On the inverse eigenvalue problem for T-alternating and T-palindromic matrix polynomials

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Abstract

The inverse eigenvalue problem for T-alternating matrix polynomials over arbitrary algebraically closed fields of characteristic different from two is considered. The main result shows that the necessary conditions obtained in [10] for a matrix polynomial to be the Smith form of a T-alternating matrix polynomial are under mild conditions also sufficient to be the Smith form of a T-alternating matrix polynomial with invertible leading coefficient which is additionally in anti-triangular form.. In particular, this result implies that any T-alternating matrix polynomial with invertible leading coefficient is equivalent to a T-alternating matrix polynomial in anti-triangular form that has the same finite and infinite elementary divisors as the original matrix polynomial. Finally, the inverse eigenvalue problem for T-palindromic matrix polynomials is considered excluding the case that both $+1$ and -1 are eigenvalues.

Key words. Matrix polynomial, matrix pencil, Smith form, alternating matrix polynomial, palindromic matrix polynomial, triangularization, anti-triangular form.

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1 Introduction

Matrix polynomials and polynomial eigenvalue problems have been studied intensively in the last few decades. Recently, two particular topics have gained considerable interest: *inverse polynomial eigenvalue problems* and *the triangularization of matrix polynomials*. The aim of this paper is to combine these two topics with special emphasis on T-alternating matrix polynomials..

A matrix polynomial of degree k, where k is a positive integer, is an expression $P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j$ with coefficient matrices $A_j \in \mathbb{F}^{n\times m}$ over an arbitrary field $\mathbb{F}$. If $n = m$ and the characteristic of $\mathbb{F}$ is different from two, such a $P(\lambda)$ is called T-*even* if $P(-\lambda) = P(\lambda)^T$,

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and it is called *T-odd* if $P(-\lambda) = -P(\lambda)^T$. Observe that these definitions are equivalent to the fact that the sequence of coefficient matrices $(A_0, A_1, \ldots, A_k)$ alternates between symmetric and skew-symmetric matrices, starting with a symmetric A_0 in the T-even case and with a skew-symmetric A_0 in the T-odd case. Therefore, the term *T-alternating* has been introduced in [15] as a hypernym for T-even and T-odd matrix polynomials.

In [9] it was observed that there exist T-alternating matrix polynomials that do not allow a T-alternating strong linearization and a detailed explanation of this effect was presented in [10] by characterizing the possible Smith forms of T-alternating matrix polynomials in terms of pairing properties of elementary divisors (see Section 2 for details). In particular, it was shown that those paring conditions for the elementary divisors of a given matrix polynomial $S(\lambda)$ in Smith form were necessary and sufficient for the existence of a T-alternating matrix polynomial $P(\lambda)$ having $S(\lambda)$ as its Smith form, but the constructed $P(\lambda)$ may have a rather high degree resulting in many infinite elementary divisors. Thus, the following question cannot be answered based on the results obtained in [10]:

Problem 1.1 *Let $S(\lambda) = \operatorname{diag}\big(d_1(\lambda), \ldots, d_n(\lambda)\big)$ be a matrix polynomial in Smith form that satisfies the necessary conditions for being a Smith form of a T-even matrix polynomial, and assume that $\sum_{j=1}^{n} \deg(d_j) = nk$. Does there exist a T-even $n \times n$ matrix polynomial $P(\lambda)$ of degree k with invertible leading coefficient that has the Smith form $S(\lambda)$?*

An important application for inverse quadratic eigenvalue problems is the design of feedback controllers for second order systems, see [1, 16]. Since in many cases the coefficient matrices of the second order system are symmetric, the inverse symmetric quadratic eigenvalue problem has been studied intensively. Important contributions to its solution have been made in [4, 5, 7] under the additional assumption that the eigenvalues of the designed systems are semisimple. While this limitation may not be of importance in applications, it turns out to be a restriction to the solution of Problem 1.1, where the eigenvalues of the given Smith form need not be semisimple. Other techniques to tackle inverse quadratic eigenvalue problems include solvents [6] and the construction of *quasi-canonical forms* [8], but it is not clear if these techniques can easily be generalized to inverse polynomial eigenvalue problems of higher degree.

A different approach to the solution of inverse polynomial eigenvalue problems involves the *triangularization of matrix polynomials*: in [19], the authors raised the question whether any regular, complex, quadratic matrix polynomial can be transformed to a quadratic matrix polynomial in triangular form with the same finite and infinite elementary divisors. Here, *triangular form* means that all coefficient matrices of the matrix polynomial are upper triangular. The question was motivated by the lack of existence of a generalized Schur form for matrix polynomials of degree greater than one: in general, it is not possible to transform a given matrix polynomial to triangular form under strict equivalence, let alone under strict unitary equivalence.

On the other hand, it was shown in the proof of [2, Theorem 1.7] that any complex matrix polynomial $P(\lambda)$ of degree k with nonsingular leading coefficient is unimodularly equivalent to a monic upper triangular matrix polynomial $T(\lambda)$ of degree k. Thus, in particular $P(\lambda)$ and $T(\lambda)$ have the same elementary divisors. Recently, the result hidden in the proof of [2, Theorem 1.7] has been generalized in [19] in the case of quadratic matrix polynomials by relaxing the condition of nonsingularity of the leading coefficient and allowing regular matrix polynomials. Finally, it was shown in [17] that any rectangular $n \times m$ matrix polynomial over an algebraically closed field is unimodularly equivalent to a matrix polynomial in triangular form if $n \leq m$. In particular, those results solve an inverse eigenvalue problem as it was highlighted in [17, Lemma 3.2]: *If* $d_1(\lambda), \ldots, d_n(\lambda)$ *are monic polynomials with entries in an algebraically closed field* $\mathbb{F}$ *such that* $d_j(\lambda)$ *divides* $d_{j+1}(\lambda)$ *for* $j = 1, \ldots, n-1$*, then there exists a monic, triangular* $n \times n$ *matrix polynomial* $P(\lambda)$ *of degree* k *over* $\mathbb{F}$ *with* $d_1(\lambda), \ldots, d_n(\lambda)$ *as invariant polynomials if and only if* $\sum_{j=1}^{n} \deg(d_j) = nk$*.*

In the case of T-alternating polynomials, triangular forms turn out to be too restrictive. Indeed, if $P(\lambda)$ is a T-alternating matrix polynomial in triangular form, then all symmetric coefficient matrices are diagonal and all skew-symmetric coefficient matrices are zero. Instead, we aim to construct *anti-triangular forms*. Recall that a matrix $A = [a_{ij}] \in \mathbb{F}^{n \times n}$ is called anti-triangular if $a_{ij} = 0$ for all (i, j) satisfying $i + j \leq n$, see, e.g., [14].

The remainder of the paper is organized as follows. In Section 2, we compile some preliminary results that will be needed in the following. In Section 3, we state and prove the main result Theorem 3.1 which gives an affirmative answer to Problem 1.1 by constructing a T-even, anti-triangular matrix polynomial whose Smith form is the given matrix polynomial $S(\lambda)$. This result is generalized in Section 4 to related structures of matrix polynomials.

Throughout the paper, $\mathbb{F}$ denotes an arbitrary field of characteristic different from two. By $\mathbb{F}[\lambda]^{n \times n}$, we will denote the set of $n \times n$ matrix polynomials over $\mathbb{F}$. Finally, R_n denotes the $n \times n$ *reverse identity*

$$R_n = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

2 Preliminaries

Recall that $P(\lambda), Q(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ are called *unimodularly equivalent* (short: *equivalent*) if there exist $E(\lambda), F(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ that are unimodular (i.e., having constant nonzero determinant) such that

$$Q(\lambda) = E(\lambda)P(\lambda)F(\lambda).$$

We will also denote equivalence of matrix polynomials by $P(\lambda) \sim Q(\lambda)$.. In some of our results, we will need a more restrictive equivalence relation than equivalence, the so-called *unimodular alternating-congruence.*

Definition 2.1 $P(\lambda), Q(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ *are called* unimodularly alternatingly-congruent *(short:* congruent*) if there exists a unimodular* $E(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ *such that*

$$Q(\lambda) = E(\lambda)P(\lambda)E^T(-\lambda).$$

It is straightforward to check that the T-alternating structure of matrix polynomials is preserved under unimodular alternating-congruence: If $P(\lambda)$ and $E(\lambda)$ are $n \times n$ matrix polynomials and $P(\lambda)$ is T-even or T-odd, then also $E(\lambda)P(\lambda)E^T(-\lambda)$ is T-even or T-odd, respectively.

Due to this feature, we added the prefix "alternating-" in order to distinguish it properly from *unimodular congruence* transformations of the form $P(\lambda) \mapsto E(\lambda)P(\lambda)E^T(\lambda)$ introduced in [13] for the sake of preserving the structure of skew-symmetric matrix polynomials. For simplicity, we will use the term *congruence* instead of *unimodular alternating-congruence* as there will be no ambiguity in this paper.

