Generic rank-k perturbations of structured matrices

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Abstract. This paper deals with the effect of generic but structured low rank perturbations on the Jordan structure and sign characteristic of matrices that have structure in an indefinite inner product space. The paper is a follow-up of earlier papers in which the effect of rank one perturbations was considered. Several results that are in contrast to the case of unstructured low rank perturbations of general matrices are presented here.

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1. Introduction

In the past two decades, the effects of generic low rank perturbations on the Jordan structure of matrices and matrix pencils with multiple eigenvalues have been extensively studied, see [5, 9, 20, 21, 23, 24]. Recently, starting with [15] the same question has been investigated for generic structure-preserving low rank perturbations of matrices that are structured with respect to some indefinite inner product. While the references [5, 9, 20, 21, 23, 24] on unstructured perturbations have dealt with the general case of rank k, [15] and the follow-up papers [16]–[19] on structure-preserving perturbations focussed on the special case $k = 1$. The reason for this restriction was the use of a particular proof technique that was based on the so-called Brunovsky form which is handy for the case $k = 1$ and may be for the case $k = 2$, but becomes rather complicated for the case $k > 2$. Nevertheless, the papers [15]–[19] (see also [6, 10]) showed that in some situations there are surprising differences in the

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changes of Jordan structure with respect to general and structure-preserving rank-one perturbations. This mainly has to do with the fact that the possible Jordan canonical forms for matrices that are structured with respect to indefinite inner products are restricted. This work has later been generalized to the case of structured matrix pencils in [1]–[3], see also [4]. Although a few questions remained open, the effect of generic structure-preserving rank-one perturbations on the Jordan structure and the sign characteristic of matrices and matrix pencils that are structured with respect to an indefinite inner product are now well understood.

In this paper, we will consider the more general case of generic structure-preserving rank k perturbations, where $k \geq 1$. Numerical experiments suggest that the following meta theorem is true:

Meta-Theorem 1.1. *Let $A \in \mathbb{F}^{n,n}$ be a matrix that is structured with respect to some indefinite inner product and let $B \in \mathbb{F}^{n,n}$ be a matrix of rank k so that $A+B$ is from the same structure class as A. Then generically the Jordan structure and sign characteristic of $A+B$ is the same that one would obtain by performing a sequence of k generic structured rank-one perturbations on A.*

Here and throughout the paper, $\mathbb{F}$ denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$. Moreover, the term generic is understood in the following way. A set $\mathcal{A} \subseteq \mathbb{F}^n$ is called *algebraic* if there exist polynomials p_j in n variables, $j = 1, \dots, k$ such that $a \in \mathcal{A}$ if and only if

$$p_j(a) = 0 \quad \text{for} \quad j = 1, \dots, k.$$

An algebraic set $\mathcal{A} \subseteq \mathbb{F}^n$ is called *proper* if $\mathcal{A} \neq \mathbb{F}^n$. Then, a set $\Omega \subseteq \mathbb{F}^n$ is called *generic* if $\mathbb{F}^n \setminus \Omega$ is contained in a proper algebraic set.

A proof of Theorem 1.1 on the meta level seems to be hard to achieve. We illustrate the difficulties for the special case of H-symmetric matrices $A \in \mathbb{C}^{n\times n}$, i.e., matrices satisfying $A^T H = HA$, where $H \in \mathbb{C}^{n\times n}$ is symmetric and invertible. An H-symmetric rank-one perturbation of A has the form $A + uu^T H$ while an H-symmetric rank-two perturbation has the form $A + [u, v][u, v]^T H = A + uu^T H + vv^T H$, where $u, v \in \mathbb{C}^n$. Here, one can immediately see that the rank-two perturbation of A can be interpreted as a sequence of two independent rank-one perturbations, so the only remaining question concerns genericity. Now the statements on generic structure-preserving rank-one perturbations of H-symmetric matrices from [15] typically have the form that they assert the existence of a generic set $\Omega(A) \subseteq \mathbb{C}^n$ such that for all $u \in \Omega(A)$ the spectrum of $A + uu^T H$ shows the generic behavior stated in the corresponding theorem. Clearly, this set $\Omega(A)$ depends on A and thus, the set of all vectors $v \in \mathbb{C}^n$ such that the spectrum of the rank-one perturbation $A + uu^T H + vv^T H$ of $A + uu^T H$ shows the generic behavior is given by $\Omega(A + uu^T H)$. On the other hand, the precise meaning of a *generic H-symmetric rank-two perturbation* $A + uu^T H + vv^T H$ of A is the existence of a generic set $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^n$ such that $(u, v) \in \Omega$. Thus, the statement of Theorem 1.1 can be translated by asserting that the set

$$\Omega = \bigcup_{u\in\Omega(A)} \Big(\{u\} \times \Omega(A + uu^T H)\Big)$$

is generic. Unfortunately, this fact cannot be proved without more detailed knowledge on the structure of the generic sets $\Omega(A)$ as the following example shows. Consider the set

$$\mathbb{C}^2 \setminus \big\{(x, e^x) \,\big|\, x \in \mathbb{C}\big\} = \bigcup_{x\in\mathbb{C}} \Big(\{x\} \times \big(\mathbb{C} \setminus \{e^x\}\big)\Big)$$

Clearly, the sets $\mathbb{C}$ and $\mathbb{C} \setminus \{e^x\}$ are generic for all $x \in \mathbb{C}$, but the set $\mathbb{C}^2 \setminus \big\{(x, e^x) \,\big|\, x \in \mathbb{C}\big\}$ is not as $\Gamma := \big\{(x, e^x) \,\big|\, x \in \mathbb{C}\big\}$ is the graph of the function exp which is not contained in a proper algebraic set.

Still, the set Γ from the previous paragraph is a *thin* set in the sense that it is a set of measure zero, so one might have the idea to weaken the term *generic* to sets whose complement is contained in a set of measure zero. However, this approach would have a significant drawback when passing to the real case. In [17, Lemma 2.2] it was shown that if $W \subseteq \mathbb{C}^n$ is a proper algebraic set in $\mathbb{C}^n$, then $W \cap \mathbb{R}^n$ is a proper algebraic set in $\mathbb{R}^n$ – a feature that allows to easily transfer results on generic rank-one perturbations from the complex to the real case. Clearly, a generalization of [17, Lemma 2.2] to sets of measure zero would be wrong as the set $\mathbb{R}^n$ itself is a set of measure zero in $\mathbb{C}^n$. Thus, using the term *generic* as defined here does not only lead to stronger statements, but also eases the discussion of the case that the matrices and perturbations under consideration are real.

The classes of structured matrices we consider in this paper are the following. Throughout the paper let $A^\star$ denote either the transpose A^T or the conjugate transpose A^* of a matrix A. Furthermore, let $H^\star = H \in \mathbb{F}^{n\times n}$ and $-J^T = J \in \mathbb{F}^{n\times n}$ be invertible. Then $A \in \mathbb{F}^{n\times n}$ is called

1. H-selfadjoint, if $\star = *$ and $A^*H = HA$;
2. H-symmetric, if $\star = T$ and $A^TH = HA$;
3. J-Hamiltonian, if $\star = T$ and $A^TJ = -JA$.

There is no need to consider H-skew-adjoint matrices A satisfying $A^\star H = -HA$ in the case $\star = *$, because this case can be reduced to the case of H-selfadjoint matrices by considering iA. Similarly, it is not necessary to discuss inner products induced by a skew-Hermitian matrix S as one can consider iS instead. On the other hand, we do not consider H-skew-symmetric matrices A satisfying $A^TH = -HA$ or J-skew-Hamiltonian matrices A satisfying $A^TJ = JA$, because in those cases rank-one perturbations do not exist and thus Theorem 1.1 cannot be applied. The investigation of these types of matrices seems to be more difficult and is referred to a later stage.

