Normal Matrices and Polar Decompositions in Indefinite Inner Products

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Abstract

Normal matrices with respect to indefinite inner products are studied using the additive decomposition into selfadjoint and skewadjoint parts. In particular, several structural properties of indecomposable normal matrices are obtained. These properties are used to describe classes of matrices that are logarithms of selfadjoint or normal matrices. In turn, we use logarithms of normal matrices to study polar decompositions with respect to indefinite inner products. It is proved, in particular, that every normal matrix with respect to an indefinite inner product defined by an invertible Hermitian matrix having at most two negative (or at most two positive) eigenvalues, admits a polar decomposition. Previously known descriptions of indecomposable normals in indefinite inner products with at most two negative eigenvalues play a key role in the proof. Both real and complex cases are considered.

Key Words: Indefinite inner product, normal matrix, polar decomposition.

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1 Introduction

Let $\mathbb{F}$ be either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. An *indefinite inner product* in $\mathbb{F}^n$ is a symmetric bilinear (if $\mathbb{F} = \mathbb{R}$) or a conjugate symmetric sesquilinear (if $\mathbb{F} = \mathbb{C}$) form $[x, y]$, $x, y \in \mathbb{F}^n$, which is assumed to be *regular*: $[x_0, y] = 0$ for all $y \in \mathbb{F}^n$ happens only when $x_0 = 0$. Every indefinite inner product is associated with a unique nonsingular Hermitian (or symmetric if $\mathbb{F} = \mathbb{R}$) matrix $H \in \mathbb{F}^{n\times n}$ such that

$$[x, y] = \langle Hx, y\rangle, \qquad x, y \in \mathbb{F}^n, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product in $\mathbb{F}^n$, and conversely, every nonsingular Hermitian matrix $H \in \mathbb{F}^{n\times n}$ defines an indefinite inner product by means of (1). Various classes of linear transformations associated with an indefinite inner product can be therefore recast in terms of matrices (representing the linear transformations with respect to the standard Euclidean orthonormal basis in $\mathbb{F}^n$) associated with H. Thus, for a matrix $X \in \mathbb{F}^{n\times n}$, we denote by $X^{[*]_H}$ or, if there is no risk of confusion, by $X^{[*]}$, the adjoint of X with respect to H, or, in short, H-*adjoint*; that is $X^{[*]} = H^{-1}X^*H$. Here and throughout the paper, X^* stands for the conjugate transpose of the matrix X. A matrix $X \in \mathbb{F}^{n\times n}$ is called H-*selfadjoint* if $X = X^{[*]}$, H-*skewadjoint* if $X = -X^{[*]}$, and H-*unitary* if X is nonsingular and $X^{[*]} = X^{-1}$. The classes of H-selfadjoint, H-skewadjoint, and H-unitary matrices are contained in a more general class of H-*normal* matrices X which are defined by the property that X commutes with $X^{[*]}$.

In recent years, normal matrices with respect to an indefinite inner product have been intensively studied, from various points of view: classification [8], [9], [10], [12], [13], [14], [19], numerical ranges [17], [18], polar decompositions [1]. The general problem of classification of H-normal matrices has been posed in [7], and several open problems concerning H-normal matrices have been stated in a recent paper [20]. Indecomposability (see [8], [13], [14]) is a key concept in studies of H-normal matrices. A matrix A is called *indecomposable*, or more precisely H-*indecomposable*, if there is no non-trivial subspace $V \in \mathbb{F}^n$ such that V is H-nondegenerate and is invariant for both A and $A^{[*]}$. Clearly, every matrix can be decomposed as a direct sum of indecomposable matrices. Moreover, A is H-normal if and only if each its indecomposable constituent is normal with respect to the indefinite inner product induced by H on the corresponding A- and $A^{[*]}$-invariant subspace. Indeed, with respect to a suitable basis, both A and the indefinite inner product can be represented by block diagonal matrices so that the diagonal blocks of the matrix representing A are exactly the indecomposable constituents of A.

In this paper we continue the study of H-normal matrices, with emphasis on polar decompositions (to be defined later on). We use the following approach for the discussion of H-normal matrices. For the case $H = I$ it is well-known that a matrix $X \in \mathbb{F}^{n\times n}$ is normal if and only if the Hermitian part $\frac{1}{2}(X + X^*)$ of X and the skew-Hermitian part $\frac{1}{2}(X - X^*)$ of X commute. This fact can be generalized to the case of indefinite inner products, as follows:

Let $\mathcal{A}_{\mathbb{F}}(H)$ and $\mathcal{S}_{\mathbb{F}}(H)$ denote the sets of H-selfadjoint and H-skewadjoint $n \times n$ matrices, respectively, with entries in $\mathbb{F}$. Then $\mathbb{F}^{n\times n} = \mathcal{A}_{\mathbb{F}}(H)\dot{+}\mathcal{S}_{\mathbb{F}}(H)$ (direct sum) via

$$X = \frac{1}{2}(X + X^{[*]}) + \frac{1}{2}(X - X^{[*]}).$$

Lemma 1 *Let $X \in \mathbb{F}^{n\times n}$ and $X = A + S$, where $A \in \mathcal{A}_{\mathbb{F}}(H)$ and $S \in \mathcal{S}_{\mathbb{F}}(H)$. Then*

$$X \text{ is } H\text{-normal} \quad \Longleftrightarrow \quad AS = SA.$$

Proof. This follows directly from

$$\begin{aligned} X^{[*]}X = H^{-1}(A^* + S^*)H(A + S) &= A^2 - SA + AS - S^2; \\ XX^{[*]} = (A + S)H^{-1}(A^* + S^*)H &= A^2 + SA - AS - S^2. \quad \square \end{aligned}$$

The approach of investigating the selfadjoint and skewadjoint parts of H-normal matrices has the advantage that one can use well-known results from Lie-theory in the context of polar decompositions.

Given a fixed nonsingular Hermitian matrix $H \in \mathbb{F}^{n\times n}$, an *$H$-polar decomposition* of $X \in \mathbb{F}^{n\times n}$ is, by definition, a representation in the form $X = UA$, where $U \in \mathbb{F}^{n\times n}$ is H-unitary and $A \in \mathbb{F}^{n\times n}$ is H-selfadjoint. Thus, we use a definition of the polar decomposition that is more general than the standard definition, since we allow A to be H-selfadjoint, not only H-nonnegative (as the standard definition requires). Polar decompositions with respect to an indefinite inner products have been studied in detail in [1], [2], [3], [4]; for historic perspective and algebraic treatment of polar decompositions see [15].

It is well known that in case of the standard positive definite inner product $H = I$, a matrix X is H-normal if and only if the factors U and A in its H-polar decomposition commute (see Section IX.12 in [6], or [11]); it is assumed here that A is positive semidefinite. Therefore, we are particularly interested in H-polar decompositions of H-normal matrices such that the factors commute. Polar decompositions of this type, as well as several problems concerning existence of polar decompositions for H-normal matrices, are studied in Section 5. In Section 6 we prove the result that every H-normal matrix admits an H-polar decomposition provided H has at most two negative (or at most two positive) eigenvalues, counted with multiplicities.

The remaining part of the paper is organized as follows. In Section 2 we recall canonical forms of H-selfadjoint and H-skew-adjoint matrices. These forms are used in Section 3 to obtain some information on indecomposable H-normal matrices. In Section 4 we study exponentials and H-selfadjoint (resp., H-normal) logarithms of H-selfadjoint (resp., H-normal) matrices.

The following notation will be fixed throughout the paper. We let H denote a Hermitian $n\times n$ nonsingular complex matrix, or an nonsingular real symmetric matrix (when $\mathbb{F} = \mathbb{R}$), unless stipulated otherwise. Standard matrices: $\mathcal{J}_p(\lambda)$ is the $p \times p$ upper triangular Jordan block with eigenvalue λ; $Z_p = [\delta_{i+j,p+1}]_{i,j=1}^{p}$ is the $p \times p$ matrix with ones on the southwest-northeast diagonal and zeros elsewhere; I_p is the $p \times p$ identity matrix. By $\sigma(A)$, we denote the spectrum of the matrix A. For $z \in \mathbb{C}$, $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ and $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$ are the real and imaginary parts, respectively. Finally, $\mathbb{N} = \{1, 2, \cdots\}$.

2 Preliminaries

As we pointed out in the introduction, the selfadjoint and skewadjoint parts of H-normal matrices will play an important role in our discussion. Therefore, we review some well-known results on canonical forms for complex or real H-selfadjoint and H-skewadjoint matrices in this section.

Theorem 2 *Let $A \in \mathbb{C}^{n \times n}$ be H-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$, such that*

$$P^{-1}AP = A_1 \oplus \ldots \oplus A_k \qquad \text{and} \qquad P^*HP = H_1 \oplus \ldots \oplus H_k, \tag{2}$$

where A_j, H_j are of the same size and each pair (A_j, H_j) has one and only one of the following forms:

1. *Blocks associated with real eigenvalues:*

$$A_j = \mathcal{J}_p(\lambda_0) \quad \text{and} \quad H_j = \varepsilon Z_p, \tag{3}$$

where $\lambda_0 \in \mathbb{R}$, $p \in \mathbb{N}$, and $\varepsilon \in \{1, -1\}$.

2. *Blocks associated with a pair of nonreal conjugate eigenvalues:*

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda_0) & 0 \\ 0 & \mathcal{J}_p(\lambda_0^*) \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \tag{4}$$

where $\lambda_0 \in \mathbb{C} \backslash \mathbb{R}$ and $p \in \mathbb{N}$.

Moreover, the form $(P^{-1}AP, P^*HP)$ of (A, H) is uniquely determined up to the permutation of blocks and called the canonical form of (A, H).

Proof. See [7], for example. $\square$

We sometimes use a slightly different form for the blocks of type (4). Multiplying the matrices from both sides by $\begin{bmatrix} I_p & 0 \\ 0 & Z_p \end{bmatrix}$, one finds that (4) takes the form

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda_0) & 0 \\ 0 & \mathcal{J}_p(\lambda_0)^* \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}. \tag{5}$$

There exists an analogue of the form (2) in the real case, see [7]. Before stating this result, we recall the Kronecker product of matrices. This will enable us to introduce a brief and useful notation for real Jordan blocks that are associated with a pair of complex conjugate eigenvalues.

Definition 3 *Let $A = [a_{ij}]_{i,j} \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{k\times l}$. Then we define*

$$A \otimes B = [a_{ij}B]_{i,j} \in \mathbb{F}^{(nk)\times(ml)}.$$

It is well-known that in the real Jordan canonical form of a matrix, a Jordan block associated with a pair of conjugate nonreal eigenvalues $\alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$ has the form

$$I_p \otimes \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} + \mathcal{J}_p(0) \otimes I_2 = \begin{bmatrix} \alpha & \beta & 1 & 0 & & & 0 \\ -\beta & \alpha & 0 & 1 & & & \\ & & \alpha & \beta & \ddots & & \\ & & -\beta & \alpha & & 1 & 0 \\ & & & & \ddots & 0 & 1 \\ & & & & & \alpha & \beta \\ 0 & & & & & -\beta & \alpha \end{bmatrix}. \tag{6}$$

The eigenvalues of (6) are those of the two-by-two blocks on the diagonal. Note that the set of all two-by-two matrices of this form is a field that is isomorphic to the field of complex numbers.

Remark 4 Define the set

$$\mathcal{M}_{\mathbb{C}} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

Then the map

$$\phi : \mathbb{C} \to \mathcal{M}_{\mathbb{C}}, \quad (\alpha + i\beta) \mapsto \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

is a field isomorphism. This isomorphism can easily be extended to an algebra isomorphism, also denoted by ϕ, between the algebra $\mathbb{C}^{n\times n}$ and the matrix algebra $\mathcal{M}_{\mathbb{C}}^{n\times n}$ with entries in $\mathcal{M}_{\mathbb{C}}$. Namely,

$$\phi\left([z_{j,k}]_{j,k=1}^n\right) = \begin{bmatrix} \operatorname{Re} z_{j,k} & \operatorname{Im} z_{j,k} \\ -\operatorname{Im} z_{j,k} & \operatorname{Re} z_{j,k} \end{bmatrix}_{j,k=1}^n, \quad z_{j,k} \in \mathbb{C}.$$

Note that $\phi(\mathcal{J}_p(\alpha+i\beta))$ is the real Jordan block (6) associated with the eigenvalues $\alpha \pm i\beta$. Furthermore, the following conditions are satisfied for $\gamma, \delta \in \mathbb{R}$.

