^{T})$, $r_{\mathbb{R},2}^{\mathcal{S}_i}(R; B)$ and $r_{\mathbb{R},2}^{\mathcal{S}_d}(R; B)$ with respect to real restricted perturbations to R, as obtained in Theorem 3.2, Theorem 3.7, and Theorem 3.11, respectively, and compared them to the corresponding complex distances $r_{\mathbb{C},2}(R; B, B^{T})$, $r_{\mathbb{C},2}^{\mathcal{S}_i}(R; B)$ and $r_{\mathbb{C},2}^{\mathcal{S}_d}(R; B)$ as obtained in [20, Theorem 3.3], [20, Theorem 4.5] and [20, Theorem 4.2], respectively.

We chose random matrices $J, R, Q, B \in \mathbb{R}^{n,n}$ for different values of $n \leq 14$ with $J^{T} = -J$, $R^{T} = R \geq 0$ and B of full rank, such that $(J - R)Q$ is asymptotically stable and all restricted stability radii were finite. The results in Table 4.1 illustrate that the stability radius $r_{\mathbb{R},2}(R, B, B^{T})$ obtained under general restricted perturbations is significantly smaller than the real stability radii obtained under structure-preserving restricted perturbations, and it also illustrates that the real stability radii may bd significantly larger than the corresponding complex stability radii.

size n	$r_{\mathbb{C},2}(R; B, B^{T})$	$r_{\mathbb{R},2}(R; B, B^{T})$	$r_{\mathbb{C},2}^{\mathcal{S}_i}(R; B)$	$r_{\mathbb{R},2}^{\mathcal{S}_i}(R; B)$	$r_{\mathbb{C},2}^{\mathcal{S}_d}(R; B)$	$r_{\mathbb{R},2}^{\mathcal{S}_d}(R; B)$
4	0.1649	0.1649	4.8237	8.2820	15.7348	17.2941
6	0.2390	0.2932	6.0391	15.0695	32.1021	38.1951
8	0.0665	0.1211	3.7859	5.1034	18.5757	24.0881
9	0.1648	0.1950	8.7892	25.3983	118.6212	227.5047
11	0.1135	0.1146	1.4003	1.6289	4.0695	4.5260
13	0.1013	0.1315	4.8191	5.6550	21.6680	43.9863
14	0.0410	0.1071	1.3693	1.7953	4.9894	6.1504

Table 4.1: *Comparison between complex and real stability radii.*

Example 4.1 As a second example consider the lumped parameter, mass-spring-damper dynamical system shown in Figure 4.1, see e.g. [32]. It has two point masses m_1 and m_2,

which are connected by a spring-damper pair with constants k_2 and c_2, respectively. Mass m_1 is linked to the ground by another spring-damper pair with constants k_1 and c_1, respectively. The system has two degrees of freedom. These are the displacements $u_1(t)$ and $u_2(t)$ of the two masses measured from their static equilibrium positions. Known dynamic forces $f_1(t)$ and $f_2(t)$ act on the masses. The equations of motion can be written in the matrix form as

$$M\ddot{u} + D\dot{u} + Ku = f,$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad D = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where the real symmetric matrices M, D and K denote the mass, damping and stiffness matrices, respectively, and $f, u, \dot{u}$ and $\ddot{u}$ are the force, displacement, velocity, and acceleration vectors, respectively. With the values $m_1 = 2$, $m_2 = 1$, $c_1 = 0.1$, $c_2 = 0.3$, $k_1 = 6$, and $k_2 = 3$
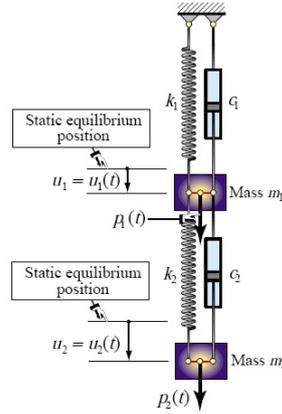

Figure 4.1: A two degree of freedom mass-spring-damper dynamical system.

we have $M, D, K > 0$ and an appropriate first order formulation has the linear DH pencil $\lambda I_4 - (J - R)Q$,

$$J = \begin{bmatrix} 0 & -K \\ K & 0 \end{bmatrix}, \quad R = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}^{-1}. \tag{4.1}$$

The eigenvalues of $(J - R)Q$ are $-0.2168 \pm 2.4361i$ and $-0.0332 \pm 1.2262i$ and thus the system is asymptotically stable. Setting $B = C^T = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \in \mathbb{R}^4$, we perturb only the damping matrix D and the corresponding real stability radii are given as follows.