The remainder of the paper is organized as follows. In section 2 we provide preliminary results that will be needed in the following. In sections 3 and 4 we consider structure-preserving rank k perturbations of H-symmetric, H-selfadjoint, and J-Hamiltonian matrices with the focus on the change of

Jordan structures in section 3 and on the change of the sign characteristig in section 4.

2. Preliminaries

We start with a series of lemmas that will be key tools in this paper. First, we recap [2, Lemma 2.2] and also give a proof for completeness.

Lemma 2.1 **([2]).** *Let $\mathcal{B} \subseteq \mathbb{F}^\ell$ not be contained in any proper algebraic subset of $\mathbb{F}^\ell$. Then, $\mathcal{B} \times \mathbb{F}^k$ is not contained in any proper algebraic subset of $\mathbb{F}^\ell \times \mathbb{F}^k$.*

Proof. First, we observe that the hypothesis that $\mathcal{B}$ is not contained in any proper algebraic subset of $\mathbb{F}^\ell$ is equivalent to the fact that for any nonzero polynomial p in ℓ variables there exists an $x \in \mathcal{B}$ (possibly depending on p) such that $p(x) \neq 0$. Letting now q be any nonzero polynomial in $\ell + k$ variables, then the assertion is equivalent to showing that there exists an $(x, y) \in \mathcal{B} \times \mathbb{F}^k$ such that $q(x, y) \neq 0$.

Thus, for any such q consider the set

$$\Gamma_q := \left\{ y \in \mathbb{F}^k \mid q(\,\cdot\,, y) \text{ is a nonzero polynomial in } \ell \text{ variables} \right\}$$

which is not empty (otherwise q would be constantly zero). Now, for any $y \in \Gamma_q$, by hypothesis there exists an $x \in \mathcal{B}$ such that $q(x, y) \neq 0$ but then $(x, y) \in \mathcal{B} \times \mathbb{F}^k$. □

Lemma 2.2 **([15]).** *Let $Y(x_1, \ldots, x_r) \in \mathbb{F}^{m\times n}[x_1, \ldots, x_r]$ be a matrix whose entries are polynomials in $x_1, \ldots, x_r$. If $\operatorname{rank} Y(a_1, \ldots, a_r) = k$ for some $[a_1, \ldots, a_r]^T \in \mathbb{F}^r$, then the set*

$$\{[b_1, \ldots, b_r]^T \in \mathbb{F}^r \;:\; \operatorname{rank} Y(b_1, \ldots, b_r) \geq k\} \tag{2.1}$$

is generic.

Lemma 2.3. *Let $H^\star = H \in \mathbb{F}^{n\times n}$ be invertible and let $A \in \mathbb{F}^{n\times n}$ have rank k. If n is even, let also $-J^T = J \in \mathbb{F}^{n\times n}$ be invertible.*

(1) *If $\mathbb{F} = \mathbb{C}$, $\star = *$ or $\mathbb{F} = \mathbb{R}$, $\star = T$, and if $A^\star H = HA$, then there exists a matrix $U \in \mathbb{F}^{n\times k}$ of rank k and a signature matrix $\Sigma = \operatorname{diag}(s_1, \ldots, s_k) \in \mathbb{R}^{k\times k}$, where $s_j \in \{+1, -1\}$, $j = 1, \ldots, n$ such that $A = U\Sigma U^\star H$.*

(2) *If $\mathbb{F} = \mathbb{C}$, $\star = T$, and A is H-symmetric, then there exists a matrix $U \in \mathbb{C}^{n\times k}$ of rank k such that $A = UU^T H$.*

(3) *If $\mathbb{F} = \mathbb{R}$ and A is J-Hamiltonian, then there exists a matrix $U \in \mathbb{R}^{n\times k}$ of rank k and a signature matrix $\Sigma = \operatorname{diag}(s_1, \ldots, s_k) \in \mathbb{R}^{k\times k}$, where $s_j \in \{+1, -1\}$, $j = 1, \ldots, n$, such that $A = U\Sigma U^T J$.*

(4) *If $\mathbb{F} = \mathbb{C}$ and A is J-Hamiltonian, then there exists a matrix $U \in \mathbb{C}^{n\times k}$ of rank k such that $A = UU^T J$.*

Proof. If $\star = *$ and A is H-selfadjoint, then AH^{-1} is Hermitian. By Sylvesters Law of Inertia, there is a nonsingular matrix $\widetilde{U} \in \mathbb{C}^{n\times n}$ and a matrix $\widetilde{\Sigma} = \mathrm{diag}(s_1,\dots,s_n) \in \mathbb{C}^{n\times n}$ such that $AH^{-1} = \widetilde{U}\widetilde{\Sigma}\widetilde{U}^*$, where $s_1,\dots,s_k \in \{+1,-1\}$ and $s_{k+1} = \cdots = s_n = 0$ as A has rank k. Letting $U \in \mathbb{C}^{n\times k}$ contain the first k columns of $\widetilde{U}$ and $\Sigma = \mathrm{diag}(s_1,\dots,s_k) \in \mathbb{C}^{k\times k}$, we obtain that $A = U\Sigma U^* H$ which proves 1. The other parts of the lemma are proved analogously using adequate factorizations like a nonunitary version of the Takagi factorization. □

Lemma 2.4. *Let $A, G, R \in \mathbb{C}^{n\times n}$, let G, R be invertible, and let A have the pairwise distinct eigenvalues $\lambda_1,\dots,\lambda_m \in \mathbb{C}$ with algebraic multiplicities $a_1,\dots,a_m$. Suppose that the matrix $A + URU^\star G$ generically (with respect to the entries of $U \in \mathbb{C}^{n\times k}$ if $\star = T$ and with respect to the real and imaginary parts of the entries of $U \in \mathbb{C}^{n\times k}$ if $\star = *$) has the eigenvalues $\lambda_1,\dots,\lambda_m$ with algebraic multiplicities $\widetilde{a}_1,\dots,\widetilde{a}_m$, where $\widetilde{a}_j \le a_j$ for $j = 1,\dots,m$.*

Furthermore, let $\varepsilon > 0$ be such that the discs

$$D_j := \left\{\mu \in \mathbb{C} \,\middle|\, |\lambda_j - \mu| < \varepsilon^{2/n}\right\}, \quad j = 1,\dots,m$$

are pairwise disjoint. If for each $j = 1,\dots,m$ there exists a matrix $U_j \in \mathbb{C}^{n\times k}$ with $\|U_j\| < \varepsilon$ such that the matrix $A + U_jRU_j^\star G$ has exactly $(a_j - \widetilde{a}_j)$ simple eigenvalues in D_j different from λ_j, then generically (with respect to the entries of U if $\star = T$ and with respect to the real and imaginary parts of the entries of U if $\star = *$) the eigenvalues of $A + URU^\star G$ that are different from the eigenvalues of A are simple.

Lemma 2.4 was proved in [18, Lemma 8.1] for the case $k = 1$, $\star = T$, and $R = I_k$, but the proof remains valid (with obvious adaptions) for the more general statement in Lemma 2.4.