Unfortunately, a canonical form for matrix polynomials under congruence seems not to be available yet; hence, we will mainly use (unimodular) equivalence transformations and the *Smith form*, which is the corresponding canonical form, see, e.g., [2] for details.

Theorem 2.2 (Smith form) *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$. Then, there exists a nonnegative integer r and unimodular $E(\lambda), F(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ such that*

$$E(\lambda)P(\lambda)F(\lambda) = \operatorname{diag}\big(d_1(\lambda), \dots, d_r(\lambda), 0, \dots, 0\big),$$

where $d_1(\lambda), \dots, d_r(\lambda)$ are monic and $d_j(\lambda) \,|\, d_{j+1}(\lambda)$ for $j = 1, \dots, r-1$. Moreover, r and $d_1(\lambda), \dots, d_r(\lambda)$ are unique.

The polynomials $d_1(\lambda), \dots, d_r(\lambda)$ are called the *invariant polynomials* of $P(\lambda)$ and can be characterized in terms of greatest common divisors (short: gcd's) of minors of $P(\lambda)$. We recall that a minor of order j of $P(\lambda)$ is defined to be the determinant of a $j \times j$ submatrix of $P(\lambda)$ that is obtained by extracting j rows and j columns of $P(\lambda)$.

Theorem 2.3 *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ have the Smith form*

$$S(\lambda) = \operatorname{diag}\big(d_1(\lambda),\, \dots,\, d_r(\lambda),\, 0, \dots, 0\big).$$

Set $p_0(\lambda) \equiv 1$ and define $p_j(\lambda) \equiv 0$ if all minors of $P(\lambda)$ of order j are zero, otherwise let $p_j(\lambda)$ be the greatest common divisor (gcd) of all minors of $P(\lambda)$ of order j. Then r is the largest integer j such that $p_j(\lambda) \not\equiv 0$ and the invariant polynomials of $P(\lambda)$ are given by

$$d_j(\lambda) = \frac{p_j(\lambda)}{p_{j-1}(\lambda)}, \quad j = 1, \dots, r.$$

In order to apply this result, we will need a few lemmas involving gcd's. The first one is well-known; a proof can be found, e.g., in [3].

Lemma 2.4 (Lemma of Bézout) *Let $p(\lambda), q(\lambda) \in \mathbb{F}[\lambda]$, then there exist polynomials $z_1(\lambda), z_2(\lambda) \in \mathbb{F}[\lambda]$ such that*

$$z_1(\lambda)p(\lambda) + z_2(\lambda)q(\lambda) = \gcd\big\{p(\lambda),\, q(\lambda)\big\}.$$

Since the matrix polynomials focused on in this paper are T-alternating, we define the *parity* $\varepsilon(p)$ of an alternating scalar polynomial $p(\lambda) \in \mathbb{F}[\lambda]$ to be $\varepsilon(p) := +1$ if $p(\lambda) = p(-\lambda)$ is even, and $\varepsilon(p) := -1$ if $p(\lambda) = -p(-\lambda)$ is odd. The proof of the following lemma can be found in [10].

Lemma 2.5 *Let $p(\lambda) \in \mathbb{F}[\lambda]$ be divided by $d(\lambda) \in \mathbb{F}[\lambda] \setminus \{0\}$ with $\deg(d) \leq \deg(p)$ to get*

$$p(\lambda) = d(\lambda)q(\lambda) + r(\lambda), \quad \deg(r) < \deg(d).$$

If $p(\lambda)$ and $d(\lambda)$ are alternating (not necessarily with the same parity), then $q(\lambda)$ and $r(\lambda)$ are alternating as well. Moreover, $p(\lambda), r(\lambda)$, and $d(\lambda)q(\lambda)$ all three have the same parity.

A key lemma in our constructions will be the following factorization result for even scalar polynomials.

Lemma 2.6 *Let $s(\lambda) \in \mathbb{F}[\lambda]$ be even and of degree $2k$, where k is a nonnegative integer, and let the field $\mathbb{F}$ be algebraically closed. Then, there exists an $x(\lambda) \in \mathbb{F}[\lambda]$ of degree k such that*

$$s(\lambda) = x(\lambda)x(-\lambda).$$

If $s(0) \neq 0$, then $x(\lambda)$ can be chosen such that

$$\gcd\big\{x(\lambda), x(-\lambda)\big\} = 1.$$

Proof. By [10, Lemma 4.1], $s(\lambda)$ admits the factorization

$$s(\lambda) = c\,\lambda^{\alpha_0}\left[(\lambda - \lambda_1)(\lambda + \lambda_1)\right]^{\alpha_1} \cdot \cdots \cdot \left[(\lambda - \lambda_r)(\lambda + \lambda_r)\right]^{\alpha_r},$$

where $\alpha_0 \in \mathbb{N}$ is even, $c \in \mathbb{F} \setminus \{0\}$, $\alpha_1, \ldots, \alpha_r \in \mathbb{N} \setminus \{0\}$, and $\lambda_1, \ldots, \lambda_r, -\lambda_1, \ldots, -\lambda_r$ are pairwise distinct. The result then follows easily by setting

$$x(\lambda) := \sqrt{(-1)^k c}\,\lambda^{\alpha_0/2}\prod_{j=1}^{r}(\lambda - \lambda_j)^{\alpha_j}. \quad \square$$

The following theorems are the main results from [10]; they completely characterize the possible Smith forms of T-alternating matrix polynomials.

Theorem 2.7 (E-Smith form) *Suppose that*

$$S(\lambda) = \operatorname{diag}\big(\lambda^{\alpha_1}p_1(\lambda),\ \lambda^{\alpha_2}p_2(\lambda),\ \ldots,\ \lambda^{\alpha_r}p_r(\lambda),\ 0, \ldots, 0\big) \in \mathbb{F}[\lambda]^{n\times n},$$

*where $0 \leq \alpha_1 \leq \cdots \leq \alpha_r$ are nonnegative integers, all $p_j(\lambda)$ are monic with $p_j(0) \neq 0$, and $p_j(\lambda)\,|\,p_{j+1}(\lambda)$ for $j = 1, \ldots, r-1$. Then $S(\lambda)$ is the Smith form of some T-**even** $n \times n$ matrix polynomial if and only if:*

1) $p_j(\lambda)$ is even for $j = 1, \ldots, r$.

*2) If ν is the number of **odd** exponents among $\alpha_1, \dots, \alpha_r$, then ν is an even integer. Letting $k_1 < k_2 < \cdots < k_\nu$ be the positions on the diagonal of $S(\lambda)$, where these **odd** exponents α_{k_j} occur, the following properties hold:*

(a) adjacency-pairing of positions:

$$k_2 = k_1 + 1, \quad k_4 = k_3 + 1, \quad \dots, \quad k_\nu = k_{\nu-1} + 1,$$

(b) equality-pairing of odd exponents:

$$\alpha_{k_2} = \alpha_{k_1}, \quad \alpha_{k_4} = \alpha_{k_3}, \quad \dots, \quad \alpha_{k_\nu} = \alpha_{k_{\nu-1}}. \tag{2.1}$$

Theorem 2.8 (O-Smith form) *Suppose that*

$$S(\lambda) = \operatorname{diag}\big(\lambda^{\alpha_1} p_1(\lambda),\, \lambda^{\alpha_2} p_2(\lambda),\, \dots,\, \lambda^{\alpha_r} p_r(\lambda),\, 0, \dots, 0\big) \in \mathbb{F}[\lambda]^{n\times n},$$

*where $0 \le \alpha_1 \le \cdots \le \alpha_r$ are nonnegative integers, all $p_j(\lambda)$ are monic with $p_j(0) \neq 0$, and $p_j(\lambda) \,|\, p_{j+1}(\lambda)$ for $j = 1, \dots, r-1$. Then $S(\lambda)$ is the Smith form of some T-**odd** $n \times n$ matrix polynomial if and only if:*

1) $p_j(\lambda)$ is even for $j = 1, \dots, r$.

*2) If ν is the number of **even** exponents among $\alpha_1, \dots, \alpha_r$, then ν is an even integer. Letting $k_1 < k_2 < \cdots < k_\nu$ be the positions on the diagonal of $S(\lambda)$ where these **even** exponents α_{k_j} occur, the following properties hold:*

(a) adjacency-pairing of positions:

$$k_2 = k_1 + 1, \quad k_4 = k_3 + 1, \quad \dots, \quad k_\nu = k_{\nu-1} + 1,$$

(b) equality-pairing of even exponents:

$$\alpha_{k_2} = \alpha_{k_1}, \quad \alpha_{k_4} = \alpha_{k_3}, \quad \dots, \quad \alpha_{k_\nu} = \alpha_{k_{\nu-1}}. \tag{2.2}$$

Finally, we review the result that was obtained in the proof of [2, Theorem 1.7] for the case $\mathbb{F} = \mathbb{C}$, but the proof easily generalizes to arbitrary algebraically closed fields. For this and for the remainder of the paper, let k always denote a positive integer.