1. If $A \in \mathbb{C}^{n\times n}$ has the eigenvalue $\gamma + i\delta$ then $\phi(A)$ has the eigenvalues $\gamma \pm i\delta$.

2. If $M \in \mathcal{M}_{\mathbb{C}}^{n\times n}$ has the eigenvalues $\gamma \pm i\delta$ then the spectrum of $\phi^{-1}(M)$ is contained in $\{\gamma \pm i\delta\}$.

This follows easily from bringing A and $\phi^{-1}(M)$ to Jordan canonical form and real Jordan canonical form, respectively, and from noting that both blocks

$$\begin{bmatrix} \gamma & \delta \\ -\delta & \gamma \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \gamma & -\delta \\ \delta & \gamma \end{bmatrix}$$

have the eigenvalues $\gamma \pm i\delta$.

Theorem 5 *Let H be real and $A \in \mathbb{R}^{n\times n}$ be H-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$, such that*

$$P^{-1}AP = A_1 \oplus \ldots \oplus A_k \qquad \text{and} \qquad P^*HP = H_1 \oplus \ldots \oplus H_k, \tag{7}$$

where A_j, H_j are of the same size and each pair (A_j, H_j) has one and only one of the following forms:

1. *Blocks associated with real eigenvalues:*

$$A_j = \mathcal{J}_p(\lambda_0) \quad \text{and} \quad H_j = \varepsilon Z_p, \tag{8}$$

where $\lambda_0 \in \mathbb{R}$, $p \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$.

2. *Blocks associated with a conjugate pair $\alpha \pm i\beta$ of nonreal eigenvalues:*

$$A_j = I_p \otimes \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} + \mathcal{J}_p(0) \otimes I_2 \quad \text{and} \quad H_j = Z_{2p}, \tag{9}$$

where $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, and $p \in \mathbb{N}$.

Moreover, the form $(P^{-1}AP, P^*HP)$ of (A, H) is uniquely determined up to the permutation of blocks and called the real canonical form of (A, H).

Proof. See [7]. $\square$

Since the real block (A_j, H_j) in the form (9) has, according to Theorem 2, the complex canonical form

$$\left(\begin{bmatrix} \mathcal{J}_p(\alpha + i\beta) & 0 \\ 0 & \mathcal{J}_p(\alpha - i\beta) \end{bmatrix}, \quad \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix} \right),$$

we find in particular that every complex pair (A, H) has a real canonical form.

Corollary 6 *Let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that $(P^{-1}AP, P^*HP)$ is in real canonical form (7).*

Next, we are interested in the canonical form for H-skewadjoint matrices. First, we note that in the complex case the canonical form for H-skewadjoint matrices is already given in Theorem 2. This follows from the obvious fact that a matrix $S \in \mathbb{C}^{n\times n}$ is H-skewadjoint if and only if the matrix iS is H-selfadjoint.

For the canonical form of real H-skewadjoint matrices, let us first recall the following notation from [16]:

$$F_j = \begin{bmatrix} 0 & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{j-1} & & 0 \end{bmatrix} \quad \text{and} \quad G_{2j} = \begin{bmatrix} 0 & & & F_2^{j-1} \\ & & -F_2^{j-1} & \\ & \iddots & & \\ (-1)^{j-1}F_2^{j-1} & & & 0 \end{bmatrix}$$

Note that $F_j \in \mathbb{R}^{j\times j}$ is symmetric if j is odd and skew-symmetric if j is even, whereas $G_{2j} \in \mathbb{R}^{2j\times 2j}$ is symmetric for all j. The following well-known result can be found, for example, in [16].

Theorem 7 *Let H be real and $S \in \mathbb{R}^{n\times n}$ be H-skewadjoint. Then there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$, such that*

$$P^{-1}SP = S_1 \oplus \ldots \oplus S_k \qquad \text{and} \qquad P^*HP = H_1 \oplus \ldots \oplus H_k, \tag{10}$$

where S_j, H_j are of the same size and each pair (S_j, H_j) has one and only one of the following forms.

1. *Odd sized blocks associated with the eigenvalue zero:*

$$S_j = \mathcal{J}_{2p+1}(0) \quad \text{and} \quad H_j = \varepsilon F_{2p+1}, \tag{11}$$

where $p \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$.

2. *Paired even sized blocks associated with the eigenvalue zero:*

$$S_j = \begin{bmatrix} \mathcal{J}_{2p}(0) & 0 \\ 0 & -\mathcal{J}_{2p}(0)^T \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & I_{2p} \\ I_{2p} & 0 \end{bmatrix}, \tag{12}$$

where $p \in \mathbb{N}$.

3. *Blocks associated with a pair of nonzero purely imaginary eigenvalues:*

$$S_j = I_p \otimes \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} + \mathcal{J}_p(0) \otimes I_2 \quad \text{and} \quad H_j = \varepsilon G_{2p}, \tag{13}$$

where $\beta > 0$, $p \in \mathbb{N}$, and $\varepsilon \in \{1, -1\}$.

4. Blocks associated with a pair of nonzero real eigenvalues:

$$S_j = \begin{bmatrix} \mathcal{J}_p(\alpha) & 0 \\ 0 & -\mathcal{J}_p(\alpha)^T \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \tag{14}$$

where $\alpha > 0$ and $p \in \mathbb{N}$.

5. Blocks associated with a quadruple of nonreal, non purely imaginary eigenvalues:

$$\begin{aligned} S_j &= \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix} \otimes \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} + \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & -\mathcal{J}_p(0)^T \end{bmatrix} \otimes I_2 \\ H_j &= \begin{bmatrix} 0 & I_{2p} \\ I_{2p} & 0 \end{bmatrix}, \end{aligned} \tag{15}$$

where $\alpha, \beta > 0$, and $p \in \mathbb{N}$.

Moreover, the form $(P^{-1}SP, P^*HP)$ of (S, H) is uniquely determined up to the permutation of blocks and called the real canonical form of (S, H).

3 On decomposability of H-normal matrices

In this section we will present a canonical form for complex or real pairs (X, H), where X is H-normal. This form is related to the known results given in [8] for the complex case and in [14] for the real case. In [8], it was shown that in the complex case every matrix X can be decomposed (in a suitably chosen basis) into an H-orthogonal direct sum of matrices X_j, where X_j has either only one eigenvalue or two distinct eigenvalues. We will give an alternative result of this nature, emphasizing knowledge of the H-selfadjoint and H-skewadjoint parts of X. Therefore, we will need the following auxiliary results concerning commuting matrices.

Proposition 8 *Let $A, S \in \mathbb{F}^{n\times n}$ such that $AS = SA$ and*

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where S is partitioned conformable to A. If A_{11} and A_{22} have no common eigenvalues, then $S_{12} = S_{21} = 0$.

Proof. Since A and S commute we obtain from

$$\begin{bmatrix} A_{11}S_{11} & A_{11}S_{12} \\ A_{22}S_{21} & A_{22}S_{22} \end{bmatrix} = AS = SA = \begin{bmatrix} S_{11}A_{11} & S_{12}A_{22} \\ S_{21}A_{11} & S_{22}A_{22} \end{bmatrix}$$

two Sylvester equations $A_{11}S_{12} = S_{12}A_{22}$ and $A_{22}S_{21} = S_{21}A_{11}$ which have only the trivial solutions, since the spectra of A_{11} and A_{22} are disjoint. □

Proposition 9 *Let $A, S \in \mathbb{F}^{n\times n}$ be such that $AS = SA$, the matrix A has the distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, and S has only one eigenvalue $\mu \in \mathbb{F}$. Then the spectrum of $A+S$ is $\{\lambda_1+\mu, \ldots, \lambda_m+\mu\}$.*

The result of Proposition 9 follows at once from the well-known fact that two commuting matrices can be simultaneously triangularized by similarity (over the field of complex numbers).

We are now able to prove the following result concerning a canonical form of H-normal matrices X. This form will not only give information on X, but also on the selfadjoint and skewadjoint parts of X.

Theorem 10 *Let $X \in \mathbb{C}^{n\times n}$ be H-normal, and let $X = A + S$, where $A \in \mathcal{A}_{\mathbb{C}}(H)$ and $S \in \mathcal{S}_{\mathbb{C}}(H)$. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{array}{rclcrcl} P^{-1}XP &=& X_1 \oplus \ldots \oplus X_k, && P^{-1}AP &=& A_1 \oplus \ldots \oplus A_k, \\ P^{*}HP &=& H_1 \oplus \ldots \oplus H_k, && P^{-1}SP &=& S_1 \oplus \ldots \oplus S_k, \end{array} \tag{16}$$

where, for each j, the matrices X_j, A_j, S_j and H_j have the same size. Furthermore, each X_j is indecomposable with respect to H_j and the corresponding blocks S_j and A_j have at most two distinct eigenvalues each. Moreover, the following conditions are satisfied.

1. *If $\sigma(A_j) = \{\lambda_0\}$ and $\sigma(S_j) = \{\mu_0\}$, then λ_0 is real, μ_0 is purely imaginary and $\sigma(X_j) = \{\lambda_0 + \mu_0\}$,*
2. *If A_j or S_j has two distinct eigenvalues, then*

$$A_j = \begin{bmatrix} A_{j1} & 0 \\ 0 & A_{j1}^* \end{bmatrix}, \quad S_j = \begin{bmatrix} S_{j1} & 0 \\ 0 & -S_{j1}^* \end{bmatrix}, \quad \text{and} \quad H_j = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

   Furthermore, we have $\sigma(A_{j1}) = \{\lambda_j\}$ and $\sigma(S_{j1}) = \{\mu_j\}$ for some $\lambda_j, \mu_j \in \mathbb{C}$ and $\sigma(X_j) = \{\lambda_j + \mu_j, \lambda_j^* - \mu_j^*\}$, where $\lambda_j + \mu_j \neq \lambda_j^* - \mu_j^*$.

Proof. From Theorem 2 we find that there exists a nonsingular matrix $Q \in \mathbb{C}^{n\times n}$, such that

$$Q^{-1}AQ = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \quad \text{and} \quad Q^*HQ = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix},$$

where $\sigma(A_{11}) \cap \sigma(A_{22}) = \emptyset$ and where A_{11} has either only one eigenvalue that is real or else only a pair (λ, λ^*) of nonreal eigenvalues. Then it follows from Proposition 8 that

$$Q^{-1}SQ = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix},$$

where $Q^{-1}SQ$ is partitioned conformable to $Q^{-1}AQ$. Thus, we may assume that A has either only one eigenvalue that is real or only a pair (λ, λ^*) of eigenvalues, where $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Analogously, applying Theorem 2 to iS, we may assume that also S has either only one eigenvalue that is purely imaginary or zero, or only a pair of eigenvalues $(\mu, -\mu^*)$, where $\mu \in \mathbb{C}\backslash(i\mathbb{R})$. Furthermore, we may decompose the corresponding X into an H-orthogonal direct sum of indecomposable matrices. Hence, it is sufficient to consider an indecomposable matrix X and the following two cases for the corresponding matrices A and S.

Case 1): Each A and S have only one eigenvalue.

Let λ denote the eigenvalue of A and μ the eigenvalue of S. From the discussion above it follows that $\lambda \in \mathbb{R}$ and $\mu \in i\mathbb{R}$. Hence, the result follows from Proposition 9.

Case 2): At least one of the matrices A or S has two distinct eigenvalues.