Definition 2.5. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two finite nonincreasing sequences of positive integers given by $n_1 \ge \cdots \ge n_m$ and $\eta_1 \ge \cdots \ge \eta_\ell$, respectively. We say that $\mathcal{L}_2$* dominates *$\mathcal{L}_1$ if $\ell \ge m$ and $\eta_j \ge n_j$ for $j = 1,\dots,m$.*

Theorem 2.6. *Let $A, G, R \in \mathbb{C}^{n\times n}$, let G, R be invertible, and let $k \in \mathbb{N}\setminus\{0\}$. Furthermore, let $\lambda \in \mathbb{C}$ be an eigenvalue of A with geometric multiplicity $m > k$ and suppose that $n_1 \ge n_2 \ge \cdots \ge n_m$ are the sizes of the Jordan blocks associated with λ in the Jordan canonical form of A, i.e., the Jordan canonical form of A takes the form*

$$\mathcal{J}_{n_1}(\lambda) \oplus \mathcal{J}_{n_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widetilde{\mathcal{J}},$$

where $\lambda \notin \sigma(\widetilde{\mathcal{J}})$. Then, the following statements hold:

(1) *If $U_0 \in \mathbb{C}^{n\times k}$, then the Jordan canonical form of $A + U_0RU_0^\star G$ is given by*

$$\mathcal{J}_{\eta_1}(\lambda) \oplus \mathcal{J}_{\eta_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{\eta_\ell}(\lambda) \oplus \widehat{\mathcal{J}}; \qquad \eta_1 \ge \cdots \ge \eta_\ell,$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$ and where $(\eta_1,\dots,\eta_\ell)$ dominates $(n_{k+1},\dots,n_m)$, that is, we have $\ell \ge m - k$, and $\eta_j \ge n_{j+k}$ for $j = 1,\dots,m-k$.

(2) *Assume that for all $U \in \mathbb{C}^{n\times k}$ the algebraic multiplicity a_U of λ as an eigenvalue of $A + URU^\star G$ satisfies $a_U \geq a_0$ for some $a_0 \in \mathbb{N}$. If there exists one matrix $U_0 \in \mathbb{C}^{n\times k}$ such that $a_{U_0} = a_0$, then the set*

$$\Omega := \{U \in \mathbb{C}^{n\times k} \,|\, U \text{ has full column rank and } a_U = a_0\}$$

is generic (with respect to the entries of U if $\star = T$ and with respect to the real and imaginary parts of the entries of U if $\star = *$).

(3) *Assume that there exists a matrix $U_0 \in \mathbb{C}^{n\times k}$ such that the Jordan canonical form of $A + U_0RU_0^\star G$ is described as in the statements* (a) *and* (b) *below:*

(a) *The Jordan structure at λ is given by*

$$\mathcal{J}_{n_{k+1}}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.

(b) *All eigenvalues of $A + U_0RU_0^\star G$ that are not eigenvalues of A are simple.*

Then, there exists a generic set $\Omega \subseteq \mathbb{C}^{n\times k}$ (with respect to the entries of $U \in \mathbb{C}^{n\times k}$ if $\star = T$ and with respect to the real and imaginary parts of the entries of $U \in \mathbb{C}^{n\times k}$ if $\star = *$) such that the Jordan canonical form of $A + URU^\star G$ is as described in (a) *and* (b) *for all $U \in \Omega$.*

Proof. (1) is a particular case of [5, Lemma 2.1].

(2) In the rest of this proof, the term *generic* is meant it the sense 'generic with respect to the real and imaginary parts of the entries of $U \in \mathbb{C}^{n\times k}$ if $\star = *$'. Using Lemma 2.2 for $Y = (A + URU^\star G - \lambda I_n)^n$ shows that the set

$$\Omega' := \{U \in \mathbb{C}^{n\times k} \,|\, U \text{ has full column rank and } a_U \leq a_0\}$$

is generic. But $\Omega = \Omega'$ as by hypothesis $a_U \geq a_0$ for all $U \in \mathbb{C}^{n\times k}$.

(3) Combining (1) and (2) shows that the set Ω_1 of all $U \in \mathbb{C}^{n\times k}$ satisfying condition (a) is generic. Moreover, by Lemma 2.4 the set Ω_2 of all $U \in \mathbb{C}^{n\times k}$ satisfying condition (b) is also generic. Thus, $\Omega = \Omega_1 \cap \Omega_2$ is the desired set. □

We end this section by collecting important facts about the canonical forms of matrices that are structured with respect to some indefinite inner products. These forms are available in many sources, see, e.g., [8, 11, 14] or [12, 13, 26] in terms of pairs of Hermitian or symmetric and/or skew-symmetric matrices. We do not need the explicit structures of the canonical forms for the purpose of this paper, but only information on paring of certain Jordan blocks and on the *sign characteristic*. The sign characteristic is an important invariant of matrices that are structured with respect to indefinite inner products, we refer the reader to [7, 8] for details. To give a brief impression, consider the following example.

Example 2.7. Let $\lambda \in \mathbb{R}$ and consider the matrices

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \mathcal{J}_2(\lambda) := \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix}.$$

Then A_1 and A_2 are both H-selfadjoint and they are similar. However, they are not equivalent as H-selfadjoint matrices in the sense that there does not exists a nonsingular matrix $S \in \mathbb{C}^{2\times 2}$ such that $S^{-1}A_1S = A_2$ and $S^*HS = H$. (Note that this transformations corresponds to a change of basis in $\mathbb{C}^2$ with transformation matrix S). Indeed, any transformation matrix S that changes A_1 into A_2 would transform H into $-H$. In fact, $(\mathcal{J}_2(\lambda), H)$ and $(\mathcal{J}_2(\lambda), -H)$ are the canonical forms of the pairs (A_1, H) and (A_2, H), respectively, and they differ by a sign $\sigma \in \{+1, -1\}$ as a scalar factor of the matrix inducing the indefinite inner product. This sign is an additional invariant that can be thought of as being attached to the partial multiplicity 2 of the eigenvalue λ of A_1 (or A_2).

In general, if $H \in \mathbb{C}^{n\times n}$ is invertible and $\lambda \in \mathbb{R}$ is an eigenvalue of the H-selfadjoint matrix $A \in \mathbb{C}^{n\times n}$, then in the canonical form of (A, H) there is a sign for any partial multiplicity n_i of λ as an eigenvalue of A. The collection of all these signs then forms the sign characteristic of the eigenvalue λ. As in the example, we interpret the sign to be attached to the particular partial multiplicity. The following theorem states which eigenvalues of matrices that are structured with respect to indefinite inner products have a sign characteristic and it also lists possible restrictions in the Jordan structure of particular eigenvalues if there are any.

Theorem 2.8 (Restriction of Jordan structures). *Let $H^\star = H \in \mathbb{F}^{n\times n}$ be invertible and let $A \in \mathbb{F}^{n\times n}$. If n is even, let also $-J^T = J \in \mathbb{F}^{n\times n}$ be invertible. Furthermore, let $\lambda \in \mathbb{C}$ be an eigenvalue of A.*

(1) *Let either $\mathbb{F} = \mathbb{C}$ and $\star = *$ or $\mathbb{F} = \mathbb{R}$ and $\star = T$, and let $A^\star H = HA$. If λ is real, then each partial multiplicity of λ has a sign in the sign characteristic of λ.*
(2) *Let $\mathbb{F} = \mathbb{C}$ and $\star = T$, and let A be H-symmetric. Then λ does not have a sign characteristic.*
(3) *Let $\mathbb{F} = \mathbb{C}$ and $\star = T$, and let A be J-Hamiltonian. Then λ does not have a sign characteristic. If $\lambda = 0$, then the partial multiplicities of λ as an eigenvalue of A of each fixed odd size n_0 occur an even number of times.*
(4) *Let $\mathbb{F} = \mathbb{R}$ and $\star = T$, and let A be J-Hamiltonian. If $\lambda \neq 0$ is purely imaginary, then each partial multiplicity of λ has a sign in the sign characteristic of λ. If $\lambda = 0$, then the partial multiplicities of λ as an eigenvalue of A of each fixed odd size n_0 occur an even number of times. Furthermore, each even partial multiplicity of the eigenvalue $\lambda = 0$ has a sign in the sign characteristic of λ.*

3. Jordan structure under rank-k perturbations

In this section, we aim to investigate the effect of structure-preserving rank-k perturbations on the Jordan structure of H-selfadjoint, H-symmetric, and J-Hamiltonian matrices.

