Indefinite Research Problem: Indefinite inner product normal matrices

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Abstract

Several open research problems are formulated concerning normal matrices with respect to indefinite inner products.

Key Words: Indefinite inner products, normal, polar decompositions.

Mathematics Subject Classification: 15A63, 15A57, 15A23.

1 Introduction

Let $\mathbb{C}$ be the field of complex numbers. An *indefinite inner product* in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form $[x, y]$, $x, y \in \mathbb{C}^n$, which is assumed to be *regular*: $[x_0, y] = 0$ for all $y \in \mathbb{C}^n$ happens only when $x_0 = 0$. Every indefinite inner product is associated with a unique invertible Hermitian matrix $H \in \mathbb{C}^{n\times n}$ such that

$$[x, y] = \langle Hx, y\rangle, \qquad x, y \in \mathbb{C}^n, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product in $\mathbb{C}^n$, and conversely, every invertible Hermitian matrix $H \in \mathbb{C}^{n\times n}$ defines an indefinite inner product by means of (1). For a matrix $X \in \mathbb{C}^{n\times n}$, we denote by $X^{[*]}$ the adjoint of X with respect to H, or, in short, H-*adjoint* (the dependence on H is suppressed in the notation $X^{[*]}$); that is $X^{[*]} = H^{-1}X^*H$. Here, X^* stands for the conjugate transpose of the matrix X. A matrix $X \in \mathbb{C}^{n\times n}$ is called

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H-*normal* if X commutes with $X^{[*]}$. The class of H-normal matrices includes H-*selfadjoint* matrices: $X^{[*]} = X$, and H-*unitary* matrices: $X^{[*]}X = I$.

Recently, H-normal matrices have been studied from several directions: indecomposability and canonical forms [6], [7], [8], [9], [10], [11], [18], polar decompositions [1], [2], [3], [16], [17], numerical ranges [13], [14], unitary invariance [4], various particular classes [20].

We fix an invertible Hermitian $n \times n$ matrix H throughout the remaining part of the paper.

2 Indecomposability

A matrix X is called *indecomposable*, or more precisely H-*indecomposable* if there is no nontrivial subspace $V \subseteq \mathbb{C}^n$ such that V is H-nondegenerate (in other words, $[x_0, y] = 0$ for a fixed $x_0 \in V$ and every $y \in V$ only if $x_0 = 0$) and invariant for both X and $X^{[*]}$. Clearly, every matrix can be decomposed as a direct sum of indecomposable matrices. Moreover, X is H-normal (resp. H-selfadjoint, or H-unitary) if and only if each its indecomposable constituent is normal (resp. selfadjoint, or unitary) with respect to the indefinite inner product induced by H on the corresponding X- and $X^{[*]}$-invariant subspace. (In the case of H-selfadjoint or H-unitary matrices X, the $X^{[*]}$-invariance of a subspace is evidently equivalent to the X-invariance.) Complete descriptions of indecomposable H-selfadjoints and H-unitaries, more precisely, simultaneous canonical forms of such matrices and of the indefinite inner product, are well known, see, e.g., [5], [7].

The problem of describing the indecomposable H-normal matrices had been posed in [5]. In full generality, this is still an open problem, and may be intractable. Indeed, every $n \times n$ matrix is normal with respect to some (regular) indefinite inner product (see Theorem I.4.8 of [5]). However, all indecomposable normal matrices with respect to indefinite inner product for which the corresponding matrix H has only one negative eigenvalue, have been described in [6] (the complex case) and in [11] (the real case). For indefinite inner products with H having two negative eigenvalues, all indecomposable normals have been described in [10], [11], in both real and complex cases. See [6], [9] for some general information concerning indecomposable H-normals.

Therefore, it makes sense to consider particular classes of H-normal matrices and seek description, for example in terms of indecomposables, of H-normal matrices within a particular class. In the next two sections we consider several such classes.