We may assume, without loss of generality, that A has two distinct eigenvalues. Otherwise, we may consider $iX = iS + iA$ and the H-selfadjoint part iS of iX. From the discussion at the beginning of the proof, we see that A has a pair (λ, λ^*) of eigenvalues, where $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Formula (5) implies that there exists a nonsingular matrix R, such that

$$R^{-1}AR = \begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J}^* \end{bmatrix} \quad \text{and} \quad R^*HR = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $\mathcal{J}$ is a matrix in Jordan canonical form, having the only eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Let $R^{-1}SR$ be partitioned conformable to $R^{-1}AR$. Then it follows from Proposition 8 that

$$R^{-1}SR = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & 0 \\ 0 & -S_{11}^* \end{bmatrix} \quad \text{(since } S \text{ is skew-adjoint)},$$

since the spectra of $\mathcal{J}$ and $\mathcal{J}^*$ are disjoint. If S_{11} has two distinct eigenvalues, then we may once again decompose A, S and H into smaller blocks by applying analogous arguments and making use of Proposition 8. Thus, we may assume without loss of generality that also S_{11} has only one eigenvalue, say μ. Now

$$R^{-1}XR = \begin{bmatrix} \mathcal{J} + S_{11} & 0 \\ 0 & \mathcal{J}^* - S_{11}^* \end{bmatrix}.$$

By Proposition 9, we find that $\sigma(X) = \{\lambda + \mu, \lambda^* - \mu^*\}$, and since $\lambda \neq \lambda^*$, we also have $\lambda + \mu \neq \lambda^* - \mu^*$. $\square$

Unfortunately, we cannot give general statements on the Jordan structures of X_j in the pairs (X_j, H_j) in Theorem 10. The following examples show that an indecomposable X_j may have more than one Jordan block associated with the same eigenvalue.

Example 11 The case of one eigenvalue: Let

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad H = Z_4.$$

It was shown in [8] that X is indecomposable.

Example 12 The case of two distinct eigenvalues: Let

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = A + S, \quad H = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^* \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & -S_1^* \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $AS = SA$ and, therefore, X is H-normal. It is easy to check that each A_1 and S_1 consists of two 2×2 Jordan blocks whereas X_1 (or X_2, respectively) consist of one 1×1 and one 3×3 Jordan block associated with the eigenvalue 1 (or -1, respectively). Assume that X is decomposable, i.e.,

$$X = \begin{bmatrix} \tilde{X}_1 & 0 \\ 0 & \tilde{X}_2 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} \tilde{H}_1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix}.$$

Then in particular X, A, and S are simultaneously decomposable into smaller blocks. Since S has only non purely imaginary eigenvalues, Theorem 10 implies that both $\tilde{X}_1$ and $\tilde{X}_2$ must have two distinct eigenvalues and that the algebraic multiplicities of these eigenvalues in the block $\tilde{X}_1$ (and $\tilde{X}_2$, respectively) are equal. It follows from the Jordan structure of X that the only nontrivial decomposition of X would be into a block of size 6 and a block of size 2. On the other hand, Theorem 2 implies that the canonical form of (S, H) consists of two 4×4 blocks that are associated with the pair $(1, -1)$ of non purely imaginary eigenvalues. Thus, X and S are not simultaneously H-decomposable.

These examples show that an H-normal matrix X may be indecomposable although both its selfadjoint and skewadjoint parts are decomposable. This illuminates the difficulty in finding a complete classification of indecomposable H-normal matrices. For the case that H has at most two negative eigenvalues, this problem was solved in [8] and [13]. In connection with this problem we mention that, as shown in [8], the problem of classification of arbitrary H-normal matrices is "wild", and therefore one cannot reasonably expect a complete solution.

Next, we will discuss the analogue of Theorem 10 in the real case. It is already known that a real pair (X, H), where X is H-normal, can be decomposed into factors (X_j, H_j) such that X_j has either one or two distinct real eigenvalues, one or two pairs of complex conjugate eigenvalues, or one real eigenvalue and one pair of complex conjugate eigenvalues, see [14]. Again, we state a new version of this result that also gives information on the selfadjoint and skewadjoint parts of X. Since in its proof we will have to deal with real Jordan blocks associated with complex conjugate eigenvalues, we will need some auxiliary results concerning matrices from the algebra $\mathcal{M}_{\mathbb{C}}^{n\times n}$ introduced in Remark 4.

Define the permutation matrix

$$\Omega_{m,n} = [e_1, e_{n+1}, \ldots, e_{(m-1)n+1}, e_2, e_{n+2}, \ldots, e_{(m-1)n+2}, \ldots, e_n, e_{2n}, \ldots, e_{mn}],$$

where e_j denotes the jth unit column vector of length mn. If $\mathcal{A} \in \mathbb{C}^{m\times m}$ and $\mathcal{B} \in \mathbb{C}^{n\times n}$ then $\Omega_{m,n}$ has the following effect:

$$\Omega_{m,n}^{-1}\Big(\mathcal{A} \otimes \mathcal{B}\Big)\Omega_{m,n} = \mathcal{B} \otimes \mathcal{A}.$$

Proposition 13 *Let $X \in \mathbb{C}^{m\times m}$ be a matrix having the only eigenvalue $\alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. If $Y \in \mathbb{R}^{2m\times 2m}$ commutes with $\phi(X)$, then $Y \in \mathcal{M}_{\mathbb{C}}^{m\times m}$.*

Proof. We may assume without loss of generality that X is in Jordan canonical form. Indeed, let $X_0 = S^{-1}XS$ be the Jordan canonical form of X, where $S \in \mathbb{C}^{m\times m}$ is some nonsingular matrix. Clearly, $\phi(S)^{-1}Y\phi(S)$ commutes with X_0, and if we already know that $\phi(S)^{-1}Y\phi(S) \in \mathcal{M}_{\mathbb{C}}^{m\times m}$, then we obviously have $Y \in \mathcal{M}_{\mathbb{C}}^{m\times m}$, too.

Thus, assume that there exists an $m\times m$ nilpotent matrix $N = \operatorname{diag}\Big(\mathcal{J}_{p_1}(0), \ldots, \mathcal{J}_{p_k}(0)\Big)$ such that

$$\phi(X) = \phi\Big((\alpha + i\beta)I_m + N\Big) = I_m \otimes \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} + N \otimes I_2.$$

Moreover, let $P = \Omega_{m,2}$ and $Q = \frac{1}{\sqrt{2}}P\begin{bmatrix} iI_m & I_m \\ I_m & iI_m \end{bmatrix}$. Then

$$P^{-1}\phi(X)P = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \otimes I_m + I_2 \otimes N = \begin{bmatrix} \alpha I_m + N & \beta I_m \\ -\beta I_m & \alpha I_m + N \end{bmatrix}.$$

Let $P^{-1}YP = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$ be partitioned conformable to $P^{-1}\phi(X)P$. Then

$$\begin{aligned} Q^{-1}\phi(X)Q &= \begin{bmatrix} (\alpha - i\beta)I_m + N & 0 \\ 0 & (\alpha + i\beta)I_m + N \end{bmatrix} \\ \text{and} \quad Q^{-1}YQ &= \frac{1}{2}\begin{bmatrix} Y_1 + Y_4 + i(Y_3 - Y_2) & Y_2 + Y_3 + i(Y_4 - Y_1) \\ Y_2 + Y_3 + i(Y_1 - Y_4) & Y_1 + Y_4 + i(Y_2 - Y_3) \end{bmatrix}. \end{aligned}$$

It follows from Proposition 8 that $Y_2 + Y_3 + i(Y_4 - Y_1) = 0$, that is $Y_4 = Y_1$ and $Y_3 = -Y_2$. From this, we find that

$$P^{-1}YP = \begin{bmatrix} Y_1 & Y_2 \\ -Y_2 & Y_1 \end{bmatrix}$$

and this implies $Y \in \mathcal{M}_{\mathbb{C}}^{m\times m}$. $\square$

The proof of the following lemma is straightforward.

Lemma 14 *Let $A \in \mathbb{C}^{n\times n}$ and $M \in \mathcal{M}_{\mathbb{C}}^{n\times n}$. Then the following statements hold.*

1. $\Big(\phi(A)\Big)^T = \phi(A^*)$ *and* $\phi^{-1}(M^T) = \Big(\phi^{-1}(M)\Big)^*$.

2. $(I_n \otimes Z_2)\phi(A)(I_n \otimes Z_2) = \phi(\overline{A})$.

We are now able to state and prove the real analogue of Theorem 10.

Theorem 15 *Let H be real and $X \in \mathbb{R}^{n\times n}$ be H-normal. Furthermore, let $X = A + S$, where $A \in \mathcal{A}_{\mathbb{R}}(H)$ and $S \in \mathcal{S}_{\mathbb{R}}(H)$. Then there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$\begin{array}{rclrcl} P^{-1}XP & = & X_1 \oplus \ldots \oplus X_k, & P^{-1}AP & = & A_1 \oplus \ldots \oplus A_k, \\ P^*HP & = & H_1 \oplus \ldots \oplus H_k, & P^{-1}SP & = & S_1 \oplus \ldots \oplus S_k, \end{array} \tag{17}$$

where X_j, A_j, S_j, and H_j have the same size, X_j is indecomposable with respect to H_j, and for each j one and only one of the following conditions is satisfied.

1. *We have $\sigma(A_j) = \{\alpha\}$, $\sigma(S_j) = \{0\}$, and $\sigma(X_j) = \{\alpha\}$ for some $\alpha \in \mathbb{R}$.*

2. *We have $\sigma(A_j) = \{\alpha\}$ and $\sigma(S_j) = \{\pm i\delta\}$ for some $\alpha \in \mathbb{R}$, $\delta > 0$. Furthermore, $\sigma(X_j) = \{\alpha \pm i\delta\}$.*

3. *We have*
$$X_j = \begin{bmatrix} X_{j1} & 0 \\ 0 & X_{j2} \end{bmatrix},\ A_j = \begin{bmatrix} A_{j1} & 0 \\ 0 & A_{j1}^T \end{bmatrix},\ S_j = \begin{bmatrix} S_{j1} & 0 \\ 0 & -S_{j1}^T \end{bmatrix},\ \text{and } H_j = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$
where $\sigma(A_{j1}) = \{\alpha\}$ and $\sigma(S_{j1}) = \{\gamma\}$ for some $\alpha \in \mathbb{R}$, $\gamma > 0$. Furthermore, $\sigma(X_{j1}) = \{\alpha + \gamma\}$ and $\sigma(X_{j2}) = \{\alpha - \gamma\}$.

4. *We have*
$$X_j = \begin{bmatrix} X_{j1} & 0 \\ 0 & X_{j2} \end{bmatrix},\ A_j = \begin{bmatrix} A_{j1} & 0 \\ 0 & A_{j1}^T \end{bmatrix},\ S_j = \begin{bmatrix} S_{j1} & 0 \\ 0 & -S_{j1}^T \end{bmatrix},\ \text{and } H_j = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$
where $\sigma(A_{j1}) = \{\alpha\}$ and $\sigma(S_{j1}) = \{\gamma \pm i\delta\}$ for some $\alpha \in \mathbb{R}$, $\gamma, \delta > 0$. Furthermore, $\sigma(X_{j1}) = \{(\alpha + \gamma) \pm i\delta\}$ and $\sigma(X_{j2}) = \{(\alpha - \gamma) \pm i\delta\}$.

5. *We have $\sigma(X_j) = \sigma(A_j) = \{\alpha \pm i\beta\}$ and $\sigma(S_j) = \{0\}$ for some $\alpha \in \mathbb{R}, \beta > 0$. Furthermore, if p is the size of A_j then p is even and H_j has $\frac{p}{2}$ positive and $\frac{p}{2}$ negative eigenvalues.*

6. *We have*
$$X_j = \begin{bmatrix} X_{j1} & 0 \\ 0 & X_{j2} \end{bmatrix},\ A_j = \begin{bmatrix} A_{j1} & 0 \\ 0 & A_{j1}^T \end{bmatrix},\ S_j = \begin{bmatrix} S_{j1} & 0 \\ 0 & -S_{j1}^T \end{bmatrix},\ \text{and } H_j = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$
where $\sigma(A_{j1}) = \{\alpha \pm i\beta\}$ and $\sigma(S_{j1}) = \{\pm i\delta\}$ for some $\alpha \in \mathbb{R}$, $\beta, \delta > 0$. Furthermore, $\sigma(X_{j1}) = \{\alpha \pm i(\beta + \delta)\}$ and $\sigma(X_{j2}) = \{\alpha \pm i(\beta - \delta)\}$.