In the first theorem, we consider both H-symmetric and H-selfadjoint matrices simultaneously. By Lemma 2.3, the form of a structure-preserving rank k perturbation of A depends on both the underlying field and on $\star$ being equal to $*$ or T.

Theorem 3.1. *Let $H \in \mathbb{F}^{n,n}$ be invertible with $H^\star = H$ and let $A \in \mathbb{F}^{n,n}$ be H-symmetric in case $\star = T$ and H-selfadjoint in case $\star = *$. Furthermore let $\Sigma = \operatorname{diag}(s_1, \ldots, s_k)$ with $s_j \in \{-1, +1\}$ for $j = 1, \ldots, k$ if A is either H-selfadjoint or real H-symmetric, and with $s_j = 1$ for $j = 1, \ldots, k$ if A is complex H-symmetric. Then, there exists a generic set $\Omega_k \subseteq \mathbb{F}^{n\times k}$ (with respect to the entries of $U \in \mathbb{F}^{n\times k}$ if $\star = T$ and with respect to the real and imaginary parts of the entries of $U \in \mathbb{C}^{n\times k}$ if $\star = *$) such that for all $U \in \Omega_k$ and $B := U\Sigma U^\star H$ the following statements hold:*

(1) *Let $\lambda \in \mathbb{C}$ be any eigenvalue of A and let m denote its geometric multiplicity. If $k \geq m$, then λ is not an eigenvalue of $A + B$. Otherwise, suppose that $n_1 \geq n_2 \geq \cdots \geq n_m$ are the sizes of the Jordan blocks associated with λ in the Jordan canonical form of A, i.e., the Jordan canonical form of A takes the form*

$$\mathcal{J}_{n_1}(\lambda) \oplus \mathcal{J}_{n_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widetilde{\mathcal{J}},$$

where $\lambda \notin \sigma(\widetilde{\mathcal{J}})$. Then, the Jordan canonical form of $A + B$ is given by

$$\mathcal{J}_{n_{k+1}}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.

(2) *If $\mu \in \mathbb{C}$ is an eigenvalue of $A + B$, but not of A, then μ is a simple eigenvalue of $A + B$.*

Proof. We prove this theorem in the complex case only, since the real case is then obtained by the fact that for a generic set $\Omega_k \subseteq \mathbb{C}^{n,k}$, the set $\Omega_k \cap \mathbb{R}^{n,k}$ is generic as well, see [17, Lemma 2.2]. We show that there exist two generic subsets $\Omega_{k,1}$ and $\Omega_{k,2}$ of $\mathbb{C}^{n,k}$ so that property (1) is satisfied on $\Omega_{k,1}$ and property (2) on $\Omega_{k,2}$. Then, $\Omega_k := \Omega_{k,1} \cap \Omega_{k,2}$ is the desired generic set.

Concerning (1): By part (3) of Theorem 2.6 it is sufficient to construct one particular H-symmetric or H-selfadjoint rank-k perturbation, respectively, such that the Jordan structure is as claimed. We do this by constructing a sequence of k rank-one perturbations with the desired properties.

Now, by [15, Theorem 5.1] if $\star = T$ and by [16, Theorem 3.3] if $\star = *$, for a generic rank-1 perturbation of the form $s_1 uu^\star H$ the perturbed matrix $A + s_1 uu^\star H$ will have the partial multiplicities $n_2, \ldots, n_m$ at each eigenvalue λ. We consider now a fixed u_1 so that $A_1 := A + s_1 u_1 u_1^\star H$ has this property. Then [15, Theorem 5.1] or [16, Theorem 3.3], respectively, can be applied anew to the matrix A_1 showing that there exists a vector u_2 such that $A_2 = A + s_1 u_1 u_1^\star H + s_2 u_2 u_2^\star H$ has the partial multiplicities $n_3, \ldots, n_m$ at each eigenvalue λ. Repeating this step $k-2$ more times results in an H-symmetric or H-selfadjoint matrix $A_k = A + s_1 u_1 u_1^\star H + \cdots + s_k u_k u_k^\star H$ that has the partial multiplicities $n_{k+1}, \ldots, n_m$ at each eigenvalue λ.

Concerning (2): We assert that the particular rank-k perturbation of the form $A + u_1u_1^\star H + \cdots + u_ku_k^\star H$ constructed above has the property that all eigenvalues different from those of A are simple. In fact, since in each step $j = 2, \ldots, k$ we generate $A_j := A_{j-1} + s_ju_ju_j^\star H$, only the eigenvalues of A_j that have been eigenvalues of A_{j-1} can be multiple, so that these have been also eigenvalues of A. Thus, the existence of the desired generic set $\Omega_{k,2}$ follows from Lemma 2.4. □

Now, we turn to J-Hamiltonian matrices. As we saw in Theorem 2.8, the Jordan blocks of Hamiltonian matrices at 0 have to be paired in a certain way. This restriction produced surprising results in the case of Hamiltonian rank-one perturbations of Hamiltonian matrices, see [15, Theorem 4.2]. We will in the following see that also in the case of rank-k perturbations, taking care of this pairing of certain blocks will be the most challenging task.

Theorem 3.2. *Let $J \in \mathbb{F}^{n,n}$ be skew-symmetric and invertible, let $A \in \mathbb{F}^{n,n}$ be J-Hamiltonian. Furthermore, let $\Sigma = \operatorname{diag}(s_1, \ldots, s_k)$ with $s_j \in \{-1, +1\}$ for $j = 1, \ldots, k$ if $\mathbb{F} = \mathbb{R}$ and with $s_j = 1$ for $j = 1, \ldots, k$ if $\mathbb{F} = \mathbb{C}$. Then, there exists a generic set $\Omega_k \subseteq \mathbb{F}^{n\times k}$ such that for all $U \in \Omega_k$ and $B := U\Sigma U^TJ$ the following statements hold:*

(1) *Let $\lambda \in \mathbb{C}$ be any eigenvalue of A and let m denote its geometric multiplicity. If $k \geq m$, then λ is not an eigenvalue of $A + B$. Otherwise, suppose that $n_1 \geq n_2 \geq \cdots \geq n_m$ are the sizes of the Jordan blocks associated with λ in the Jordan canonical form of A, i.e., the Jordan canonical form of A takes the form*

$$\mathcal{J}_{n_1}(\lambda) \oplus \mathcal{J}_{n_2}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widetilde{\mathcal{J}},$$

where $\lambda \notin \sigma(\widetilde{\mathcal{J}})$. Then:

(1a) *If either $\lambda \neq 0$ or $\lambda = 0$ and $n_1 + \cdots + n_k$ is even, then the Jordan canonical form of $A + B$ is given by*

$$\mathcal{J}_{n_{k+1}}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.

(1b) *If $\lambda = 0$ and $n_1 + \cdots + n_k$ is odd, then the Jordan canonical form of $A + B$ is given by*

$$\mathcal{J}_{n_{k+1}+1}(\lambda) \oplus \mathcal{J}_{n_{k+2}}(\lambda) \oplus \cdots \oplus \mathcal{J}_{n_m}(\lambda) \oplus \widehat{\mathcal{J}},$$

where $\lambda \notin \sigma(\widehat{\mathcal{J}})$.

(2) *If $\mu \in \mathbb{C}$ is an eigenvalue of $A + B$, but not of A, then μ is a simple eigenvalue of $A + B$.*

Proof. We show that there exist two generic sets $\Omega_{k,1}$ and $\Omega_{k,2}$ so that property (1) is satisfied on $\Omega_{k,1}$ and property (2) on $\Omega_{k,2}$. Then, $\Omega_k := \Omega_{k,1} \cap \Omega_{k,2}$ is the desired generic set.

