Eigenvalue perturbation theory of structured real matrices and their sign characteristics under generic structured rank-one perturbations*

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Abstract

An eigenvalue perturbation theory under rank-one perturbations is developed for classes of real matrices that are symmetric with respect to a non-degenerate bilinear form, or Hamiltonian with respect to a non-degenerate skew-symmetric form. In contrast to the case of complex matrices, the sign characteristic is a crucial feature of matrices in these classes. The behavior of the sign characteristic under generic rank-one perturbations is analyzed in each of these two classes of matrices. Partial results are presented, but some questions remain open. Applications include boundedness and robust boundedness for solutions of structured systems of linear differential equations with respect to general perturbations as well as with respect to structured rank perturbations of the coefficients.

Key Words: real J-Hamiltonian matrices, real H-symmetric matrices, indefinite inner product, perturbation analysis, generic rank-one perturbation, T-even matrix polynomial, symmetric matrix polynomial, bounded solution of differential equations, robustly bounded solution of differential equations, invariant Lagrangian subspaces.

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[‖]College of William and Mary, Department of Mathematics, P.O.Box 8795, Williamsburg, VA 23187-8795, USA. We are very sad that between finalizing this paper and its publication our dear friend Leiba Rodman passed away on March 2, 2015. Leiba was the driving force behind the research presented in this paper, and its main author.

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1 Introduction

We consider the perturbation theory for Jordan structures associated with several classes of structured real matrices under generic perturbations that have rank one and are structure preserving. We continue the investigations in [25], where the focus was on general results on classes of structured complex matrices, and in [26], where complex matrices that are selfadjoint in an indefinite inner product were studied. Here, we mainly focus on the real case, which is the most relevant case in applications, and where with the sign characteristic of certain eigenvalues an extra invariant is occurring that plays a crucial role in perturbation theory, as shown in [13, 26] for selfadjoint matrices with respect to a non-degenerate sesquilinear form.

The structure classes that we consider in this paper are defined as follows. Let $\mathbb{F}$ denote either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$ and let I_n denote the $n \times n$ identity matrix. The superscript ^{T} denotes the transpose and * denotes the conjugate transpose of a matrix or vector; thus $X^* = X^T$ for $X \in \mathbb{R}^{n\times n}$.

Definition 1.1 *Let $J \in \mathbb{R}^{2n\times 2n}$ be a nonsingular skew-symmetric matrix. A matrix $A \in \mathbb{R}^{2n\times 2n}$ is called* J-Hamiltonian *if $JA = (JA)^T$.*

The perturbation analysis for real and complex Hamiltonian matrices has recently been studied in several sources and contexts, see, e.g., [2, 25, 28, 31]. A detailed motivation for the analysis of these classes and a review over the literature is given in [25]. The classical and most important class of J-Hamiltonian matrices is obtained with

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Another type of symmetry arises when using a nonsingular symmetric matrix instead of a skew-symmetric matrix J as in Definition 1.1.

Definition 1.2 *Let $H \in \mathbb{R}^{n\times n}$ be an invertible symmetric matrix. A matrix $A \in \mathbb{R}^{n\times n}$ is called* H-symmetric *if $HA = (HA)^T$.*

In this paper we focus on real matrices that are J-Hamiltonian or H-symmetric. We do not consider H-skew-Hermitian and J-skew-Hamiltonian matrices, as there are no rank-one matrices in these classes.

Besides the introduction and conclusion, the paper consists of eight sections. Section 2 contains preliminaries, including the Jordan forms of unstructured matrices under generic rank-one perturbations, and the canonical forms of real J-Hamiltonian and H-symmetric matrices.

Sections 3, 4, and 5 deal with Jordan forms and the sign characteristic arising in J-Hamiltonian matrices under rank-one J-Hamiltonian perturbations. The main results in Sections 3, 4, and 5 are Theorem 3.1, which gives the Jordan canonical form under generic rank-one perturbations, Corollary 3.2 that provides inequalities of Jordan canonical forms under rank-one perturbations (not necessarily generic), and

Theorem 4.7 which presents properties of the sign characteristic related to generic rank-one perturbations. In particular, we show that the sign characteristic is constant within every connected component of the set of generic rank-one J-Hamiltonian perturbations. The analysis of the sign characteristic is continued in Section 5, where the focus is on rank-one perturbations with small norm. In this context, Theorems 5.1 and 5.2 establish the behavior of perturbed eigenvalues with real parts zero, provided that the generic rank-one J-Hamiltonian perturbations are small in norm, with special emphasis on the sign characteristics. These results are applied in Section 7 to boundedness and robust boundedness of solutions to systems of differential equations of the form

$$A_\ell x^{(\ell)} + A_{\ell-1}x^{(\ell-1)} + \cdots + A_1\dot{x} + A_0 x = 0, \tag{1.1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is an unknown ℓ-times continuously differentiable vector function, and the A_k's are constant real $n \times n$ matrices such that A_k is skew-symmetric if k is odd, A_k is symmetric if k is even, and A_ℓ is invertible. The matrix polynomial

$$P(\lambda) = \sum_{j=0}^{\ell} A_j \lambda^j$$

associated with the differential equation (1.1) is called a *T-even matrix polynomial*, since $P(-\lambda) = P(\lambda)^T$, and is a special case of a so-called *alternating matrix polynomial*, because its coefficient matrices alternate between symmetric and skew-symmetric structure. Therefore, we call a differential equation of the form (1.1) a *T-even differential equation*. T-even differential equations and their associated T-even matrix polynomials have many applications in optimal control and finite element analysis of structures, see [1, 7, 21, 23, 30].

A similar analysis as the one in Section 5 is given in Section 6 for rank-one perturbations of real H-symmetric matrices, and again the sign characteristic plays an important role. These results are applied in Section 8 to study the boundedness and robust boundedness of solutions to linear differential equations of the form

$$i^\ell A_\ell x^{(\ell)} + i^{\ell-1} A_{\ell-1}x^{(\ell-1)} + \cdots + iA_1\dot{x} + A_0 x = 0,$$

where all coefficients A_k are real symmetric matrices.

Finally, in Section 9 we discuss the existence of invariant Lagrangian subspaces for J-Hamiltonian matrices under rank one perturbation.

The following notation will be used throughout the paper. $\mathbb{N}$ stands for the set of positive integers. The real, and imaginary parts of a complex number λ will be denoted by $\mathrm{Re}(\lambda) = \frac{\lambda+\overline{\lambda}}{2}$, $\mathrm{Im}(\lambda) = \frac{\lambda-\overline{\lambda}}{2i}$, respectively. The vector space of $n \times m$ matrices with entries in $\mathbb{F}$ (either $\mathbb{C}$ or $\mathbb{R}$) is denoted by $\mathbb{F}^{n\times m}$, and we will frequently identify $\mathbb{F}^{n\times 1}$ with $\mathbb{F}^n$.

An $m \times m$ upper triangular Jordan block associated with an eigenvalue λ is denoted by $\mathcal{J}_m(\lambda)$, and by $\mathcal{J}_m(a \pm ib)$ we denote a quasi-upper triangular $m \times m$ real Jordan block with non-real complex conjugate eigenvalues $\lambda = a + ib, \overline{\lambda} = a - ib$, $a, b \in \mathbb{R}$, $b \neq 0$, i.e., m is even and the 2×2 blocks on the main block diagonal of $\mathcal{J}_m(a \pm ib)$ have the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

The spectrum of a matrix $A \in \mathbb{F}^{2n \times 2n}$, i.e., the set of eigenvalues including possibly non-real eigenvalues of real matrices, is denoted by $\sigma(A)$. An eigenvalue of A is called *simple* if it has algebraic multiplicity one, i.e., it is a simple root of the characteristic polynomial of A. The *root subspace* of a matrix $A \in \mathbb{F}^{n \times n}$ corresponding to an eigenvalue $\lambda \in \mathbb{F}$ is defined as $\operatorname{Ker}(A - \lambda I_n)^n \subseteq \mathbb{F}^n$. If $\mathbb{F} = \mathbb{R}$ and $\lambda = a + ib \in \mathbb{C} \setminus \mathbb{R}$ with $a, b \in \mathbb{R}$, $b > 0$, then $\operatorname{Ker}\left(A^2 - 2aA + (a^2 + b^2)I_n\right)^n \subseteq \mathbb{R}^n$ is the *real root subspace* of A corresponding to the pair of conjugate complex eigenvalues $a \pm ib$ of A.

A block diagonal matrix with diagonal blocks $X_1, \ldots, X_q$ (in that order) is denoted by $X_1 \oplus X_2 \oplus \cdots \oplus X_q$; also, $X \oplus \cdots \oplus X$ (q times) is abbreviated to $X^{\oplus q}$. We also introduce the anti-diagonal matrices

$$\Sigma_k = \begin{bmatrix} 0 & \cdots & 0 & (-1)^0 \\ \vdots & \iddots & (-1)^1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ (-1)^{k-1} & 0 & \cdots & 0 \end{bmatrix} = (-1)^{k-1}\Sigma_k^T \quad \text{and} \quad R_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Then R_n is symmetric, Σ_k is symmetric if k is odd, and skew-symmetric if k is even. Moreover, we use the skew-symmetric matrices $\Sigma_k \otimes \Sigma_2^k$, where $\otimes$ denotes the Kronecker (tensor) product $[a_{ij}] \otimes B = [a_{ij}B]$. For example, for $k = 1, 2, 3, 4$, we have

$$\begin{aligned} \Sigma_1 \otimes \Sigma_2^1 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Sigma_2 \otimes \Sigma_2^2 = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}, \\ \Sigma_3 \otimes \Sigma_2^3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; \Sigma_4 \otimes \Sigma_2^4 = \begin{bmatrix} 0 & 0 & 0 & I_2 \\ 0 & 0 & -I_2 & 0 \\ 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

We make frequent use of the standard bilinear and in the case $\mathbb{F} = \mathbb{C}$ sesquilinear form on $\mathbb{F}^n$ given by

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y}_j, \quad x = [x_1, \ldots, x_n]^T, \quad y = [y_1, \ldots, y_n]^T \in \mathbb{F}^n.$$

2 Preliminaries

In this section we recall some preliminary results, beginning with a general result on unstructured generic rank-one perturbations. We say that a set $\Omega \subseteq \mathbb{F}^n$ is *algebraic* if there exist finitely many polynomials $f_1(x_1,\ldots,x_n),\ldots,f_k(x_1,\ldots,x_n)$ with coefficients in $\mathbb{F}$ such that a vector $[a_1,\ldots,a_n]^T \in \mathbb{F}^n$ belongs to Ω if and only if

$$f_j(a_1,\ldots,a_n)=0, \quad j=1,2,\ldots,k.$$

In particular, the empty set is algebraic and $\mathbb{F}^n$ is algebraic. We say that a set $\Omega \subseteq \mathbb{F}^n$ is *generic* if the complement $\mathbb{F}^n \setminus \Omega$ is contained in an algebraic set which is not $\mathbb{F}^n$. Note that the union of finitely many algebraic sets is again algebraic. Note also that the genericity of a matrix set $\Omega \subseteq \mathbb{F}^{n\times n} \cong \mathbb{F}^{n^2}$ is preserved under similarity $X \mapsto S^{-1}XS$, where $S \in \mathbb{F}^n$ is invertible and independent of $X \in \Omega$.

The general perturbation analysis for generic low-rank perturbations has been studied in [16, 32, 36, 37], and specifically for rank-one perturbations in [4, 15, 25]. For the case of rank-one perturbations we have the following result on the Jordan structure of generic perturbations of real as well as complex matrices.

Theorem 2.1 *Let $A \in \mathbb{F}^{n\times n}$ be a matrix having pairwise distinct (real and non-real) eigenvalues $\lambda_1,\ldots,\lambda_p$ and suppose that the Jordan blocks associated with the eigenvalue λ_j in the (complex) Jordan form of A have dimensions $n_1^{(j)} > n_2^{(j)} > \cdots > n_{m_j}^{(j)}$ repeated $r_1^{(j)}, r_2^{(j)},\ldots,r_{m_j}^{(j)}$ times, respectively, and geometric multiplicity $g^{(j)}$; thus*

$$g^{(j)} = r_1^{(j)} + \cdots + r_{m_j}^{(j)}, \qquad j=1,2,\ldots,p.$$

If $B = vu^T$, $v,u \in \mathbb{F}^n \setminus \{0\}$, is a rank-one matrix, then generically (with respect to the entries of u and v) the Jordan blocks of $A+B$ with eigenvalue λ_j are just the $g^{(j)}-1$ smallest Jordan blocks of A with eigenvalue λ_j, for $j=1,2,\ldots,p$, and, moreover, the eigenvalues of $A+B$ which are distinct from any of $\lambda_1,\ldots,\lambda_p$ are simple.

