Corrigendum to ‘Numerical methods for eigenvalues of singular polynomial eigenvalue problems’ (LAA 719 (2025) 1–33)

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In the appendix of [1], the following lemma was formulated.

Lemma A.1. *Let $p_1(\lambda), \dots, p_k(\lambda)$ be complex polynomials such that their greatest common divisor $q(\lambda) = \gcd(p_1(\lambda), \dots, p_k(\lambda))$ has simple roots. Then there exists a generic set $\Omega \subseteq \mathbb{C}^k$ such that for all $s = (s_1, \dots, s_k) \in \Omega$ the polynomial $s_1\, p_1(\lambda) + \cdots + s_k\, p_k(\lambda)$ has only simple roots.*

In the proof given in [1], the argument only works for the particular case $k = 2$. Below, we present a corrected proof. Case 1) mainly follows the lines from the original proof, but is adapted to the special case $k = 2$, while the proof of Case 2) is new.

Proof. First we note that the polynomial $\widetilde{p}(\lambda) := s_1\, p_1(\lambda) + \cdots + s_k\, p_k(\lambda)$ has simple roots if and only if the determinant of the Sylvester matrix $S\big(\widetilde{p}(\lambda), \widetilde{p}\,'(\lambda)\big)$ of $\widetilde{p}(\lambda)$ and its derivative $\widetilde{p}\,'(\lambda)$ is nonzero. Since this determinant is a polynomial in the variables $s_1, \dots, s_k$ it remains to show that $\det S\big(\widetilde{p}(\lambda), \widetilde{p}\,'(\lambda)\big)$ is not the zero polynomial and to this end it is sufficient to find one example $s = (s_1, \dots, s_k)$ such that the roots of the polynomial $\widetilde{p}(\lambda)$ are simple.

Let $p_i(\lambda) = q(\lambda)\, h_i(\lambda)$ for $i = 1, \dots, k$, where $\gcd\big(h_1(\lambda), \dots, h_k(\lambda)\big) = 1$, and let $\eta_1, \dots, \eta_\ell \in \mathbb{C}$ be the roots of $q(\lambda)$ that are simple by assumption.

Case 1): $k = 2$.

If $h_1(\lambda)$ and $h_2(\lambda)$ are both constant polynomials, then $p_1(\lambda)$ and $p_2(\lambda)$ are both multiples of $q(\lambda)$ and hence for any $s = (s_1, s_2) \in \mathbb{C}^2$ such that $s_1\, h_1(\lambda) + s_2\, h_2(\lambda) \neq 0$ the roots of $s_1\, p_1(\lambda) + s_2\, p_2(\lambda)$ are $\eta_1, \dots, \eta_\ell$ and thus simple. Thus, in the following we may assume that one of the polynomials $h_1(\lambda)$ and $h_2(\lambda)$ is not constant, without loss of generality let this be the case for $h_1(\lambda)$.

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Suppose that ξ is a multiple zero of $s_1\, h_1(\lambda) + s_2\, h_2(\lambda)$ for some $s = (s_1, s_2) \in \mathbb{C}^2$. Then we have $s_1\, h_1(\xi) + s_2\, h_2(\xi) = 0$ and $s_1\, h_1'(\xi) + s_2\, h_2'(\xi) = 0$. It follows that s is in the kernel of the matrix $H(\xi)$, where

$$H(\lambda) := \begin{bmatrix} h_1(\lambda) & h_2(\lambda) \\ h_1'(\lambda) & h_2'(\lambda) \end{bmatrix}$$

is defined as a 2×2 polynomial matrix and it does not depend on s. We know that $\big(h_1(\xi), h_2(\xi)\big) \neq (0, 0)$ for each $\xi \in \mathbb{C}$ because $\gcd\big(h_1(\lambda), h_2(\lambda)\big) = 1$, so $\operatorname{rank}\big(H(\xi)\big) \geq 1$ for all $\xi \in \mathbb{C}$.

Let us show that $\operatorname{nrank}\big(H(\lambda)\big) = 2$. To this end, consider the polynomial

$$d(\lambda) := h_1(\lambda)\, h_2'(\lambda) - h_1'(\lambda)\, h_2(\lambda).$$

Suppose that $d(\lambda)$ is identically zero. This implies that $h_1(\lambda)\, h_2'(\lambda) = h_1'(\lambda)\, h_2(\lambda)$. But then it follows from $\gcd\big(h_1(\lambda), h_2(\lambda)\big) = 1$ that $h_1(\lambda)$ divides its derivative $h_1'(\lambda)$, which is not possible, since the degree of $h_1(\lambda)$ is at least one. Therefore $d(\lambda)$ is not identically zero and this shows that $\operatorname{nrank}\big(H(\lambda)\big) = 2$.

Then there are only finitely many values $\xi_1, \ldots, \xi_r$ such that $\operatorname{rank}\big(H(\xi_i)\big) = 1$ for $i = 1, \ldots, r$. Now consider the $r + \ell$ lines $s_1\, h_1(\xi_i) + s_2\, h_2(\xi_i) = 0$, $i = 1, \ldots, r$, and $s_1\, h_1(\eta_j) + s_2\, h_2(\eta_j) = 0$, $j = 1, \ldots, \ell$, in $\mathbb{C}^2$. If s is not on any of these lines, the roots of $s_1\, h_1(\lambda) + s_2\, h_2(\lambda)$ are simple and different from $\eta_1, \ldots, \eta_\ell$. But then also the roots of $s_1\, p_1(\lambda) + s_2\, p_2(\lambda)$ are simple. This proves the claim for the case $k = 2$.

Case 2): $k > 2$. Let $\zeta_1, \ldots, \zeta_m \in \mathbb{C}$ be the roots of $h_k(\lambda)$. Consider the polynomial matrix

$$\widetilde{H}(\lambda) = \begin{bmatrix} h_1(\lambda) & \ldots & h_{k-1}(\lambda) \end{bmatrix}$$

Since $\gcd\big(h_1(\lambda), \ldots, h_k(\lambda)\big) = 1$ it follows that $\widetilde{H}(\zeta_i) \neq 0$ for $i = 1, \ldots, m$. Thus, if $t = (t_1, \ldots, t_{k-1})$ is not from any of the m hyperplanes $\operatorname{Ker} \widetilde{H}(\zeta_i)$ for $i = 1, \ldots, m$, it follows that $\widetilde{h}(\lambda) := t_1\, h_1(\lambda) + \cdots + t_{k-1}\, h_{k-1}(\lambda)$ has no roots in common with $h_k(\lambda)$, or, in other words, $\gcd\big(\widetilde{h}(\lambda), h_k(\lambda)\big) = 1$. But then we can apply Case 1) to find a linear combination of $s_1\, \widetilde{h}(\lambda) + s_2\, h_k(\lambda)$ that has only simple roots that are distinct from the roots of $q(\lambda)$. Then we conclude that also the roots of $s_1\, t_1\, p_1(\lambda) + \cdots + s_1\, t_{k-1}\, p_{k-1}(\lambda) + s_2\, p_k(\lambda)$ are simple. □

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References

[1] M. E. Hochstenbach, C. Mehl, and B. Plestenjak. Numerical methods for eigenvalues of singular polynomial eigenvalue problems. *Linear Algebra Appl.*, 719:1–33, 2025.