Proof of (1): We first mention that in the case $\lambda = 0$, all odd-sized multiplicities have to occur an even number of times by Theorem 2.8. This implies in paticular that $n_1 + \cdots + n_m$ is even. Therefore, if the number $n_1 + \cdots + n_k$

is even, then odd entries in both subsequences $n_1, \dots, n_k$ and $n_{k+1}, \dots, n_m$ occur an even number of times so that, in particular, there is no fundamental obstruction to the sequence $n_{k+1}, \dots, n_m$ of partial multiplicities occurring in some Hamiltonian matrix at 0.

On the other hand, if $n_1 + \cdots + n_k$ is odd, then there must occur an odd number of blocks of size $n_k = n_{k+1}$ in both subsequences $n_1, \dots, n_k$ and $n_{k+1}, \dots, n_m$. In particular, it is thus not possible for the partial multiplicities $n_{k+1}, \dots, n_m$ to be realized in some Hamiltonian matrix at 0.

Case (1a): By part (3) of Theorem 2.6 it is sufficient to construct a sequence of k Hamiltonian rank-one perturbations such that the Jordan structure is as claimed.

If $\lambda \neq 0$, then by [15, Theorem 4.2] for a generic rank-1 perturbation of the form $s_1 uu^T J$, the perturbed matrix $A + s_1 uu^T J$ will have the partial multiplicities $n_2, \dots, n_m$ at λ (if $s_1 = -1$, this also holds by passing from u to iu). We consider now a fixed u_1 so that $A_1 := A + s_1 u_1 u_1^T J$ has this property. Then [15, Theorem 4.2] can be applied anew to the matrix A_1. Repeating this step $k-1$ more times results in a Hamiltonian matrix $A_k = A + s_1 u_1 u_1^T J + \cdots + s_k u_k u_k^T J$ that has the partial multiplicities $n_{k+1}, \dots, n_m$ at λ.

Next, let us consider the case that $\lambda = 0$ but $n_1 + \cdots + n_k$ is even. We aim to proceed as for $\lambda \neq 0$ applying [15, Theorem 4.2]. In this case, a generic rank-1 perturbation of the form $s_1 uu^T J$ will in the perturbed matrix $A + s_1 uu^T J$ at 0 create the partial multiplicities $n_2, \dots, n_m$ if n_1 is even and $n_2 + 1, n_3, \dots, n_m$ if n_1 is odd. We now fix u_1 so that

$$A_1 := A + s_1 u_1 u_1^T J$$

has this property. Again, by [15, Theorem 4.2] for generic v the matrix $A_1 + s_2 vv^T J$ will at 0 have the partial multiplicities $n_3, \dots, n_m$ if $n_1 + n_2$ is even (this includes the case that $n_1 = n_2$ are odd as in this case the block of size $n_2 + 1$ will simply disappear) and $n_3 + 1, n_4, \dots, n_m$ if $n_1 + n_2$ is odd. We fix u_2 with this property setting

$$A_2 := A + s_1 u_1 u_1^T J + s_2 u_2 u_2^T J.$$

After $k-2$ more steps of this procedure, we obtain a Hamiltonian matrix $A_k = A + s_1 u_1 u_1^T J + \cdots + s_k u_k u_k^T J$ with the partial multiplicities $n_{k+1}, \dots, n_m$ at 0 as $n_1 + \cdots + n_k$ is even.

Case (1b): Let us assume that $\lambda = 0$ and $n_1 + \cdots + n_k$ is odd, which immediately implies $k + 1 \leq m$. As mentioned above, the partial multiplicity sequence $n_{k+1}, \dots, n_m$ contains the odd entry n_{k+1} an odd number of times, and thus cannot be realized in a Hamiltonian matrix at 0. Hence, the minimum algebraic multiplicity of $A + B$ at zero is $n_{k+1} + \cdots + n_m + 1$. By Theorem 2.6(2), this is the generic algebraic multiplicity of $A + B$ at 0 if we can find a particular perturbation that creates this algebraic multiplicity. However, such a perturbation is easily constructed as in Case (1a), $\lambda = 0$, using [15, Theorem 4.2].

In order to determine the precise partial multiplicities of $A+B$ in this case, we employ an argument that was initally used to prove [2, Theorem 3.4]: In the following we assume that $B = U\Sigma U^T J$, where U is an element of a generic set $\widetilde{\Omega}_k \subseteq \mathbb{C}^{n\times k}$ such that (1a) holds for all nonzero eigenvalues of A and the algebraic multiplicity of $A+B$ at zero is $n_{k+1}+\cdots+n_m+1$. It remains to determine the generic partial multiplicities of $A+B$ at 0. Let us group together Jordan blocks of the same size, i.e., let

$$(n_1, n_2, n_3, \ldots, n_m) = (\underbrace{s_1, \ldots, s_1}_{t_1 \text{ times}}, \underbrace{s_2, \ldots, s_2}_{t_2 \text{ times}}, \ldots, \underbrace{s_\nu, \ldots, s_\nu}_{t_\nu \text{ times}}),$$

and let ℓ be such that $s_\ell = n_k = n_{k+1}$. Then ℓ is odd, t_ℓ is even, and

$$(n_{k+1}, \ldots, n_m) = (\underbrace{s_\ell, \ldots, s_\ell}_{d \text{ times}}, \underbrace{s_{\ell+1}, \ldots, s_{\ell+1}}_{t_{\ell+1} \text{ times}}, \ldots, \underbrace{s_\nu \ldots, s_\nu}_{t_\nu \text{ times}}),$$

where d is odd. Now, $A+B$ has the algebraic multiplicity $n_{k+1}+\cdots+n_m+1$ at zero and by Theorem 2.6, the list of descending partial multiplicities of $A+B$ at zero dominates $(n_{k+1}, \ldots, n_m)$. Therefore, either one of the blocks corresponding to the partial multiplicities $n_k, \ldots, n_m$ has grown in size by exactly one, or a new block of size one has been created. Moreover, the Hamiltonian matrix $A+B$ must have an even number of Jordan blocks of size s_ℓ at 0. If $\nu > \ell$ and $s_{\ell+1} < s_\ell - 1$ then these restrictions can only be realized by the list of partial multiplicities given by

$$(s_\ell + 1, \underbrace{s_\ell, \ldots, s_\ell}_{(d-1) \text{ times}}, \underbrace{s_{\ell+1}, \ldots, s_{\ell+1}}_{t_{\ell+1} \text{ times}}, \ldots, \underbrace{s_\nu \ldots, s_\nu}_{t_\nu \text{ times}}). \tag{3.2}$$

Only when $\nu > \ell$ and $s_{\ell+1} = s_\ell - 1$, or when $\nu = \ell$ and $s_\ell = 1$ then also a list different from (3.2) can be realized, namely

$$(\underbrace{s_\ell, \ldots, s_\ell}_{(d+1) \text{ times}}, \underbrace{s_{\ell+1}, \ldots, s_{\ell+1}}_{(t_{\ell+1}-1) \text{ times}}, \ldots, \underbrace{s_\nu \ldots, s_\nu}_{t_\nu \text{ times}}). \tag{3.3}$$