More precisely, there is a generic set $\Omega \subseteq \mathbb{F}^{2n}$, where $\mathbb{F}^{2n}$ is identified with the $2n$ independent entries of u and v, such that for every $(u,v) \in \Omega$, the Jordan structure of $A+B$ corresponding to the eigenvalue λ_j consists of $r_1^{(j)}-1$ Jordan blocks of size $n_1^{(j)}$ and $r_k^{(j)}$ Jordan blocks of size $n_k^{(j)}$ for $k=2,\ldots,m_j$, for $j=1,2,\ldots,p$, and moreover, for every $(u,v)\in\Omega$, the eigenvalues of $A+B$ which are distinct from any of $\lambda_1,\ldots,\lambda_p$ are simple.

For the complex case, various parts of of Theorem 2.1 were proved in [15, 25, 32, 36], and for the real case in [27]. Introducing the slightly changed notation that for a matrix A the values

$$w_1^{(\lambda)}(A) \geq w_2^{(\lambda)}(A) \geq \cdots \geq w_k^{(\lambda)}(A) \geq \cdots$$

are the partial multiplicities of λ as an eigenvalue of A in non-increasing order, and by convention $w_k^{(\lambda)}(A) = 0$ if k is greater than the geometric multiplicity of λ, the following corollary of Theorem 2.1 provides information on the Jordan structure of matrices perturbed by (a not necessarily generic) rank-one perturbation.

Corollary 2.2 *Let $A \in \mathbb{F}^{n\times n}$ be given as in Theorem 2.1. Then for all rank-one matrices $B \in \mathbb{F}^{n\times n}$ and for $j = 1, 2, \dots, p$ we have*

$$\sum_{i=1}^{\infty} \min\{k, w_i^{(\lambda_j)}(A+B)\} \geq \sum_{i=2}^{\infty} \min\{k, w_i^{(\lambda_j)}(A)\}, \quad k = 1, 2, \dots .$$

(We emphasize that the second sum starts with $i = 2$.)

Proof. We consider the complex case only; the proof for the real case is analogous. For a fixed k, consider the set

$$\Omega_k := \left\{ (u, v) \in \mathbb{C}^n \times \mathbb{C}^n \,\middle|\, \dim \operatorname{Ker} (A + uv^T - \lambda_j I)^k < \sum_{i=2}^{\infty} \min\{k, w_i^{(\lambda_j)}(A)\} \right\}.$$

Clearly, Ω_k is an open set in $\mathbb{C}^n \times \mathbb{C}^n$, because of the lower semicontinuity of the rank as a function of matrices. On the other hand, the generic set Ω from Theorem 2.1 satisfies

$$(u, v) \in \Omega \quad \Longrightarrow \quad \dim \operatorname{Ker} (A + uv^T - \lambda_j I)^k = \sum_{i=2}^{\infty} \min\{k, w_i^{(\lambda_j)}(A)\}$$

and is thus contained in the complement of Ω_k. Therefore, Ω_k must be empty and the proof is complete. □

Corollary 2.2 is a generalized version of [27, Theorem 4.3] (stated for the complex case only).

We will frequently make use of the structured canonical form for H-symmetric and J-Hamiltonian matrices which is available in many sources, see, e.g., [12, 13] or [17, 19, 20, 38] in the framework of pairs of symmetric and skew-symmetric matrices.

Theorem 2.3 *Let $H \in \mathbb{F}^{n\times n}$ be symmetric (if $\mathbb{F} = \mathbb{R}$) or Hermitian (if $\mathbb{F} = \mathbb{C}$) and invertible, and let $A \in \mathbb{F}^{n\times n}$ be H-selfadjoint, i.e., $HA = A^*H$. Then there exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ such that $P^{-1}AP$ and P^*HP are real block diagonal matrices*

$$P^{-1}AP = A_1 \oplus A_2, \quad P^*HP = H_1 \oplus H_2, \tag{2.1}$$

where the block structure is partitioned further as

(i) $$A_1 = A_{1,1} \oplus \cdots \oplus A_{1,\mu}, \quad H_1 = H_{1,1} \oplus \cdots \oplus H_{1,\mu},$$

where

$$A_{1,j} = \left(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}}\right) \oplus \left(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}}\right) \oplus \cdots \oplus \left(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}}\right),$$

with $\lambda_j \in \mathbb{R}$ pairwise distinct, $n_{1,j} > \cdots > n_{m_j,j}$, $j = 1, \dots, \mu$, and

$$H_{1,j} = \left(\bigoplus_{s=1}^{\ell_{1,j}} \sigma_{1,s,j} R_{n_{1,j}}\right) \oplus \left(\bigoplus_{s=1}^{\ell_{2,j}} \sigma_{2,s,j} R_{n_{2,j}}\right) \oplus \cdots \oplus \left(\bigoplus_{s=1}^{\ell_{m_j,j}} \sigma_{m_j,s,j} R_{n_{m_j,j}}\right),$$

where $\sigma_{i,s,j} \in \{+1, -1\}$, $s = 1, \dots, \ell_{i,j}$, $i = 1, \dots, m_j$, $j = 1, \dots, \mu$;

(ii)
$$A_2 = A_{2,1} \oplus \cdots \oplus A_{2,\nu}, \qquad H_2 = H_{2,1} \oplus \cdots \oplus H_{2,\nu}, \tag{2.2}$$

where
$$A_{2,j} = \mathcal{J}_{r_{j,1}}(a_j \pm ib_j) \oplus \cdots \oplus \mathcal{J}_{r_{j,q_j}}(a_j \pm ib_j),$$
$$H_{2,j} = R_{r_{j,1}} \oplus \cdots \oplus R_{r_{j,q_j}},$$

with even integers $r_{j,1} \geq \cdots \geq r_{j,q_j} \in \mathbb{N}$ *and* $a_j, b_j \in \mathbb{R}$ *with* $b_j > 0$ *for* $j = 1, \ldots, \nu$. *Moreover,* $a_1 + ib_1, \ldots, a_\nu + ib_\nu$ *are pairwise distinct.*

The form (2.1) *is uniquely determined by the pair* (A, H)*, up to a simultaneous permutation of diagonal blocks in the right hand sides of* (2.1)*.*

The signs $\sigma_{i,s,j} \in \{+1, -1\}$ attached to every Jordan block corresponding to a real eigenvalue in the Jordan form of an H-selfadjoint matrix A form the *sign characteristic* of the pair (A, H). The sign characteristic was introduced in [9] in the context of H-selfadjoint matrices with respect to sesquilinear forms, see also [8, 10, 13, 19, 20] for variants.

Theorem 2.4 *Let* J *be a real nonsingular skew-symmetric matrix and let* A *be real* J*-Hamiltonian. Then there exists a real, invertible matrix* P *such that* $P^{-1}AP$ *and* P^TJP *are block diagonal matrices*

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_s, \quad P^TJP = J_1 \oplus \cdots \oplus J_s, \tag{2.3}$$

where each diagonal block (A_j, J_j) *is of one of the following five types:*

(i)
$$A_j = \mathcal{J}_{2n_1}(0) \oplus \cdots \oplus \mathcal{J}_{2n_p}(0), \quad J_j = \kappa_1 \Sigma_{2n_1} \oplus \cdots \oplus \kappa_p \Sigma_{2n_p},$$
with $\kappa_1, \ldots, \kappa_p \in \{+1, -1\}$*;*

(ii)
$$A_j = \begin{bmatrix} \mathcal{J}_{2m_1+1}(0) & 0 \\ 0 & -\mathcal{J}_{2m_1+1}(0)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{2m_q+1}(0) & 0 \\ 0 & -\mathcal{J}_{2m_q+1}(0)^T \end{bmatrix},$$
$$J_j = \begin{bmatrix} 0 & I_{2m_1+1} \\ -I_{2m_1+1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2m_q+1} \\ -I_{2m_q+1} & 0 \end{bmatrix};$$

(iii)
$$A_j = \begin{bmatrix} \mathcal{J}_{\ell_1}(a) & 0 \\ 0 & -\mathcal{J}_{\ell_1}(a)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\ell_r}(a) & 0 \\ 0 & -\mathcal{J}_{\ell_r}(a)^T \end{bmatrix},$$
$$J_j = \begin{bmatrix} 0 & I_{\ell_1} \\ -I_{\ell_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{\ell_r} \\ -I_{\ell_r} & 0 \end{bmatrix},$$

where $a > 0$*, and the number* a*, the total number* $2r$ *of Jordan blocks, and the sizes* $\ell_1, \ldots, \ell_r$ *depend on the particular diagonal block* (A_j, J_j)*;*

(iv) $A_j = \begin{bmatrix} \mathcal{J}_{2k_1}(a \pm ib) & 0 \\ 0 & -\mathcal{J}_{2k_1}(a \pm ib)^T \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{2k_s}(a \pm ib) & 0 \\ 0 & -\mathcal{J}_{2k_s}(a \pm ib)^T \end{bmatrix}$,

$$J_j = \begin{bmatrix} 0 & I_{2k_1} \\ -I_{2k_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I_{2k_s} \\ -I_{2k_s} & 0 \end{bmatrix},$$

where $a, b > 0$, and again the numbers a and b, the total number $2s$ of Jordan blocks, and the sizes $2k_1, \ldots, 2k_s$ depend on (A_j, J_j);

(v) $A_j = \mathcal{J}_{2h_1}(\pm ib) \oplus \cdots \oplus \mathcal{J}_{2h_t}(\pm ib), \quad J_j = \eta_1(\Sigma_{h_1} \otimes \Sigma_2^{h_1}) \oplus \cdots \oplus \eta_t(\Sigma_{h_t} \otimes \Sigma_2^{h_t})$,

where $b > 0$ and $\eta_1, \ldots, \eta_t$ are signs ± 1. Again, the parameters b, t, $h_1, \ldots, h_t$, and $\eta_1, \ldots, \eta_t$ depend on the particular diagonal block (A_j, J_j).

There is at most one block each of type (i) *and* (ii)*. Furthermore, two blocks A_i and A_j of one of the types* (iii)–(v) *have nonintersecting spectra if $i \neq j$. Moreover, the form* (2.3) *is uniquely determined by the pair (A, J), up to a simultaneous permutation of diagonal blocks in the right hand sides of* (2.3)*.*

The signs $\kappa_i, \eta_j \in \{+1, -1\}$ associated with each even partial multiplicity of the zero eigenvalue and with each partial multiplicity corresponding to purely imaginary eigenvalues ib of A with $b > 0$ form the *sign characteristic* of the pair (A, J).

We indicate a useful property of positive definiteness related to real J-Hamiltonian matrices to be used in Section 7. For this, we need the following definition.

