# Anti-triangular and anti-$m$-Hessenberg forms for Hermitian matrices and pencils

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**Abstract**

Hermitian pencils, i.e., pairs of Hermitian matrices, arise in many applications, such as linear quadratic optimal control or quadratic eigenvalue problems. We derive conditions from which anti-triangular and anti-$m$-Hessenberg forms for general (including singular) Hermitian pencils can be obtained under unitary equivalence transformations.

## 1 Introduction

In this paper, we discuss necessary and sufficient conditions for the existence of particular condensed forms for Hermitian matrices and pencils from which eigenvalues and nested sets of invariant subspaces can be obtained. It is the main purpose to include the discussion of singular pencils.

Canonical forms for Hermitian pencils or for related pairs of quadratic or Hermitian forms are well-known and have been widely discussed in literature, starting with the results of Weierstraß for the regular case (see [24]) and the results of Kronecker for the singular case (see [10]). For a complete discussion of canonical forms for Hermitian pencils, see [22], and for a large list of references, see [23].

For the sake of numerical stability, we are interested in finding condensed forms for Hermitian pencils under unitary transformations. In other words, we try to reduce both matrices of the pencil via a simultaneous unitary similarity transformation.

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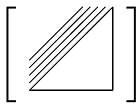
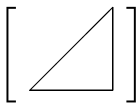
One possible condensed form for Hermitian pencils is the diagonal form. However, the problem of computing this form reduces to the problem of diagonalizing two Hermitian matrices simultaneously. It is well known that this is possible if and only if the matrices commute (see, e.g., [21]).

Other possible condensed forms are the so-called anti-triangular or more general anti-$m$-Hessenberg forms.

**Definition 1** *Let $X = (x_{jk}) \in \mathbb{C}^{n\times n}$ and $m \in \mathbb{N}$. We say that $X$ is* **lower anti-$m$-Hessenberg** *if $x_{jk} = 0$ for all $j, k$ such that $j + k \leq n - m$, i.e., $X$ has the pattern*

.

*Analogously, we say that $X$ is upper anti-$m$-Hessenberg if we have $x_{j,k} = 0$ for all $j, k$ with $j + k > n + m + 1$. If $X$ is lower anti-0-Hessenberg, i.e., $X$ has the pattern*

,

*we also say that $X$ is* **lower anti-triangular**. *If $X$ is lower anti-1-Hessenberg, we also say that $X$ is* **lower anti-Hessenberg**. *Analogously, we define upper anti-triangular and upper anti-Hessenberg matrices.*

As long as it is not stated otherwise, 'anti-triangular' and 'anti-$m$-Hessenberg' always means 'lower anti-triangular' and 'lower anti-$m$-Hessenberg', respectively. Analogous to the matrix case, we define anti-triangular and anti-$m$-Hessenberg forms for pencils.

In this paper we will discuss necessary and sufficient conditions for the existence of anti-triangular and anti-$m$-Hessenberg forms for (possibly singular) Hermitian pencils. In this task, it is sufficient to discuss the existence of these forms under simultaneous congruence, for if $P$ is a nonsingular matrix such that $P^*(\lambda G - H)P$ is in anti-triangular form (or in anti-$m$-Hessenberg form), then $P$ can be chosen to be unitary. This follows easily by applying a QR-decomposition on $P$, see also Lemma 2 in the following section. Hence, both $G$ and $H$ are simultaneously unitarily similar to anti-triangular matrices (or to anti-$m$-Hessenberg matrices, respectively).

It will turn out that the existence of anti-triangular forms for singular Hermitian pencils is equivalent to the existence of anti-$m$-Hessenberg forms for certain regular Hermitian pencils. This motivates our interest in anti-$m$-Hessenberg forms in addition to anti-triangular forms.

But besides this, the special case of anti-1-Hessenberg forms of Hermitian pencils is of interest itself. During the numerical computation of the Schur form of a matrix, the matrix is usually reduced to Hessenberg form in the first step (see, e.g., [8]). Anti-Hessenberg forms in the Hermitian case seem to be the analogue of Hessenberg forms in the general case.