7. *We have*

$$X_j = \begin{bmatrix} X_{j1} & 0 \\ 0 & X_{j2} \end{bmatrix}, \; A_j = \begin{bmatrix} A_{j1} & 0 \\ 0 & A_{j1}^T \end{bmatrix}, \; S_j = \begin{bmatrix} S_{j1} & 0 \\ 0 & -S_{j1}^T \end{bmatrix}, \; \textit{and } H_j = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $\sigma(A_{j1}) = \{\alpha \pm i\beta\}$ *and* $\sigma(S_{j1}) = \{\gamma\}$ *for some* $\alpha \in \mathbb{R}$, $\beta, \gamma > 0$. *Furthermore,* $\sigma(X_{j1}) = \{(\alpha+\gamma) \pm i\beta\}$ *and* $\sigma(X_{j2}) = \{(\alpha-\gamma) \pm i\beta\}$.

8. *We have*

$$X_j = \begin{bmatrix} X_{j1} & 0 \\ 0 & X_{j2} \end{bmatrix}, \; A_j = \begin{bmatrix} A_{j1} & 0 \\ 0 & A_{j1}^T \end{bmatrix}, \; S_j = \begin{bmatrix} S_{j1} & 0 \\ 0 & -S_{j1}^T \end{bmatrix}, \; \textit{and } H_j = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $\sigma(A_{j1}) = \{\alpha \pm i\beta\}$ *and* $\sigma(S_{j1}) = \{\gamma \pm i\delta\}$ *for some* $\alpha \in \mathbb{R}$, $\gamma > 0$, *and* $\beta, \delta > 0$. *Furthermore,* $\sigma(X_{j1}) = \{(\alpha+\gamma) \pm i(\beta+\delta)\}$ *and* $\sigma(X_{j2}) = \{(\alpha-\gamma) \pm i(\delta-\beta)\}$.

Proof. It is sufficient to show that every H-normal matrix can be decomposed into a direct sum of blocks of type 1.–8. That these blocks can be chosen to be indecomposable then follows from the fact that we may always start with the case that the matrix X under consideration is already indecomposable. According to Proposition 8 and in view of Theorems 5 and 7, it is sufficient to consider the following cases.

Case (a): A has the only eigenvalue $\alpha \in \mathbb{R}$.

Considering the four different subcases that S may only have the eigenvalue zero, a pair of purely imaginary eigenvalues, a pair of real eigenvalues, or a quadruple of nonreal and non purely imaginary eigenvalues, respectively, it follows from Theorem 7 and Propositions 8 and 9 that A, S, and H can be brought to the forms 1., 2., 3., or 4. of the theorem, respectively.

Case (b): The spectrum of A is $\{\alpha \pm i\beta\}$, where $\alpha \in \mathbb{R}$ and $\beta > 0$. This means in particular that the size of A is even, i.e., $n = 2q$ for some $q \in \mathbb{N}$. Considering the four different subcases for the spectrum of S according to Theorem 7, we obtain the forms 5.–8. of the theorem.

5.: Assume $\sigma(S) = \{0\}$. This subcase follows from Theorem 5 and Proposition 8.

6.: Assume $\sigma(S) = \{\pm i\delta\}$ for some $\delta > 0$. We show in the following that X necessarily has two pairs of conjugate complex eigenvalues (unless $|\beta| = |\delta|$, in which case X has one pair of conjugate complex eigenvalues and one real eigenvalue).

Assume that (A, H) is in real canonical form (7), i.e., there exists a nilpotent matrix N in Jordan canonical form, such that

$$A = I_m \otimes \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} + N \otimes I_2 \quad \text{and} \quad H = Z_{2m}.$$

Now it follows from Proposition 13 that $S \in \mathcal{M}_{\mathbb{C}}^{m\times m}$. Therefore, $\tilde{A} = \phi^{-1}(A)$ and $\tilde{S} = \phi^{-1}(S)$ (and also $\phi^{-1}(H)$) are well-defined. By construction, $\tilde{A}$ has only the eigenvalue $\alpha + i\beta$ and according to Remark 4, $\tilde{S}$ may have the eigenvalues $i\delta$ and/or $-i\delta$. Now let $Q \in \mathbb{C}^{m\times m}$ be such that

$$Q^{-1}\tilde{S}Q = \begin{bmatrix} \tilde{S}_{11} & 0 \\ 0 & \tilde{S}_{22} \end{bmatrix},$$

where $\tilde{S}_{11} \in \mathbb{C}^{k\times k}$ has only the eigenvalue $i\delta$ and $\tilde{S}_{22} \in \mathbb{C}^{(m-k)\times(m-k)}$ has only the eigenvalue $-i\delta$. Then it follows from Proposition 8 that

$$Q^{-1}\tilde{A}Q = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix},$$

where the blocks have sizes corresponding to $\tilde{S}_{11}$ and $\tilde{S}_{22}$. Now consider the images of these matrices under ϕ. We obtain

$$\hat{A} := \phi(Q^{-1}\tilde{A}Q) = \begin{bmatrix} \phi(\tilde{A}_{11}) & 0 \\ 0 & \phi(\tilde{A}_{22}) \end{bmatrix}, \quad \hat{S} := \phi(Q^{-1}\tilde{S}Q) = \begin{bmatrix} \phi(\tilde{S}_{11}) & 0 \\ 0 & \phi(\tilde{S}_{22}) \end{bmatrix},$$

$$\text{and} \quad \hat{H} := \phi(Q)^* H \phi(Q) = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix},$$

where $\hat{H}$ is partitioned conformable to $\hat{S}$ and $\hat{A}$. Since every complex matrix is similar to its transpose, there exists a nonsingular matrix $W \in \mathbb{C}^{m\times m}$ such that $\tilde{A}_{11}^T = W^{-1}\tilde{A}_{11}W$. In view of Lemma 14 we have

$$\Big(\phi(\tilde{A}_{11})\Big)^T = (I_q \otimes Z_2)\phi(\tilde{A}_{11}^T)(I_q \otimes Z_2) = (I_q \otimes Z_2)\phi(W)^{-1}\phi(\tilde{A}_{11})\phi(W)(I_q \otimes Z_2). \quad (18)$$

Note that $\hat{A}$ is still $\hat{H}$-selfadjoint, i.e., $\Big(\phi(\tilde{A}_{11})\Big)^T H_{11} = H_{11}\phi(\tilde{A}_{11})$. This together with (18) implies

$$\phi(\tilde{A}_{11})\phi(W)(I_q \otimes Z_2)H_{11} = \phi(W)(I_q \otimes Z_2)H_{11}\phi(\tilde{A}_{11}).$$

Thus, Proposition 13 implies

$$\phi(W)(I_q \otimes Z_2)H_{11} \in \mathcal{M}_{\mathbb{C}}^{m\times m}.$$

This means that we also have $(I_q \otimes Z_2)H_{11} \in \mathcal{M}_{\mathbb{C}}^{m\times m}$, i.e., there exists $G \in \mathbb{C}^{m\times m}$, such that

$$\phi(G) = (I_q \otimes Z_2)H_{11}.$$

Since $\hat{S}$ is $\hat{H}$-skewadjoint, we have:

$$-\Big(\phi(\tilde{S}_{11})\Big)^T H_{11} = H_{11}\phi(\tilde{S}_{11}).$$

This equation is equivalent to the equation

$$-\phi(\tilde{S}_{11}^T)(I_q \otimes Z_2)H_{11} = (I_q \otimes Z_2)H_{11}\phi(\tilde{S}_{11}),$$

and this implies: $-\tilde{S}_{11}^T G = G\tilde{S}_{11}$. But $\tilde{S}_{11}$ has the only eigenvalue $i\delta$ and the only eigenvalue of $-\tilde{S}_{11}^T$ is $-i\delta$, hence this equation has only the solution $G = 0$. This implies $H_{11} = 0$. In an analogous way, we show that $H_{22} = 0$. Since H is nonsingular, it follows that m is even, $k = \frac{m}{2}$, and H_{12} is nonsingular. Setting finally

$$R = \phi(Q)\begin{bmatrix} I & 0 \\ 0 & H_{12}^{-1} \end{bmatrix},$$

we obtain that the matrices $R^{-1}AR$, $R^{-1}XR$, $R^{-1}SR$, R^*HR have the following forms:

$$R^{-1}AR = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad R^{-1}SR = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix},$$

$$R^{-1}XR = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}, \quad R^*HR = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Moreover, we have by symmetry that $A_{22} = A_{11}^T$ and $S_{22} = -S_{11}^T$, and by construction that $\sigma(A_{11}) = \{\alpha \pm i\beta\}$, $\sigma(S_{11}) = \{\pm i\delta\}$ and furthermore $\sigma(X_{11}) = \{\alpha \pm i(\beta+\delta)\}$ and $\sigma(X_{22}) = \{\alpha \pm i(\beta-\delta)\}$. (This follows from the fact that $\tilde{S}_{11}$ has only one eigenvalue $i\delta$ and $\tilde{A}_{11}$ has only the eigenvalue $\alpha + i\beta$. Thus, $\sigma(\tilde{A}_{11} + \tilde{S}_{11}) = \{\alpha + i(\beta+\delta)\}$ and it follows from Remark 4 that X_{11} has the eigenvalues $\alpha \pm i(\beta+\delta)$. A similar argument holds for X_{22}.)

7.: Assume $\sigma(S) = \{\pm\gamma\}$, where $\gamma > 0$. The proof in this case follows easily from the structure of S given in Theorem 7 and from Proposition 8.

8.: Assume $\sigma(S) = \{\gamma \pm i\delta, -\gamma \pm i\delta\}$, $\gamma > 0$. Then, we may assume in view of Theorem 7 and Proposition 8 that

$$X = A + S, \quad A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{11}^T \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & 0 \\ 0 & -S_{11}^T \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} 0 & I_q \\ I_q & 0 \end{bmatrix},$$

where $\sigma(S_{11}) = \{\gamma \pm i\delta\}$. It follows in particular that the sizes of A_{11} and S_{11} are even, i.e., $q = 2p$ for some $p \in \mathbb{N}$ and thus, $n = 2q = 4p$. Assuming furthermore that $A_{11} \in \mathbb{C}^{q\times q}$ is in real Jordan canonical form, we obtain by Proposition 13 that $S_{11} \in \mathcal{M}_{\mathbb{C}}^{\frac{q}{2}\times\frac{q}{2}}$. Moreover, we also have $X, A, S, H \in \mathcal{M}_{\mathbb{C}}^{n\times n}$ and their images under ϕ^{-1} have the following forms:

$$\phi^{-1}(A) = \begin{bmatrix} \phi^{-1}(A_{11}) & 0 \\ 0 & \left(\phi^{-1}(A_{11})\right)^* \end{bmatrix}, \quad \phi^{-1}(S) = \begin{bmatrix} \phi^{-1}(S_{11}) & 0 \\ 0 & -\left(\phi^{-1}(S_{11})\right)^* \end{bmatrix},$$

$$\phi^{-1}(X) = \phi^{-1}(A) + \phi^{-1}(S), \quad \phi^{-1}(H) = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}.$$

Note that $\phi^{-1}(A)$ and $\phi^{-1}(S)$ are $\phi^{-1}(H)$-selfadjoint and -skewadjoint, respectively. Furthermore, $\phi^{-1}(A)$ has a pair of conjugate complex eigenvalues, namely $\alpha \pm i\beta$. Therefore, we may apply Theorem 10 and we find that $\phi^{-1}(A)$, $\phi^{-1}(S)$ and $\phi^{-1}(H)$ can be decomposed into blocks of type (2) of Theorem 10. Retranslating this result to the real case via ϕ, we obtain the form stated in 8.. $\square$

Remark 16 Comparing our result with the result in [14], we find the five cases listed there for an indecomposable block X:

Eigenvalues of X	Case number in Theorem 15
one real eigenvalue	1.
two distinct real eigenvalues	3.
one pair of complex conjugate eigenvalues	2. and 5.
two pairs of complex conjugate eigenvalues	4., 7., and 6. and 8. for $\lvert\beta\rvert \neq \lvert\delta\rvert$
one real eigenvalue and one pair of complex conjugate eigenvalues	6. and 8. for $\lvert\beta\rvert = \lvert\delta\rvert$

An interesting special case is when the matrix X has blocks of type 2. and 5. for a fixed α and $\delta = \beta$. Although in this case both blocks display the same pair of eigenvalues, namely $\alpha \pm i\beta$, Theorem 15 shows that X can still be decomposed, using the knowledge on the structure of the H-selfadjoint and H-skewadjoint parts of X. A similar situation occurs if X has blocks of type 4. and 7. with a fixed α and $\beta = \delta$. Again, a further decomposition is possible although both blocks of both types 4. and 7. display the eigenvalues $(\alpha+\gamma)\pm i\beta$ and $(\alpha - \gamma) \pm i\beta$.