Definition 2.5 *Let $H \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then a subspace $\mathcal{M} \subseteq \mathbb{R}^n$ is called H-*definite *if either $x^T H x > 0$ for all $x \in \mathcal{M} \setminus \{0\}$ or $x^T H x < 0$ for all $x \in \mathcal{M} \setminus \{0\}$.*

Theorem 2.6 *Let $J \in \mathbb{R}^{n \times n}$ be a nonsingular skew-symmetric matrix, let $A \in \mathbb{R}^{n \times n}$ be J-Hamiltonian, and let Λ be a set of eigenvalues of A which is either of the form $\{0\}$, $\{a, -a\}$, $\{bi, -bi\}$, or $\{\pm a \pm ib\}$ with $a, b \in \mathbb{R}$. Denote by $\mathcal{M}_\Lambda \subseteq \mathbb{R}^n$ the sum of root subspaces of A that correspond to eigenvalues in Λ. Then the following statements are equivalent.*

(1) *Every root subspace $\mathcal{M}$ in $\mathcal{M}_\Lambda$ is JA-definite;*

(2) *All eigenvalues in Λ are nonzero purely imaginary and semisimple (i.e., its algebraic multiplicity coincides with its geometric multiplicity), and for every eigenvalue in Λ, the signs in the sign characteristic corresponding to that eigenvalue are the same. (However, signs corresponding to different eigenvalues may be different.)*

Proof. Observe that JA is symmetric. The proof of Theorem 2.6 then follows by inspection, assuming (without loss of generality) that J and A are given by the canonical form (2.3). □

Example 2.7 Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad J_1 = A, \quad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then A is J_1-Hamiltonian as well as J_2-Hamiltonian, and the sign characteristic of the J_1-Hamiltonian matrix A consists of two pluses, whereas the sign characteristic of the J_2-Hamiltonian matrix A consists of one plus and one minus. The (only) root subspace of A is $\mathcal{M} = \mathbb{R}^4$, and the symmetric matrix J_iA is $-I_4$ for $i = 1$ and $(-I_2) \oplus I_2$ for $i = 2$. Thus $\mathcal{M}$ is J_1A-definite, but not J_2A-definite, as predicted by Theorem 2.6.

3 Structure-preserving rank-one perturbations of real J-Hamiltonian matrices

In this section we address the behavior of the Jordan form under generic rank-one perturbations. The more subtle question of the behavior of the sign characteristic will be addressed in Sections 4 and 5. Our first main theorem is the real analogue of [25, Theorem 4.2].

We only consider J-Hamiltonian rank-one perturbations of the form uu^TJ, where $u \in \mathbb{R}^n \setminus \{0\}$. The results concerning perturbations of the form $-uu^TJ$ are completely analogous, and can be obtained by applying the results for uu^TJ, where J is replaced with $-J$.

Theorem 3.1 *Let $J \in \mathbb{R}^{n\times n}$ be skew-symmetric and invertible, and let $A \in \mathbb{R}^{n\times n}$ be J-Hamiltonian. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$, be the set of pairwise distinct nonzero eigenvalues of A having nonnegative imaginary parts, and let $\lambda_{p+1} = 0$. For every λ_j, $j = 1, 2, \ldots, p+1$, let $n_{1,j} > n_{2,j} > \ldots > n_{m_j,j}$ be the sizes of Jordan blocks in the real Jordan form of A associated with the eigenvalue λ_j and let there be exactly $\ell_{k,j}$ Jordan blocks of size $n_{k,j}$ associated with λ_j in the real Jordan form of A, for $k = 1, 2, \ldots, m_j$.*

Consider a J-Hamiltonian rank-one perturbation of the form $B = uu^TJ \in \mathbb{R}^{n\times n}$.

(1) *If for the eigenvalue $\lambda_{p+1} = 0$, $n_{1,p+1}$ is even (in particular, if A is invertible), then generically with respect to the components of u, the matrix $A + B$ has the Jordan canonical form*

$$\oplus_{j=1}^{p+1} \left(\left(\oplus_{s=1}^{\ell_{1,j}-1} \mathcal{J}_{n_{1,j}}(\lambda_j) \right) \oplus \left(\oplus_{s=1}^{\ell_{2,j}} \mathcal{J}_{n_{2,j}}(\lambda_j) \right) \oplus \cdots \oplus \left(\oplus_{s=1}^{\ell_{m_j,j}} \mathcal{J}_{n_{m_j,j}}(\lambda_j) \right) \right) \oplus \widetilde{\mathcal{J}},$$

where $\mathcal{J}_{n_{j,k}}(\lambda_j)$ is replaced with $\mathcal{J}_{n_{j,k}}(\operatorname{Re}\lambda_j \pm i \operatorname{Im}\lambda_j)$ if λ_j is non-real, and where $\widetilde{\mathcal{J}}$ contains all the real Jordan blocks of $A+B$ associated with eigenvalues different from any of $\lambda_1, \ldots, \lambda_{p+1}$.

(2) *If for the eigenvalue* $\lambda_{p+1} = 0$, $n_{1,p+1}$ *is odd (in this case* $\ell_{1,p+1}$ *is even), then generically with respect to the components of* u, *the matrix* $A+B$ *has the Jordan canonical form*

$$\begin{aligned}
&\oplus_{j=1}^{p} \left(\left(\oplus_{s=1}^{\ell_{1,j}-1} \mathcal{J}_{n_{1,j}}(\lambda_j) \right) \oplus \left(\oplus_{s=1}^{\ell_{2,j}} \mathcal{J}_{n_{2,j}}(\lambda_j) \right) \oplus \cdots \oplus \left(\oplus_{s=1}^{\ell_{m_j,j}} \mathcal{J}_{n_{m_j,j}}(\lambda_j) \right) \right) \\
\oplus &\left(\oplus_{s=1}^{\ell_{1,p+1}-2} \mathcal{J}_{n_{1,p+1}}(0) \right) \oplus \left(\oplus_{s=1}^{\ell_{2,p+1}} \mathcal{J}_{n_{2,p+1}}(0) \right) \oplus \cdots \oplus \left(\oplus_{s=1}^{\ell_{m_{p+1},p+1}} \mathcal{J}_{n_{m_{p+1},p+1}}(0) \right) \\
\oplus & \qquad\qquad \mathcal{J}_{n_{1,p+1}+1}(0) \oplus \widetilde{\mathcal{J}},
\end{aligned}$$

where $\mathcal{J}_{n_{j,k}}(\lambda_j)$ *is replaced with* $\mathcal{J}_{n_{j,k}}(\operatorname{Re}\lambda_j \pm i \operatorname{Im}\lambda_j)$ *if* λ_j *is non-real, and where* $\widetilde{\mathcal{J}}$ *contains all the real Jordan blocks of* $A+B$ *associated with eigenvalues different from any of* $\lambda_1, \ldots, \lambda_{p+1}$.

(3) *In either case* (1) *or* (2)*, generically the part* $\widetilde{\mathcal{J}}$ *has simple eigenvalues.*

Proof. We use the corresponding result for the complex case ([25, Theorem 4.2]). (A standard transformation to the complex canonical form of the pair (J, A) is used here; we will not display this transformation explicitly.) Accordingly, there is a generic set $\Omega \in \mathbb{C}^n$ such that every (complex) matrix of the form $A + uu^T J$, $u \in \Omega$, has the properties described in Theorem 3.1. Thus, $\mathbb{C}^n \setminus \Omega \subseteq \mathcal{Q}$, where $\mathcal{Q}$ is a proper algebraic set, i.e., different from $\mathbb{C}^n$. Therefore

$$\mathbb{R}^n \setminus (\Omega \cap \mathbb{R}^n) \subseteq \mathcal{Q} \cap \mathbb{R}^n.$$

By [27, Lemma 2.2], $\mathcal{Q} \cap \mathbb{R}^n$ is a proper algebraic set in $\mathbb{R}^n$. So Theorem 3.1 holds with the generic set $\Omega \cap \mathbb{R}^n$. □

Observe that if A is invertible or if 0 is an eigenvalue of A with a single even size Jordan block, then for every $u \in \mathbb{R}^n$ in the generic set such that $A + uu^T J$ has the properties (1)–(3) of Theorem 3.1, the matrix $A + uu^T J$ is invertible. Indeed, by Theorem 2.4 a real J-Hamiltonian matrix cannot have a simple eigenvalue at zero.

The analogue of Corollary 2.2 also holds in the context of rank-one perturbations of J-Hamiltonian matrices.

Corollary 3.2 *Let* $J \in \mathbb{R}^{n\times n}$ *be skew-symmetric and invertible, and let* $A \in \mathbb{R}^{n\times n}$ *be* J*-Hamiltonian. Then, for all rank-one* J*-Hamiltonian matrices* B *we have*

$$\sum_{i=1}^{\infty} \min\{k, w_i^{(\lambda_j)}(A+B)\} \geq \sum_{i=2}^{\infty} \min\{k, w_i^{(\lambda_j)}(A)\}, \quad k = 1, 2, \ldots,$$

for $j = 1, 2, \ldots, p+1$ *if* $n_{1,p+1}$ *is even (in particular, if* 0 *is not an eigenvalue of* A*), and if* $n_{1,p+1}$ *is odd then*

$$\sum_{i=1}^{\infty} \min\{k, w_i^{(\lambda_j)}(A+B)\} \geq \sum_{i=2}^{\infty} \min\{k, w_i^{(\lambda_j)}(A)\}, \quad k = 1, 2, \ldots,$$

for $j = 1, 2, \ldots, p$ *and*

$$\sum_{i=1}^{\infty} \min\{k, w_i^{(0)}(A+B)\} \geq \min\{k, w_2^{(0)}(A)+1\} + \sum_{i=3}^{\infty} \min\{k, w_i^{(0)}(A)\}, \quad k = 1, 2, \ldots.$$

Proof. The proof of Corollary 3.2 is analogous to that of Corollary 2.2, but using Theorem 3.1 instead of Theorem 2.1. □

4 Rank-one perturbations of the sign characteristic of real J-Hamiltonian matrices

In this section, we study the behavior of the sign characteristic of real J-Hamiltonian matrices under structure-preserving rank-one perturbations. Note that if a real matrix A_0 is J_0-Hamiltonian, where J_0 is a real invertible skew-symmetric matrix, then iA_0 is iJ_0-selfadjoint, where the matrix iJ_0 is (complex) Hermitian. As such, there is the sign characteristic of the pair (iA_0, iJ_0) that attaches a sign $+1$ or -1 to every partial multiplicity of real eigenvalues of iA_0 (see, for example, [12, 13, 18]). The sign characteristic of (iA_0, iJ_0) relates to the sign characteristic of (A_0, J_0) as follows.

Theorem 4.1 *Let* $J \in \mathbb{R}^{n\times n}$ *be skew-symmetric and invertible, and let* $A \in \mathbb{R}^{n\times n}$ *be* J*-Hamiltonian.*

(a) *If* $A_0 = J_{2n_0}(0)$, $J_0 = \kappa\Sigma_{2n_0}$, $\kappa = \pm 1$, *then the sign characteristic of* (iA_0, iJ_0) *is* κ *if* n_0 *is odd, and* $-\kappa$ *if* n_0 *is even.*

(b) *If*

$$A_0 = J_{2m_0+1}(0) \oplus (-J_{2m_0+1}(0))^T, \quad J_0 = \begin{bmatrix} 0 & I_{2m_0+1} \\ -I_{2m_0+1} & 0 \end{bmatrix}$$

and m_0 *is a nonnegative integer, then the sign characteristic of* (iA_0, iJ_0) *consists of opposite signs attached to the two partial multiplicities* $2m_0+1, 2m_0+1$ *of* iA.

(c) *If*

$$A_0 = J_{2h_0}(\pm ib), \quad J_0 = \eta(\Sigma_{h_0} \otimes \Sigma_2^{h_0}), \tag{4.1}$$

where $b > 0$, $\eta \in \{+1, -1\}$, *then the sign characteristic of* (iA_0, iJ_0) *consists of* $-\eta$ *attached to each of the eigenvalues* $\pm b$ *of* iA_0 *if* h_0 *is even, and consists of* η *attached to the eigenvalue* $\pm b$ *of* iA_0 *if* h_0 *is odd.*

Theorem 4.1 was proved in [33] (note that in [33] lower triangular Jordan blocks are used rather than upper triangular ones used here), see also [17, Theorem 3.4.1] and [34]. In the proof of Theorem 4.1 the following proposition is used, that provides a convenient alternative description of the sign characteristic for complex H-selfadjoint matrices, see [12, 13], see Theorem 2.3.

Let $H \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix, and let $X \in \mathbb{C}^{n\times n}$ be H-selfadjoint, i.e., $HX = X^*H$. Let λ_0 be a fixed real eigenvalue of X, and let $\Psi_1 \subseteq \mathbb{C}^n$ be the subspace spanned by the eigenvectors of X corresponding to λ_0, i.e., $\Psi_1 = \operatorname{Ker}(X - \lambda_0 I)$. For $x \in \Psi_1 \setminus \{0\}$, denote by $\nu(x)$ the maximal length of a Jordan chain of X beginning with the eigenvector x, i.e., there exists a chain of $\nu(x)$ vectors $y_1 = x, y_2, \dots, y_{\nu(x)}$ such that

$$(X - \lambda_0 I)y_j = y_{j-1} \quad \text{for} \quad j = 2, 3, \dots, \nu(x) \quad \text{and} \quad (X - \lambda_0 I)y_1 = 0,$$

and there is no chain of $\nu(x) + 1$ vectors with analogous properties. Let, furthermore, Ψ_i, $i = 1, 2, \dots, \gamma$ $(\gamma = \max\{\nu(x) \mid x \in \Psi_1 \setminus \{0\}\})$ be the subspace of Ψ_1 spanned by all $x \in \Psi_1$ with $\nu(x) \geq i$.