The motivation for the research in this paper arises from structured eigenvalue problems in control theory and in the numerical simulation of mechanical systems.

The first application is the linear quadratic optimal control problem, see [12, 13, 18] and the references therein. This is the problem of minimizing the cost functional

$$\frac{1}{2}\int_{t_0}^{\infty}\Big(x(t)^*Qx(t)+u(t)^*Ru(t)+u(t)^*S^*x(t)+x(t)^*Su(t)\Big)dt \tag{1}$$

subject to the dynamics

$$\begin{aligned} E\dot{x}(t) &= Ax(t)+Bu(t), \qquad t_0<t &(2)\\ x(t_0) &= x_0, &(3)\end{aligned}$$

where $A, E, Q \in \mathbb{C}^{n\times n}, B, S \in \mathbb{C}^{n\times m}, R \in \mathbb{C}^{m\times m}$, $Q$, $R$ Hermitian, $x_0, x(t), u(t) \in \mathbb{C}^n$, and $t_0, t \in \mathbb{R}$. It is known that solutions of (1)–(3) can be obtained via the solution of a boundary value problem, see [17, 18] and the references therein. For the solution of this boundary value problem one has to compute deflating subspaces of the matrix pencil

$$\lambda\begin{bmatrix} E & 0 & 0\\ 0 & -E^* & 0\\ 0 & 0 & 0\end{bmatrix}-\begin{bmatrix} A & 0 & B\\ Q & A^* & S\\ S^* & B^* & R\end{bmatrix}. \tag{4}$$

Applying a row permutation, we see that the pencil (4) is equivalent to the pencil

$$\lambda\mathcal{A}-\mathcal{B}=\lambda\begin{bmatrix} 0 & -E^* & 0\\ E & 0 & 0\\ 0 & 0 & 0\end{bmatrix}-\begin{bmatrix} Q & A^* & S\\ A & 0 & B\\ S^* & B^* & R\end{bmatrix}. \tag{5}$$

Multiplying $\mathcal{A}$ by $i$, we find that $\lambda i\mathcal{A}-\mathcal{B}$ is a Hermitian pencil, i.e., both $i\mathcal{A}$ and $\mathcal{B}$ are Hermitian. Clearly, both pencils $\lambda\mathcal{A}-\mathcal{B}$ and $\lambda i\mathcal{A}-\mathcal{B}$ have the same right deflating subspaces and the eigenvalues of $\lambda i\mathcal{A}-\mathcal{B}$ coincide with the eigenvalues of $\lambda\mathcal{A}-\mathcal{B}$ multiplied by $i$. Therefore, to analyze and compute eigenvalues and deflating subspaces, it is sufficient to consider the Hermitian pencil $\lambda i\mathcal{A}-\mathcal{B}$. It should be noted, however, that if the original problem is real, then we have obtained an Hermitian nonreal problem in this way. For the real case one has to discuss 'skew-Hermitian/Hermitian' pencils $\lambda\mathcal{S}-\mathcal{H}$, i.e., pencils where $\mathcal{S}$ is skew Hermitian and $\mathcal{H}$ is Hermitian. This case is more complicated, because one has to deal with an additional symmetry. It is well known that the spectra of skew-Hermitian/Hermitian pencils are symmetric with respect to the imaginary axis (see [23]). In the real case, the spectra have an additional symmetry with respect to the real axis. In this paper, we only consider the complex case. The real case is referred to a later discussion.