Theorem 2.9 *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ satisfy $\deg\big(\det P(\lambda)\big) = nk$ and assume the field $\mathbb{F}$ to be algebraically closed. Then $P(\lambda)$ is equivalent to an upper triangular matrix polynomial, whose diagonal elements have degree k.*

We note that the matrix polynomial $P(\lambda)$ in Theorem 2.9 may have off-diagonal entries of arbitrary degree, which can by a simple procedure be reduced to degree $k-1$ or less, which is shown in [17]. However, for our purpose the statement in Theorem 2.9 is sufficient.

3 The inverse T-even polynomial eigenvalue problem

In this section, we will prove our main theorem.

Theorem 3.1 *Let the field $\mathbb{F}$ be algebraically closed and let $S(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be in E-Smith form as in Theorem* 2.7*. If* $\deg\big(\det S(\lambda)\big) = nk$*, then $S(\lambda)$ is equivalent to a T-even, lower anti-triangular $n \times n$ matrix polynomial of degree k, whose leading coefficient is anti-diagonal.*

The proof of Theorem 3.1 will be carried out in the following subsections and proceed in two main steps:

I) Construct a T-even, lower anti-triangular matrix polynomial such that all entries on the anti-diagonal have degree k. (The entries below the main anti-diagonal may have arbitrary degree.)

II) Reduce the degrees of the entries in the strict lower anti-triangular part so that the resulting matrix polynomial has degree k.

We will begin by describing the procedure that will be used to carry out Step II in Subsection 3.1. In the following subsections, we will then prove Theorem 3.1 by executing Step I in the case $n = 2$ in Subsection 3.2, in the case of even n in Subsection 3.3, and in the case of odd n in Subsection 3.4 thereby completing the proof of Theorem 3.1.

3.1 Reducing the degree

In the following result, the field $\mathbb{F}$ need not be algebraically closed.

Theorem 3.2 *Let* $P(\lambda) = \big[\, p_{ij}(\lambda) \,\big]_{i,j=1}^{n} \in \mathbb{F}[\lambda]^{n\times n}$ *be T-even and lower anti-triangular, and let its anti-diagonal elements all have degree k. Then, $P(\lambda)$ is congruent to a T-even, lower anti-triangular matrix polynomial of degree k with anti-diagonal leading coefficient. More precisely, there exists a unimodular $E(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ such that*

$$E(\lambda)P(\lambda)E(-\lambda)^T =: \breve{P}(\lambda) = \big[\, \breve{p}_{ij}(\lambda) \,\big]_{i,j=1}^{n}$$

is lower anti-triangular and has the same anti-diagonal elements as $P(\lambda)$ and all other elements have degrees not exceeding $k-1$, i.e.,

$$\breve{p}_{i,n+1-i}(\lambda) = p_{i,n+1-i}(\lambda) \quad \textit{and} \quad \deg(\breve{p}_{ij}) \le k-1 \quad \textit{for} \quad i+j > n+1, \quad i,j = 1,\dots,n.$$

Proof. Letting $\kappa := \lfloor k/2 \rfloor$, we will construct $\breve{P}(\lambda)$ in two steps. In the first step, the degrees of the elements in the diagonal positions $(n-\kappa+1, n-\kappa+1), \dots, (n,n)$ will be reduced to $k-1$ or less, in the second step, all other elements in the strict lower anti-triangular part will be considered.

Step 1: reducing the diagonal elements. For each $i = n - \kappa + 1, \dots, n$ we aim to reduce the degree of the element in the (i, i) position to $k - 1$ or less. Hence, for any i let $\widetilde{p}(\lambda) := p_{i,n-i+1}(\lambda)$ be the anti-diagonal element in the same row as $p_{ii}(\lambda)$; since $P(\lambda)$ is T-even, we obtain $p_{n-i+1,i}(\lambda) = \widetilde{p}(-\lambda)$. If $\deg(p_{ii}) \geq k$ (else there is nothing to do), consider polynomials $q(\lambda), r(\lambda)$ such that

$$\widetilde{p}(\lambda)q(\lambda) + r(\lambda) = p_{ii}(\lambda) = p_{ii}(-\lambda) = \widetilde{p}(-\lambda)q(-\lambda) + r(-\lambda),$$

where $\deg(r) < \deg(\widetilde{p}) \leq k$. Then, adding the $\big(-q(\lambda)/2\big)$-multiple of the $(n + i - 1)$st column to the ith column and then the $\big(-q(-\lambda)/2\big)$-multiple of the $(n+i-1)$st row to the ith row (this is a congruence transformation), the element in the (i, i) position is changed to

$$p_{ii}(\lambda) - \frac{q(\lambda)\widetilde{p}(\lambda)}{2} - \frac{q(-\lambda)\widetilde{p}(-\lambda)}{2} = \frac{r(\lambda) + r(-\lambda)}{2},$$

which is even and of degree $k - 1$ or less. Observe that no anti-diagonal elements and no diagonal elements other than $p_{ii}(\lambda)$ have been changed. Thus, executing this procedure for $i = n - \kappa + 1, \dots, n$, we obtain a matrix polynomial again denoted by $P(\lambda)$ which is T-even, lower anti-triangular, and whose diagonal elements have degree $k - 1$ or less.

Step 2: reducing the off-diagonal elements. Now, we will use an induction argument to simultaneously reduce the degrees of the elements in the (i, j) and (j, i) positions for $j = n-i+2, \dots, i-1$ and for $i = \kappa+2, \dots, n$. To illustrate this, consider the 7×7 scheme

$$\begin{bmatrix}
 & & & & & & \bullet \\
 & & & & & \bullet & \circ \\
 & & & & \bullet & \star & \circ \\
 & & & \bullet & \diamond & \star & \circ \\
 & & \bullet & \diamond & \bullet & \star & \circ \\
 & \bullet & \star & \star & \star & \bullet & \circ \\
\bullet & \circ & \circ & \circ & \circ & \circ & \bullet
\end{bmatrix}.$$

In this case, the above loops mean that first the $\diamond$-elements are treated simultaneously, then the $\star$-elements are reduced simultaneously from top to bottom and from left to right, and then the $\circ$-elements are operated on simultaneously from top to bottom and from left to right.

Back to the general case, assume that we are currently considering the elements in the (i, j) and (j, i) positions and denote the matrix polynomial resulting from the previous steps again by $P(\lambda)$. For simplicity, set $\widehat{p}(\lambda) := p_{ij}(\lambda)$ and $\widetilde{p}(\lambda) := p_{i,n-i+1}(\lambda)$. Since $P(\lambda)$ is T-even, we obtain

$$p_{ji}(\lambda) = \widehat{p}(-\lambda) \quad \text{and} \quad p_{n-i+1,i}(\lambda) = \widetilde{p}(-\lambda).$$

If $\deg(\widehat{p}) \geq k$ (else there is nothing to do), let $\widehat{p}(\lambda) = \widetilde{p}(\lambda)q(\lambda) + r(\lambda)$, where $\deg(r) < \deg(\widetilde{p}) = k$. Adding the $\big(-q(-\lambda)\big)$-multiple of the $(n - i + 1)$st row to the jth row and the $\big(-q(\lambda)\big)$-multiple of the $(n - i + 1)$st column to the jth column (note that this is a congruence transformation), we obtain a matrix polynomial again denoted by $P(\lambda)$ with elements in the (i, j) and (j, i) positions now given by

$$\widehat{p}(\lambda) - q(\lambda)\widetilde{p}(\lambda) = r(\lambda) \quad \text{and} \quad \widehat{p}(-\lambda) - q(-\lambda)\widetilde{p}(-\lambda) = r(-\lambda),$$

respectively. These polynomials have degrees less than or equal to $k-1$. Furthermore, $P(\lambda)$ is T-even and lower anti-triangular and all elements on the anti-diagonal, as well as those on the diagonal and those previously reduced are unchanged. Completing the induction, we finally obtain a matrix polynomial in the desired form that is congruent to the matrix polynomial we started with. □

3.2 The 2×2 case

For the remainder of this paper we will assume the additional condition that the field $\mathbb{F}$ is algebraically closed.

Let $S(\lambda) = \operatorname{diag}\big(\lambda^\alpha p_1(\lambda), \lambda^\beta p_2(\lambda)\big)$ be a possible Smith form of a T-even matrix polynomial, where $p_1(0), p_2(0) \neq 0$. Then, by Theorem 2.7 both $p_1(\lambda)$ and $p_2(\lambda)$ are even polynomials, and the exponents α and β are either both even (including the case $\alpha = \beta = 0$) or they are both odd and equal. Furthermore, since $p_1(\lambda)$ divides $p_2(\lambda)$, the latter can be factorized as $p_2(\lambda) = p_1(\lambda)s(\lambda)$, where by Lemma 2.5 also $s(\lambda)$ is even. The following lemma therefore covers Theorem 3.1 in the case $n = 2$.