Proposition 4.2 *Let $H \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix, and let $X \in \mathbb{C}^{n\times n}$ be H-selfadjoint. For $i = 1, \dots, \gamma$, let*

$$f_i(x, y) = \langle x, Hy^{(i)}\rangle, \quad x \in \Psi_i, \quad y \in \Psi_i \setminus \{0\},$$

where $y = y^{(1)}, y^{(2)}, \dots, y^{(i)}$ is a Jordan chain of X corresponding to a real eigenvalue λ_0 with eigenvector y, and let $f_i(x, 0) = 0$. Then,

(i) *$f_i(x, y)$ does not depend on the choice of $y^{(2)}, \dots, y^{(i)}$;*

(ii) *for some selfadjoint linear transformation $G_i : \Psi_i \to \Psi_i$,*

$$f_i(x, y) = \langle x, G_i y\rangle, \quad x, y \in \Psi_i;$$

(iii) *for the transformation G_i of* (ii)*, $\Psi_{i+1} = \operatorname{Ker} G_i$ (by definition $\Psi_{\gamma+1} = \{0\}$);*

(iv) *the number of positive (negative) eigenvalues of G_i of* (ii) *counting multiplicities, coincides with the number of positive (negative) signs in the sign characteristic of (X, H) corresponding to the Jordan blocks of size i associated with the eigenvalue λ_0 of X.*

Note that Proposition 4.2 is also valid for real matrices, with obvious changes (e.g., Ψ_i are now subspaces of $\mathbb{R}^n$) and an analogous proof.

In view of Theorem 4.1 many properties of the sign characteristic of perturbed J-Hamiltonian matrices A will follow from the corresponding results on the sign characteristic of the pair (iA, iJ) studied in [26].

Example 4.3 Let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2.$$

Then A is J-Hamiltonian and one immediately checks that $A \pm uu^T J = \pm uu^T J$ has the Jordan form $\mathcal{J}_2(0)$ if and only if $u \neq 0$. Furthermore, assuming that $u \neq 0$, one checks easily that the sign characteristic of the iJ-selfadjoint matrix $\pm iuu^T J$ is ± 1. Thus, by Theorem 4.1 the sign characteristic of the J-Hamiltonian matrix $\pm uu^T J$ is ± 1.

Conjecture 4.4 (a) Assume $n_{1,p+1} = 1$ in the notation of Theorem 3.1. Then the sign of the block $\mathcal{J}_2(0)$ in Theorem 3.1 that arises from the combination of a pair of 1×1 Jordan blocks with eigenvalue 0 of the original matrix coincides with the sign of rank-one symmetric real matrix $\pm Juu^T J$.

(b) The sign corresponding to the even size block $\mathcal{J}_{n_{1,p+1}+1}(0)$ in Theorem 3.1 of size at least 4 that arises from the combination of the two largest odd size blocks corresponding to the eigenvalue 0 of the original matrix is the same for $A + uu^T J$ and $A - uu^T J$ for any u, but depends on u. (Compare example 4.5 below.)

The following example (cf. [25, Example 4.1]) indicates a potential way to compute the sign discussed in part (b) of Conjecture 4.4. The example also shows that part (a) of Conjecture 4.4 is wrong for even sizes bigger than 2.

Example 4.5 Consider the matrix

$$Z(w) = \begin{bmatrix} \mathcal{J}_{2m+1}(0) & 0 \\ 0 & \mathcal{J}_{2m+1}(0) \end{bmatrix} + ww^T \begin{bmatrix} 0 & \Sigma_{2m+1} \\ -\Sigma_{2m+1} & 0 \end{bmatrix} \in \mathbb{R}^{(4m+2)\times(4m+2)},$$

with $m > 0$, the case $m = 0$ was considered in Example 4.3. Adopting the notation used in [25], in particular, making use of the matrix $\Upsilon = \operatorname{diag}(1, -1, 1, \cdots, \pm 1)$, Theorem 3.1 shows that generically (with respect to the components of $w \in \mathbb{R}^{2m+1}$) $Z(w)$ has the Jordan from $\mathcal{J}_{2m+2}(0) \oplus \mathcal{J}_1(z_1) \oplus \mathcal{J}_1(z_2) \oplus \cdots \oplus \mathcal{J}_1(z_{2m})$, where the z_j are pairwise distinct nonzero complex numbers.

We transform $Z(w)$ to the matrix

$$M := \begin{bmatrix} \mathcal{J}_{2m+1}(0) & 0 \\ 0 & -\mathcal{J}_{2m+1}(0)^T \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} -v^T & u^T \end{bmatrix},$$

where we introduce

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \text{with} \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_{2m+1} \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ \vdots \\ v_{2m+1} \end{bmatrix}.$$

Then M is Hamiltonian with respect to $J = \begin{bmatrix} 0 & I_{2m+1} \\ -I_{2m+1} & 0 \end{bmatrix}$.

The construction of a Jordan chain $x_1, \ldots, x_{2m+2}$ of M is given in [25]. Specifically, the vectors x_1 and x_{2m+2} are given by

$$x_1 = \begin{bmatrix} \mathcal{J}_{2m+1}^{2m} u \\ (\mathcal{J}_{2m+1}^{2m})^T v \end{bmatrix} = \begin{bmatrix} u_{2m+1} \\ 0 \\ \vdots \\ 0 \\ v_1 \end{bmatrix}, \quad x_{2m+2} = \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} -(I + \Upsilon)\mathcal{J}_{2m+1}^T u \\ (I + \Upsilon)\mathcal{J}_{2m+1} v \end{bmatrix},$$

where we used the abbreviation $\mathcal{J}_{2m+1} := \mathcal{J}_{2m+1}(0)$ and where a and b are chosen to satisfy[1]

$$-av_1 + bu_{2m+1} + 2u^T \mathcal{J}_{2m+1} v = 1.$$

The sign κ_1 corresponding to the nilpotent Jordan block of size $2m+2$ is necessarily given by the sign of

$$x_{2m+2}^* J x_1 = x_{2m+2}^* \begin{bmatrix} 0 & I_{2m+1} \\ -I_{2m+1} & 0 \end{bmatrix} \begin{bmatrix} u_{2m+1} \\ 0 \\ \vdots \\ 0 \\ v_1 \end{bmatrix} = -2(v_1 u_{2m} + v_2 u_{2m+1}).$$

Observing that generically this sign is nonzero, it can be both $+1$ or -1.

Considering instead of $M = A + ww^T J$ the matrix $M' = A - ww^T J$, then

$$M' = \begin{bmatrix} \mathcal{J}_{2m+1} & 0 \\ 0 & -\mathcal{J}_{2m+1}^T \end{bmatrix} - \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} -v^T & u \end{bmatrix}.$$

Following the line of argument in Example 4.1 in [25], we obtain that the vectors in a Jordan chain corresponding to the eigenvalue 0 of M' of length $2m+2$ can be obtained as follows: the first $2m+1$ vectors are the same as those for M, whereas the last one, which we shall denote by y_{2m+2}, is now constructed by choosing α and β such that

$$\alpha v_1 - \beta u_{2m+1} - 2u^T \mathcal{J}_{2m+1} v = 1,$$

and setting

$$y_{2m+2} = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \beta \end{bmatrix} + \begin{bmatrix} -(I+\Upsilon)\mathcal{J}_{2m+1}^T u \\ (I+\Upsilon)\mathcal{J}_{2m+1} v \end{bmatrix}.$$

Then the sign in the sign characteristic of (M', J) corresponding to the nilpotent Jordan block of size $2m+2$ is given by the sign of the number

$$y_{2m+2}^* J x_1 = -2(v_1 u_{2m} + v_2 u_{2m+1}),$$

which is the same as the sign in the sign characteristic of (M, J) corresponding to the nilpotent Jordan block of size $2m+2$.

This proves part (b) of Conjecture 4.4 for this particular example.

[1]We point out an error in Example 4.1 in [25], namely: the expression $u^T(I+\Upsilon)\mathcal{J}_{2m+1}v + v^T(I+\Upsilon)\mathcal{J}_{2m+1}^T u$ was simplified to $2u^T(I+\Upsilon)\mathcal{J}_{2m+1}v$, which is incorrect as Υ and $\mathcal{J}_{2m+1}$ do not commute but anti-commute. Nevertheless, the formulas for the sign characteristic in the same example are correct.

In the next theorem we use the notation introduced in Theorem 3.1 and we assume that the J-Hamiltonian matrix A has the Jordan form as described in Theorem 3.1. For the proof we need a version of [26, Theorem 2.3(c)] that does not explicitly involve a genericity hypothesis.

Theorem 4.6 *Let $H \in \mathbb{C}^{n\times n}$ be Hermitian and invertible and let $A \in \mathbb{C}^{n\times n}$ be H-self-adjoint. Suppose that the pair (A, H) has the canonical form $(A_1 \oplus A_2, H_1 \oplus H_2)$ as in Theorem 2.3 and that, furthermore, $B = uu^*H \in \mathbb{C}^{n\times n}$ has the following properties:*

(a) *the pair $(A + B, H)$ has the canonical form (A', H'), given by*

$$\begin{aligned} A' &= \bigoplus_{j=1}^{\mu}\Big(\big(\mathcal{J}_{n_{1,j}}(\lambda_j)^{\oplus \ell_{1,j}-1}\big) \oplus \big(\mathcal{J}_{n_{2,j}}(\lambda_j)^{\oplus \ell_{2,j}}\big) \oplus \cdots \oplus \Big(\mathcal{J}_{n_{m_j,j}}(\lambda_j)^{\oplus \ell_{m_j,j}}\Big)\Big) \\ &\quad \oplus \bigoplus_{j=1}^{\nu}\left(\bigoplus_{s=2}^{q_j} \mathcal{J}_{r_{j,s}}(a_j \pm ib_j)\right) \oplus A_3', \end{aligned}$$

$$\begin{aligned} H' &= \bigoplus_{j=1}^{\mu}\left(\left(\bigoplus_{s=1}^{\ell_{1,j}-1} \sigma'_{1,s,j} R_{n_{1,j}}\right) \oplus \left(\bigoplus_{s=1}^{\ell_{2,j}} \sigma'_{2,s,j} R_{n_{2,j}}\right) \oplus \cdots \oplus \left(\bigoplus_{s=1}^{\ell_{m_j,j}} \sigma'_{m_j,s,j} R_{n_{m_j,j}}\right)\right) \\ &\quad \oplus \bigoplus_{j=1}^{\nu}\left(\bigoplus_{s=2}^{q_j} R_{r_{j,s}}\right) \oplus H_3', \end{aligned}$$

where A_3' consists of Jordan blocks with eigenvalues different from the eigenvalues of A, and $\sigma'_{i,s,j} \in \{+1, -1\}$;

(b) *eigenvalues of $A + uu^*H$ which are not eigenvalues of A are all simple.*

Then

$$\sum_{s=1}^{\ell_{i,j}} \sigma'_{i,s,j} = \sum_{s=1}^{\ell_{i,j}} \sigma_{i,s,j}, \quad \text{for } \ i = 2, \ldots, m_j \ \text{ and } \ j = 1, \ldots, \mu,$$

and the list $(\sigma'_{1,1,j}, \ldots, \sigma'_{1,\ell_{1,j}-1,j})$ is obtained from $(\sigma_{1,1,j}, \ldots, \sigma_{1,\ell_{1,j},j})$ by removing either exactly one sign $+1$ or exactly one sign -1.