Hereby, in the latter case of $\nu = \ell$ and $s_\ell = 1$, the above list is given by $(s_\ell, \ldots, s_\ell)$ (repeated $(d+1)$ times), and this interpretation shall be applied to the following lists as well. Then, aiming to prove that the partial multiplicities in (3.2) are generically realized in $A+B$ at 0, let us assume the opposite: assume for some Hamiltonian matrix A that $A+B$ has the partial multiplicities from (3.3) at 0 for all $U \in \mathcal{B}$, where $\mathcal{B}$ is not contained in any proper algebraic subset of $\mathbb{C}^{n,k}$. Then, we apply a further Hamiltonian rank-1 perturbation $suu^T J$ to $A+B$ (again, $s \in \{-1, +1\}$). By Theorem 2.6(1), for all $[U, u] \in \mathcal{B} \times \mathbb{C}^n$, the sequence of partial multiplicities at 0 of the Hamiltonian matrix $A+B+suu^T J$ dominates

$$(\underbrace{s_\ell, \ldots, s_\ell}_{d \text{ times}}, \underbrace{s_{\ell+1}, \ldots, s_{\ell+1}}_{(t_{\ell+1}-1) \text{ times}}, \ldots, \underbrace{s_\nu \ldots, s_\nu}_{t_\nu \text{ times}}). \tag{3.4}$$

On the other hand, applying the already proved part (1a) to the case $k+1$, we find that there exists a generic set $\Gamma \subseteq \mathbb{C}^{n\times(k+1)}$ such that the partial multiplicities of $A + [U, u](\Sigma \oplus [s])[U, u]^T J$ at 0 are given by

$$(\underbrace{s_\ell, \dots, s_\ell}_{(d-1)\ \text{times}}, \underbrace{s_{\ell+1}, \dots, s_{\ell+1}}_{t_{\ell+1}\ \text{times}}, \dots, \underbrace{s_\nu \dots, s_\nu}_{t_\nu\ \text{times}}),$$

for all $[U, u] \in \Gamma$. Observe that the latter sequence does not dominate the one in (3.4). Thus, a contradiction is obtained as by Lemma 2.1 the set $\mathcal{B} \times \mathbb{C}^n$ is not contained in any proper algebraic subset of $\mathbb{C}^{n,k+1}$ and thus, clearly, $(\mathcal{B} \times \mathbb{C}^n) \cap \Gamma$ is not empty.

Proof of (2): Analogous to (2) of Theorem 3.1. □

4. Sign characteristic under rank-k perturbations

Since the behavior of the Jordan structure of matrices under rank-k perturbations was already established in the previous section, we now turn to the question of the change of the sign characteristic of H-selfadjoint, real H-symmetric, and real J-Hamiltonian matrices. We recall that by Theorem 2.8 each partial multiplicity n_{ij} of a real eigenvalue λ_i of a matrix A that is H-selfadjoint or real H-symmetric has a sign $\sigma_{ij} \in \{+1, -1\}$ in the *sign characteristic* of λ_i.

We go on to prove a theorem without an explicit genericity hypothesis that will hence be applicable to both H-selfadjoint and real H-symmetric matrices.

Theorem 4.1. *Let $H \in \mathbb{C}^{n\times n}$ be invertible and Hermitian let $A \in \mathbb{C}^{n\times n}$ be H-selfadjoint. Let $\Sigma = \operatorname{diag}(s_1, \dots, s_k)$ with $s_j \in \{-1, +1\}$ and let $\lambda_1, \dots, \lambda_p$ be the pairwise distinct real eigenvalues of A and $\lambda_{p+1}, \dots, \lambda_q$ be the pairwise distinct nonreal eigenvalues of A. Further, (in difference to before) let $n_{1,j} > \cdots > n_{m_j,j}$ be the distinct block sizes of A at some eigenvalue λ_j such that there exist $\ell_{i,j}$ blocks of size $n_{i,j}$ at λ_j and, whenever $j \in \{1, \dots, p\}$, let A have the signs $\{\sigma_{1,i,j}, \dots, \sigma_{\ell_{i,j},i,j}\}$ attached to its blocks of size $n_{i,j}$ at λ_j.*

Then, whenever $U \in \mathbb{C}^{n,k}$ is such that for $B := U\Sigma U^* H$ the statement (1) below is satisfied, also (2) holds.

(1) *The perturbed matrix $A+B$ has the Jordan structure as described in (1) of Theorem 3.1. More precisely, for each $j = 1, \dots, q$, the matrix $A+B$ has the distinct block sizes $n_{\kappa_j,j} > n_{\kappa_j+1,j} > \cdots > n_{m_j,j}$ occuring $\ell'_{\kappa_j,j}, \ell_{\kappa_j+1,j}, \dots, \ell_{m_j,j}$ times, respectively, at λ_j, where $\ell'_{\kappa_j,j} = \ell_{1,j} + \cdots + \ell_{\kappa_j,j} - k$ and κ_j is the smallest integer with $\ell'_{\kappa_j,j} \geq 1$.*
(2) *For each $j = 1, \dots, p$, let $\{\sigma'_{1,\kappa_j,j}, \dots, \sigma'_{\ell'_{\kappa_j,j},\kappa_j,j}\}$ be the signs of $A+B$ at blocks of size $n_{\kappa_j,j}$ at λ_j and let $\{\sigma'_{1,i,j}, \dots, \sigma'_{\ell_{i,j},i,j}\}$ be the signs at blocks of size $n_{i,j}$ at λ_j for $i = \kappa_j + 1, \dots, m_j$. Then,*

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j}, \qquad i = \kappa_j + 1, \dots, m_j, \quad j = 1, \dots, p \tag{4.5}$$

and

$$\left|\sum_{s=1}^{\ell_{\kappa_j,j}} \sigma_{s,\kappa_j,j} - \sum_{s=1}^{\ell'_{\kappa_j,j}} \sigma'_{s,\kappa_j,j}\right| \leq \ell_{\kappa_j,j} - \ell'_{\kappa_j,j}, \quad j = 1,\ldots,p. \tag{4.6}$$

Proof. In the first step of the proof, we show that there exists some $\Omega'_k \subseteq \mathbb{C}^{n,k}$, that is generic with respect to the real and imaginary parts of its entries, so that for all $U \in \Omega'_k$, the statements from (1) and (2) above hold.

Letting $\Omega_1,\ldots,\Omega_k$ be the sets constructed in Theorem 3.1, that are generic with respect to the real and imaginary parts of their entries, we define

$$\Omega'_k := (\Omega_1 \times \mathbb{C}^{n,k-1}) \cap (\Omega_2 \times \mathbb{C}^{n,k-2}) \cap \cdots \cap \Omega_k,$$

which is (as the intersection of finitely many generic sets) clearly a generic subset of $\mathbb{C}^{n,k}$ (with respect to the real and imaginary parts of its entries). Now, let $U := [u_1,\ldots,u_k] \in \Omega'_k$, then clearly the Jordan structure of $A_1 := A+s_1u_1u_1^*H$ is as described in (1) and (2) of Theorem 3.1 for $k=1$. Therefore, by [19, Theorem 4.6] for all $j=1,\ldots,p$ all signs of A attached to blocks at λ_j of size $n_{2,j},\ldots,n_{m_j,j}$ are preserved, i.e., they are the same in A and A_1. Further, of the $\ell_{1,j}$ signs attached to blocks of size $n_{1,j}$ in A at λ_j, exactly $\ell_{1,j}-1$ are attached to blocks of size $n_{1,j}$ in A_1, i.e., if η is the sum of the $\ell_{1,j}$ signs attached to blocks of size $n_{1,j}$ in A at λ_j and if $\widetilde{\eta}$ is the sum of the $\ell_{1,j}-1$ signs attached to blocks of size $n_{1,j}$ in A_1, then $|\eta-\widetilde{\eta}|=1$. (If there are both signs $+1$ and -1 among the list of $\ell_{1,j}$ signs attached to the blocks of size $n_{1,j}$, then it depends on the particular perturbations whether the sign that has been dropped to obtain the list of $\ell_{1,j}-1$ signs is positive or negative).