3 Classes Related to Polar Decompositions

Given a fixed invertible Hermitian matrix $H \in \mathbb{C}^{n \times n}$, an H-*polar decomposition* of $X \in \mathbb{C}^{n \times n}$ is, by definition, a representation in the form $X = UA$, where $U \in \mathbb{C}^{n \times n}$ is H-unitary and $A \in \mathbb{C}^{n\times n}$ is H-selfadjoint. Polar decompositions with respect to an indefinite inner product have been studied in detail in [1], [2], [3], [4]. In contrast to the standard positive definite inner product, not every matrix has an H-polar decomposition (Examples 1.1 and 1.2 in [1]).

It was proved in [1] that if H has only one negative eigenvalue, then every H-normal matrix admits an H-polar decomposition. For H having two negative eigenvalues, this result was proved in [16]. Also, a nonsingular H-normal matrix always admits an H-polar decomposition [1]. In general, this is an open problem first formulated in [1]:

Problem 1 *Has every H-normal matrix an H-polar decomposition?*

A necessary and sufficient condition for the existence of an H-polar decomposition was given in [1], [4]. A matrix $X \in \mathbb{C}^{n\times n}$ admits an H-polar decomposition if and only if there exists an H-selfadjoint matrix $A \in \mathbb{C}^{n\times n}$ such that $A^2 = X^{[*]}X$ and $\mathrm{Ker}(X) = \mathrm{Ker}(A)$. If X is an H-normal matrix, then it is easy to check that its H-selfadjoint and H-skewadjoint parts

$$A_X = \frac{1}{2}(X + X^{[*]}) \quad \text{and} \quad S_X = \frac{1}{2}(X - X^{[*]})$$

commute. Thus, setting $A_1 := A_X$ and $A_2 := -iS_X$, the matrices A_1 and A_2 are H-selfadjoint and satisfy

$$X^{[*]}X = (A_1 - iA_2)(A_1 + iA_2) = A_1^2 + A_2^2,$$

i.e., $X^{[*]}X$ is the sum of two squares of H-selfadjoints. As mentioned above, it is a necessary condition for the existence of an H-polar decomposition of X that $X^{[*]}X$ can be written as a square of an H-selfadjoint matrix A.

Problem 2 *If A_1 and A_2 are two commuting H-selfadjoint matrices, does there exist an H-selfadjoint matrix A such that $A_1^2 + A_2^2 = A^2$?*

In general, the set $\{A^2 \,:\, A \text{ is } H\text{−selfadjoint}\}$ (where H is fixed) is not convex, in contrast to the convexity of the cone of positive semidefinite matrices with respect to the Euclidean inner product, as the following example shows: Let

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $A_1^2 + A_2^2 = \begin{bmatrix} -3 & 2 \\ 0 & -3 \end{bmatrix}$ is not a square of any H-selfadjoint matrix, since $A_1^2 + A_2^2$ has only one Jordan block associated with the eigenvalue -3. This contradicts the conditions for the existence of an H-selfadjoint square root, see Theorem 3.1 in [19]. As pointed out by the referee, in this example the matrix $X = A_1 + iA_2$ is not H-normal and admits an H-polar decomposition $X = UA$, where

$$A = \begin{bmatrix} a+ib & b \\ 0 & a-ib \end{bmatrix}, \; a = \sqrt{\frac{\sqrt{13}-3}{2}}, \; b = a = \sqrt{\frac{\sqrt{13}+3}{2}}, \quad U = XA^{-1}.$$

The next problem is a natural extension of Problem 2.

Problem 3 *Describe the set of pairs of H-selfadjoint matrices $A_1, A_2 \in \mathbb{C}^{n\times n}$ for which $A^2 = A_1^2 + A_2^2$ for some H-selfadjoint A.*

Concerning commutativity of the factors in H-polar decomposition of H-normal matrices, note that in the complex case the factors can always be chosen such that they commute as long as the H-normal matrix is nonsingular (see [16]). This is not always true in the real case (see Example 28 in [16]) or in the case of singular X (see Example 27 in [16]).