Lemma 3.3 *Let $S(\lambda) \in \mathbb{F}[\lambda]^{2\times 2}$ be in Smith form*

$$S(\lambda) = \operatorname{diag}\big(\lambda^\alpha p(\lambda), \lambda^\beta p(\lambda)s(\lambda)\big),$$

where $0 \leq \alpha \leq \beta$ and where $p(\lambda)$ and $s(\lambda)$ are monic, even polynomials with $p(0), s(0) \neq 0$, and assume $\deg\big(\det S(\lambda)\big) = 2k$.

1) *If $\beta \neq \alpha$, but $\beta - \alpha$ is even, then $S(\lambda)$ is equivalent to a lower anti-triangular matrix polynomial of degree k of the form*

$$\begin{bmatrix} 0 & (-\lambda)^{(\alpha+\beta)/2} p(\lambda)x(-\lambda) \\ \lambda^{(\alpha+\beta)/2} p(\lambda)x(\lambda) & \lambda^\alpha p(\lambda) \end{bmatrix}, \tag{3.1}$$

where $x(\lambda)x(-\lambda) = s(\lambda)$. Moreover, the matrix polynomial (3.1) is T-even if α (and therefore also β) is even.

2) *If $\beta = \alpha$, then $S(\lambda)$ is equivalent to a T-even, lower anti-diagonal matrix polynomial of degree k of the form*

$$\begin{bmatrix} 0 & (-\lambda)^\alpha p(\lambda)x(-\lambda) \\ \lambda^\alpha p(\lambda)x(\lambda) & 0 \end{bmatrix}, \tag{3.2}$$

where $x(\lambda)x(-\lambda) = s(\lambda)$.

Proof. Since $s(\lambda)$ is even with $s(0) \neq 0$, it follows from Lemma 2.6 that it can be factorized as $s(\lambda) = x(\lambda)x(-\lambda)$ for some polynomial $x(\lambda)$ satisfying $\gcd\{x(\lambda),\, x(-\lambda)\} = 1$.

We will first prove 1). Observe that

$$\begin{aligned}\begin{bmatrix}\lambda^\alpha p(\lambda) & 0\\ 0 & \lambda^\beta p(\lambda)s(\lambda)\end{bmatrix} &\sim \begin{bmatrix}\lambda^\alpha p(\lambda) & 0\\ -\lambda^{(\alpha+\beta)/2}p(\lambda)x(-\lambda) & \lambda^\beta p(\lambda)s(\lambda)\end{bmatrix}\\ &\sim \begin{bmatrix}\lambda^\alpha p(\lambda) & \lambda^{(\alpha+\beta)/2}p(\lambda)x(\lambda)\\ -\lambda^{(\alpha+\beta)/2}p(\lambda)x(-\lambda) & 0\end{bmatrix}.\end{aligned}$$

Indeed, the first equivalence follows by adding the $\big(-\lambda^{(\beta-\alpha)/2}x(-\lambda)\big)$-multiple of the first row of $S(\lambda)$ to its second row, and the second equivalence follows by adding the $\big(\lambda^{(\beta-\alpha)/2}x(\lambda)\big)$-multiple of the first column of the resulting matrix polynomial to its second column. The latter matrix polynomial is easily seen to be equivalent to the desired shape (3.1) by multiplying the second row with a suitable power of -1 followed by a row and column permutation. It is now straightforward to check that the constructed matrix polynomial has degree k and that it is T-even if α is even.

Next, let us prove 2) assuming $\beta=\alpha$. As in 1), it follows that $S(\lambda)$ is equivalent to

$$\begin{bmatrix}0 & (-\lambda)^\alpha p(\lambda)x(-\lambda)\\ \lambda^\alpha p(\lambda)x(\lambda) & \lambda^\alpha p(\lambda)\end{bmatrix}$$

with $\gcd\{x(\lambda),\, x(-\lambda)\}=1$. Hence, by Lemma 2.4 there exist polynomials $z_1(\lambda), z_2(\lambda)$ such that

$$z_1(\lambda)\,\lambda^\alpha p(\lambda)x(\lambda)+z_2(\lambda)\,(-\lambda)^\alpha p(\lambda)x(-\lambda)=\lambda^\alpha p(\lambda).$$

Subtracting the $z_1(\lambda)$-multiple of the first column from the second column and the $z_2(\lambda)$-multiple of the first row from the second row we obtain the desired form (3.2). ☐

3.3 The case of even n

In this subsection, we prove a result that in particular covers Theorem 3.1 in the case that n is even but greater than two.

Lemma 3.4 *Let $n\in\mathbb{N}$ be even and let $S(\lambda)\in\mathbb{F}[\lambda]^{n\times n}$ be in E-Smith form as in Theorem 2.7. Then, $S(\lambda)$ is equivalent to a T-even, lower anti-triangular matrix polynomial. If additionally $\deg\big(\det S(\lambda)\big)=nk$, this matrix polynomial can be chosen to have degree k and an anti-diagonal leading coefficient.*

Proof. We will proceed with the proof in two steps. In the first step we will transform $S(\lambda)$ to a lower anti-triangular matrix polynomial that is T-even, and in the second step we will reduce its degree to k if the additional hypothesis is satisfied.

Step 1: reduction to anti-triangular form. Let

$$S(\lambda)=\operatorname{diag}\big(\lambda^{\alpha_1}p_1(\lambda),\lambda^{\alpha_2}p_2(\lambda),\ldots,\lambda^{\alpha_n}p_n(\lambda)\big),$$

where $p_j(0)\neq 0$ for $j=1,\ldots,n$. Further, let $k_1<\cdots<k_\nu$ denote the indices j for which α_j is odd and let $\ell_1<\cdots<\ell_\mu$ denote the indices j for which α_j is even. Then, by Theorem 2.7 the integer ν is even and thus with n also μ must be even. We go on to define

$$S_1(\lambda):=\operatorname{diag}\big(\lambda^{\alpha_{k_1}}p_{k_1}(\lambda),\ldots,\lambda^{\alpha_{k_\nu}}p_{k_\nu}(\lambda)\big)\quad\text{and}\quad S_2(\lambda):=\operatorname{diag}\big(\lambda^{\alpha_{\ell_1}}p_{\ell_1}(\lambda),\ldots,\lambda^{\alpha_{\ell_\mu}}p_{\ell_\mu}(\lambda)\big),$$

then $S_1(\lambda)$ and $S_2(\lambda)$ are both in E-Smith form as in Theorem 2.7, and both of them have even size. Further, applying row and column permutations we obtain $S(\lambda) \sim S_1(\lambda) \oplus S_2(\lambda)$ which we continue to modify as follows:

Partitioning both $S_1(\lambda)$ and $S_2(\lambda)$ into 2×2 diagonal blocks, each occuring 2×2 block is also in E-Smith form, as the occuring exponents of λ are either both even or odd and equal. Hence, applying Lemma 3.3 we find that each such block is equivalent to a T-even matrix polynomial of the form (3.1) or (3.2). Executing this step for all 2×2 diagonal blocks of $S_1(\lambda)$ and $S_2(\lambda)$, we find that $S_1(\lambda) \oplus S_2(\lambda)$ is equivalent to a T-even matrix polynomial of the form

$$\begin{bmatrix} 0 & q_1(-\lambda) \\ q_1(\lambda) & r_1(\lambda) \end{bmatrix} \oplus \begin{bmatrix} 0 & q_2(-\lambda) \\ q_2(\lambda) & r_2(\lambda) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & q_\kappa(-\lambda) \\ q_\kappa(\lambda) & r_\kappa(\lambda) \end{bmatrix},$$

defining $\kappa := n/2$. We apply a row and column permutation to obtain the T-even matrix polynomial

$$\begin{bmatrix} 0 & T(-\lambda)^T \\ T(\lambda) & R(\lambda) \end{bmatrix}, \quad \text{where} \quad T(\lambda) = \begin{bmatrix} 0 & & q_\kappa(\lambda) \\ & \iddots & \\ q_1(\lambda) & & 0 \end{bmatrix}$$

is anti-diagonal and $R(\lambda) = \operatorname{diag}\big(r_1(\lambda), \ldots, r_\kappa(\lambda)\big)$ is diagonal.