4 Exponential and logarithmic functions

In this section we develop some results concerning exponential and logarithmic functions of H-selfadjoint and H-normal matrices. Besides the uses in the present paper for H-polar decompositions with commuting factors, these results are independently interesting.

We start with the following well-known result from Lie theory (it is easily deduced from the fact that the range of the exponential map from a Lie algebra into the corresponding Lie group contains a neighborhood of identity).

Proposition 17 *The set*

$$\{e^X \; : \; X \textit{ is } H\textit{-skewadjoint}\} \tag{19}$$

coincides with the connected component of identity of the group of H-unitary matrices.

Furthermore, we immediately obtain the following lemma from the fact that the exponential of a matrix M is a power series in M.

Lemma 18 *Let $\mathcal{A}_{\mathbb{F}}(H)$ and $\mathcal{N}_{\mathbb{F}}(H)$ denote the sets of H-selfadjoint and H-normal $n \times n$ matrices, respectively, with entries in $\mathbb{F}$. Then*

$$\exp\Big(\mathcal{A}_{\mathbb{F}}(H)\Big) \subseteq \mathcal{A}_{\mathbb{F}}(H) \quad \textit{and} \quad \exp\Big(\mathcal{N}_{\mathbb{F}}(H)\Big) \subseteq \mathcal{N}_{\mathbb{F}}(H).$$

These results offer the following approach for the discussion of H-polar decompositions. If $X = A + S$ is an H-normal matrix given as a sum of its selfadjoint and skewadjoint parts A and S, respectively, we obtain the following equation by applying the exponential map, using the fact that A and S commute.

$$\exp(X) = \exp(S + A) = \exp(S)\exp(A)$$

Note that this is a polar decomposition of $\exp(X)$ with commuting factors. Hence, the question arises which H-selfadjoint and H-normal matrices possess a logarithm within each class. We consider both real and complex cases.

First let us look at H-selfadjoint matrices. The answer which intrinsic conditions determine the set $\exp(\mathcal{A}_{\mathbb{F}}(H))$ is given in the next result.

Proposition 19 *Let A be an H-selfadjoint matrix with entries in $\mathbb{F}$. Then the following statements are equivalent.*

1. *There exists an H-selfadjoint matrix $B \in \mathbb{F}^{n\times n}$ such that $A = \exp(B)$.*
2. *A is nonsingular and has an H-selfadjoint square root, i.e., there exists an H-selfadjoint matrix $C \in \mathbb{F}^{n\times n}$ such that $A = C^2$.*
3. *A is nonsingular, and if $\lambda_0 < 0$ is a negative real eigenvalue of A, then the part in the canonical form of (A, H) associated with λ_0 takes the form*

$$\left(\bigoplus_{j=1}^{r}\begin{bmatrix} \mathcal{J}_{p_j}(\lambda_0) & 0 \\ 0 & \mathcal{J}_{p_j}(\lambda_0)\end{bmatrix}, \bigoplus_{j=1}^{r}\begin{bmatrix} Z_{p_j} & 0 \\ 0 & -Z_{p_j}\end{bmatrix}\right).$$

Proof. '1) $\Rightarrow$ 2)': This is obvious, take $C = \exp\left(\frac{1}{2}B\right)$.

'2) $\Leftrightarrow$ 3)': Follows from Lemmas 7.7 and 7.8 of [1].

'3) $\Rightarrow$ 1)': Consider first the complex case. We may assume that (A, H) is in canonical form (2). Thus, it is sufficient to consider the blocks separately.

Case a): A is a block associated with a pair of nonreal eigenvalues, i.e., using formula (5),

$$A = \begin{bmatrix} \mathcal{J}_p(\lambda_e) & 0 \\ 0 & \mathcal{J}_p(\lambda_e)^* \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix},$$

where $\lambda_e \in \mathbb{C}\backslash\mathbb{R}$ and $p \in \mathbb{N}$.

Choose $\lambda_0 \in \mathbb{C}$, such that $\lambda_e = e^{\lambda_0}$. Then $\exp(\mathcal{J}_p(\lambda_0))$ is similar to $\mathcal{J}_p(e^{\lambda_0})$, i.e., there exists a nonsingular matrix $T \in \mathbb{C}^{p\times p}$, such that

$$\exp\Big(T^{-1}\mathcal{J}_p(\lambda_0)T\Big) = T^{-1}\exp\Big(\mathcal{J}_p(\lambda_0)\Big)T = \mathcal{J}_p(e^{\lambda_0}) = \mathcal{J}_p(\lambda_e).$$

Define $B = \begin{bmatrix} T^{-1}\mathcal{J}_p(\lambda_0)T & 0 \\ 0 & T^*\mathcal{J}_p(\lambda_0)^*(T^{-1})^* \end{bmatrix}$. Then B is H-selfadjoint and

$$\exp(B) = \begin{bmatrix} \exp\Big(T^{-1}\mathcal{J}_p(\lambda_0)T\Big) & 0 \\ 0 & \exp\Big(T^{-1}\mathcal{J}_p(\lambda_0)T\Big)^* \end{bmatrix} = A.$$

Case b): A is a block associated with a positive real eigenvalue, i.e., $A = \mathcal{J}_p(\lambda_e)$ and $H = \varepsilon Z_p$, where $\lambda_e > 0$ and $\varepsilon \in \{1, -1\}$. Choose $\lambda_0 \in \mathbb{R}$, such that $e^{\lambda_0} = \lambda_e$. Then $\exp(\mathcal{J}_p(\lambda_0))$ is similar to $\mathcal{J}_p(e^{\lambda_0})$. Since $\mathcal{J}_p(\lambda_0)$ is H-selfadjoint, by Lemma 18 the matrix $\exp(\mathcal{J}_p(\lambda_0))$ is also H-selfadjoint. This together with Theorem 2 applied to $\exp(\mathcal{J}_p(\lambda_0))$ implies that there exists a nonsingular matrix $T \in \mathbb{C}^{p\times p}$, such that

$$\exp\Big(T^{-1}\mathcal{J}_p(\lambda_0)T\Big) = \mathcal{J}_p(\lambda_e) \quad \text{and} \quad T^*HT = \delta Z_p = \pm H,$$

where $\delta = \pm 1$. Letting $B = T^{-1}\mathcal{J}_p(\lambda_0)T$, we see that the matrix

$$HB = HT^{-1}J_p(\lambda_0)T = \pm T^*HJ_p(\lambda_0)T$$

is Hermitian. Thus, B is H-selfadjoint and satisfies $\exp(B) = A$.

Case c): According to 3), the remaining case is now

$$A = \begin{bmatrix} \mathcal{J}_p(\lambda_e) & 0 \\ 0 & \mathcal{J}_p(\lambda_e) \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} -Z_p & 0 \\ 0 & Z_p \end{bmatrix}, \quad \lambda_e < 0.$$

Define $X = \frac{1}{\sqrt{2}}\begin{bmatrix} I_p & -Z_p \\ I_p & Z_p \end{bmatrix}$. Then

$$X^{-1}AX = \begin{bmatrix} \mathcal{J}_p(\lambda_e) & 0 \\ 0 & \mathcal{J}_p(\lambda_e)^* \end{bmatrix} \quad \text{and} \quad X^*HX = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}.$$

Choose $\lambda_0 = \log(-\lambda_e) + i\pi$. Then $e^{\lambda_0} = \lambda_e$. Analogously to Case a), we find a nonsingular matrix T, such that $B = \begin{bmatrix} T^{-1}\mathcal{J}_p(\lambda_0)T & 0 \\ 0 & T^*\mathcal{J}_p(\lambda_0)^*(T^{-1})^* \end{bmatrix}$ is H-selfadjoint and $e^B = A$.

Consider now the real case. From the discussion of the complex case, we know that there exists a possibly complex H-selfadjoint matrix $\tilde{B}$ such that $\exp(\tilde{B}) = A$. Now

Corollary 6 implies that there exists a nonsingular matrix $S \in \mathbb{C}^{n\times n}$ such that $S^{-1}\tilde{B}S$ and S^*HS are real. But then, also $\exp(S^{-1}\tilde{B}S) = S^{-1}AS$ is real. Since the real pairs (A, H) and $(S^{-1}AS, S^*HS)$ have the same complex canonical form, they also have the same real canonical form. This follows from Theorems 2 and 5 if we observe that in both the real and the complex case, the canonical forms are uniquely determined by the same sets of invariants. Hence, we find that there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that $A = T^{-1}S^{-1}AST$ and $H = T^*S^*HST$. Now choose the real H-selfadjoint matrix $B = T^{-1}S^{-1}\tilde{B}ST \in \mathbb{R}^{n\times n}$. Then $e^B = A$. $\square$

Remark 20 It follows from Corollary 2.1 of [4] that the matrix A satisfying the equivalent statements of Proposition 19 is H-*consistent*, i.e., the number of positive (resp., negative) eigenvalues of HA does not exceed the number of positive (resp., negative) eigenvalues of H; the eigenvalues are counted according to their multiplicities. Equivalently, an H-selfadjoint matrix A is H-consistent if and only if $HA = Y^*HY$ for some (not necessarily invertible) matrix $Y \in \mathbb{F}^{n\times n}$.

For the case of H-normal matrices, the following general fact concerning functions of H-normal matrices is useful.

Lemma 21 *Let $X \in \mathbb{C}^{n\times n}$ be an H-normal matrix, and let Γ be a simple (i.e., without self-intersections) closed rectifiable contour in the complex plane such that the eigenvalues of X are inside Γ. Then for every (single-valued) function $f(z)$ which is analytic on Γ and in the interior of Γ, the matrix $f(X)$ defined by the functional calculus*

$$f(X) = \frac{1}{2\pi i}\int_\Gamma f(z)(zI - X)^{-1}dz \tag{20}$$

is also H-normal.

If in addition X is real, the contour Γ is symmetric with respect to the real line, and the function $f(z)$ is such that $\overline{f(z)} = f(\overline{z})$, then $f(X)$ is real as well.

Proof. For the first part of the lemma, approximate the integral in (20) by Riemann sums

$$\sum_{j=0}^{N-1}(z_{j+1} - z_j)f(z_j)(z_jI - X)^{-1}; \tag{21}$$

here $z_0, \cdots, z_{N-1}$ are consecutive partition points on Γ in the counterclockwise direction, and we let $z_N = z_0$. Now using the easily verified property

$$(zI - X)^{-1}H^{-1}(wI - X^*)^{-1}H = H^{-1}(wI - X^*)^{-1}H(zI - X)^{-1}, \quad z, w \in \mathbb{C},$$

which follows from H-normality of X, one obtains that $f(X)$ is H-normal. For the second part, assuming without loss of generality that Γ is connected and therefore intersects the real axis, observe that one can choose the points z_j in (21) so that N is odd, $z_j = \overline{z_{N-j}}$, $(j = 1, \cdots, N-1)$, and $z_0 = z_N$ is real. Applying the complex conjugation to (21), we see that $-2\pi i \overline{f(X)}$ can be approximated by the sums

$$\sum_{j=0}^{N-1} (\overline{z_{j+1}} - \overline{z_j}) f(\overline{z_j})(\overline{z_j} I - X)^{-1} = \sum_{j=0}^{N-1} (z_{N-j-1} - z_{N-j}) f(z_{N-j})(z_{N-j} I - X)^{-1},$$

which, upon the change of index $k = N - j$, take the form

$$\sum_{k=1}^{N} (z_{k-1} - z_k) f(z_k)(z_k I - X)^{-1} = -\sum_{k=1}^{N} (z_k - z_{k-1}) f(z_k)(z_k I - X)^{-1}.$$

In view of (20), the latter sums (with the sign minus in front) approximate $-2\pi i f(X)$, so we indeed have $\overline{f(X)} = f(X)$. ☐

Using Lemma 21 with a contour Γ that does not intersect any ray from the origin that contains eigenvalues of X, and with $f(z) = \log(z)$, a (single valued) analytic function defined on the closure of the interior of Γ, we have in the complex case:

Proposition 22 *In the complex case, the set*

$$\exp\Big(\mathcal{N}_{\mathbb{F}}(H)\Big) = \{e^X \ : \ X \text{ is } H\text{-normal}\} \tag{22}$$

coincides with the set of nonsingular H-normal matrices.