Other applications of Hermitian pencils arise in the numerical treatment of quadratic eigenvalue problems in mechanics. In quadratic eigenvalue problems one is interested in computing $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \backslash \{0\}$ such that

$$(A + \lambda B + \lambda^2 C)x = 0,$$

where typically $A, C \in \mathbb{C}^{n\times n}$ are Hermitian and $B$ is Hermitian or skew Hermitian. Hermitian quadratic eigenvalue problems arise for example in the analysis of geometrical nonlinear buckling structures with finite element methods (see [3, 9]) or in the theory of damped oscillatory systems (see [6, 11]). With the substitution $\mu = \frac{1}{\lambda}$ for $\lambda \neq 0$, the problem can be linearized such that it reduces to the generalized Hermitian eigenvalue problem

$$\mu \begin{bmatrix} B & A \\ A & 0 \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \begin{bmatrix} -C & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix}, \tag{6}$$

see, e.g., [9]. Quadratic eigenvalue problems with $B$ skew Hermitian arise in numerical simulation of the deformation of anisotropic materials (see [14]) and the acoustic simulation of poroelastic materials (see [20]). In this case, the substitution $\mu = i\lambda$ leads to the linearized eigenvalue problem

$$\mu \begin{bmatrix} 0 & iC \\ -iC & -iB \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix} = \begin{bmatrix} -C & 0 \\ 0 & -A \end{bmatrix} \begin{bmatrix} \lambda x \\ x \end{bmatrix}, \tag{7}$$

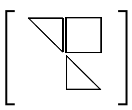
For a detailed study of Hermitian quadratic eigenvalue problems, and more general, of matrix polynomials see [6].

Anti-triangular forms for Hermitian pencils are related to Schur-like forms for skew-Hamiltonian/Hamiltonian pencils that are discussed in [16]. A skew-Hamiltonian/Hamiltonian pencil is a pencil $\lambda S - H$ such that $S$ is skew-Hamiltonian, that is $SJ - JS^* = 0$, and such that $H$ is Hamiltonian, that is $HJ + JH^* = 0$, where

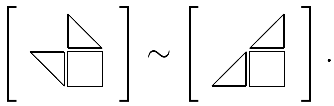
$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Thus, skew-Hamiltonian/Hamiltonian pencils are structured with respect to an indefinite inner product, defined by the matix $J$. Condensed forms for matrices and pencils that are structured with respect to indefinite inner products have been widely discussed in the literature, see [4, 5, 7, 12, 15, 19, 25], to name a few.

If $\lambda S - H$ is a skew-Hamiltonian/Hamiltonian pencil, then the pencil $\lambda iJS - JH$ is Hermitian. Furthermore, if $\lambda S - H$ is in Schur-like form, i.e., $\lambda S - H$ has the pattern

 ,

then the corresponding Hermitian pencil $\lambda iJS - JH$ is congruent to a pencil in anti-triangular form and has the pattern



(Here, $\sim$ denotes congruence.) From this point of view, it seems that anti-triangular forms for Hermitian pencils are the natural forms to look for if one is interested in obtaining condensed forms under unitary transformations.

Hessenberg-like forms for Hamiltonian matrices have been discussed in, e.g., [1, 4]. Anti-Hessenberg forms for Hermitian matrices correspond to Hessenberg-like forms for Hamiltonian matrices.

In [16] it was shown that not every regular skew-Hamiltonian/Hamiltonian pencil can be reduced to Schur-like form. This generalizes a result on Hamiltonian matrices (see [15]). The reason why a Schur-like form does not always exist is because certain conditions on the purely imaginary eigenvalues have to be satisfied. This comes from the fact that purely imaginary eigenvalues of Hamiltonian matrices have signs $\varepsilon = \pm 1$ that are invariant under structure-preserving transformations, see [15], or [6, 12] for a more general setting. An analogous situation holds in the pencil case (see [16, 22]).

However, the consideration of Hermitian pencils is more general than the consideration of skew-Hamiltonian/Hamiltonian pencils, since the case of odd-sized pencils is included in the context of Hermitian pencils. Furthermore, only the case of regular pencils is discussed in [16], and it is the purpose of this paper to include the singular case. This case is of interest as well; see for example [18] for applications when the pencil (5) is singular.