Step 2: reduction to degree k. If also $\deg\big(\det S(\lambda)\big) = nk$ is satisfied, it follows that $\deg\big(\det T(\lambda)\big) = \kappa k$. Thus, by Theorem 2.9 there are unimodular $\kappa \times \kappa$ matrix polynomials $E(\lambda), F(\lambda)$ such that $E(\lambda)\, T(\lambda)\, F(\lambda)$ is an upper triangular matrix polynomial of degree k. Hence, setting $\widetilde{E}(\lambda) := R_\kappa E(\lambda)$ and $M(\lambda) := \widetilde{E}(\lambda) T(\lambda) F(\lambda)$, we obtain that

$$\begin{bmatrix} F(-\lambda)^T & 0 \\ 0 & \widetilde{E}(\lambda) \end{bmatrix} \begin{bmatrix} 0 & T(-\lambda)^T \\ T(\lambda) & R(\lambda) \end{bmatrix} \begin{bmatrix} F(\lambda) & 0 \\ 0 & \widetilde{E}(-\lambda)^T \end{bmatrix} = \begin{bmatrix} 0 & M(-\lambda)^T \\ M(\lambda) & \widetilde{E}(\lambda) R(\lambda) \widetilde{E}(-\lambda)^T \end{bmatrix}$$

is a T-even, lower anti-triangular matrix polynomial equivalent to $S(\lambda)$ with anti-diagonal entries of degree k. Thus, by Theorem 3.2 it is congruent to a T-even, lower anti-triangular matrix polynomial of degree k, whose leading coefficient is anti-diagonal. $\square$

3.4 The case of odd n

The proof of Theorem 3.1 in the case that n is odd is more involved. The key difficulty is that a T-even, lower anti-triangular matrix polynomial of odd dimension has a middle element in the $\big((n+1)/2, (n+1)/2\big)$-position that is exceptional as it does not have a symmetrically placed counterpart on the anti-diagonal unlike all other anti-diagonal elements. If we aim to proceed as in the proof of Lemma 3.4 and decompose $S(\lambda)$ into 2×2 blocks, then one element will remain because n is odd. If this polynomial has degree k, then it could be used as the exceptional middle element, but in general $S(\lambda)$ may not contain such an entry. The next natural step would be to try to create that polynomial from a 3×3 submatrix of $S(\lambda)$. The following lemma provides conditions when this is indeed possible.

Lemma 3.5 *Let* $S(\lambda) = \operatorname{diag}\big(d_1(\lambda), d_2(\lambda), d_3(\lambda)\big) \in \mathbb{F}[\lambda]^{3\times 3}$ *be in E-Smith form as in Theorem* 2.7 *and let* $k \in \mathbb{N}\setminus\{0\}$ *be even. If* $\deg(d_1) \le k \le \deg(d_3)$*, then* $S(\lambda)$ *is equivalent to a* T*-even, lower anti-triangular matrix polynomial, whose middle entry has degree* k.

Proof. By Theorem 2.7 we may assume that $S(\lambda)$ has the form

$$S(\lambda) = \operatorname{diag}\big(\lambda^{\alpha} p(\lambda),\ \lambda^{\beta} p(\lambda)s(\lambda),\ \lambda^{\gamma} p(\lambda)s(\lambda)t(\lambda)\big),$$

where $0 \le \alpha \le \beta \le \gamma$ and where $p(\lambda), s(\lambda), t(\lambda)$ are monic, even polynomials with $p(0), s(0), t(0) \ne 0$. Thus, the condition on k now translates into

$$\alpha + \deg(p) \le k \le \gamma + \deg(p) + \deg(s) + \deg(t), \tag{3.3}$$

from which we can distinguish two cases:

Case (1): α *is even.* Then, by Theorem 2.7 either both β and γ are even or they are both odd and equal. We set $\delta := \beta - \alpha$, $\varepsilon := \gamma - \alpha$, and

$$\widehat{S}(\lambda) := \operatorname{diag}\big(1, \lambda^{\delta} s(\lambda), \lambda^{\varepsilon} s(\lambda)t(\lambda)\big)$$

in order to obtain $\lambda^{\alpha} p(\lambda)\widehat{S}(\lambda) = S(\lambda)$. Note that either both δ and ε are even or they are both odd and equal. This lemma is now equivalent to showing that there is a T-even, lower anti-triangular matrix polynomial $T(\lambda)$ equivalent to $\widehat{S}(\lambda)$, whose middle entry has degree $m := k - \alpha - \deg(p)$. It is clear that m is even and from (3.3) we obtain

$$0 \le m \le \varepsilon + \deg(s) + \deg(t).$$

Since $\deg(s)$ and $\deg(t)$ are both even, there exist nonnegative, even integers $\ell, \widetilde{\ell}, \widehat{\ell}$ with $m = \ell + \widetilde{\ell} + \widehat{\ell}$ such that also $\ell \le \varepsilon$, $\widetilde{\ell} \le \deg(s)$, and $\widehat{\ell} \le \deg(t)$ are satisfied. Since $\mathbb{F}$ is algebraically closed, there exist monic even polynomials $\widehat{s}(\lambda)$ of degree $(\deg(s) - \widetilde{\ell}\,)$ and $\widehat{t}(\lambda)$ of degree $\widehat{\ell}$, such that $\widehat{s}(\lambda) \,|\, s(\lambda)$ and $\widehat{t}(\lambda) \,|\, t(\lambda)$. Then, by Lemma 2.5 the corresponding quotients are again even and thus Lemma 2.6 can be applied, so there exist polynomials $x(\lambda), y(\lambda)$ such that

$$s(\lambda) = \widehat{s}(\lambda)x(\lambda)x(-\lambda) \quad \text{and} \quad t(\lambda) = \widehat{t}(\lambda)y(\lambda)y(-\lambda). \tag{3.4}$$

Finally, if δ is even, we use Lemma 2.6 to obtain a polynomial $z(\lambda)$ such that

$$\lambda^{\ell}\, \widehat{t}(\lambda)x(\lambda)x(-\lambda) - \lambda^{\delta} s(\lambda) = z(\lambda)z(-\lambda), \tag{3.5}$$

since the polynomial on the left hand side is even. If δ is odd, we apply Lemma 2.6 to obtain a polynomial $a(\lambda)$ such that

$$t(\lambda) = a(\lambda)y(\lambda)\, a(-\lambda)y(-\lambda) \quad \text{and} \quad \gcd\big\{a(\lambda)y(\lambda),\ a(-\lambda)y(-\lambda)\big\} = 1, \tag{3.6}$$

to define $z(\lambda) := \lambda^{\ell/2} a(-\lambda)x(\lambda)$ in this case. We remark that in both cases, $z(\lambda)$ contains the root zero with multiplicity $\min\{\delta, \ell\}/2$. (Indeed, if δ is odd, then $\ell < \varepsilon = \delta$.) We now aim to show that

$$T(\lambda) := \begin{bmatrix} 0 & 0 & (-\lambda)^n\, \widehat{s}(\lambda)x(-\lambda)y(-\lambda) \\ 0 & \lambda^\ell\, \widehat{t}(\lambda)x(\lambda)x(-\lambda) & z(-\lambda) \\ \lambda^n\, \widehat{s}(\lambda)x(\lambda)y(\lambda) & z(\lambda) & 1 \end{bmatrix},$$

where we set $n := (\delta + \varepsilon - \ell)/2$, is the desired matrix polynomial. It is straightforward that $T(\lambda)$ is T-even and that its middle entry has degree m. Thus, we remain to show that the Smith form of $T(\lambda)$ is given by $\widehat{S}(\lambda)$. Applying Theorem 2.3, we immediately find $p_1(\lambda) = 1$. Moreover,

$$\begin{aligned} p_2(\lambda) = \gcd\big\{&0,\ 0,\ \lambda^{n+\ell}\, \widehat{t}(\lambda)x(\lambda)x(-\lambda)^2\, \widehat{s}(\lambda)y(-\lambda),\ 0,\ \lambda^{2n}\, \widehat{s}(\lambda)^2 x(\lambda)y(\lambda)x(-\lambda)y(-\lambda), \\ &\lambda^n z(\lambda)\widehat{s}(\lambda)x(-\lambda)y(-\lambda),\ \lambda^{n+\ell}\, \widehat{s}(\lambda)x(\lambda)^2 y(\lambda)\widehat{t}(\lambda)x(-\lambda), \\ &\lambda^n\, \widehat{s}(\lambda)x(\lambda)y(\lambda)z(-\lambda),\ \lambda^\ell\, \widehat{t}(\lambda)x(\lambda)x(-\lambda) - z(\lambda)z(-\lambda)\big\}, \end{aligned}$$

where the 2×2 minors of $T(\lambda)$ are given in lexicographical order and powers of -1 are ignored. First, we compute that the last minor is equal to $\lambda^\delta s(\lambda)$ by (3.5) if δ is even and that it is equal to 0 by (3.6) otherwise. Further observe that:

$$2n = \delta + \varepsilon - \ell \geq \delta, \quad n + \frac{\ell}{2} = \frac{\delta + \varepsilon}{2} \geq \delta, \quad \text{and} \quad n + \frac{\delta}{2} = \delta + \frac{\varepsilon - \ell}{2} \geq \delta.$$

Hence, the multiplicities of the root zero occuring in all but the last above 2×2 minors (including the multiplicity $n + \min\{\delta, \ell\}/2$ in the 6th and 8th minor) are greater than or equal to δ. Because of (3.4), also all 2×2 minors are divisible by $s(\lambda)$; since powers of λ and $s(\lambda)$ are relatively prime, $\lambda^\delta s(\lambda)$ divides all 2×2 minors of $T(\lambda)$. Thus, we obtain $p_2(\lambda) = \lambda^\delta s(\lambda)$ in the case that δ is even; otherwise $p_2(\lambda)$ is a divisor of the gcd of the 6th and 8th minor, i.e.,

$$p_2(\lambda) \mid \gcd\big\{\lambda^{n+\ell/2} s(\lambda)a(-\lambda)y(-\lambda),\ \lambda^{n+\ell/2} s(\lambda)a(\lambda)y(\lambda)\big\} = \lambda^\delta s(\lambda),$$

using (3.6) and the definitions of n and $z(\lambda)$, resulting in $p_2(\lambda) = \lambda^\delta s(\lambda)$ in this case as well. Finally, $p_3(\lambda)$ is the normalized determinant of $T(\lambda)$:

$$p_3(\lambda) = \gcd\big\{\lambda^{\ell+2n}\, \widehat{s}(\lambda)^2\, \widehat{t}(\lambda)x(\lambda)^2 x(-\lambda)^2 y(\lambda)y(-\lambda)\big\} = \lambda^{\delta+\varepsilon} s(\lambda)^2 t(\lambda),$$

where the last equality follows from the definition of n and (3.4). The invariant polynomials of $T(\lambda)$ thus match the diagonal entries of $\widehat{S}(\lambda)$ from which we obtain the assertion.