Now, we consider the perturbed matrix $A_2 := A_1 + s_2u_2u_2^*H$. Since $[u_1,u_2] \in \Omega_2$, clearly A_2 has the Jordan structure as described in (1) and (2) of Theorem 3.1 for $k=1$, whereby we consider A_1 instead of A as the unperturbed matrix in that theorem. Hence, again applying [19, Theorem 4.6] for all $j=1,\ldots,p$, all signs of A_1 attached to blocks of size $n_{3,j},\ldots,n_{m_j,j}$ are preserved, i.e., they are the same in A_2 and A_1. Further, if $\ell_{1,j}\geq 2$, then also all signs of A_1 at blocks of size $n_{2,j}$ at λ_j are preserved and of the $\ell_{1,j}-1$ signs of A_1 at blocks of size $n_{1,j}$, exactly $\ell_{1,j}-2$ are preserved, i.e., attached to blocks of size $n_{1,j}$ in A_2 (the remaining sign does not occur any more since the corresponding block was destroyed under perturbation). In the remaining case $\ell_{1,j}=1$, the matrix A_1 does not have a Jordan block of size $n_{1,j}$ at λ_j, thus of its $\ell_{2,j}$ signs attached to blocks of size $n_{2,j}$, exactly $\ell_{2,j}-1$ are attached to blocks of size $n_{2,j}$ in A_2.

Now, repeating this argument $k-2$ more times, we arrive at $A_k = A+B$ letting the largest Jordan block of A_k at λ_j have size $n_{\kappa_j,j}$ with exactly $\ell'_{\kappa_j,j} = \ell_{1,j}+\cdots+\ell_{\kappa_j,j}-k$ copies. Then, the signs at blocks of size $n_{\kappa_j+1,j},\ldots,n_{m_j,j}$ are preserved, i.e., they are the same in A and in A_k (4.5), and of the signs attached to blocks of size $n_{\kappa_j,j}$ in A, exactly $\ell_{\kappa'_j,j}$ are attached to blocks of size $n_{\kappa_j,j}$ in A_k which is equivalent to (4.6).

At last, we turn to the second step of the proof by following the lines of the proof of [19, Theorem 4.6]. Thus, let us assume for some $U \in \mathbb{C}^{n,k}$ that the property (1) from above holds but $U \notin \Omega_k'$. Then, by [22, Theorem 3.4], there exists $\delta > 0$ such that for every $U_0 \in \mathbb{C}^{n,k}$ with $\|U - U_0\| < \delta$ and with $(A + U_0\Sigma U_0 H, H)$ satisfying property (1) (where B is replaced by $U_0\Sigma U_0 H$), the sign characteristic of $(A+U_0\Sigma U_0 H, H)$ coincides with that of $(A + U\Sigma UH, H)$. It remains to choose $U_0 \in \Omega_k'$, which is possible in view of the genericity of Ω_k'. □

Now, if A is H-selfadjoint, it is immediately clear that for the generic (with respect to the real and imaginary parts of the entries) set $\Omega_k \subseteq \mathbb{C}^{n,k}$ from Theorem 3.1, both (1) and (2) from the above Theorem hold. Then again, if H is real symmetric and A is real H-symmetric, the same is true for the real generic set $\Omega_k \subseteq \mathbb{R}^{n,k}$ predicted by Theorem 3.1 in the case $\mathbb{F} = \mathbb{R}$. Note that since there was no explicit genericity hypothesis in Theorem 4.1, it is applicable in both the H-selfadjoint and the H-symmetric case, despite the two different notions of genericity.

Next, let us turn to rank-k perturbations of real J-Hamiltonian matrices, whereby Theorem 4.1 will be a key ingredient. Again, the J-Hamiltonian case will prove to be more difficult since the partial multiplicities of a J-Hamiltonian matrix behave differently under structured low-rank perturbations. By Theorem 2.8, if $\lambda = 0$ is an eigenvalue of a J-Hamiltonian matrix, then only even partial multiplicities will have a sign in the sign characteristic. In order to allow a unified treatment of purely imaginary eigenvalues including the eigenvalue $\lambda = 0$, we will extend the notion of sign characteristic and define each odd partial multiplicity at the eigenvalue zero to have the "sign" zero in the sign characteristic.

Theorem 4.2. *Let $J \in \mathbb{R}^{n\times n}$ be invertible and skew-symmetric and let $A \in \mathbb{R}^{n\times n}$ be J-Hamiltonian. Let $\Sigma = \mathrm{diag}(s_1, \ldots, s_k)$ with $s_j \in \{-1, +1\}$ and let $\lambda_1, \ldots, \lambda_p$ be the purely imaginary eigenvalues of A and $\lambda_{p+1}, \ldots, \lambda_q$ be the non purely imaginary eigenvalues of A. Further, let $n_{1,j} > \cdots > n_{m_j,j}$ be the distinct block sizes of A at some eigenvalue λ_j such that there exist $\ell_{i,j}$ blocks of size $n_{i,j}$ at λ_j and, whenever $j \in \{1, \ldots, p\}$, let A have the signs $\{\sigma_{1,i,j}, \ldots, \sigma_{\ell_{i,j},i,j}\}$ attached to its blocks of size $n_{i,j}$ at λ_j.*

Then, whenever $U \in \mathbb{C}^{n,k}$ is such that for $B := U\Sigma U^*H$ the statement (1) *below is satisfied, also* (2) *holds.*

(1) *The perturbed matrix $A + B$ has the Jordan structure as described in* (1) *of Theorem* 3.2*. More precisely, for each $j = 1, \ldots, q$, letting $\ell'_{\kappa_j,j} = \ell_{1,j} + \cdots + \ell_{\kappa_j,j} - k$ and letting κ_j be the smallest integer with $\ell'_{\kappa_j,j} \geq 1$, then:*

   (a) *If either $\lambda_j \neq 0$ or $\lambda_j = 0$ and $\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ is even, $A + B$ has the distinct block sizes*

$$n_{\kappa_j,j} > n_{\kappa_j+1,j} > \cdots > n_{m_j,j} \tag{4.7}$$

   occuring $\ell'_{\kappa_j,j}, \ell_{\kappa_j+1,j}, \ldots, \ell_{m_j,j}$ times, respectively, at λ_j.

(b) *If* $\lambda_j = 0$ *and* $\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ *is odd,* $A + B$ *has the distinct block sizes*

$$n_{\kappa_j,j} + 1 > n_{\kappa_j,j} > n_{\kappa_j+1,j} > \cdots > n_{m_j,j} \tag{4.8}$$

occuring $1, (\ell'_{\kappa_j,j} - 1), \ell_{\kappa_j+1,j}, \ldots, \ell_{m_j,j}$ *times, respectively, at* 0.

(2) *For each* $j = 1, \ldots, p$, *let* $\{\sigma'_{1,\kappa_j,j}, \ldots, \sigma'_{\ell'_{\kappa_j,j},\kappa_j,j}\}$ *be the signs of* $A+B$ *at blocks of size* $n_{\kappa_j,j}$ *at* λ_j *and let* $\{\sigma'_{1,i,j}, \ldots, \sigma'_{\ell_{i,j},i,j}\}$ *be the signs at blocks of size* $n_{i,j}$ *at* λ_j *for* $i = \kappa_j + 1, \ldots, m_j$. *Then the following statements hold for* $j = 1, \ldots, p$:

(a1) *If* $\lambda_j \neq 0$, *then the signs of* $A + B$ *satisfy*

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j}, \qquad i = \kappa_j + 1, \ldots, m_j,$$

and

$$\left| \sum_{s=1}^{\ell_{\kappa_j,j}} \sigma_{s,\kappa_j,j} - \sum_{s=1}^{\ell'_{\kappa_j,j}} \sigma'_{s,\kappa_j,j} \right| \leq \ell_{\kappa_j,j} - \ell'_{\kappa_j,j}.$$

(a2) *If* $\lambda_j = 0$ *and* $\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ *is even, the signs of* $A + B$ *satisfy*