Clearly, this result does not hold in the real case, because it is well known that a real matrix Y has a real logarithm if and only if for every negative eigenvalue λ_0, and every size p, the number of Jordan blocks $J_p(\lambda_0)$ in the Jordan form of Y is even (see, e.g., [5]). However, Proposition 22 could be restated in the following way: in the complex case every H-normal matrix that has a logarithm has also an H-normal logarithm. But this formulation does not hold in the real case, either. We give the following two counterexamples.

Example 23 $\mathbb{F} = \mathbb{R}$. Let $0 < \lambda \in \mathbb{R}$ and consider

$$X = \begin{bmatrix} -\lambda & 0 & \cos(\gamma) & \sin(\gamma) \\ 0 & -\lambda & -\sin(\gamma) & \cos(\gamma) \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}, \quad Y = \begin{bmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & r & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}, \tag{23}$$

$$\text{and} \quad H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \tag{24}$$

where $0 < \gamma < \pi$ and $r \in \mathbb{R}$ such that $|r| > 1$. Then it was shown in [14] that X and Y are indecomposable H-normal matrices that are not H-unitarily similar. On the other hand, is easy to see that X and Y have the same Jordan forms. However, X has an H-normal logarithm, and Y does not. Setting

$$L_X = \begin{bmatrix} \log(\lambda) & \pi & -\frac{\cos\gamma}{\lambda} & -\frac{\sin\gamma}{\lambda} \\ -\pi & \log(\lambda) & \frac{\sin\gamma}{\lambda} & -\frac{\cos\gamma}{\lambda} \\ 0 & 0 & \log(\lambda) & \pi \\ 0 & 0 & -\pi & \log(\lambda) \end{bmatrix}, \tag{25}$$

one finds that L_X is H-normal and that $\exp(L_X) = X$. On the other hand, one can show that Y has no real H-normal logarithm. Indeed, we first note that an H-normal logarithm L_Y of Y necessarily has to be indecomposable, for if L_Y is decomposable, i.e., there exists a nonsingular matrix $P \in \mathbb{C}^{4\times 4}$ such that

$$P^{-1}L_Y P = L_1 \oplus \ldots \oplus L_m \quad \text{and} \quad P^* H P = H_1 \oplus \ldots \oplus H_m,$$

then clearly also $\exp(L_Y) = Y$ is decomposable. Furthermore, the spectrum of L_Y has to contain only pairs of complex conjugate numbers of the form $\log(\lambda) + i(2k+1)\pi$, k an integer, since the spectrum of Y is $\{-\lambda\}$. Checking the list of real indecomposable H-normal matrices in [14] for the given H, one finds that L_Y necessarily has to be H-unitarily similar to one of the following types of matrices.

$$\text{type 1}: \quad L_Y = \begin{bmatrix} \log(\lambda) & \beta & 0 & 0 \\ -\beta & \log(\lambda) & 0 & 0 \\ 0 & 0 & \log(\lambda) & \hat{\beta} \\ 0 & 0 & -\hat{\beta} & \log(\lambda) \end{bmatrix},$$

$$\text{type 2}: \quad L_Y = \begin{bmatrix} \log(\lambda) & \beta & \cos(\delta) & \sin(\delta) \\ -\beta & \log(\lambda) & -\sin(\delta) & \cos(\delta) \\ 0 & 0 & \log(\lambda) & \beta \\ 0 & 0 & -\beta & \log(\lambda) \end{bmatrix},$$

$$\text{type 3}: \quad L_Y = \begin{bmatrix} \log(\lambda) & \beta & 0 & 1 \\ -\beta & \log(\lambda) & 1 & 0 \\ 0 & 0 & \log(\lambda) & -\beta \\ 0 & 0 & \beta & \log(\lambda) \end{bmatrix},$$

where $\beta = \pi + 2z_1\pi$, $\hat{\beta} = \pi + 2z_2\pi$ for some integers z_1, z_2 and $0 \le \delta < 2\pi$. For type 1, the matrix $\exp(L_Y)$ has a Jordan structure different from Y. For type 2, one finds that $\exp(L_Y)$ is H-unitarily similar to a matrix of the same type of X in (23) for some parameter γ, and for type 3, one finds that L_Y and $\exp(L_Y)$ are H-selfadjoint in contrast to Y. Thus, Y cannot have a real H-normal logarithm.

Example 24 Consider the pair (X, H) in the canonical form

$$X = [-1] \oplus \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad H = [1] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that H has one negative eigenvalue, and that X is the direct sum of two indecomposable blocks ([8]). Suppose that $Y \in \mathbb{R}^{3\times 3}$ is H-normal and $\exp(Y) = X$. Given the canonical forms for H-normal operators for the case that H has one negative eigenvalue (see [14]), we may assume that there exists a nonsingular matrix $P \in \mathbb{R}^{3\times 3}$ such that

$$P^{-1}YP = Y_1 \oplus \ldots \oplus Y_m \quad \text{and} \quad P^*HP = H_1 \oplus \ldots \oplus H_m.$$

From this, we see that $(X, H) = (\exp(Y), H)$ can be decomposed into at least m blocks. Thus, the uniqueness of the decomposition into indecomposable blocks (see [8]) implies $m \leq 2$. If $m = 1$ then it follows from [14] that Y has only one eigenvalue, hence so does X, which is a contradiction. This implies $m = 2$. But then

$$P^{-1}XP = P^{-1}\exp(Y)P = \exp(Y_1) \oplus \exp(Y_2).$$

Thus, the uniqueness of the decomposition into indecomposable blocks implies that both $[-1]$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ have a real logarithm. This is a contradiction.

However, using Lemma 21 with $f(z) = \log(z)$, an analytic function in $\mathbb{C} \setminus (-\infty, 0]$, we obtain the following sufficient condition.

Lemma 25 *Let H be real. Then every nonsingular real H-normal matrix with no negative eigenvalues has a real H-normal logarithm.*

Example 23 shows that the fact whether a real H-normal matrix X has a real H-normal logarithm or not is determined not only by the Jordan structure of X, but by some further invariants. Since a complete classification of H-normal matrices up to H-unitary similarity is unknown (this is a "wild" problem as shown in [8]), the problem of characterizing the set of H-normal matrices that have an H-normal logarithm remains unsolved for the case $\mathbb{F} = \mathbb{R}$.

5 H-polar decompositions

In this section we discuss H-polar decompositions of H-normal matrices. As it was pointed out in the introduction, it is well-known that in the case $H = I$, a matrix X is normal if and only if X has a polar decomposition with commuting factors (assuming the selfadjoint factor is in fact positive semidefinite). Thus, the question arises whether this is still true for the case that H is indefinite. Therefore, we will be interested in the following two questions.

- Does every H-normal matrix have an H-polar decomposition?

- If an H-normal matrix has an H-polar decomposition, does it have an H-polar decomposition with commuting factors?

First, let us consider the second question. We immediately obtain the following result.

Lemma 26 *Assume that X admits an H-polar decomposition $X = UA$. Then X is H-normal if and only if $UA^2 = A^2U$.*

The proof is straightforward.

However, simple examples show that an H-normal matrix need not have an H-polar decomposition with commuting factors even when it admits an H-polar decomposition:

Example 27 Complex case: Let

$$X = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}, \quad H = Z_2.$$

It is easy to see that X is H-normal. In fact, $X^{[*]}X = XX^{[*]} = 0$. It is easy to see that all H-polar decompositions $X = UA$ of X are described by the formulas

$$U = \begin{bmatrix} 0 & ix \\ ix^{-1} & y \end{bmatrix}, \quad A = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix},$$

where $x \neq 0$ and y are arbitrary real numbers. Clearly, U and A do not commute for any values of the parameters x and y.

Next, we show in general that *every indecomposable H-normal matrix $X \in \mathbb{C}^{2n\times 2n}$ that has two distinct eigenvalues $\lambda = 0$ and $\mu \neq 0$ and that allows an H-polar decomposition cannot have an H-polar decomposition with commuting factors.* First, we may assume that (X, H) is in the canonical form of Theorem 10, i.e.,

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad \text{and} \quad X^{[*]} = \begin{bmatrix} X_2^* & 0 \\ 0 & X_1^* \end{bmatrix},$$

where $X_1 \in \mathbb{C}^{n\times n}$ is singular and $X_2 \in \mathbb{C}^{n\times n}$ is nonsingular. Let $X = UA$ be an H-polar decomposition, where $U \in \mathbb{C}^{2n\times 2n}$ is H-unitary and $A \in \mathbb{C}^{n\times n}$ is H-selfadjoint and assume that $UA = AU$. This implies

$$X^{[*]} = A^{[*]}U^{[*]} = AU^{-1} = U^{-1}A = U^{-2}X.$$

Since X_1 is singular, there exists a vector $x \in \mathbb{C}^n\backslash\{0\}$ such that $X_1x = 0$. Since X_2 is nonsingular, we obtain that

$$0 \neq X_2^*x = X^{[*]}\begin{bmatrix} x \\ 0 \end{bmatrix} = U^{-2}X\begin{bmatrix} x \\ 0 \end{bmatrix} = U^{-2}X_1x = 0;$$

and this is a contradiction. Hence, $UA \neq AU$.

Example 28 Real case: Let

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H = Z_2.$$

Then X is H-normal and admits an H-polar decomposition $X = U_0 A_0$, where

$$U_0 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Again, U_0 and A_0 do not commute. Note that we have in every H-polar decomposition $X = UA$ with $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$ that $U^{-1} = AX^{-1} = \begin{bmatrix} a & -b \\ c & -a \end{bmatrix}$. Since U^{-1} is H-unitary, we obtain that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = H = (U^{-1})^* H U^{-1} = \begin{bmatrix} 2ac & -a^2 - bc \\ -a^2 - bc & 2ab \end{bmatrix}.$$

If $a \neq 0$, then $b = c = 0$ and $-a^2 = 1$ which is a contradiction. Hence $a = 0$ and

$$A = \begin{bmatrix} 0 & b \\ -b^{-1} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -b \\ -b^{-1} & 0 \end{bmatrix}$$

for some $b \in \mathbb{R} \setminus \{0\}$. But U and A do not commute.

Based on our results from the previous section, we can prove the following theorem that establishes a sufficient condition for the existence of H-polar decompositions with commuting factors.

Theorem 29 $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{F} = \mathbb{R}$. *Let X be a nonsingular H-normal matrix, and in the real case assume in addition that X has no negative eigenvalues. Then X admits an H-polar decomposition $X = UA$ with commuting factors $U, A \in \mathbb{F}^{n\times n}$, and such that U belongs to the connected component of identity in the $\mathbb{F}$-group of H-unitary matrices and A belongs to the set of H-selfadjoint matrices that possess an H-selfadjoint logarithm.*

Proof. Let X be a nonsingular H-normal matrix. By Proposition 22, or Lemma 25 in the real case, we have $X = e^Y$ for some H-normal matrix Y. Write $Y = Y_1 + Y_2$, where Y_1 is H-selfadjoint and Y_2 is H-skewadjoint. Then $X = UA$ with $U = e^{Y_2}$, $A = e^{Y_1}$ is an H-polar decomposition of X with the required properties; the connected component property of U follows from Proposition 17. $\square$

It follows from Remark 20 that the matrix A in Theorem 29 is H-consistent (in the terminology of [4]).