Problem 4 *Characterize H-normal matrices X that admit an H-polar decomposition $X = UA$ with commuting factors U and A.*

There is another way to interpret commutativity: If $X = UA$ is an H-polar decomposition of X, i.e., U is H-unitary and A is H-selfadjoint, then

$$UA = AU \iff UX = XU \implies XA = AX.$$

Of course, if A is invertible, then $XA = AX \implies UA = AU$, but in general

$$XA = AX \not\Longrightarrow UA = AU. \tag{2}$$

To illustrate (2), consider the following example borrowed from [17]:

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 \\ & \lambda & 0 & r & z \\ & & \lambda & 1 & 0 \\ & & & \lambda & 0 \\ & & & & \lambda \end{bmatrix}, \; \lambda \text{ real}, \; r > 0, \; z = \pm 1, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 & \frac{r}{2} \\ & \lambda & 0 & \frac{r}{2} & z \\ & & \lambda & 1 & 0 \\ & & & \lambda & 0 \\ & & & & \lambda \end{bmatrix}; \tag{4}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{r}{2\lambda} \\ 0 & 1 & 0 & \frac{r}{2\lambda} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{if} \quad \lambda \neq 0, \quad U = \begin{bmatrix} 1 & -\frac{r}{2z} & \frac{r^2}{4z} & -\frac{r^4}{32} & 0 \\ 0 & 1 & \frac{r}{2} & -\frac{3r^3}{16z} & -\frac{r^2}{8} \\ 0 & 0 & 1 & -\frac{r^2}{2z} & -\frac{r}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{r}{2z} & 1 \end{bmatrix}, \quad \text{if} \quad \lambda = 0. \tag{5}$$

Here X is H-normal, A is H-selfadjoint, U is H-unitary, and the equality $X = UA$ holds. Moreover, A in (4) and X commute; A and U in (5) do not commute if $\lambda = 0$ and commute if $\lambda \neq 0$.

Problem 5 *Characterize H-normal matrices X that admit H-polar decompositions $X = UA$ such that $XA = AX$.*

In the positive definite case, it is well known that the selfadjoint factor A can be chosen to be positive semidefinite. In the case of indefinite H, there are at least three generally non-equivalent ways to define positive semidefiniteness: An $n \times n$ H-selfadjoint matrix B is called *H-nonnegative* if (1) HB is positive semidefinite; or if (2) there exists an H-selfadjoint matrix C such that $B = C^2$; or if (3) the number of positive (resp. negative) eigenvalues of HB, counted with multiplicities, does not exceed the number of positive (resp. negative) eigenvalues of H, also counted with multiplicities. H-selfadjoint matrices B that satisfy (3) are called *H-consistent* in [4].

Problem 6 *Identify those H-normal matrices X that admit H-polar decompositions $X = UA$ with the H-selfadjoint factor A belonging to one of the H-nonnegative classes described by (1) or (2).*

In connection with Problem 6 note that if X admits an H-polar decomposition $X = UA$, then X also admits an H-polar decomposition in which A is H-consistent ([4], Lemma 4.7 of [1]). Some results in the direction of Problem 6 are found in [2].

4 Other Classes

Consider the following three classes of H-normal matrices X:

(i) *block Toeplitz*: Every indecomposable block of X has either only one Jordan block or exactly two Jordan blocks in which case their eigenvalues are distinct;

(ii) *polynomially normal*: There is a polynomial p such that $X^{[*]} = p(X)$;

(iii) *polynomials of selfadjoints*: There exists a polynomial p and an H-selfadjoint matrix A such that $X = p(A)$.

Block Toeplitz H-normal matrices have been studied in [7], [8], where their canonical forms were obtained. Polynomially normals and polynomials of selfadjoints, as well as many other classes of H-normal matrices have been studied in [20]. Notice that X is polynomially normal if and only if X satisfies the property that $XB = BX$ implies $X^{[*]}B = BX^{[*]}$, for every B (this follows from the fact that the algebra generated by the identity and one linear transformation on a finite dimensional vector space coincides with the double commutant of the algebra (see [12], p. 113). It was proved in [20] that every polynomially normal matrix is block Toeplitz, and every block Toeplitz matrix is polynomial of selfadjoint.

Problem 7 *Develop canonical forms for the classes of polynomially normal matrices and of polynomials of selfadjoints.*

Finally, we remark that the problems posed in this paper are applicable to the real case as well, i.e, real $n \times n$ matrices that are H-normal, where H is a fixed real symmetric invertible $n \times n$ matrix. In this case, the factors in the H-polar decomposition are also assumed to be real.

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