In Section 2 we will discuss basic properties of Hermitian anti-triangular and anti-$m$-Hessenberg matrices and in Section 3 we discuss corresponding forms for the case of regular Hermitian pencils. In Section 3 another important condensed form for Hermitian pencils is derived, the so-called sign condensed form. In a certain sense, this form displays 'how far away' a Hermitian pencil is from being congruent to anti-triangular or anti-$m$-Hessenberg form. The case of singular pencils will be discussed in section 4.

Throughout the paper we use the following notation.

1. Given two square matrices $A$, $B$, we define the *direct sum* $A \oplus B$ of $A$ and $B$ by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

   Analogously we define the direct sum of square pencils.

2. By $Z_p$ we denote the $p \times p$ zip matrix $Z_p = [\delta_{i+j,p+1}]_{i,j=1}^{p}$ with ones on the anti-diagonal and zeros elsewhere. By $\mathcal{O}_p$ we denote the $p \times p$ zero matrix.

3. By $\sigma(\lambda)$ we denote the sign of $\lambda \in \mathbb{R}$, that is

$$\sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda = 0, \\ -1 & \text{if } \lambda < 0. \end{cases}$$

4. By $A \sim B$ we denote that the matrices $A$ and $B$ are congruent.
5. By $\mathrm{spec}(A)$ we denote the spectrum of a square matrix $A$.
6. By $e_j$ we denote the $j$th unit vector.
7. The abbreviation "w.l.o.g." for "without loss of generality" will be frequently used.

## 2 Anti-triangular and anti-$m$-Hessenberg forms

In this section we discuss conditions when Hermitian matrices can be transformed to anti-triangular and anti-$m$-Hessenberg matrices via unitary congruence transformations. It turns out that the conditions for unitary congruence are the same as for congruence.

**Lemma 2** *Let $A \in \mathbb{C}^{n\times n}$. If $A$ is congruent to an anti-$m$-Hessenberg matrix for some $m \in \mathbb{N}$ then $A$ is unitarily similar to an anti-$m$-Hessenberg matrix.*

**Proof**. Let $\tilde{A}$ be in anti-$m$-Hessenberg form and let $\tilde{A}$ and $A$ be congruent, i.e., there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$, such that $P^*AP = \tilde{A}$. Let $P = QR$ be a QR-decomposition (see [8]) of $P$. Then $Q^*AQ = R^{-*}\tilde{A}R^{-1}$ is still anti-$m$-Hessenberg. □

Let us recall that the **inertia index** of a Hermitian matrix $G$ is

$$\mathrm{Ind}(G) = (\nu_+, \nu_-, \nu_0),$$

where $\nu_+, \nu_-, \nu_0$ are the numbers of positive, negative and zero eigenvalues of $G$, respectively. Conditions for the existence of both anti-triangular and anti-$m$-Hessenberg forms will be based on the following lemma.

**Lemma 3** *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and let $\mathrm{Ind}(A) = (\nu_+, \nu_-, \nu_0)$. Then $A$ is congruent to a matrix of the form*

$$\begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix}, \tag{8}$$

*where $A_3 \in \mathbb{C}^{k\times k}$, $A_2 \in \mathbb{C}^{(n-k)\times k}$ if and only if $|\nu_+ - \nu_-| \le 2k + \nu_0 - n$.*

**Proof**. ($\Rightarrow$): Let $A$ be in the form (8). Then there exist $S \in \mathbb{C}^{(n-k)\times(n-k)}$ and $T \in \mathbb{C}^{k\times k}$ nonsingular such that

$$SA_2T = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix},$$

where $m \leq k, n-k$. From this we obtain that

$$\begin{bmatrix} S & 0 \\ 0 & T^* \end{bmatrix} A \begin{bmatrix} S^* & 0 \\ 0 & T \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} m & n-k-m & m & k-m \end{array} \\ \begin{bmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \\ I_m & 0 & A_{31} & A_{32} \\ 0 & 0 & A_{32}^* & A_{33} \end{bmatrix} \end{array},$$

for some $A_{31}, A_{32}$, and $A_{33}$. Furthermore, we obtain

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -\frac{1}{2}A_{31} & 0 & I & 0 \\ -A_{32}^* & 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \\ I_m & 0 & A_{31} & A_{32} \\ 0 & 0 & A_{32}^* & A_{33} \end{bmatrix} \begin{bmatrix} I & 0 & -\frac{1}{2}A_{31} & -A_{32} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{33} \end{bmatrix}.$$