Case (2): α is odd. In this case, we have $\beta = \alpha$ and that γ is even. Also, $S(\lambda) = \lambda^{\alpha-1} p(\lambda)\widehat{S}(\lambda)$ holds for $\widehat{S}(\lambda) := \operatorname{diag}\big(\lambda, \lambda\, s(\lambda), \lambda^\delta s(\lambda)t(\lambda)\big)$ as we define $\delta := \gamma - \alpha + 1$. In this case, setting $m := k - \alpha + 1 - \deg(p)$, clearly m is even and we obtain $1 \leq m \leq \delta + \deg(s) + \deg(t)$ from (3.3). As 1 is odd and m is even, there exist nonnegative, even integers $\ell, \widetilde{\ell}, \widehat{\ell}$ with $m = \ell + \widetilde{\ell} + \widehat{\ell}$ such that also $2 \leq \ell \leq \delta$, $\widetilde{\ell} \leq \deg(s)$, and $\widehat{\ell} \leq \deg(t)$ are satisfied. As in the previous case, we obtain even polynomials $\widehat{s}(\lambda)$ of degree $(\deg(s) - \widetilde{\ell})$ and $\widehat{t}(\lambda)$ of degree $\widehat{\ell}$, such that (3.4) holds for certain polynomials $x(\lambda), y(\lambda)$.

Finally, since $s(\lambda)$ is even with $s(0) \neq 0$, using Lemma 2.6 we factorize $s(\lambda) = z(\lambda)z(-\lambda)$ such that $\gcd\{z(\lambda),\, z(-\lambda)\} = 1$. Now, let us show that $\lambda^{\alpha-1}p(\lambda)T(\lambda)$ with

$$T(\lambda) := \begin{bmatrix} 0 & 0 & (-\lambda)^n\, \widehat{s}(\lambda)x(-\lambda)y(-\lambda) \\ 0 & \lambda^\ell\, \widehat{t}(\lambda)x(\lambda)x(-\lambda) & -\lambda\, z(-\lambda) \\ \lambda^n\, \widehat{s}(\lambda)x(\lambda)y(\lambda) & \lambda\, z(\lambda) & 0 \end{bmatrix},$$

where we set $n := 1 + (\delta - \ell)/2$, is the desired matrix polynomial. It is straightforward that $T(\lambda)$ is T-even and that its middle entry has degree m. Thus, we remain to show that the Smith form of $T(\lambda)$ is given by $\widehat{S}(\lambda)$. Applying Theorem 2.3, since $z(\lambda)$ and $z(-\lambda)$ are relatively prime, it is $p_1(\lambda) = \lambda$. Moreover,

$$\begin{aligned} p_2(\lambda) = \gcd \big\{&0,\ 0,\ \lambda^{n+\ell}\, \widehat{t}(\lambda)x(\lambda)x(-\lambda)^2\, \widehat{s}(\lambda)y(-\lambda),\ 0, \\ &\lambda^{2n}\, \widehat{s}(\lambda)^2 x(\lambda)y(\lambda)x(-\lambda)y(-\lambda),\ \lambda^{n+1}z(\lambda)\widehat{s}(\lambda)x(-\lambda)y(-\lambda), \\ &\lambda^{n+\ell}\, \widehat{s}(\lambda)x(\lambda)^2 y(\lambda)\widehat{t}(\lambda)x(-\lambda),\ \lambda^{n+1}\, \widehat{s}(\lambda)x(\lambda)y(\lambda)z(-\lambda),\ \lambda^2 z(\lambda)z(-\lambda)\big\}, \end{aligned}$$

where again the 2×2 minors of $T(\lambda)$ are given in lexicographical order and powers of -1 are ignored. Because of (3.4), clearly $p_2(\lambda) = \lambda^2 s(\lambda)$ holds. Finally, $p_3(\lambda)$ is the normalized determinant of $T(\lambda)$ given by $p_3(\lambda) = \lambda^{\delta+2}s(\lambda)^2 t(\lambda)$, which is computed using the definition of n and (3.4); thus $T(\lambda)$ has the Smith form $\widehat{S}(\lambda)$. □

We are now in the position to prove Theorem 3.1 in the case that n is odd.

Lemma 3.6 *Let $n \in \mathbb{N}$ be odd and let $S(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be in E-Smith form as in Theorem 2.7 satisfying $\deg\big(\det S(\lambda)\big) = nk$. Then, $S(\lambda)$ is equivalent to a T-even, lower anti-triangular matrix polynomial of degree k, whose leading coefficient is anti-diagonal.*

Proof. First, we highlight that k is even because by [10, Lemma 3.4] the determinant of a T-even matrix polynomial is an even scalar polynomial, so it has even degree. Thus, as n is odd, k must be even. As in the proof of Lemma 3.4, we now proceed in two steps.

Step 1: reduction to anti-triangular form. Let

$$S(\lambda) = \operatorname{diag}\big(\lambda^{\alpha_1}p_1(\lambda), \lambda^{\alpha_2}p_2(\lambda), \ldots, \lambda^{\alpha_n}p_n(\lambda)\big),$$

where $p_j(0) \neq 0$ for $j = 1, \ldots, n$. As in the proof of Lemma 3.4, let $k_1 < \cdots < k_\nu$ denote the indices j for which α_j is odd and let $\ell_1 < \cdots < \ell_\mu$ denote the indices j for which α_j is even. Then ν is even and thus μ is odd as n is odd. Now, we aim to apply Lemma 3.5 to a suitable 3×3 submatrix of $S(\lambda)$ that we construct depending on the degree of $\lambda^{\alpha_{\ell_1}}p_{\ell_1}(\lambda)$. If on the one hand $\alpha_{\ell_1} + \deg(p_{\ell_1}) \leq k$, we define

$$\widehat{S}(\lambda) := \operatorname{diag}\big(\lambda^{\alpha_{\ell_1}}p_{\ell_1}(\lambda),\ \lambda^{\alpha_{\widehat{n}}}p_{\widehat{n}}(\lambda),\ \lambda^{\alpha_n}p_n(\lambda)\big),$$

where $\widehat{n} := k_{\nu-1}$ if $n = k_\nu$ and $\widehat{n} := \ell_{\mu-1}$ if $n = \ell_\mu$, and we claim that $\widehat{S}(\lambda)$ fulfills the hypotheses of Lemma 3.5. First, by our definition of $\widehat{n}$, it is clear that $\widehat{S}(\lambda)$ is in

Smith form. Then, $\alpha_n + \deg(p_n) \geq k$ holds as well, since otherwise a contradiction to $\deg\big(\det S(\lambda)\big) = nk$ would be obtained.

If on the other hand $\alpha_{\ell_1} + \deg(p_{\ell_1}) > k$, then

$$\widehat{S}(\lambda) := \operatorname{diag}\big(\lambda^{\alpha_1}p_1(\lambda),\, \lambda^{\alpha_2}p_2(\lambda),\, \lambda^{\alpha_{\ell_1}}p_{\ell_1}(\lambda)\big),$$

that we will also show to satisfy the hypotheses of Lemma 3.5. First, clearly $\alpha_1 = \alpha_2$ is odd, further $\alpha_1 + \deg(p_1) < k$, as otherwise we would obtain a contradiction to $\deg\big(\det S(\lambda)\big) = nk$. Therefore, the diagonal entries of $\widehat{S}(\lambda)$, that are invariant polynomials of $S(\lambda)$, are in the right order so that $\widehat{S}(\lambda)$ is in Smith form.