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j}$$

for $i = \kappa_j + 1, \ldots, m_j$, *where both sums are zero whenever* $n_{i,j}$ *is odd. Furthermore, if* $n_{\kappa_j,j}$ *is odd, then the above also holds for* $i = n_{\kappa_j,j}$ *(as in that case both sums are zero), and if* $n_{\kappa_j,j}$ *is even, then*

$$\left| \sum_{s=1}^{\ell_{\kappa_j,j}} \sigma_{s,\kappa_j,j} - \sum_{s=1}^{\ell'_{\kappa_j,j}} \sigma'_{s,\kappa_j,j} \right| \leq \ell_{\kappa_j,j} - \ell'_{\kappa_j,j}.$$

(b) *If* $\lambda_j = 0$ *and* $\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ *is odd, the signs of* $A + B$ *satisfy*

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j}$$

for $i = \kappa_j + 1, \ldots, m_j$, *where both sums are zero whenever* $n_{i,j}$ *is odd. (In particular,* $n_{\kappa_j,j}$ *is odd, so all corresponding signs are zero.)*

Proof. We proceed using [19, Theorem 4.1] in order to identify the signs attached to blocks in (A, J) with ones attached to blocks in (iA, iJ), where iA is an iJ-selfadjoint (complex) matrix.

We first consider the case (a1), i.e., $\lambda_j = i\alpha$ is different from zero. Now, for any $U \in \mathbb{R}^{n,k}$ such that the perturbed matrix $A + U\Sigma U^T J$ has the partial multiplicities in (4.7) at λ_j, also $iA + iU\Sigma U^T J$, which is iJ-selfadjoint, has these multiplicities at $-\alpha$. Hence, by Theorem 4.1, the signs of $iA+iU\Sigma U^T J$ at $-\alpha$ are obtained as follows: All signs at blocks of sizes $n_{\kappa_j+1,j}, \ldots, n_{m_j,j}$ are preserved, and of the signs at blocks of size $n_{\kappa_j,j}$, exactly $\ell'_{\kappa_j,j}$ ones are preserved. Now, the same procedure applies to the signs of $A+U\Sigma U^T J$ by [19, Theorem 4.1], i.e., the signs satisfy the assertion in (a1).

The next case is (a2), i.e, we have $\lambda_j = 0$ and the number

$$\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$$

is even. This number is the sum of the sizes of all blocks at λ_j that are destroyed under perturbation in this case. Since $\ell_{1,j}n_{1,j}, \ldots, \ell_{\kappa_j-1,j}n_{\kappa_j-1,j}$ are all even, this implies that either $\ell_{\kappa_j,j} - \ell'_{\kappa_j,j}$ or $n_{\kappa_j,j}$ is even (or both), i.e., an even number of odd-sized blocks is destroyed under perturbation.

Again, let $U \in \mathbb{R}^{n,k}$ be such that the perturbed matrix $A+U\Sigma U^T J$ has the partial multiplicities from (4.7) at 0. Then the same is true for the iJ-selfadjoint matrix $iA+iU\Sigma U^T J$ at 0. Hence, by Theorem 4.1, the signs of $iA+iU\Sigma U^T J$ are obtained as follows: All signs at blocks of sizes $n_{\kappa_j+1,j}, \ldots, n_{m_j,j}$ are preserved, and of the signs at blocks of size $n_{\kappa_j,j}$, exactly $\ell'_{\kappa_j,j}$ ones are preserved. By [19, Theorem 4.1] this translates to the signs of $A+U\Sigma U^T J$ at 0: All signs at blocks of even sizes smaller than $n_{\kappa_j,j}$ are preserved. Further, if $n_{\kappa_j,j}$ is even, then exactly $\ell'_{\kappa_j,j}$ signs at this block size are preserved, i.e., the signs satisfy the assertion in (a2).

Finally, let $\lambda_j = 0$ and let $\ell_{1,j}n_{1,j} + \cdots + \ell_{\kappa_j-1,j}n_{\kappa_j-1,j} + (\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ be odd. From this immediately follows that $(\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})n_{\kappa_j,j}$ must be odd, i.e., $n_{\kappa_j,j}$ and $(\ell_{\kappa_j,j} - \ell'_{\kappa_j,j})$ are both odd, and since $\ell_{\kappa_j,j}$ is even, $\ell'_{\kappa_j,j}$ is odd. In particular, as $n_{\kappa_j,j}$ is odd, there are no signs attached to blocks of this size in neither A nor $A+U\Sigma U^T J$. Also, we note that $\ell'_{\kappa_j,j} - 1$ may be 0 so that in the perturbed pencil, there do not occur blocks of this size.

Concerning the Jordan structure of the perturbed matrix, again we assume that $U \in \mathbb{R}^{n\times k}$ is such that the perturbed matrix $A + U\Sigma U^T J$ has the partial multiplicities in (4.8) at 0. Concerning the sign characteristic, we cannot apply Theorem 4.1 in this case (note that the partial multiplicities in (4.8) differ from the ones required in Theorem 4.1) so that we continue with a strategy similar to the one from the proof of Theorem 3.2:

Let $s_{k+1} \in \{-1, +1\}$ and let $u \in \Omega_1$ be a vector from the generic set Ω_1 in Theorem 3.1 applied for the case $k = 1$ to the matrix $A + B$. Then at the eigenvalue $\lambda_j = 0$, the matrix $A+B+s_{k+1}uu^T J$ has the partial multiplicities

$$n_{\kappa_j,j} > n_{\kappa_j+1,j} > \cdots > n_{m_j,j}$$

occurring $(\ell'_{\kappa_j,j} - 1), \ell_{\kappa_j+1,j}, \ldots, \ell_{m_j,j}$ times, respectively, i.e., only the newly generated block of size $n_{\kappa_j,j} + 1$ at $\lambda_j = 0$ in $A + B$ has vanished.

Let $\{\sigma''_{1,i,j}, \ldots, \sigma''_{\ell_{i,j},i,j}\}$ be the signs of $A + B + s_{k+1}uu^T J$ at blocks of size $n_{i,j}$ at $\lambda_j = 0$ for $i = \kappa_j + 1, \ldots, m_j$. (The signs on the blocks of size $n_{\kappa_j,j}$ are zero by definition as $n_{\kappa_j,j}$ is odd, so there is no need for considering these signs in the following.) Observe that $A + B + s_{k+1}uu^T J$ is a rank-one perturbation of $A + B$ that satisfies the hypotheses of (1) and (a2), so applying the already proved part (a2) for the rank-one case to the matrix $A + B$, we obtain that

$$\sum_{s=1}^{\ell_{i,j}} \sigma'_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma''_{s,i,j}, \quad i = \kappa_j + 1, \ldots, m_j. \tag{4.9}$$

On the other hand, the matrix $A+B+s_{k+1}uu^T J$ is a rank-$(k+1)$ perturbation of A that also satisfies the hypotheses of (1) and (a2), so applying the already proved part (a2) for the rank-$(k+1)$ case to the matrix A, we obtain that

$$\sum_{s=1}^{\ell_{i,j}} \sigma_{s,i,j} = \sum_{s=1}^{\ell_{i,j}} \sigma''_{s,i,j}, \quad i = \kappa_j + 1, \ldots, m_j. \tag{4.10}$$

Combining (4.9) and (4.10), we see that the assertion in (b) is satisfied. □

In particular, since the case $k = 1$ is included in the above theorem, we have hereby proved [19, Conjecture 4.8]. Then again, in the above theorem, there is no statement on the sign at the newly generated block of (even) size $n_{\kappa_j,j} + 1$ in the case (2b). Examples show that this sign can either be $+1$ or -1 depending on the particular perturbation; see also [19, Conjecture 4.4], [4].

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