This implies $\mathrm{Ind}(A) = (m, m, n-k-m) + \mathrm{Ind}(A_{33})$. Moreover, since $A_{33}$ is a $(k-m)\times(k-m)$ matrix, we obtain from $n-k-m \leq \nu_0$ that

$$|\nu_+ - \nu_-| \leq k - m = 2k + n - k - m - n \leq 2k + \nu_0 - n.$$

($\Leftarrow$): Assume w.l.o.g. that $\nu_+ - \nu_- \geq 0$; otherwise consider $-A$. Then the matrix

$$\tilde{A} = \begin{bmatrix} 0 & 0 & I_{\nu_-} & 0 \\ 0 & O_{\nu_0} & 0 & 0 \\ I_{\nu_-} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\nu_+ - \nu_-} \end{bmatrix}$$

is congruent to $A$, since $\mathrm{Ind}(\tilde{A}) = (\nu_+, \nu_-, \nu_0)$. It remains to show that $\nu_- + \nu_0 \geq n - k$ and this follows from

$$\begin{aligned} \nu_- + \nu_0 &= n - \nu_+ = n - \nu_- - (\nu_+ - \nu_-) \\ &\geq n - \nu_- - (2k + \nu_0 - n) = 2(n-k) - (\nu_- + \nu_0). \quad \square \end{aligned}$$

**Corollary 4** *Let* $A \in \mathbb{C}^{n\times n}$ *be Hermitian,* $\mathrm{Ind}(A) = (\nu_+, \nu_-, \nu_0)$*, and* $m \in \mathbb{N}$*, where* $m < n$*.*

1. *If $n-m$ is even, then $A$ is congruent to an anti-$m$-Hessenberg matrix if and only if*

$$|\nu_+ - \nu_-| \le \nu_0 + m.$$

2. *If $n-m$ is odd, then $A$ is congruent to an anti-$m$-Hessenberg matrix if and only if*

$$|\nu_+ - \nu_-| \le \nu_0 + m + 1.$$

**Proof**. Let us first consider the case that $n-m$ is even. If $A$ is congruent to an anti-$m$-Hessenberg matrix, then in particular $A$ is congruent to a matrix of the form

$$\begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix},$$

where $A_3 \in \mathbb{C}^{k\times k}$ and $A_2 \in \mathbb{C}^{(n-k)\times k}$ with $k := \frac{n+m}{2}$. Hence, Lemma 3 implies that

$$|\nu_+ - \nu_-| \le 2k + \nu_0 - n = \nu_0 + m.$$

Conversely assume that $|\nu_+ - \nu_-| \le \nu_0 + m$. Then Lemma 3 implies that $A$ is congruent to a matrix of the form

$$\begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix},$$

where $A_2 \in \mathbb{C}^{(n-k)\times k}$ and $A_3 \in \mathbb{C}^{k\times k}$. Let $S \in \mathbb{C}^{(n-k)\times(n-k)}$ and $T \in \mathbb{C}^{k\times k}$ be nonsingular, such that

$$SA_2T = \begin{bmatrix} 0 & \tilde{A}_2 \end{bmatrix},$$

where $\tilde{A}_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ is anti-triangular. Clearly such matrices always exist. It follows that

$$\begin{bmatrix} S & 0 \\ 0 & T^* \end{bmatrix} \begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix} \begin{bmatrix} S^* & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} 0 & SA_2T \\ (SA_2T)^* & T^*A_3T \end{bmatrix}$$

is anti-triangular, and thus, $A$ is congruent to an anti-triangular matrix. The case that $n-m$ is odd follows in an analogous way, noting that in this case an anti-$m$-Hessenberg form of $A$ has the structure

$$\begin{bmatrix} 0 & A_2 \\ A_2^* & A_3 \end{bmatrix},$$

where $A_3 \in \mathbb{C}^{k\times k}$ and $A_2 \in \mathbb{C}^{(n-k)\times k}$ with $k := \frac{n+m+1}{2}$. $\square$

The next result is a special case of Corollary 4.