In both cases, by Lemma 3.5 we have that $\widehat{S}(\lambda)$ is equivalent to some T-even, lower anti-triangular matrix polynomial $\widehat{T}(\lambda)$, whose middle entry has degree k. Also, clearly $S(\lambda) \sim \widehat{S}(\lambda) \oplus \widetilde{S}(\lambda)$ for a certain $(n-3)\times(n-3)$ matrix polynomial $\widetilde{S}(\lambda)$ that is in E-Smith form as in Theorem 2.7. Thus, by Lemma 3.4 we obtain that $\widetilde{S}(\lambda)$ is equivalent to a T-even, lower anti-triangular matrix polynomial $Q(\lambda)$, therefore

$$S(\lambda) \sim \widehat{S}(\lambda) \oplus \widetilde{S}(\lambda) \sim \widehat{T}(\lambda) \oplus Q(\lambda).$$

Denoting the middle entry of $\widehat{T}(\lambda)$ by $\widehat{q}(\lambda)$, it is $\deg(\widehat{q}\,) = k$; applying row and column permutations, we obtain a T-even matrix polynomial of the form

$$\begin{bmatrix} 0 & 0 & T(-\lambda)^T \\ 0 & \widehat{q}(\lambda) & a(-\lambda)^T \\ T(\lambda) & a(\lambda) & R(\lambda) \end{bmatrix}. \tag{3.7}$$

Step 2: Reduction to degree k. Clearly, the determinant of (3.7) has degree nk. Since $\widehat{q}$ has degree k, it follows that $\det T(\lambda)$ has degree κk, where $\kappa := (n-1)/2$. Thus, Theorem 2.9 implies that there are unimodular $\kappa \times \kappa$ matrix polynomials $E(\lambda), F(\lambda)$ such that $E(\lambda)\,T(\lambda)\,F(\lambda)$ is an upper triangular matrix polynomial of degree k. Hence, setting $\widetilde{E}(\lambda) := R_\kappa E(\lambda)$ and $M(\lambda) := \widetilde{E}(\lambda)T(\lambda)F(\lambda)$, we obtain that

$$\begin{aligned} &\begin{bmatrix} F(-\lambda)^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \widetilde{E}(\lambda) \end{bmatrix} \begin{bmatrix} 0 & 0 & T(-\lambda)^T \\ 0 & \widehat{q}(\lambda) & a(-\lambda)^T \\ T(\lambda) & a(\lambda) & X(\lambda) \end{bmatrix} \begin{bmatrix} F(\lambda) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \widetilde{E}(-\lambda)^T \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & M(-\lambda)^T \\ 0 & \widehat{q}(\lambda) & a(-\lambda)^T\widetilde{E}(-\lambda)^T \\ M(\lambda) & \widetilde{E}(\lambda)a(\lambda) & \widetilde{E}(\lambda)X(\lambda)\widetilde{E}(-\lambda)^T \end{bmatrix} \end{aligned}$$

is a T-even, lower anti-triangular matrix polynomial equivalent to $S(\lambda)$ with anti-diagonal entries of degree k. Hence, by Theorem 3.2, it is congruent to a T-even, lower anti-triangular matrix polynomial of degree k, whose leading coefficient is anti-diagonal. $\square$

4 Inverse polynomial eigenvalue problems for related structures

It is straightforward to generalize the results derived in the last section to T-odd matrix polynomials.

Theorem 4.1 *Let $S(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be in O-Smith form as in Theorem 2.8 satisfying $\deg\big(\det S(\lambda)\big) = nk$. Then $S(\lambda)$ is equivalent to a T-odd, lower anti-triangular matrix polynomial of degree k, whose leading coefficient is anti-diagonal.*

Proof. Comparing the conditions of Theorems 2.7 and 2.8, we observe that if $S(\lambda)$ is in Smith form satisfying the conditions of Theorem 2.8, then $\widehat{S}(\lambda) := \lambda S(\lambda)$ is also in Smith form and satisfies the conditions of Theorem 2.7. Moreover, $\deg\big(\det \widehat{S}(\lambda)\big) = n(k+1)$. Hence, by Theorem 3.1, there exist unimodular matrix polynomials $E(\lambda), F(\lambda)$ such that

$$E(\lambda)\widehat{S}(\lambda)F(\lambda) = \lambda E(\lambda)S(\lambda)F(\lambda)$$

is a T-even, lower anti-triangular matrix polynomial of degree $k+1$. Then $E(\lambda)S(\lambda)F(\lambda)$ is the desired T-odd, lower anti-triangular matrix polynomial of degree k. $\square$

Let us now consider Theorem 3.1 and Theorem 4.1 in a different way: If $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is T-alternating with nonsingular leading coefficient, we can apply Theorem 3.1 or 4.1 to its Smith form to obtain a T-alternating anti-triangular form $T(\lambda)$. Since the leading coefficient of $T(\lambda)$ is anti-diagonal, it can be 'normalized' by a final congruence transformation $T(\lambda) \mapsto M^T T(\lambda) M$ as follows:

Corollary 4.2 *Let $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be a T-alternating matrix polynomial of degree k with invertible leading coefficient. Then $P(\lambda)$ is equivalent to a T-alternating matrix polynomial of degree k that is in lower anti-triangular form and has either the leading coefficient R_n or*

$$\begin{bmatrix} 0 & R_{n/2} \\ -R_{n/2} & 0 \end{bmatrix}.$$

Next, let us turn to palindromic matrix polynomials. Recall that a matrix polynomial $P(\lambda) = \sum_{j=0}^{k} \lambda^j A_j$ (where leading coefficients $A_k, A_{k-1}, \ldots$ are allowed to be zero) is called *T-palindromic* (or T-palindromic of type $+1$) if $\operatorname{rev} P(\lambda) = P(\lambda)^T$ and that it is called *T-anti-palindromic* (or T-palindromic of type -1) if $\operatorname{rev} P(\lambda) = -P(\lambda)^T$, where

$$\operatorname{rev} P(\lambda) = \sum_{j=0}^{k} \lambda^j A_{k-j}$$

is the *reversal* of $P(\lambda)$. We will further call $P(\lambda)$ *palindromic* if it is either T-palindromic or T-anti-palindromic. In [11, Theorem 7.6] necessary conditions for a Smith form to be that of a palindromic matrix polynomial were presented consisting of pairing conditions for the elementary divisors associated with $+1$ and -1 that are parallel to the pairing conditions for the elementary divisors associated with zero in Theorem 2.7 and Theorem 2.8. However, these conditions were not sufficient. In particular, it was highlighted that the problem of finding conditions that are both necessary and sufficient for a matrix polynomial in Smith form to be the Smith form of a palindromic matrix polynomial remains an open problem. Based on Theorem 3.1, we are able to get one step closer to solving this open problem.

As a tool, we will consider the Cayley transformations of matrix polynomials that were used in [9] to relate palindromic and T-alternating matrix polynomials. Recall that for a matrix polynomial $P(\lambda)$ the two Cayley transformations with pole at -1 and $+1$ are given by

$$\mathcal{C}_{-1}(P)(\mu) := (\mu+1)^k P\left(\frac{\mu-1}{\mu+1}\right) \quad \text{and} \quad \mathcal{C}_{+1}(P)(\mu) := (1-\mu)^k P\left(\frac{1+\mu}{1-\mu}\right), \tag{4.1}$$

respectively. The following table is extracted from [9] and adapted to our focus, it displays the correspondence in the structures of the matrix polynomial $P(\lambda)$ and its Cayley transforms.

Table 4.1: Cayley transforms of structured matrix polynomials.

$P(\lambda)$	$\mathcal{C}_{-1}(P)(\mu)$		$\mathcal{C}_{+1}(P)(\mu)$	
	k even	k odd	k even	k odd
T-palindromic	T-even	T-odd	T-even	
T-anti-palindromic	T-odd	T-even	T-odd	
T-even	T-palindromic		T-palindromic	T-anti-palindromic
T-odd	T-anti-palindromic		T-anti-palindromic	T-palindromic

In order to keep the discussion short, we do not review in detail the necessary conditions for a matrix polynomial to be in *P-Smith form*, instead we refer the reader to [11, Theorem 7.6]. Then, we obtain the following result parallel to Theorem 3.1.

Theorem 4.3 *Let $S(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ be regular with $\deg\big(\det S(\lambda)\big) = nk - m$, where m is the algebraic multiplicity of the eigenvalue $\lambda_0 = 0$. Further, let $S(\lambda)$ be in P-Smith form as in* [11, Theorem 7.6]*, i.e., it is the possible Smith form of a T-palindromic or T-anti-palindromic matrix polynomial.*
Then, $S(\lambda)$ is equivalent to a T-palindromic or T-anti-palindromic, respectively, matrix polynomial in lower anti-triangular form of degree k if $+1$ and -1 are not both eigenvalues of $S(\lambda)$.

Proof. We will only prove the theorem in the case that $S(\lambda)$ is the possible Smith form of a T-palindromic matrix polynomial and that -1 is not an eigenvalue of $S(\lambda)$. The other cases, i.e., $S(\lambda)$ is the possible Smith form of a T-anti-palindromic matrix polynomial and/or $+1$ is not an eigenvalue of $S(\lambda)$ can be shown analogously.