Proof. By [26, Theorem 2.3], there is a generic (with respect to the real and imaginary parts of the components of u) set $\Omega \subseteq \mathbb{C}^n$ such that the result of Theorem 4.6 holds under the additional hypothesis that $u \in \Omega$. Suppose now that $u \notin \Omega$. Then by [34, Theorem 3.6], there exists $\delta > 0$ such that for every $u_0 \in \mathbb{C}^n$ with $\|u - u_0\| < \delta$ and with $(A + u_0u_0^*H, H)$ satisfying properties (a) and (b) (where u is replaced by u_0), the sign characteristic of $(A + u_0u_0^*H, H)$ coincides with that of $(A + uu^*H, H)$. It remains to choose $u_0 \in \Omega$, which is possible in view of the genericity of Ω. □

Thus, in Theorem 4.6, the sign characteristic of the pair $(A+B, H)$ for the eigenvalue λ_j is the same as that for (A, H), except that for the set of Jordan blocks with eigenvalue λ_j and maximal size, one sign is dropped. We use this result to prove the following Theorem.

Theorem 4.7 *Let Ω be the generic set of all vectors $u \in \mathbb{R}^n$ for which the Jordan form of the J-Hamiltonian matrix $A + uu^T J$ has the properties described in parts* (1)–(3) *of Theorem* 3.1. *Let Λ be the set of those pairwise distinct numbers among $\{\lambda_1, \ldots, \lambda_p\}$ that have zero real parts and include $\lambda_{p+1} = 0$ in Λ if $0 \in \sigma(A)$ and if at least one partial multiplicity of* 0 *is even. For $j = 1, 2, \ldots, p$ (if $0 \notin \Lambda$) or for $j = 1, 2, \ldots, p+1$ (if $0 \in \Lambda$), let $\xi_{k,j}^{(1)}, \ldots, \xi_{k,j}^{(\ell_{k,j})}$ be the signs in the sign characteristic of (A, J) associated with the eigenvalue $\lambda_j \in \Lambda$ and the partial multiplicities $n_{k,j}$ repeated $\ell_{k,j}$ times; if $\lambda_{p+1} = 0 \in \Lambda$ we assume in addition that $n_{1,p+1}$ is even. Then we have the following properties of the sign characteristic:*

(a) *Within each connected component Ω_0 of Ω, the sign characteristic of the pair $(A + uu^T J, J)$, $u \in \Omega_0$, corresponding to those λ_j's in Λ that are eigenvalues of $A + uu^T J$, is constant, and the sign characteristic of any simple purely imaginary eigenvalue $\gamma = \gamma(u)$ of $A + uu^T J$ which is different from any of the λ_j is also constant, assuming $\gamma(u)$ is taken a continuous function of $u \in \Omega_0$.*

(b) *Suppose that A satisfies one of the following two mutually exclusive conditions*

  (b1) *A is invertible;*

  (b2) *A is not invertible and $n_{1,p+1}$ is even.*

  Then for every $u \in \Omega$, the signs in the sign characteristic of $(A + uu^T J, J)$ that correspond to $\lambda_j \in \Lambda$ and partial multiplicities smaller than $n_{1,j}$, coincide with the corresponding signs in the sign characteristic of (A, J), whereas the signs $\eta_{1,j}^{(1)}, \ldots, \eta_{1,j}^{(\ell_{1,j}-1)}$ in the sign characteristic of $(A + uu^T J, J)$ that correspond to $\lambda_j \in \Lambda$ and $\ell_{1,j} - 1$ partial multiplicities equal to $n_{1,j}$, satisfy

$$\eta_{1,j}^{(1)} + \cdots + \eta_{1,j}^{(\ell_{1,j}-1)} = \left(\xi_{1,j}^{(1)} + \cdots + \xi_{1,j}^{(\ell_{1,j})}\right) - \xi_{1,j}^{(k_0)}$$

  for some k_0, $1 \leq k_0 \leq \ell_{1,j}$.

Proof. Part (a) follows from [34, Theorem 3.6] (or from [5]) which asserts the persistence of the sign characteristic under suitable structure-preserving small norm perturbations in the context of H-selfadjoint matrices, combined with Theorem 4.1. For part (b) we can proceed as in part (a) using Theorem 4.6. □

Note that for the proof of part (b) of Theorem 4.7 one cannot simply use the corresponding result for selfadjoint matrices iA with respect to the indefinite inner product induced by iJ (Theorem 2.3 part (c) of [26]), since the notion of "generic" is different in [26, Theorem 2.3(c)] and in Theorem 4.7.

For the missing case in Theorem 4.7 we have the following conjecture.

Conjecture 4.8 Consider the hypotheses and notation of Theorem 4.7, and suppose that $0 \in \Lambda$ and that the largest partial multiplicity $n_{1,p+1}$ corresponding to the zero eigenvalue is odd. Then the statement in Theorem 4.7(b) holds for every $\lambda_j \in \Lambda \setminus \{0\}$. Moreover, for $\lambda_{p+1} = 0$ we have that the signs in the sign characteristic of $(A+uu^T J, J)$ that correspond to λ_{p+1} and even partial multiplicities smaller than $n_{1,p+1}$, coincide with the corresponding signs in the sign characteristic of (A, J).

In the last few theorems of this section we have restricted attention to perturbations of the form $+uu^T J$. Similar results hold for perturbations of the form $-uu^T J$.

5 Rank-one perturbations of small norm for real J-Hamiltonian matrices

We continue our study of the local behavior of the sign characteristic of real J-Hamiltonian matrices under generic structure-preserving rank-one perturbations and consider newly arising purely imaginary eigenvalues in the perturbed matrix, i.e., those that are not eigenvalues of the original matrix, assuming perturbations of sufficiently small norm. It will be convenient to distinguish the cases that the unperturbed eigenvalue is nonzero, and that the unperturbed eigenvalue is zero.

Consider an eigenvalue $\lambda_0 = ib$ of A with $b > 0$, let $n_1 > \cdots > n_p$ be the distinct partial multiplicities of A corresponding to λ_0, and suppose there are ℓ_j blocks in the real Jordan form of A having size $2n_j$ and eigenvalues $\pm\lambda_0$, for $j = 1, 2, \ldots, p$, with the signs $\xi_{j,k} = \pm 1$, $k = 1, 2, \ldots, \ell_j$ attached to the blocks of size $2n_j$ (repeated ℓ_j times) in the sign characteristic of (A, J) associated with the eigenvalues $\pm\lambda_0$. Recall from Theorem 2.4 that the signs $\xi_{j,k}$, for every fixed j, are uniquely determined up to a permutation of the blocks. For the purpose of our analysis, it will be convenient to single out $\xi_{1,1}$ and to classify the various possibilities according to the value $\xi_{1,1} = 1$ or $\xi_{1,1} = -1$.

According to Theorem 4.7, for a generic set (with respect to the components of u) of vectors $u \in \mathbb{R}^n$, we have one of the following four (not necessarily mutually exclusive) situations.

(E+) n_1 is even, $\xi_{1,1} = 1$, and for the eigenvalue λ_0 the J-Hamiltonian matrix $A - uu^T J$ has distinct partial multiplicities $n_1 > \cdots > n_p$ repeated $\ell_1 - 1, \ell_2, \ldots, \ell_p$ times respectively (if $\ell_1 = 1$, then n_1 is omitted), with signs in the sign characteristic $\xi_{1,k}$, $k = 2, \ldots, \ell_1$ corresponding to the partial multiplicities n_1 (repeated $\ell_1 - 1$ times) and $\xi_{j,k}$, $k = 1, 2, \ldots, \ell_j$ corresponding to the partial multiplicities n_j (repeated ℓ_j times) for $j = 2, 3, \ldots, p$;

(E−) n_1 is even, $\xi_{1,1} = -1$, and all other properties as described in (E+);

(O+) n_1 is odd, $\xi_{1,1} = 1$, and all other properties as described in (E+);

(O−) n_1 is odd, $\xi_{1,1} = -1$, and all other properties as described in (E+).

We assume in addition that $\|u\|$ is sufficiently small, so that $A - uu^TJ$ has generically n_1 eigenvalues $\nu_1, \ldots, \nu_{n_1}$ different from λ_0, (which may be purely imaginary or not) that are clustered around λ_0. By Theorem 4.7, we may assume that generically the eigenvalues $\nu_1, \ldots \nu_{n_1}$ are all simple. Renumbering these eigenvalues so that $\nu_1, \ldots, \nu_m$ are purely imaginary and the remaining eigenvalues are non-purely imaginary, we use the notation $\nu_j = ir_j$, $j = 1, 2, \ldots, m$, where $r_1 < \cdots < r_m$ are real. Note that m may depend on u, but this dependence is not reflected in the notation. Thus, there is a sign η_j associated with ν_j, $j = 1, 2, \ldots, m$, in the sign characteristic of $(A - uu^TJ, J)$, and obviously, $m \leq n_1$.

Theorem 5.1 *Let $J \in \mathbb{R}^{n\times n}$ be skew-symmetric and invertible, let $A \in \mathbb{R}^{n\times n}$ be J-Hamiltonian, and let Ω be the open generic set of vectors $u \in \mathbb{R}^n$, for which one of the cases* (E+), (E−), (O+), (O−) *holds and for which the eigenvalues $\nu_1, \ldots, \nu_{n_1}$ of $A - uu^TJ$ are all distinct, simple, and none of them is equal to λ_0.*

(a) *Suppose that $u \in \Omega$ and $\|u\|$ is sufficiently small (the sufficiency of the smallness of $\|u\|$ is determined by the pair (A, J) only). Then we have that m is even and $\eta_1 + \cdots + \eta_m = 0$ in cases* (E+) *and* (E−), *and m is odd and $\eta_1 + \cdots + \eta_m = \pm 1$ in cases* (O+) *and* (O−).

(b) *Suppose in addition to the assumption in* (a) *that the geometric multiplicity of λ_0 as eigenvalue of A is equal to one. Then we have the following cases.*

  (b1) *If* (E+) *holds, then $m \neq 0$ and for some odd $k < m$, we have*

$$r_1 < \cdots < r_k < i^{-1}\lambda_0 < r_{k+1} < \cdots < r_m,$$

  and $\eta_q = (-1)^q$, for $q = 1, 2, \ldots, m$.

  (b2) *If* (E−) *holds, then none of the new eigenvalues ν_q, $q = 1, 2, \ldots, m$, is purely imaginary, i.e., $m = 0$.*

  (b3) *If* (O+) *holds, then $i^{-1}\lambda_0 < r_1 < \cdots < r_m$, with $\eta_q = (-1)^{q-1}$, for $q = 1, 2, \ldots, m$.*

  (b4) *If* (O−) *holds, then $r_1 < \cdots < r_m < i^{-1}\lambda_0$, with $\eta_q = (-1)^q$, for $q = 1, 2, \ldots, m$.*

Proof. The assertion follows from Theorem 5.3 of [26] applied to the iJ-selfadjoint matrix iA, and taking into account Theorem 4.1. □

Finally, we consider perturbations of the eigenvalue zero. Let $n_1 > \cdots > n_p$ be the pairwise distinct partial multiplicities of A corresponding to the eigenvalue zero, and suppose that n_j is repeated ℓ_j times. Thus, ℓ_j is even if n_j is odd, and for every j for which n_j is even, there are signs $\xi_{k,j} = \pm 1$ for $k = 1, 2, \ldots, \ell_j$ in the sign characteristic of (A, J) associated with the ℓ_j nilpotent Jordan blocks of the even size n_j. As in the case of nonzero purely imaginary eigenvalues, we distinguish the following cases for $\xi_{1,1}$ (in case n_1 is even).

(E0+) n_1 is even, $\xi_{1,1} = 1$, and at the eigenvalue zero the J-Hamiltonian matrix $A - uu^T J$ has distinct partial multiplicities $n_1 > \cdots > n_p$ repeated $\ell_1 - 1, \ell_2, \ldots, \ell_p$ times respectively (if $\ell_1 = 1$, then n_1 is omitted), with signs $\xi_{1,k}$, $k = 2, \ldots, \ell_1$ in the sign characteristic, corresponding to the partial multiplicities n_1 (repeated $\ell_1 - 1$ times), and $\xi_{j,k}$, $k = 1, 2, \ldots, \ell_j$ corresponding to the partial multiplicities n_j (repeated ℓ_j times) for those indices j, $j = 2, 3, \ldots, p$, for which n_j is even.

(E0−) n_1 is even, $\xi_{1,1} = -1$, and all other properties are as described in (E0+).