$r_{\mathbb{R},2}(R, B, B^T)$	$r_{\mathbb{R},2}^{\mathcal{S}_i}(R, B)$	$r_{\mathbb{R},2}^{\mathcal{S}_d}(R, B)$
0.0796	0.1612	0.3250

As long as the norm of perturbation in damping matrix D is less than the stability radius the system remains asymptotically stable. We also see that the stability radii $r_{\mathbb{R},2}^{\mathcal{S}_i}(R, B)$ and $r_{\mathbb{R},2}^{\mathcal{S}_d}(R, B)$ that preserve the semidefiniteness of R are significantly larger than $r_{\mathbb{R},2}(R, B, B^T)$.

In Table 4.2, we list the values of various stability radii for mass-spring-damper system [32] of increasing size. The corresponding masses, damping constants and spring constants were chosen from the top of the vectors

$$m = [0.6857\ 1.7812\ 0.3785\ 0.2350\ 2.6719\ 0.7919\ 1.0132\ 1.3703]^T,$$
$$c = [0.6231\ 1.3050\ 2.3721\ 1.5574\ 1.0474\ 1.8343\ 0.3242\ 1.7115]^T$$

and

$$k = [0.2637\ 1.5203\ 0.8644\ 0.2485\ 0.7850\ 0.4135\ 2.3963\ 0.1022]^T,$$

respectively, i.e., four dimensional ($n = 4$) DH pencil as in (4.1) is corresponding to the first two entries of mass vector m, damping vector c and spring vector k, similarly $n = 6$ is corresponding to the first three entries from the vectors m, c and k, and so on. The restriction matrices $B = C^T = [e_1\ e_2 \cdots e_{n/2}] \in \mathbb{R}^{n,\frac{n}{2}}$ are such that only the damping matrix D in R is perturbed. In addition to the conclusions of Example 4.1, we found that as expected, the stability radius with respect to general perturbations decreases as the system dimension increases, while this is much less pronounced for the stability radii with respect to structure preserving perturbations.

size n	$r_{\mathbb{R},2}(R;B,B^T)$	$r_{\mathbb{R},2}^{\mathcal{S}_i}(R;B)$	$r_{\mathbb{R},2}^{\mathcal{S}_d}(R;B)$
4	0.2827	0.3213	0.3642
6	0.1755	0.2417	0.3299
8	0.1220	0.1995	0.3221
10	0.1013	0.3221	0.9009
12	0.0772	0.2817	0.9308
14	0.0618	0.2577	0.9938
16	0.0524	0.2560	1.1537

Table 4.2: *Various stability radii for mass-spring-damper system of increasing size.*

Conclusions

We have presented formulas for the stability radii under real restricted structure-preserving perturbations to the dissipation term R in dissipative Hamiltonian systems. The results and the numerical examples show that the system is much more robustly asymptotically stable under structure-preserving perturbations than when the structure is ignored. Open problems include the computation of the real stability radii when the energy functional Q or the structure matrix J, or all three matrices R, Q, and J are perturbed.

References

[1] B. Adhikari. *Backward perturbation and sensitivity analysis of structured polynomial eigenvalue problem.* PhD thesis, Dept. of Math., IIT Guwahati, Assam, India, 2008.

[2] R. Byers. A bisection method for measuring the distance of a stable to unstable matrices. *SIAM J. Sci. Statist. Comput.*, 9:875–881, 1988.

[3] M. Dalsmo and A.J. van der Schaft. On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM J. Control Optim.*, 37:54–91, 1999.

[4] C. Davis, W. Kahan, and H. Weinberger. Norm-preserving dialations and their applications to optimal error bounds. 19:445–469, 1982.

[5] M.A. Freitag and A. Spence. A newton-based method for the calculation of the distance to instability. 435(12):3189–3205, 2011.

[6] F.R. Gantmacher. *Theory of Matrices*, volume 1. Chelsea, New York, 1959.

[7] Israel Gohberg, Peter Lancaster, and Leiba Rodman. *Indefinite linear algebra and applications*. Birkhäuser, Basel, 2006.

[8] G. Golo, A.J. van der Schaft, P.C. Breedveld, and B.M. Maschke. Hamiltonian formulation of bond graphs. In A. Rantzer R. Johansson, editor, *Nonlinear and Hybrid Systems in Automotive Control*, pages 351–372. Springer, Heidelberg, 2003.

[9] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Johns Hopkins Univ. Press, Baltimore, 3rd edition, 1996.