By [11, Theorem 7.6] we have

$$S(\lambda) = \operatorname{diag}\left(\lambda^{\alpha_1}(\lambda-1)^{\beta_1}p_1(\lambda), \ldots, \lambda^{\alpha_n}(\lambda-1)^{\beta_n}p_n(\lambda)\right), \tag{4.2}$$

where $0 \le \alpha_1 \le \cdots \le \alpha_n$, $0 \le \beta_1 \le \cdots \le \beta_n$, and $p_j(-1), p_j(0), p_j(1) \neq 0$ for $j = 1, \ldots, n$. Furthermore, all $p_j(\lambda)$ are palindromic (of type $+1$) and all odd exponents β_j occur in equal pairs in adjacent diagonal positions.

Now, consider the list of all elementary divisors of $S(\lambda)$ including infinite elementary divisors of degrees $\alpha_1, \ldots, \alpha_n$. (More precisely, the degrees of the infinite elementary divisors in this list are exactly the nonzero entries of $(\alpha_1, \ldots, \alpha_n)$ as elementary divisors have positive degree.) Then, because of $\sum_{j=1}^{n} \alpha_j = m$, the sum of the degrees of all (finite and infinite) elementary divisors in this list is nk.

Hence, by [17, Section 5] this list of elementary divisors is realizable by an $n \times n$ matrix polynomial $Q(\lambda)$ of degree k, i.e., the Smith form of $Q(\lambda)$ is $S(\lambda)$ and its infinite elementary divisors have the degrees $\alpha_1, \ldots, \alpha_n$. Also, since all $p_j(\lambda)$ from (4.2) are palindromic, for all $\widehat{\lambda} \in \mathbb{F} \setminus \{-1, 0, 1\}$ the elementary divisors of $Q(\lambda)$ associated with $\widehat{\lambda}$ and $\widehat{\lambda}^{-1}$ are paired by [11, Corollary 5.9], i.e., if $(\lambda - \widehat{\lambda})^{\gamma_1}, \ldots, (\lambda - \widehat{\lambda})^{\gamma_\nu}$ are the elementary divisors associated with $\widehat{\lambda}$, then $(\lambda - \widehat{\lambda}^{-1})^{\gamma_1}, \ldots, (\lambda - \widehat{\lambda}^{-1})^{\gamma_\nu}$ are the elementary divisors associated with $\widehat{\lambda}^{-1}$.

Then, the Cayley transformation $\mathcal{C}_{+1}$ transforms $Q(\lambda)$ into the matrix polynomial $\widehat{Q}(\lambda) := \mathcal{C}_{+1}(Q)(\lambda)$ of degree k. As by [18, Theorem 3.4], only the eigenvalue -1 is transformed to an eigenvalue ∞ under $\mathcal{C}_{+1}$, $\widehat{Q}(\lambda)$ only has finite eigenvalues. Let

$$\widehat{S}(\lambda) = \operatorname{diag}\left(\lambda^{\widetilde{\beta}_1}\widetilde{p}_1(\lambda), \ldots, \lambda^{\widetilde{\beta}_n}\widetilde{p}_n(\lambda)\right)$$

be the Smith form of $\widehat{Q}(\lambda)$, where $0 \le \widetilde{\beta}_1 \le \cdots \le \widetilde{\beta}_n$ and $\widetilde{p}_j(0) \neq 0$ for $j = 1, \ldots, n$. Now, we will show that $\widehat{S}(\lambda)$ is in E-Smith form as in Theorem 2.7. Indeed, by [18, Theorem 3.4] (see also [12]), the Cayley transform $\mathcal{C}_{+1}$ transports elementary divisors of $Q(\lambda)$ at $\widehat{\lambda}$ to elementary divisors of $\widehat{Q}(\lambda)$ at $\widehat{\mu}$, where $\widehat{\mu} = (\widehat{\lambda} - 1)/(\widehat{\lambda} + 1)$ as $\widehat{\lambda} \neq -1$. Thus, the degrees of the elementary divisors of $\widehat{Q}(\lambda)$ at $\widehat{\mu}$ and $-\widehat{\mu}$ are given by $\gamma_1, \ldots, \gamma_\nu$ if $\widehat{\mu} \in \mathbb{F} \setminus \{-1, 0, 1\}$ and by $\alpha_1, \ldots, \alpha_n$ if $\widehat{\mu} \in \{-1, 1\}$. Hence, $\widetilde{p}_1(\lambda), \ldots, \widetilde{p}_n(\lambda)$ are necessarily even. Finally, the degrees of the elementary divisors of $Q(\lambda)$ at 1 are those of the elementary divisors of $\widehat{Q}(\lambda)$ at 0, hence $\widetilde{\beta}_j = \beta_j$ for $j = 1, \ldots, n$. As all odd β_j's occur in equal pairs in adjacent diagonal positions, $\widehat{S}(\lambda)$ is indeed in E-Smith form as in Theorem 2.7.

Moreover, $\deg\left(\det \widehat{S}(\lambda)\right) = \deg\left(\det \widehat{Q}(\lambda)\right) = nk$ as $\widehat{Q}(\lambda)$ is regular and does not have the eigenvalue ∞. Thus, by Theorem 3.1 there exist unimodular matrix polynomials $E(\lambda), F(\lambda)$ such that $\widehat{P}(\lambda) := E(\lambda)\widehat{Q}(\lambda)F(\lambda)$ is a T-even, lower anti-triangular matrix polynomial of degree k. Now applying the inverse Cayley transform $\mathcal{C}_{-1}$, we obtain that $P(\lambda) := \mathcal{C}_{-1}(\widehat{P})(\lambda)$ is a T-palindromic matrix polynomial, which is in anti-triangular form. Clearly, the Smith form of $P(\lambda)$ is $S(\lambda)$ (using again [18, Theorem 3.4]) which concludes the proof. □

5 Conclusion

We have studied the inverse T-alternating and T-palindromic polynomial eigenvalue problem over arbitrary algebraically closed fields of characteristic different from two. In particular, we have developed sufficient conditions under which an $n \times n$ matrix polynomial is the Smith form of a T-alternating $n \times n$ matrix polynomial with nonsingular leading coefficient of degree k. The analogous problem for T-palindromic matrix polynomials was considered in the case that not both $+1$ and -1 are eigenvalues. Additionally, the constructed matrix polynomials are in lower anti-triangular form. It remains an open problem to consider the inverse T-alternating eigenvalue problem for the case that not only finite elementary divisors, but also infinite elementary divisors are prescribed $-$ then the techniques developed in this paper cannot be applied. Similarly, the inverse T-palindromic eigenvalue problem remains unsolved if both $+1$ and -1 are prescribed eigenvalues.

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References

[1] E.K. Chu and B.N. Datta. Numerically robust pole assignment for second-order systems. *Int. J. Control*, 64:1113–1127, 1996.

[2] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* Academic Press, New York, 1982.

[3] F.M. Hall. *An Introduction to Abstract Algebra*, volume 1. Cambridge University Press, Cambridge, 1972.

[4] P. Lancaster. Inverse spectral problems for semisimple damped vibrating systems. *SIAM J. Matrix Anal. Appl.*, 29:279–301, 2007.

[5] P. Lancaster and U. Prells. Inverse problems for damped vibrating systems. *J. Sound Vibration*, 283:891–914, 2005.

[6] P. Lancaster and F. Tisseur. Hermitian quadratic matrix polynomials: solvents and inverse problems. *Linear Algebra Appl.*, 436:4017–4026, 2012.

[7] P. Lancaster and I. Zaballa. On the inverse symmetric quadratic eigenvalue problem. Submitted, 2013.

[8] D.S. Mackey. Private communication. 2013.

[9] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28(4):1029–1051, 2006.

[10] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Jordan structures of alternating matrix polynomials. *Linear Algebra Appl.*, 432(4):867–891, 2010.

[11] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Smith forms for palindromic matrix polynomials. *Electron. J. Linear Algebra*, 22:53–91, 2011.

[12] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Möbius transformations of matrix polynomials. In preparation. 2013.

[13] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Skew-symmetric matrix polynomials and their Smith forms. *Linear Algebra Appl.*, 438:4625–4653, 2013.

[14] C. Mehl. Jacobi-like algorithms for the indefinite generalized Hermitian eigenvalue problem. *SIAM J. Matrix Anal. Appl.*, 25:964–985, 2004.

[15] V. Mehrmann and D. Watkins. Polynomial eigenvalue problems with Hamiltonian structure. *Electr. Trans. Num. Anal.*, 13:106–118, 2002.

[16] N.K. Nichols and J. Kautsky. Robust eigenstructure assignment in quadratic matrix polynomials: Nonsingular case. *SIAM J. Matrix Anal. Appl.*, 23:77–102, 2001.

[17] L. Taslaman, F. Tisseur, and I. Zaballa. Triangularizing matrix polynomials. *Linear Algebra Appl.*, 439:1679–1699, 2013.

[18] F. Tisseur and I. Zaballa. *Finite and infinite elementary divisors of matrix polynomials: a global approach.* MIMS EPrint 2012.78, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2012.

[19] F. Tisseur and I. Zaballa. Triangularizing quadratic matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 34:312–337, 2013.