(O0) n_1 is odd, and at the eigenvalue zero the J-Hamiltonian matrix $A - uu^T J$ has distinct partial multiplicities $n_1 + 1 > n_1 > \cdots > n_p$ repeated $1, \ell_1 - 2, \ell_2, \ldots \ell_p$ times, respectively (if $\ell_1 = 2$, then n_1 is omitted), with signs in the sign characteristic $\xi = \pm 1$ corresponding to the partial multiplicity $n_1 + 1$ and $\xi_{j,k}$, $k = 1, 2, \ldots, \ell_j$ corresponding to the partial multiplicities n_j (repeated ℓ_j times) for those indices j, $j = 2, 3, \ldots, p$, for which n_j is even.

According to Theorem 4.7, for a generic set of vectors $u \in \mathbb{R}^n$, one of the cases (E0+), (E0−), or (O0) occurs. In addition, we assume that $\|u\|$ is sufficiently small, so that $A - uu^T J$ has generically n_1 eigenvalues $\nu_1, \ldots, \nu_{n_1}$ different from zero that are clustered around zero (which may be purely imaginary or not) when n_1 is even, and $n_1 - 1$ eigenvalues $\nu_1, \ldots, \nu_{n_1 - 1}$ different from zero that are clustered around zero (which may be purely imaginary or not) when n_1 is odd. By Theorem 4.7, we may assume that generically these eigenvalues ν_j are all simple. Renumbering these eigenvalues so that $\nu_1, \ldots, \nu_m$ are purely imaginary with positive imaginary parts and the remaining eigenvalues are either the negatives of ν_j's or non-purely imaginary, we set $\nu_j = ir_j$, $j = 1, 2, \ldots, m$, where $0 < r_1 < \cdots < r_m$ are real. Thus, in the sign characteristic of $(A - uu^T J, J)$ there is a sign η_q associated with ν_q, $q = 1, 2, \ldots, m$.

In general, we cannot say anything specific about the number m and the signs η_q (except the obvious fact that $m \leq n_1/2$ if n_1 is even and $m \leq (n_1 - 1)/2$ if n_1 is odd). However, in the particular case when the geometric multiplicity of A at zero is equal to one (in this case n_1 is necessarily even), we have additional information.

Theorem 5.2 *Let $J \in \mathbb{R}^{n \times n}$ be skew-symmetric and invertible, and let $A \in \mathbb{R}^{n \times n}$ be J-Hamiltonian. Assume that the geometric multiplicity of A at zero is equal to one. Let Ω be the open generic set of vectors $u \in \mathbb{R}^n$, for which one of the cases* (E0+) *or* (E0−) *occurs, and assume that $\|u\|$ is sufficiently small. Then we have the following statements.*

(1) *If* (E0+) *holds and $n_1/2$ is odd, or if* (E0−) *holds and $n_1/2$ is even, then $A - uu^T J$ has no purely imaginary nonzero eigenvalues close to zero.*

(2) *If* (E0+) *holds and $n_1/2$ is even, or if* (E0−) *holds and $n_1/2$ is odd, then m is odd, and $\nu_j = (-1)^{j-1}$, for $j = 1, 2, \ldots, m$.*

Proof. The proof follows by combining Theorem 5.3(b) of [26] applied to the iJ-selfadjoint matrix iA and taking into account Theorem 4.1. □

In this section we focused on perturbations of the form $-uu^T J$; similar results hold for perturbations of the form $+uu^T J$.

6 Generic structure-preserving rank-one perturbations for real H-symmetric matrices

Let $H \in \mathbb{R}^{n\times n}$ be symmetric and invertible. A real analogue of [26, Theorem 3.3] that relates all eigenvalues of an H-selfadjoint matrix to the perturbed eigenvalues under generic H-selfadjoint rank-one perturbations at once, not just to one of them, and that describes the behavior of the sign characteristic as well, is given by the following result.

Theorem 6.1 *Let $H \in \mathbb{R}^{n\times n}$ be symmetric and invertible and let $A \in \mathbb{R}^{n\times n}$ be H-selfadjoint. Suppose that the pair (A, H) has the canonical form as in Theorem 2.3. If $B = uu^T H \in \mathbb{R}^{n\times n}$, then the following statements hold.*

(a) *Generically (with respect to the components of u) the pair $(A + B, H)$ has the canonical form (A', H'), where*

$$\begin{aligned} A' &= \oplus_{j=1}^{\mu} \left(\left(\oplus_{s=1}^{\ell_{1,j}-1} \mathcal{J}_{n_{1,j}}(\lambda_j) \right) \oplus \left(\oplus_{s=1}^{\ell_{2,j}} \mathcal{J}_{n_{2,j}}(\lambda_j) \right) \oplus \cdots \oplus \left(\oplus_{s=1}^{\ell_{m_j,j}} \mathcal{J}_{n_{m_j,j}}(\lambda_j) \right) \right) \\ &\quad \oplus \oplus_{j=1}^{\nu} \left(\oplus_{s=2}^{q_j} \mathcal{J}_{r_{s,j}}(a_j \pm ib_j) \right) \oplus A_3', \\ H' &= \oplus_{j=1}^{\mu} \left(\left(\oplus_{s=1}^{\ell_{1,j}-1} \sigma'_{1,s,j} R_{n_{1,j}} \right) \oplus \left(\oplus_{s=1}^{\ell_{2,j}} \sigma_{2,s,j} R_{n_{2,j}} \right) \oplus \cdots \oplus \left(\oplus_{s=1}^{\ell_{m_j,j}} \sigma_{m_j,s,j} R_{n_{m_j,j}} \right) \right) \\ &\quad \oplus \oplus_{j=1}^{\nu} \left(\oplus_{s=2}^{q_j} R_{r_{s,j}} \right) \oplus H_3', \end{aligned}$$

where A_3' consists of Jordan blocks with eigenvalues different from the λ_j's and the $a_j \pm ib_j$'s, and where the list $(\sigma'_{1,1,j}, \ldots, \sigma'_{1,\ell_{1,j}-1,j})$ is obtained from the list $(\sigma_{1,1,j}, \ldots, \sigma_{1,\ell_{1,j},j})$ by removing either exactly one sign $+1$ or exactly one sign -1.

(b) *Generically all eigenvalues of $A + uu^T H$ which are not also eigenvalues of A are simple.*

(c) *Let $\Omega \subseteq \mathbb{C}^n$ be the generic set such that for every $u \in \Omega$ the properties* (a) *and* (b) *hold. Then, within each connected component Ω_0 of Ω, the sign characteristic of the pair $(A + uu^T H, H)$, $u \in \Omega_0$, corresponding to those among the λ_j's that are eigenvalues of $A + uu^T H$, is constant, and the sign characteristic of any simple real eigenvalue $\gamma = \gamma(u)$ of $A + uu^T H$, which is different from the λ_j's is also constant, assuming that $\gamma(u)$ is taken a continuous function of $u \in \Omega_0$.*

Proof. We will employ the corresponding theorem for the complex case, given by [26, Theorem 3.3]. In this result, however, the result was stated for a slightly different canonical form of the pair (A, H), where $H = H^* \in \mathbb{C}^{n\times n}$ is invertible and $A \in \mathbb{C}^{n\times n}$ is H-selfadjoint. Namely, in [26], the pair of blocks

$$\begin{bmatrix} \mathcal{J}_{\frac{1}{2}r_{j,s}}(a_j + ib_j) & 0 \\ 0 & \mathcal{J}_{\frac{1}{2}r_{j,s}}(a_j + ib_j))^* \end{bmatrix}, \qquad \begin{bmatrix} 0 & I_{\frac{1}{2}r_{j,s}} \\ I_{\frac{1}{2}r_{j,s}} & 0 \end{bmatrix} \tag{6.1}$$

is used in place of the pair

$$\mathcal{J}_{r_{j,s}}(a_j \pm ib_j), \qquad R_{r_{j,s}} \tag{6.2}$$

in (2.2).

Let $S \in \mathbb{C}^{n\times n}$ be the transformation matrix from the form of (A, H) as given in Theorem 6.1 to the form as used in [26], so that

$$A' = S^{-1}AS, \;\; H' = S^*HS,$$

where (A', H') is the canonical form obtained from the canonical form of Theorem 2.3 by using (6.1) in place of (6.2). For any $u' \in \mathbb{C}^n$ we have

$$A' + u'(u')^*H' = S^{-1}(A + uu^*H)S,$$

where $u = Su'$. Thus, the canonical form of $(A+uu^*H, H)$ coincides with the canonical form of $(A' + u'(u')^*H', H')$. Clearly, a set Ω' of vectors in $\mathbb{C}^n$ is generic if and only if the set $S\Omega'$ is. In this way, we can apply [26, Theorem 3,3] to the pair (A, H), although it is given in a different form. We have that the assertion of Theorem 6.1 holds for complex rank-one perturbations of the form $B = uu^*H$, where $\Omega \subseteq \mathbb{C}^n$ is a generic set. As in the proof of Theorem 3.1 we conclude that $\Omega \cap \mathbb{R}^n$ is a generic subset of $\mathbb{R}^n$, and the proof is complete. □

The results of [26] for complex matrices, which concern the sign characteristic of new eigenvalues (i.e., those eigenvalues of the perturbed matrix $A+uu^TH$ that are not eigenvalues of A), in particular Theorem 5.3 there, are valid verbatim, as well as their proofs, for the real case. We will not reproduce these results here.

We mention also that the analogues of Corollary 2.2 and Theorem 4.6 hold in the context of real H-symmetric matrices. The statements and proofs are similar to those of Corollary 2.2 and Theorem 4.6, and are therefore omitted.

7 Application to T-even matrix polynomials

Consider *T-even matrix polynomials* of the form

$$L(\lambda) = A_\ell \lambda^\ell + \cdots + A_1\lambda + A_0, \tag{7.1}$$

where $A_\ell, \ldots, A_1, A_0 \in \mathbb{R}^{n\times n}$ and A_k is skew-symmetric if k is odd, A_k is symmetric if k is even, and A_ℓ is invertible. In this section we consider the application of the previous results to the structured perturbation theory of these matrix polynomials and we do this via appropriate structure preserving linearizations [10, 21, 22].

Define the real $n\ell \times n\ell$ matrices

$$\begin{aligned} C &:= \begin{bmatrix} 0 & I_n & 0 & \ldots & 0 \\ 0 & 0 & I_n & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I_n \\ -A_\ell^{-1}A_0 & -A_\ell^{-1}A_1 & -A_\ell^{-1}A_2 & \ldots & -A_\ell^{-1}A_{\ell-1} \end{bmatrix}, \\ G &:= \begin{bmatrix} A_1 & A_2 & A_3 & \ldots & A_\ell \\ -A_2 & -A_3 & \ldots & -A_\ell & 0 \\ (-1)^2 A_3 & \iddots & \iddots & 0 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ (-1)^{\ell-1}A_\ell & 0 & 0 & \ldots & 0 \end{bmatrix}. \end{aligned} \tag{7.2}$$

We see that G is skew-symmetric and invertible and C is G-Hamiltonian. The matrix C is known as the *companion matrix linearization* of the matrix polynomial (7.1), and is ubiquitous in studies of matrix polynomials, see e.g. [10]; linearizations in the context of structured matrix polynomials have been studied in [7, 12, 13, 14, 21, 23].

Matrix polynomials arise in the study of systems of differential equations with constant coefficients

$$A_\ell x^{(\ell)} + A_{\ell-1}x^{(\ell-1)} + \cdots + A_1\dot{x} + A_0 x = 0, \tag{7.3}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is an unknown ℓ-times continuously differentiable vector function of the real independent variable t. If the matrix polynomial (7.3) is T-even then we call the differential equation T-even as well.

Using the usual transformation to first order form, there is a well-known correspondence between the solutions x of (7.3) and the solutions X of the linear first-order differential equation

$$\dot{X} = CX; \tag{7.4}$$

for

$$X = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(\ell-1)} \end{bmatrix}. \tag{7.5}$$

We say that system (7.3) is *forward*, resp., *backward bounded*, if all solutions are bounded as $t \longrightarrow +\infty$, resp., $t \longrightarrow -\infty$. The system (7.3) is said to be *bounded* if all solutions are bounded on the real line.