Compare Theorem 29 with a general result on H-polar decompositions on H-normal matrices proved in [1]:

Theorem 30 $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{F} = \mathbb{R}$. *Every nonsingular H-normal matrix admits an H-polar decomposition.*

Theorem 30 does not assert that the factors of the H-polar decomposition commute; as we have seen in Example 28 the factors need not commute under the hypotheses of Theorem 30.

Furthermore, we note that for the case $H = I$, we obtain from Theorem 29 and Proposition 19 the well-known fact (see [6], for example) that a nonsingular normal matrix X has a polar decomposition $X = UA$ with commuting factors such that A is positive definite. Moreover, the polar decomposition is unique in this case. Therefore, the question arises if a decomposition as in Theorem 29 is unique as well. The answer is negative, as the following example shows:

Example 31 Consider the $(2n) \times (2n)$ matrices

$$X = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

It is easy to see that X is H-normal, in fact, X is H-selfadjoint. Note that X has two different H-polar decompositions.

$$\begin{aligned} X &= U_1A_1 = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \\ &= U_2A_2 = (-U_1)(-A_1) = \begin{bmatrix} -I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \end{aligned}$$

Both A_1 and A_2 satisfy the conditions of Proposition 19 and thus, they have H-selfadjoint logarithms. On the other hand both U_1 and U_2 have H-skewadjoint logarithms in the complex case: $0 = \log U_1$; $i\pi I = \log U_2$. In the real case consider the case that $n = 2m$ is even. Then

$$\begin{bmatrix} 0 & \pi I_m & 0 & 0 \\ -\pi I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -\pi I_m \\ 0 & 0 & \pi I_m & 0 \end{bmatrix}$$

is a real H-selfadjoint logarithm of U_2.

We now consider the question if any H-normal matrix has an H-polar decomposition (with not necessarily commuting factors). Therefore, let us review the following criterion from [1], [4]. In Propositions 32 and 33 the matrix X is not assumed to be H-normal.

Proposition 32 *Let $X \in \mathbb{F}^{n\times n}$ and let $A \in \mathbb{F}^{n\times n}$ be H-selfadjoint such that $A^2 = X^{[*]}X$ and* $\mathrm{Ker}X = \mathrm{Ker}A$. *Then X admits an H-polar decomposition.*

We quote also another result from [1] (Lemma 4.2):

Proposition 33 *If $X \in \mathbb{R}^{n\times n}$ admits an H-polar decomposition $X = U_c A_c$ with H-unitary $U_c \in \mathbb{C}^{n\times n}$ and H-selfadjoint $A_c \in \mathbb{C}^{n\times n}$, then X admits also an H-polar decomposition $X = U_r A_r$ with H-unitary $U_r \in \mathbb{R}^{n\times n}$ and H-selfadjoint $A_r \in \mathbb{R}^{n\times n}$. (It is assumed that H is real.)*

Thus, it is sufficient to consider the complex case. We already know from Theorem 30 that every nonsingular H-normal matrix has an H-polar decomposition. We generalize this result as follows.

Theorem 34 $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$. *Let X be an H-normal matrix such that each singular H-indecomposable block over $\mathbb{C}$ of X (if any) either:*

(i) has two distinct complex eigenvalues (one of them must be zero), or:

(ii) is similar to one (necessarily nilpotent) Jordan block.

Then X admits an H-polar decomposition over $\mathbb{F}$.

Proof. In view of Proposition 33 we may assume $\mathbb{F} = \mathbb{C}$. It suffices to prove existence of an H-polar decomposition for each indecomposable block of X. By Theorem 30 it is sufficient to consider singular blocks, i.e., blocks that have the eigenvalue zero. We then have to consider two different cases.

Consider first the case when the indecomposable block has two distinct eigenvalues $\lambda = 0$ and $\mu \neq 0$. By Theorem 10 we may assume that

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $\sigma(X_1) = \{0\}$, $\sigma(X_2) = \{\mu\}$. In view of Proposition 32, it is sufficient to construct an H-selfadjoint matrix A such that $A^2 = X^{[*]}X = XX^{[*]}$ and $\mathrm{Ker}X = \mathrm{Ker}A$.

Since $X^{[*]} = \begin{bmatrix} X_2^* & 0 \\ 0 & X_1^* \end{bmatrix}$, we obtain that

$$XX^{[*]} = \begin{bmatrix} X_1X_2^* & 0 \\ 0 & X_2X_1^* \end{bmatrix}.$$

Since X_1 and X_2^* commute, it follows that $X_1X_2^*$ is nilpotent and thus, the only eigenvalue of $XX^{[*]}$ is zero. This implies that there exists a nonsingular matrix P such that we obtain, setting $\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & (P^{-1})^* \end{bmatrix}$:

$$\mathcal{P}^{-1}XX^{[*]}\mathcal{P} = \begin{bmatrix} P^{-1}X_1X_2^*P & 0 \\ 0 & P^*X_2X_1^*(P^{-1})^* \end{bmatrix} = \begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J}^* \end{bmatrix} \quad \text{and} \quad \mathcal{P}^*H\mathcal{P} = H, \tag{26}$$

where $\mathcal{J} = \mathcal{J}_{p_1}(0) \oplus \ldots \oplus \mathcal{J}_{p_k}(0)$ is a nilpotent matrix in Jordan form. We note that if $x_1, \ldots, x_k$ is a basis of $\mathrm{Ker}(P^{-1}X_1X_2^*P)$ then $\left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} x_k \\ 0 \end{smallmatrix}\right]$ is a basis of $\mathrm{Ker}(\mathcal{P}^{-1}X\mathcal{P})$. From this and the obvious fact that there exists a permutation matrix $\mathcal{Q}$ such that

$$\mathcal{Q}^{-1}\mathcal{P}^{-1}XX^{[*]}\mathcal{P}\mathcal{Q} = \bigoplus_{j=1}^{k} \begin{bmatrix} \mathcal{J}_{p_j}(0) & 0 \\ 0 & \mathcal{J}_{p_j}(0)^* \end{bmatrix} \quad \text{and} \quad \mathcal{Q}^*\mathcal{P}^*H\mathcal{P}\mathcal{Q} = \bigoplus_{j=1}^{k} \begin{bmatrix} 0 & I_{p_j} \\ I_{p_j} & 0 \end{bmatrix},$$

we see that it is sufficient to consider the case

$$XX^{[*]} = \begin{bmatrix} \mathcal{J}_m(0) & 0 \\ 0 & \mathcal{J}_m(0)^* \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$$

and to construct an H-selfadjoint matrix A such that $A^2 = XX^{[*]}$ and

$$\mathrm{Ker}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T \right\}.$$

Hence, the proof in the case of two distinct eigenvalues is concluded by choosing

$$A = \begin{bmatrix} 0 & Z_m \\ Z_m\mathcal{J}_m(0) & 0 \end{bmatrix}.$$

Consider the case of an indecomposable X having only one Jordan block associated with the eigenvalue $\lambda = 0$. By Theorem 1 of [10], we may further assume that X is an upper triangular Toeplitz matrix and that $H = \pm Z_n$. Therefore, it is clear that at least one of the matrices

$$A = \frac{1}{2}(X + X^{[*]}) \quad \text{and} \quad S = \frac{1}{2}(X - X^{[*]})$$

has only one Jordan block associated with the eigenvalue zero, say A. Thus, applying a transformation

$$A \to T^{-1}AT, \; S \to T^{-1}ST, \; H \to T^*HT$$

for some nonsingular T, we may assume that $A = \mathcal{J}_n(0)$. Since S commutes with A and is H-skewadjoint, it follows that

$$S = s_1\mathcal{J}_n(0) + s_2\mathcal{J}_n(0)^2 + \ldots + s_{n-1}\mathcal{J}_n(0)^{n-1},$$

where $s_1, \ldots, s_{n-1} \in i\mathbb{R}$. According to Proposition 32, it is sufficient to show that $X^{[*]}X$ has an H-selfadjoint square root R such that $\mathrm{Ker}(R) = \mathrm{Ker}(X)$. We find that $X^{[*]}X$ has the form

$$X^{[*]}X = A^2 - S^2 = x_2\mathcal{J}_n(0)^2 + \ldots + x_{n-1}\mathcal{J}_n(0)^{n-1}, \tag{27}$$

where the x_j are real, nonnegative and $x_2 \neq 0$. Let us try

$$R = \sum_{k=1}^{n-1} r_k\mathcal{J}_n(0)^k = r_1\mathcal{J}_n(0) + r_2\mathcal{J}_n(0)^2 + \ldots + r_{n-1}\mathcal{J}_n(0)^{n-1}.$$

Then clearly $\mathrm{Ker}(R) = \mathrm{Ker}(X)$, if $r_1 \neq 0$. Assuming $n \geq 3$ (the cases $n = 1$ and $n = 2$ being trivial), we obtain

$$\begin{aligned} R^2 &= \sum_{k=2}^{n-1} \left(\sum_{l=1}^{k-1} r_l r_{k-l} \right) \mathcal{J}_n(0)^k \\ &= r_1^2 \mathcal{J}_n(0)^2 + \sum_{k=3}^{n-1} \left(2r_1 r_{k-1} + \sum_{l=2}^{k-2} r_l r_{k-l} \right) \mathcal{J}_n(0)^k. \end{aligned} \tag{28}$$

Note that the coefficient of $\mathcal{J}_n(0)^k$ in (28) is linear in r_{k-1} for $k > 2$. Comparing (27) and (28), we can now find the square root R in the following way. First, we solve the equation $r_1^2 = x_2$ over $\mathbb{R}$ (recall that $x_2 > 0$). Then we compute $r_2, \ldots, r_{n-2}$ subsequently as the unique solution of nontrivial linear equations $r_1 r_{k-1} = b_{k-1}$, for some $b_{k-1} \in \mathbb{R}$, and finally we choose an arbitrary $r_{n-1} \in \mathbb{R}$. This completes the proof. □

6 The cases of small numbers of negative eigenvalues of H

In this section we prove the following result:

Theorem 35 *Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. If the nonsingular real symmetric (in the real case) or complex Hermitian (in the complex case) $n \times n$ matrix H has at most two negative eigenvalues, counted with multiplicities, then every H-normal matrix admits an H-polar decomposition.*

Proof. In the case of H having only one negative eigenvalue, the result of Theorem 35 was proved in [1], Theorem 5.2.

We will use the forms for indecomposable normal matrices obtained in [13] for the proof of Theorem 35. Leaving aside the well-known situation when H is positive definite, and the already proved result of Theorem 35 in the case of H having only one negative eigenvalue, we have to consider two cases: (1) $\mathbb{F} = \mathbb{C}$ and H has exactly two negative eigenvalues; (2) $\mathbb{F} = \mathbb{R}$ and H has exactly two negative eigenvalues. In view of Proposition 33, we need to consider the complex case (1) only. We use a complete list of relevant indecomposable normal matrices X (obtained in [13]), and for each matrix on the list write explicitly the factors A and U from an H-polar decomposition $X = UA$. By Proposition 32, existence of an H-polar decomposition of X is guaranteed if and only if there exists an H-selfadjoint matrix A such that

$$X^{[*]} X = A^2 \quad \text{and} \quad \mathrm{Ker}\, A = \mathrm{Ker}\, X. \tag{29}$$

Verification that in every case the matrix A indeed has the indicated properties is straightforward.

We assume therefore $\mathbb{F} = \mathbb{C}$. We leave aside indecomposable normals of size 1×1 (for those, existence of polar decomposition is trivial), indecomposable normals for which the corresponding matrix H has only one negative eigenvalue (this case was proved in Theorem 5.2 of [1]), and those forms that have either two distinct eigenvalues or only one Jordan block for all allowable values of parameters (these are taken care of by Theorem 34). We are left with the following list of indecomposable normals. In the list, we give the form X of the indecomposable normals, the matrix H that describes the corresponding indefinite inner product, the H-selfadjoint matrix A having the properties (29), and the H-unitary matrix U such that an H-polar decomposition $X = UA$ holds. Only the case $\lambda = 0$ will be considered, and the matrices A and U will be given under this assumption. (If $\lambda \neq 0$, then X is nonsingular; this case is taken care of by Theorem 30.)