**Corollary 5** *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and let* $\mathrm{Ind}(A) = (\nu_+, \nu_-, \nu_0)$.

1. *If $n$ is even, $A$ is congruent to an anti-triangular matrix if and only if*

$$|\nu_+ - \nu_-| \le \nu_0.$$

2. *If $n$ is odd, $A$ is congruent to an anti-triangular matrix if and only if*

$$|\nu_+ - \nu_-| \leq \nu_0 + 1.$$

We see from these results that the inertia indices of Hermitian matrices play a key role in the discussion of anti-triangular and anti-$m$-Hessenberg forms. The following lemma establishes an auxiliary result for the computation of the inertia index of some special Hermitian matrices.

**Lemma 6** *Let $A \in \mathbb{C}^{n \times n}$ be an Hermitian matrix of the form*

$$A = \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33}, \end{bmatrix},$$

*where $A_{13} \in \mathbb{C}^{m \times k}$ and $A_{22} \in \mathbb{C}^{(n-m-k)\times(n-m-k)}$.*

1. *If $m = k$ and $A_{13}$ is invertible, then* $\mathrm{Ind}(A) = (m, m, 0) + \mathrm{Ind}(A_{22})$.
2. *If $A_{22} \in \mathbb{C}^{(n-m-k)\times(n-m-k)}$ is invertible, then*

$$\mathrm{Ind}(A) = \mathrm{Ind}\left(\begin{bmatrix} 0 & A_{13} \\ A_{13}^* & \tilde{A}_{33} \end{bmatrix}\right) + \mathrm{Ind}(A_{22}),$$

*where $\tilde{A}_{33} = A_{33} - A_{23}^* A_{22}^{-1} A_{23}$.*

**Proof**. This follows easily using Schur complements. □

# 3 Condensed forms for regular Hermitian pencils

In this section we discuss condensed forms for regular Hermitian pencils, that is, pencils $\lambda G - H \in \mathbb{C}^{n \times n}$ such that both $G$ and $H$ are Hermitian and such that $\det(\lambda G - H) \not\equiv 0$. These forms are the canonical form, anti-triangular forms that can be obtained via a unitary similarity transformation that operates simultaneously on $G$ and $H$, anti-$m$-Hessenberg forms, and the so-called sign condensed form. First let us recall the well-known canonical form for Hermitian pencils (see [22]).

**Theorem 7** *Let $\lambda G - H$ be a regular Hermitian pencil. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that*

$$P^*(\lambda G - H)P = (\lambda G_1 - H_1) \oplus \ldots \oplus (\lambda G_l - H_l), \tag{9}$$

*where the blocks $\lambda G_j - H_j$ have one and only one of the following forms.*

1. *Blocks associated with paired nonreal eigenvalues $\lambda_0$, $\lambda_0^*$:*

$$\lambda \begin{bmatrix} 0 & Z_r \\ Z_r & 0 \end{bmatrix} - \begin{bmatrix} 0 & Z_r\mathcal{J}_r(\lambda_0) \\ \mathcal{J}_r(\lambda_0)^* Z_r & 0 \end{bmatrix}.$$

2. *Blocks associated with real eigenvalues $\lambda_0$ and sign $\varepsilon \in \{1, -1\}$:*

$$\lambda\varepsilon Z_r - \varepsilon Z_r \mathcal{J}_r(\lambda_0) = \lambda\varepsilon \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & & & \lambda_0 \\ & & \lambda_0 & 1 \\ & \iddots & \iddots & \\ \lambda_0 & 1 & & 0 \end{bmatrix}.$$