[10] N. Gräbner, V. Mehrmann, S. Quraishi, C. Schröder, and U. von Wagner. Numerical methods for parametric model reduction in the simulation of disc brake squeal. *Z. Angew. Math. Mech.*, (appeared online, DOI: 10.1002/zamm.201500217), 2016.

[11] C. He and G.A. Watson. An algorithm for computing the distance to instability. *SIAM J. Matrix Anal. Appl.*, 20(1):101–116, 1998.

[12] D. Hinrichsen and A. J. Pritchard. Stability radii of linear systems. *Systems Control Lett.*, 7:1–10, 1986.

[13] D. Hinrichsen and A. J. Pritchard. Stability radius for structured perturbations and the algebraic riccati equation. *Systems Control Lett.*, 8:105–113, 1986.

[14] D. Hinrichsen and A. J. Pritchard. *Mathematical Systems Theory I. Modelling, State Space Analysis, Stability and Robustness*. Springer-Verlag, New York, NY, 2005.

[15] W. Kahan, B. N. Parlett, and E. Jiang. Residual bounds on approximate eigensystems of nonnormal matrices. *SIAM J. Numer. Anal.*, 19:470–484, 1982.

[16] D.S. Mackey, N. Mackey, and F. Tisseur. Structured mapping problems for matrices associated with scalar products. part I: Lie and jordan algebras. *SIAM J. Matrix Anal. Appl.*, 29(4):1389–1410, 2008.

[17] N. Martins and L. Lima. Determination of suitable locations for power system stabilizers and static var compensators for damping electromechanical oscillations in large scale power systems. *IEEE Trans. on Power Systems*, 5:1455–1469, 1990.

[18] N. Martins, P.C. Pellanda, and J. Rommes. Computation of transfer function dominant zeros with applications to oscillation damping control of large power systems. *IEEE Trans. on Power Systems*, 22:1657–1664, 2007.

[19] B.M. Maschke, A.J. van der Schaft, and P.C. Breedveld. An intrinsic Hamiltonian formulation of network dynamics: non-standard poisson structures and gyrators. *J. Franklin Inst.*, 329:923–966, 1992.

[20] C. Mehl, V. Mehrmann, and P. Sharma. Structured distances to instability for linear Hamiltonian systems with dissipation. Technical Report 1377, DFG Research Center MATHEON, *Mathematics for key technologies* in Berlin, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany, 2016. `url:` `http://www.matheon.de/`, submitted for publication.

[21] R. Ortega, A.J. van der Schaft, Y. Mareels, and B.M. Maschke. Putting energy back in control. *Control Syst. Mag.*, 21:18–33, 2001.

[22] R. Ortega, A.J. van der Schaft, B.M. Maschke, and G. Escobar. Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, 38:585–596, 2002.

[23] L. Qiu, B. Bernhardsson, A. Rantzer, E. Davison, P. Young, and J. Doyle. A formula for computation of the real stability radius. *Automatica*, 31:879–890, 1995.

[24] J. Rommes and N. Martins. Exploiting structure in large-scale electrical circuit and power system problems. *Linear Algebra Appl.*, 431:318–333, 2009.

[25] A.J. van der Schaft. Port-Hamiltonian systems: an introductory survey. In J.L. Verona M. Sanz-Sole and J. Verdura, editors, *Proc. of the International Congress of Mathematicians, vol. III, Invited Lectures*, pages 1339–1365, Madrid, Spain.

[26] A.J. van der Schaft. Port-Hamiltonian systems: network modeling and control of nonlinear physical systems. In *Advanced Dynamics and Control of Structures and Machines*, CISM Courses and Lectures, Vol. 444. Springer Verlag, New York, N.Y., 2004.

[27] A.J. van der Schaft and B.M. Maschke. The Hamiltonian formulation of energy conserving physical systems with external ports. *Arch. Elektron. Übertragungstech.*, 45:362–371, 1995.

[28] A.J. van der Schaft and B.M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *J. Geom. Phys.*, 42:166–194, 2002.

[29] A.J. van der Schaft and B.M. Maschke. Port-Hamiltonian systems on graphs. *SIAM J. Control Optim.*, 51:906–937, 2013.

[30] W. Schiehlen. *Multibody Systems Handbook*. Springer-Verlag, Heidelberg, Germany, 1990.

[31] C.F. Van Loan. How near is a matrix to an unstable matrix? *Contemp. Math. AMS*, 47:465–479, 1984.

[32] K. Veselić. *Damped oscillations of linear systems: a mathematical introduction*. Springer-Verlag, Heidelberg, 2011.