The search for suitable matrices A and U was conducted using MAPLE, assuming in many cases that A is upper triangular. When the matrices X, A, or U are upper triangular, this is indicated by blanks in the strictly lower triangular part.

I.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ & \lambda & 0 & z \\ & & \lambda & 0 \\ & & & \lambda \end{bmatrix}; \quad |z| = 1, \quad H = Z_4. \tag{30}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}.$$

II.

$$X = \begin{bmatrix} \lambda & 1 & \kappa & 0 \\ & \lambda & 0 & z \\ & & \lambda & \kappa(1+ir)z \\ & & & \lambda \end{bmatrix}; \quad |z| = 1,\ r > 0,\ \kappa = \pm 1; \quad H = Z_4. \tag{31}$$

$$A = \begin{bmatrix} 0 & 1 & \kappa & 0 \\ 0 & 0 & 0 & \kappa \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa z & 0 & 0 \\ 0 & irz & \kappa z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{32}$$

III.

$$X = \begin{bmatrix} \lambda & 1 & 0 & \frac{r^2}{2} + is & 0 \\ & \lambda & 0 & z & 0 \\ & & \lambda & 0 & r \\ & & & \lambda & z^2 \\ & & & & \lambda \end{bmatrix}, \quad |z| = 1,\ r > 0,\ s \in \mathbb{R}, \quad H = Z_5. \tag{33}$$

$$A=\begin{bmatrix} 0 & \overline{z} & 0 & \frac{1}{2}r^2\overline{z} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2}r^2z \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad U=\begin{bmatrix} z & is & -iszr & -\frac{1}{2}isr^2 & 0 \\ 0 & z & -rz^2 & -\frac{1}{2}r^2z & 0 \\ 0 & 0 & 1 & \overline{z}r & 0 \\ 0 & 0 & 0 & z & iz^2s \\ 0 & 0 & 0 & 0 & z \end{bmatrix}. \tag{34}$$

IV.

$$X=\begin{bmatrix} \lambda & 1 & 2ir & 0 & 0 & 0 \\ & \lambda & 1 & ir & 0 & 2r^2-\frac{1}{2}s^2+it \\ & & \lambda & 1 & 0 & 0 \\ & & & \lambda & 0 & 1 \\ & & & & \lambda & s \\ & & & & & \lambda \end{bmatrix}, \quad r,t\in\mathbb{R},\ s>0, \quad H=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & Z_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{35}$$

$$A=\begin{bmatrix} 0 & 1 & ir & \frac{3}{2}r^2-it & 0 & rt \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2}r^2+it \\ 0 & 0 & 0 & 1 & 0 & -ir \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

$$U=\begin{bmatrix} 1 & ir & -\frac{3}{2}r^2+it & -3ir^3-rt & -\frac{1}{2}irs & \frac{15}{8}r^4-\frac{1}{2}t^2-\frac{1}{8}r^2s^2 \\ 0 & 1 & ir & -\frac{1}{2}r^2-\frac{1}{2}s^2 & -s & -2ir^3 \\ 0 & 0 & 1 & ir & 0 & \frac{1}{2}r^2+it \\ 0 & 0 & 0 & 1 & 0 & ir \\ 0 & 0 & 0 & s & 1 & \frac{1}{2}irs \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

V.

$$X=\begin{bmatrix} \lambda & 1 & -2ir(\operatorname{Im} z) & 0 & 0 & 0 \\ & \lambda & z & r & 0 & (2r^2(\operatorname{Im} z)^2-\frac{s^2}{2}+it)z^2 \\ & & \lambda & z & 0 & 0 \\ & & & \lambda & 0 & z^2 \\ & & & & \lambda & s \\ & & & & & \lambda \end{bmatrix}, \quad H=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & Z_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{38}$$

where

$$|z|=1, \quad 0<\arg z<\pi, \quad r,s,t\in\mathbb{R}, \quad s>0.$$

$$A=\begin{bmatrix} 0 & \overline{z} & -r & \left(-\frac{1}{2}r^2+2r^2(\operatorname{Im}z)^2-it\right)\overline{z} & 0 & -2r(\operatorname{Im}z)t \\ 0 & 0 & 1 & r\overline{z}+zr & 0 & \left(2r^2(\operatorname{Im}z)^2+it-\frac{1}{2}r^2\right)z \\ 0 & 0 & 0 & 1 & 0 & -r \\ 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{39}$$

$$U=\begin{bmatrix} z & zr-2ir(\operatorname{Im}z) & q_3 & q_1 & -z^2sr & q_2 \\ 0 & z & -rz^2 & -\frac{z}{2}(r^2+s^2) & -sz^2 & q_4 \\ 0 & 0 & z & r & 0 & z^2(2r^2(\operatorname{Im}z)^2+it+\frac{1}{2}r^2)+2iz^3r^2(\operatorname{Im}z) \\ 0 & 0 & 0 & z & 0 & -zr-2iz^2r(\operatorname{Im}z) \\ 0 & 0 & 0 & \overline{z}s & 1 & -2isr(\operatorname{Im}z) \\ 0 & 0 & 0 & 0 & 0 & z \end{bmatrix}, \tag{40}$$

where

$$q_1=4ir^3(\operatorname{Im}z)^3+ir^3(\operatorname{Im}z)+\left(-\frac{1}{2}r^3-2r^3(\operatorname{Im}z)^2-irt\right)z$$
$$+2ir^3\overline{z}^2(\operatorname{Im}z)+\left(-\frac{1}{2}r^3-2r^3(\operatorname{Im}z)^2+irt\right)\overline{z},$$

$$q_2=-4ir^4(\operatorname{Im}z)^3+ir^4(\operatorname{Im}z)-4r^2t(\operatorname{Im}z)+\left(-\frac{1}{8}r^4+6r^4(\operatorname{Im}z)^4-r^4(\operatorname{Im}z)^2-\frac{1}{2}s^2r^2-\frac{1}{2}t^2\right)z,$$

$$q_3=2izr^2(\operatorname{Im}z)-z^2r^2+2ir^2(\operatorname{Im}z)\overline{z}-2r^2(\operatorname{Im}z)^2+it-\frac{1}{2}r^2,$$

$$q_4=z\left(-irt+2r^3(\operatorname{Im}z)^2-\frac{1}{2}s^2r\right)+z^2\left(is^2r(\operatorname{Im}z)+4ir^3(\operatorname{Im}z)^3\right).$$

VI.

$$X=\begin{bmatrix} \lambda & 0 & z & r\frac{1}{2}(1-i\sqrt{3})z \\ & \lambda & 0 & \frac{1}{2}(1+i\sqrt{3})z \\ & & \lambda & 0 \\ & & & \lambda \end{bmatrix},\quad |z|=1, r\in\mathbb{R}, r\geq\sqrt{3},\quad H=\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{41}$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad U=\begin{bmatrix} \frac{r}{2}(-1-i\sqrt{3})z & z & 0 & 0 \\ \frac{1}{2}(1+i\sqrt{3})z & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & \frac{1}{2}(1+i\sqrt{3})z & zr \end{bmatrix}. \tag{42}$$

VII.

$$X=\begin{bmatrix} \lambda & 0 & 0 & 0 \\ & \lambda & 1 & 0 \\ & & \lambda & 0 \\ & & & \lambda \end{bmatrix},\quad H=\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{43}$$

$$U = Z_2 \oplus Z_2, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

VIII.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 \\ & \lambda & 0 & 1 & 0 \\ & & \lambda & z & 0 \\ & & & \lambda & 0 \\ & & & & \lambda \end{bmatrix}, \quad |z| = 1, \quad H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & 1 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \tag{45}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\overline{z} & 1 & 0 & \overline{z} \\ 0 & -1 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \overline{z} & 0 & 0 & 0 \end{bmatrix}. \tag{46}$$

IX.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 \\ & \lambda & 0 & r & z \\ & & \lambda & z^2 & 0 \\ & & & \lambda & 0 \\ & & & & \lambda \end{bmatrix}, \quad |z| = 1, \ r > 0. \quad H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & 1 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \tag{47}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & r^2 & zr \\ 0 & 0 & 0 & r\overline{z} & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -zr & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r & z \end{bmatrix}. \tag{48}$$

X.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & ir & 0 \\ & \lambda & 0 & 1 & s & ir \\ & & \lambda & 0 & z & 0 \\ & & & \lambda & 0 & z \\ & & & & \lambda & 0 \\ & & & & & \lambda \end{bmatrix}, \quad |z| = 1, \ z \neq -1, \ r, s \in \mathbb{R}, \ s > 0, \quad H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \tag{49}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & s & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 0 & ir & 0 & -\frac{1}{2}r^2 \\ 1 & 0 & ir & 0 & -\frac{1}{2}r^2 & 0 \\ 0 & 0 & 0 & z & 0 & irz \\ 0 & 0 & z & 0 & irz & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{50}$$

XI.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ & \lambda & 0 & 1 & 0 & 0 & 0 \\ & & \lambda & 0 & 0 & -z_1\overline{z_2}\cos\alpha & \sin\alpha\,\cos\beta \\ & & & \lambda & 0 & z_1\sin\alpha & z_2\cos\alpha\,\cos\beta \\ & & & & \lambda & 0 & \sin\beta \\ & & & & & \lambda & 0 \\ & & & & & & \lambda \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_3 & 0 \\ I_2 & 0 & 0 \end{bmatrix}, \tag{51}$$

where

$$|z_1| = |z_2| = 1, \; 0 < \alpha, \beta \leq \frac{\pi}{2}.$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{z_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{52}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -z_1\overline{z_2}\cos\alpha & \overline{z_2}\sin\alpha\,\cos\beta & -\sin\alpha\,\sin\beta & 0 & 0 \\ 0 & 0 & z_1\sin\alpha & \cos\alpha\,\cos\beta & -z_2\cos\alpha\,\sin\beta & 0 & 0 \\ 0 & 0 & 0 & \overline{z_2}\sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_2 \end{bmatrix}. \tag{53}$$

XII.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ & & \lambda & 0 & 0 & 0 & -z_1\overline{z_2}\sin\alpha\,\cos\beta & \cos\alpha\,\cos\gamma \\ & & & \lambda & 0 & 0 & z_1\cos\alpha\,\cos\beta & z_2\sin\alpha\,\cos\gamma \\ & & & & \lambda & 0 & \sin\beta & 0 \\ & & & & & \lambda & 0 & \sin\gamma \\ & & & & & & \lambda & 0 \\ & & & & & & & \lambda \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{bmatrix}, \tag{54}$$

where

$$|z_1| = |z_2| = 1, \; 0 \le \alpha < \frac{\pi}{2}, \; 0 < \beta < \gamma \le \frac{\pi}{2}.$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{55}$$

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\alpha\cos\gamma & -z_1\overline{z_2}\sin\alpha\cos\beta & z_1\overline{z_2}\sin\alpha\sin\beta & -\cos\alpha\sin\gamma & 0 & 0 \\ 0 & 0 & z_2\sin\alpha\cos\gamma & z_1\cos\alpha\cos\beta & -z_1\cos\alpha\sin\beta & -z_2\sin\alpha\sin\gamma & 0 & 0 \\ 0 & 0 & 0 & \sin\beta & \cos\beta & 0 & 0 & 0 \\ 0 & 0 & \sin\gamma & 0 & 0 & \cos\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{56}$$

This concludes the proof of Theorem 35. ◻

Remark 36 We note that in each case I through XII the corresponding matrices A and U do not commute.

Remark 37 The example of an indecomposable X in a space with indefinite inner product of rank 2 that was given in [8] by

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad H = Z_4$$

appears in (VII.) in the forms (43) and (44). Indeed, setting

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

one finds that

$$P^{-1}XP = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P^*HP = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.$$

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