Theorem 7.1 *Consider a T-even system of differential equations* (7.3) *with constant coefficients and associated T-even matrix polynomial* (7.1)*, and assume that A_ℓ is invertible. Then the following statements are equivalent.*

(a) *The system is forward bounded.*

(b) *The system is backward bounded.*

(c) *The system is bounded.*

(d) *All eigenvalues of C have zero real parts, and for every eigenvalue the geometric multiplicity coincides with the algebraic multiplicity.*

Proof. Clearly (c) implies both (a) and (b). By the standard theory of systems of differential equations with constant coefficients, (d) is equivalent to (7.4) having all solutions bounded, which in turn, by the correspondence (7.5) between the solutions of (7.4) and those of (7.3), is equivalent to (c). It remains to prove that (a) or (b) implies (d). We show that (a) implies (d); the proof of the statement that (b) implies (d) is completely analogous. If (a) holds, then by the standard theory of systems of differential equations with constant coefficients [6], and in view of the correspondence (7.5), all eigenvalues of C have non-positive real parts, and for eigenvalues with zero real parts the geometric and algebraic multiplicities coincide. But then the canonical form (2.3) shows that the G-Hamiltonian matrix C cannot have eigenvalues with nonzero real parts. $\square$

In many applications it is important to decide if a given system of differential equations is not only bounded but all nearby systems are also bounded as well. And if the differential equation is T-even then, because this is a property of the underlying physical problem, the nearby systems should be also considered to have a similar structure. This concept of *robust boundedness* is defined as follows. The T-even system of differential equations (7.3) is said to be *robustly bounded* if there exists $\varepsilon > 0$ such that every T-even system of differential equations

$$A'_\ell x^{(\ell)} + A'_{\ell-1} x^{(\ell-1)} + \cdots + A'_1 \dot{x} + A'_0 x = 0, \tag{7.6}$$

with coefficients that satisfy $\max_{j=0,1,\dots,\ell} \|A'_j - A_j\| < \varepsilon$ is bounded as well. Note that this means that the perturbation is structured and the perturbed polynomial stays within the set of T-even matrix polynomials. Moreover, we may assume that ε is small enough such that A'_ℓ is invertible if $\|A'_\ell - A_\ell\| < \varepsilon$ holds.

It is obvious (arguing by contradiction) that if (7.3) is robustly bounded, then there exists $\varepsilon > 0$ such that all T-even systems (7.6) satisfying $\max_{j=0,1,\dots,\ell} \|A'_j - A_j\| < \varepsilon$ are robustly bounded. This is not necessarily the case for systems that are just bounded. A sufficient condition for robust boundedness is given in the following theorem.

Theorem 7.2 *Consider a T-even differential equation* (7.3) *with constant coefficients and suppose that A_ℓ is invertible. Assume that the companion matrix C is invertible, all eigenvalues of C have zero real part and the root subspace corresponding to every pair of complex conjugate eigenvalues of C is GC-definite. Then the system* (7.3) *is robustly bounded.*

If $\ell = 1$, then the converse also holds, namely, if the T-even first order system

$$A_1\dot{x} + A_0 x = 0, \tag{7.7}$$

is robustly bounded, then every root subspace for $-A_1^{-1}A_0$ is A_0-definite.

Proof. By combining Theorems 2.6 and 7.1, we see that (7.3) is bounded. Since every root subspace $\mathcal{M}$ of C is GC-definite, it follows by a proof similar to the one in [35, Section 13.6] that the same property holds for any Hermitian matrix Y sufficiently close to GC and any subspace $\mathcal{X}$ sufficiently close to $\mathcal{M}$ in the gap metric

$$\theta(\mathcal{X}, \mathcal{M}) := \|P_{\mathcal{X}} - P_{\mathcal{M}}\|,$$

where $P_{\mathcal{X}}$ and $P_{\mathcal{M}}$ denote the orthogonal projection on $\mathcal{X}$ and $\mathcal{M}$, respectively; (for basic properties of the gap metric, see for example [10]). Indeed, we have $|x^*GCx| > 0$ for every nonzero $x \in \mathcal{M}$. Therefore, by compactness of the unit sphere $S_{\mathcal{M}}$ in $\mathcal{M}$, we have the inequality

$$\min_{x \in S_{\mathcal{M}}} \{|x^*GCx|\} \geq \varepsilon > 0$$

for some $\varepsilon > 0$. Now suppose that a subspace $\mathcal{X} \subseteq \mathbb{R}^n$ is close to $\mathcal{M}$, in the sense that

$$\theta(\mathcal{X}, \mathcal{M}) = \|P_{\mathcal{X}} - P_{\mathcal{M}}\| \leq \delta,$$

where $\delta > 0$ is small. Take $x \in \mathcal{X}$, $\|x\| = 1$. Then

$$\|x - P_{\mathcal{M}}x\| = \|P_{\mathcal{X}}x - P_{\mathcal{M}}x\| \leq \delta, \tag{7.8}$$

so for $y := P_{\mathcal{M}}x$ we have $\|y\| \geq 1 - \delta$ and

$$|y^*GCy| = (1-\delta)^2 \left|\big(y/(1-\delta)\big)^* GC\big(y/(1-\delta)\big)\right| \geq (1-\delta)^2\varepsilon.$$

On the other hand,

$$x^*GCx = (x-y)^*GC(x-y) + y^*GC(x-y) + (x-y)^*GCy + y^*GCy;$$

and hence,

$$|x^*GCx - y^*GCy| \leq \delta^2\|GC\| + 2\delta\|GC\|\|y\| \leq \delta^2\|GC\| + 2\delta(1+\delta)\|GC\|,$$

where (7.8) was used. Thus,

$$|x^*GCx| \geq (1-\delta)^2\varepsilon - \big(\delta^2\|GC\| + 2\delta(1+\delta)\|GC\|\big),$$

which is greater than $\varepsilon/2$ if δ is sufficiently small. Furthermore,

$$|x^*Yx - x^*GCx| \leq \|Y - GC\|, \quad \text{for all} \;\; x \in \mathcal{X}, \; \|x\| = 1,$$

and so $|x^*Yx| \geq \varepsilon/4$ if $\|Y - GC\| < \varepsilon/4$.

Now, it is well known that root subspaces are continuous (even Lipschitz continuous) functions of the entries of a matrix (see, e.g., [3, Section 14.2] for the complex case). So, for every given root subspace $\mathcal{M} \subseteq \mathbb{R}^n$ corresponding either the eigenvalue $\lambda = 0$ or to a pair of non-real complex conjugate eigenvalues $\pm i\mu$ for C (recall that by assumption all eigenvalues of C have zero real part), every matrix C' which is sufficiently close to C, has an invariant subspace $\mathcal{M}'$ as close as we wish to $\mathcal{M}$. Moreover, in fact $\mathcal{M}'$ is the sum of root subspaces for C' corresponding to the eigenvalues which are close to $\lambda = 0$ or to $\pm i\mu$, as the case may be. If in addition C' is G'-Hamiltonian for some real skew-symmetric matrix G' sufficiently close to G, then $\mathcal{M}'$ is $G'C'$-definite by the observation in the preceding paragraph.

It follows that every root subspace of C' is $G'C'$-definite. Combining Theorems 2.6 and 7.1, we see that the perturbed T-even system (7.6) is bounded provided that the leading coefficient stays nonsingular, which holds if ε is sufficiently small. This proves the robust boundedness of (7.3).

Consider now the case $\ell = 1$, namely system (7.7), and assume (7.7) is robustly bounded. We will prove that every root subspace of $-A_1^{-1}A_0$ corresponding either to the eigenvalue zero or to a pair of complex conjugate nonzero purely imaginary eigenvalues is A_0-definite. Suppose not, and there is a root subspace of $-A_1^{-1}A_0$ which is not A_0-definite. We will produce an arbitrarily small perturbation of A_0 that will result in a system that is not bounded, thereby obtaining a contradiction with the robust boundedness of (7.7).

Since (7.7) is robustly bounded, it is in particular bounded, and we take advantage of Theorem 7.1. Since there is a root subspace of $-A_1^{-1}A_0$ which is not A_0-definite, we can only have two possible situations for the Jordan blocks, when passing to the canonical form of $(A_1, -A_1^{-1}A_0)$: either $-A_1^{-1}A_0$ has at least one purely imaginary eigenvalue ib, $b > 0$ with mixed sign characteristic (i.e., the part of the sign characteristic corresponding to ib consists of both pluses and minuses), or $-A_1^{-1}A_0$ has the eigenvalue zero (and then with geometric multiplicity no less than two, because Jordan blocks of size one must occur an even number of times). Thus, without loss of generality, we may assume that A_0 and $-A_1^{-1}A_0$ take one of the two following forms.

(a)

$$-A_1^{-1}A_0 = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where $b > 0$, cf. Example 2.7;

(b)

$$-A_1^{-1}A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In case (b), by Corollary 3.2, for every rank-one A_1-Hamiltonian matrix B we have that $-A_1^{-1}A_0 + B$ has a Jordan block of size two associated with the eigenvalue zero, so the system $A_1\dot{x} + (A_0 - A_1 B)x = 0$ is not bounded. In case (a), without loss of generality we may assume $b = 1$. Then we will produce a small perturbation of the form $uu^T A_1$, $u \in \mathbb{R}^4$ and u is close to zero, such that the matrix

$$D(u) := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + uu^T \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

has eigenvalues $\pm i$ of geometric multiplicity one and algebraic multiplicity two. This will again give raise to a perturbed system which is not bounded. By Corollary 3.2, $D(u)$ will still have the eigenvalues $i, -i$ both with geometric multiplicity at least one, and if $D(u)$ has eigenvalues different from $i, -i$ for u sufficiently small, then by Theorem 2.4 those must be purely imaginary. Thus, to achieve the desired properties of $D(u)$, we need that

(1) $\det D(u) = 1$ (to exclude that $D(u)$ has eigenvalues $\pm ib$ different from $i, -i$);

(2) $D(u)^2 + I \neq 0$ (to exclude that the geometric multiplicity of i and $-i$ is two).

A straightforward computation shows that the diagonal entries of $D(u)^2 + I$ are given by $-u_1^2 - u_2^2$, $-u_1^2 - u_2^2$, $u_3^2 + u_4^2$, and $u_3^2 + u_4^2$, since $u^T A_1 u = 0$. Thus condition (2) boils down to $u \neq 0$. Moreover, we have that $\det D(u) = 1 + u_1^2 + u_2^2 - u_3^2 - u_4^2$, so we only need to choose u so that $u_1^2 + u_2^2 = u_3^2 + u_4^2$ to satisfy (1). □

We do not know whether or not the converse statement in Theorem 7.2 holds in the case $\ell > 1$.

For rank-one perturbations of T-even first order systems, we have more precise information.

Theorem 7.3 *Suppose that the T-even first order system* (7.7) *is bounded, but not robustly bounded. Then the following statements hold.*

(1) *There exist A_1-Hamiltonian matrices $B = \pm uu^T A_1$ with $u \in \mathbb{R}^n \setminus \{0\}$ arbitrarily close to zero such that the T-even system*

$$A_1\dot{x} + (A_0 + A_1 B)x = 0 \tag{7.9}$$

is not bounded.

(2) *If A_0 is singular, then generically (with respect to the entries of $u \in \mathbb{R}^n$) the system* (7.9) *is not bounded, for every generic u with $\|u\|$ sufficiently small.*

(3) *If A_0 is nonsingular, then generically (with respect to the entries of $u \in \mathbb{R}^n$) the system* (7.9) *is bounded, for every generic u with $\|u\|$ sufficiently small.*

Proof. Part (1) follows from the consideration of case (a) in the proof of Theorem 7.2 if A_0 is nonsingular, and from the consideration of case (b) in the proof of Theorem 7.2 if A_0 is singular (the perturbations there have rank one).

For the proof of part (2) note that by Theorems 2.4 and 7.1, the A_1-Hamiltonian matrix $A_1^{-1}A_0$ must have the eigenvalue zero with algebraic and geometric multiplicity both equal to ν, where $\nu \geq 2$ is even. The result then follows immediately from Theorem 3.1 as the A_1-Hamiltonian matrix $A_1^{-1}A_0 + B$ will have a Jordan block of size two associated with zero in its Jordan canonical form.