3. *Blocks associated with the eigenvalue $\infty$ and sign $\varepsilon \in \{1, -1\}$:*

$$\lambda\varepsilon Z_r \mathcal{J}_r(0) - \varepsilon Z_r = \lambda\varepsilon \begin{bmatrix} 0 & & & 0 \\ & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

**Proof**. See [22]. □

**Definition 8** *Let $\lambda G - H$ be a regular Hermitian pencil and let $\lambda G_j - H_j$ be a single block of the canonical form (9) of $\lambda G - H$. If $\lambda G_j - H_j$ is a block of type* (2) *or* (3) *then the parameter $\varepsilon$ that appears in the canonical form (9) is called the* **sign** *associated with the block $\lambda G_j - H_j$.*

Besides the eigenvalues of a Hermitian pencil, the signs associated with blocks to real eigenvalues or the eigenvalue $\infty$ are invariants under congruence. The collection of these signs is sometimes referred to as the **sign characteristic** (see, e.g., [7, 12] for related work on $H$-selfadjoint matrices, where $H$ is a nonsingular Hermitian matrix). It will turn out that especially the signs of odd-sized blocks play a key role in our investigation of condensed forms. This motivates the following definition of the sign sum.

**Definition 9** *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a regular Hermitian pencil and let $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ be a real eigenvalue of $\lambda G - H$ with partial multiplicities $(p_1, \ldots, p_r, p_{r+1}, \ldots, p_m)$, where $p_1, \ldots, p_r$ are odd and $p_{r+1}, \ldots, p_m$ are even.*

1. *The tupel $(\varepsilon_1, \ldots, \varepsilon_m)$ is called the* **sign characteristic** *of $\lambda_0$, where $\varepsilon_j$ is the sign associated with the block in the canonical form (9) that corresponds to $\lambda_0$ and $p_j$.*

2. *The integer* $\mathrm{Signsum}(\lambda_0, G, H) := \varepsilon_1 + \ldots + \varepsilon_r$ *is called the* **sign sum** *of $\lambda_0$ with respect to $\lambda G - H$. If there is no risk of confusion we write* $\mathrm{Signsum}(\lambda_0)$ *instead of* $\mathrm{Signsum}(\lambda_0, G, H)$.

In addition, we set $\mathrm{Signsum}(\lambda_0, G, H) = 0$, whenever $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ is not an eigenvalue of $\lambda G - H$. We note that if in the canonical form (9) there are only even-sized blocks associated with $\lambda_0$, then $\mathrm{Signsum}(\lambda_0) = 0$, since the sign sum is obtained by the sum of the signs that correspond to odd-sized blocks. The following theorem allows to 'split' a regular Hermitian pencil into an anti-triangular part and a diagonal part. Furthermore, all the information on the sign sum, i.e., all information on the signs that is needed in the following, can be read off the diagonal part. For the proof of this result, we first state the following auxiliary remark.

**Remark 10** Let $A \in \mathbb{C}^{n\times n}$ be Hermitian.

1. If $A = \begin{bmatrix} 0 & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$, then $A$ is congruent to $\begin{bmatrix} 0 & 0 & A_{12} \\ 0 & A_{33} & 0 \\ A_{12}^* & 0 & A_{22} \end{bmatrix}$.

2. If $A = \begin{bmatrix} 0 & 0 & A_{13} & 0 \\ 0 & A_{22} & A_{23} & 0 \\ A_{13}^* & A_{23}^* & A_{33} & 0 \\ 0 & 0 & 0 & A_{44} \end{bmatrix}$, then $A$ is congruent to $\begin{bmatrix} 0 & 0 & 0 & A_{13} \\ 0 & A_{22} & 0 & A_{23} \\ 0 & 0 & A_{44} & 0 \\ A_{13}^* & A_{23}^* & 0 & A_{33} \end{bmatrix}$.