Part (3). The A_1-Hamiltonian matrix $C := -A_1^{-1}A_0$ has only purely imaginary nonzero eigenvalues, and for every eigenvalue the geometric multiplicity and the algebraic multiplicity are the same. Let $\lambda_1, \ldots, \lambda_p$ be all distinct eigenvalues of C with positive imaginary parts and multiplicities $n_1, \ldots, n_p$, respectively. By Theorem 3.1, generically the eigenvalues of $C \pm uu^T A_1$ are $\lambda_1, \ldots, \lambda_p$ with algebraic multiplicities $n_1 - 1, \ldots, n_p - 1$, respectively, and simple eigenvalues $\mu_1, \ldots, \mu_p$ different from any of the λ_j. If u is sufficiently small, then exactly one of the μ_k will be in the vicinity of λ_j, for every $j = 1, 2, \ldots, p$. Renumbering the μ_k if necessary, we may assume that μ_j is in the vicinity of λ_j, for $j = 1, 2, \ldots, p$. It is easy to see (because of the symmetry of $\sigma(C \pm uu^T A)$ with respect to the imaginary axis, and because the total algebraic multiplicity of eigenvalues of $C \pm uu^T A_1$ which are close to λ_j is equal to n_j, for $j = 1, 2, \ldots, p$) that μ_j is purely imaginary. Moreover, by [27, Theorem 4.3], the eigenvalue λ_j of $C \pm uu^T A_1$ has geometric multiplicity equal to $n_j - 1$, for generic u of sufficiently small norm. Thus, generically and for u sufficiently small in norm, all eigenvalues of $C \pm uu^T A_1$ are purely imaginary with the geometric multiplicity equal to the algebraic multiplicity for each one of them. Then the result follows from Theorem 7.1.
□

8 Application to symmetric matrix polynomials

Similar results as those for T-even matrix polynomials also hold for *real symmetric matrix polynomials* of the form

$$L(\lambda) = A_\ell \lambda^\ell + \cdots + A_1 \lambda + A_0, \tag{8.1}$$

where the coefficients $A_\ell, \ldots, A_1, A_0 \in \mathbb{R}^{n \times n}$ are symmetric and A_ℓ is invertible, and the associated *complex symmetric system of differential equations*

$$i^\ell A_\ell \, x^{(\ell)} + i^{\ell-1} A_{\ell-1} x^{(\ell-1)} + \cdots + i A_1 \dot{x} + A_0 x = 0, \tag{8.2}$$

for which complex valued solution functions $x : \mathbb{R} \to \mathbb{C}^n$ are sought. See [14] and [13] for background on symmetric matrix polynomials and associated systems of linear differential equations.

Note that in this case the companion matrix C is $\widetilde{G}$-symmetric, where

$$\widetilde{G} := \begin{bmatrix} A_1 & A_2 & A_3 & \dots & A_\ell \\ A_2 & A_3 & \dots & A_\ell & 0 \\ A_3 & & \iddots & 0 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ A_\ell & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n\ell \times n\ell}. \tag{8.3}$$

Systems of the form (8.2) have been studied in [11, 12, 13, 30]. In particular, the analogue of Theorem 7.1 and the following criterion for robust boundedness hold for (8.2) (the definitions of boundedness and robust boundedness are completely analogous to those given in Section 7).

Theorem 8.1 *The following statements are equivalent for the system* (8.2) *and the associated symmetric matrix polynomial* (8.1)*.*

(1) *The system is robustly bounded.*

(2) *Every root subspace for C, corresponding either to a real eigenvalue or to a pair of non-real complex conjugate eigenvalues, is $\widetilde{G}$-definite.*

(3) $\det L(\lambda_0) \neq 0$ *for all non-real λ_0, and for the derivative $L'(\lambda)$ with respect to λ of $L(\lambda)$, the quadratic form $x^* L'(\lambda_0) x$ is positive definite or negative definite on* $\operatorname{Ker} L(\lambda_0) \subseteq \mathbb{R}^n$*, for every real zero λ_0 of $L(\lambda)$.*

(4) *All eigenvalues of C are real, and for every eigenvalue the geometric multiplicity is equal to the algebraic multiplicity, and moreover for every eigenvalue the signs in the sign characteristic of $(C, \widetilde{G})$ are the same (but the signs corresponding to different eigenvalues may be different).*

Proof. The equivalence of statements (1), (2), and (3) in Theorem 8.1 for complex matrices is given in [13, Theorems 13.3.2, 9.2.2], [12, Section III.2.2], and was originally proved in [9], [10]. In the real case the proof is essentially the same as indicated in [13] (see Section 9.5 there). The equivalence of (2) and (4) follows from the canonical form of Theorem 2.3 for the pair $(C, \widetilde{G})$. □

The analogues of Theorem 7.3, parts (1) and (3), also hold for first order systems of type (8.2).

Note that the conditions in Theorem 8.1 are equivalent conditions but in Theorem 7.2 they are not. The difference is caused by different classes of matrices under consideration in Sections 7 and 8; in Section 7 we consider J-Hamiltonian matrices, whereas in Section 8 the matrices are H-symmetric.

Theorem 8.2 *Consider a first-order system of the form* (8.2) *given by the system $iA_1\dot{x} + A_0 x = 0$, where A_1 is invertible, and that is bounded but not robustly bounded. Then the following statements hold.*

(1) *There exist A_1-selfadjoint matrices B of rank one and norm arbitrarily close to zero such that the system*

$$iA_1\dot{x} + (A_0 + A_1 B)x = 0$$

is not robustly bounded.

(2) *Generically (with respect to the entries of $u \in \mathbb{R}^n$) the system $iA_1\dot{x} + (A_0 + A_1 uu^T A_1)x = 0$ is bounded, for u of sufficiently small norm.*

Proof. The proof follows the same approach as that of Theorem 7.3, using Theorem 8.1 and the analogue of Theorem 7.1. The proof of part (1) reduces to the case that

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1^{-1}A_0 = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}, \qquad \lambda_0 \in \mathbb{R}.$$

Letting

$$B = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} [u_1 \quad u_2]A_1, \ u_1, u_2 \in \mathbb{R},$$

we see that $A_1^{-1}A_0 + B$ has the Jordan form $\mathcal{J}_2(\lambda_0)$ if $u_1^2 = u_2^2 \neq 0$ and the Jordan form $\lambda_0 \oplus (u_1^2 - u_2^2 - \lambda_0)$ if $u_1^2 - u_2^2 \neq 0$. Taking u_1, u_2 so that $u_1^2 = u_2^2 \neq 0$ yields (1).

For statement (2) one can argue as in the proof of Theorem 7.3 using the result of Theorem 6.1. □

9 Application to invariant Lagrangian subspaces

Let A be a real J-Hamiltonian matrix of size $2n \times 2n$. A subspace $\mathcal{M} \subset \mathbb{R}^{2n}$ is called J-Lagrangian when $\mathcal{M}^\perp = J\mathcal{M}$ (i.e., $x^T Jy = 0$ for all $x, y \in \mathcal{M}$) and $\dim \mathcal{M} = n$. Such subspaces play an important role in applications, for instance in the study of linear-quadratic optimal control theory, leading to the algebraic Riccati equation, see, e.g., [17, 29].

The existence of A-invariant J-Lagrangian subspaces was discussed in [8, 33]. Using the results of Theorem 3.1 and Section 5 we are able to show that in many cases existence of invariant Lagrangian subspaces is not persistent under some rank-one perturbation of arbitrary small norm.

Theorem 9.1 *Let $J \in \mathbb{R}^{2n\times 2n}$ be skew-symmetric and invertible and let $A \in \mathbb{R}^{2n\times 2n}$ be a J-Hamiltonian matrix that has an invariant J-Lagrangian subspace. Then the following statements hold.*

(1) *Assume that A has a nonzero purely imaginary eigenvalue, or that zero is the only purely imaginary eigenvalue of A, and in this case at least one partial multiplicity of A corresponding to zero is larger than one. Then there exists a rank-one J-Hamiltonian matrix B of arbitrary small norm such that $A + B$ has no invariant Lagrangian subspace.*

(2) *Assume that A has no purely imaginary eigenvalues, or zero is the only purely imaginary eigenvalue and all partial multiplicities of A corresponding to zero are equal to one. Then for all rank-one J-Hamiltonian matrices of sufficiently small norm, the matrix $A + B$ will have an invariant Lagrangian subspace.*

Proof. In the proof we use the following criterion for existence of Lagrangian invariant subspaces for a J-Hamiltonian matrix X: for every non-zero purely imaginary eigenvalue the number of odd partial multiplicities is even, and the corresponding signs sum to zero [8, 24, 33]. In [24] the statements are made for symplectic rather than Hamiltonian matrices. It follows from these references that the existence of invariant Lagrangian subspaces is a local property.

(1) In view of the canonical form of J-Hamiltonian matrices, we need to distinguish several cases.

Case 1. For some purely imaginary nonzero eigenvalue λ the largest partial multiplicity is odd. Then by Theorem 5.1 (a) generically for a small rank-one Hamiltonian B the matrix $A + B$ has at least one purely imaginary eigenvalue, and by item (3) in Theorem 3.1 generically this eigenvalue is simple. Hence, $A + B$ does not have an invariant Lagrangian subspace.

Case 2. There exists a purely imaginary nonzero eigenvalue λ such that the largest partial multiplicity is even. Then without loss of generality we may assume that

$$A = \mathcal{J}_{2n}(\pm ib), \qquad J = \pm(\Sigma_h \otimes \Sigma_2^h).$$

Then the result follows by using Theorem 5.1 (b), and replacing B by $-B$ if necessary.

Case 3. The largest partial multiplicity corresponding to zero is odd. So without loss of generality we may assume that

$$A = \mathcal{J}_{2k+1}(0) \oplus -\mathcal{J}_{2k+1}(0)^T, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

with $k > 0$. We use the computation in Example 4.1 in [25]. Consider

$$M(u, v) = A + \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} -v^T & u^T \end{bmatrix},$$

with u and v having zero coordinates, except for the first two coordinates v_1 and v_2 of v and the last two coordinates u_{2k} and u_{2k+1} of u. Then

$$\det(M(u, v) - \lambda I) = \lambda^{2k+2}(\lambda^{2k} + 2(v_1 u_{2k} + v_2 u_{2k+1})).$$

Obviously, taking $v_1 u_{2k} + v_2 u_{2k+1} > 0$, the matrix $M(u, v)$ has a simple purely imaginary eigenvalue, and hence cannot have an invariant Lagrangian subspace.

Case 4. There exist a partial multiplicity of A corresponding to the zero eigenvalue that is even. In that case, without loss of generality let

$$A = \mathcal{J}_{2k}(0), \qquad J = \Sigma_{2k}.$$

Consider the matrix $A(\varepsilon)$ with $-\varepsilon$ in the $(2k, 1)$ position and all other entries equal to those of A. Then $A(\varepsilon)$ is J-Hamiltonian, and again, $A(\varepsilon)$ has a simple purely imaginary eigenvalue and hence no invariant Lagrangian subspace.

(2) First suppose that A has no purely imaginary eigenvalues. In this case the theorem follows from [33, Theorem 3.4]. In fact, in this case sufficiently small perturbations of any rank will still have an invariant Lagrangian subspace. Next, suppose that zero is the only purely imaginary eigenvalue, and all partial multiplicities of A at zero are equal to one. In this case, any sufficiently small rank-one perturbation will have a Jordan block of size two corresponding to zero, or the same blocks as the unperturbed case. Existence of an invariant Lagrangian subspace then follows from the canonical form. □

10 Conclusion

We have presented several results on Jordan structures and sign characteristics of real J-Hamiltonian and real H-symmetric matrices under structured rank-one perturbations. The main new findings include persistence of the sign characteristics within a connected component of the set of generic real J-Hamiltonian (or H-symmetric) rank-one perturbations. We also studied behavior of eigenvalues of particular interest of the perturbed matrix that are not eigenvalues of the original matrix, namely, real eigenvalues for real H-symmetric matrices and purely imaginary eigenvalues for real J-Hamiltonian matrices. The obtained results are applied to the analysis of the boundedness and robust boundedness of solutions to systems of structured linear differential equations, and to the existence of invariant Lagrangian subspaces.

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