**Theorem 11 (Sign condensed form)** *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a regular Hermitian pencil. Then there exists $m \in \mathbb{N}$ and a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^*(\lambda G - H)P = \lambda \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & G_{23} \\ G_{13}^* & G_{23}^* & G_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix}, \tag{10}$$

*where $G_{13}, H_{13} \in \mathbb{C}^{m\times m}$ are anti-triangular and*

$$\begin{aligned} &\lambda G_{22} - H_{22} \\ = \ &\lambda \begin{bmatrix} \varepsilon_1 I_{p_1} & & 0 \\ & \ddots & \\ & & \varepsilon_k I_{p_k} \\ 0 & & 0 \end{bmatrix} - \begin{bmatrix} \varepsilon_1 \lambda_1 I_{p_1} & & & 0 \\ & \ddots & & \\ & & \varepsilon_k \lambda_k I_{p_k} & \\ 0 & & & \varepsilon_{k+1} I_{p_{k+1}} \end{bmatrix}, \end{aligned} \tag{11}$$

*where $\lambda_1 < \ldots < \lambda_k$ and $\varepsilon_1, \ldots, \varepsilon_{k+1} \in \{1, -1\}$. Furthermore, we have for all $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ that*

$$\mathrm{Signsum}(\lambda_0, G, H) = \mathrm{Signsum}(\lambda_0, G_{22}, H_{22}).$$

**Proof**. Assume, w.l.o.g., that $\lambda G - H$ is in the canonical form (9). The proof now proceeds by induction on the number $l$ of distinct real eigenvalues, including the eigenvalue $\infty$.

$l = 0$: If $\lambda G - H$ has neither real eigenvalues nor the eigenvalue $\infty$, then clearly all the blocks in the canonical form (9) have even sizes. Thus, applying Remark 10 part 1 repeatedly, we find that $\lambda G - H$ is congruent to a pencil in form (10), where the block $\lambda G_{22} - H_{22}$ does not appear.

$l \Rightarrow l+1$: Let us pick an eigenvalue $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ of $\lambda G - H$. For the sake of briefness of notation, we consider only the case $\lambda_0 \in \mathbb{R}$. The case $\lambda_0 = \infty$ can be proved analogously. (This can be seen easily by interchanging the roles of $G$ and $H$.) After a possible reordering of blocks, we may assume that

$$\lambda G - H = \lambda \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} - \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix},$$

where $\lambda G_1 - H_1$ contains all the blocks associated with $\lambda_0$ and $\lambda G_2 - H_2$ contains all the other blocks. We assume furthermore that $\lambda G_1 - H_1$ contains $p_+$ odd-sized blocks with sign $+1$ and $p_-$ odd-sized blocks with sign $-1$, i.e., in particular we have $\mathrm{Signsum}(\lambda_0) = p_+ - p_-$. Then, applying Remark 10 several times to $\lambda G_1 - H_1$ and possibly reordering some blocks, we find that

$$\begin{aligned} &\lambda G - H \\ \sim\ &\lambda \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{G}_{15} \\ 0 & I_{p_+} & 0 & 0 & 0 \\ 0 & 0 & -I_{p_-} & 0 & 0 \\ 0 & 0 & 0 & G_2 & 0 \\ \hat{G}_{15}^* & 0 & 0 & 0 & \hat{G}_{55} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{H}_{15} \\ 0 & \lambda_0 I_{p_+} & 0 & 0 & \hat{H}_{25} \\ 0 & 0 & -\lambda_0 I_{p_-} & 0 & \hat{H}_{35} \\ 0 & 0 & 0 & H_2 & 0 \\ \hat{H}_{15}^* & \hat{H}_{25}^* & \hat{H}_{35}^* & 0 & \hat{H}_{55} \end{bmatrix}, \end{aligned}$$

where $\hat{G}_{15}$ and $\hat{H}_{15}$ are anti-triangular. Let us assume, w.l.o.g., that $p_+ \geq p_-$. Setting

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} I_{p_+ - p_-} & 0 & 0 \\ 0 & I_{p_-} & I_{p_-} \\ 0 & -I_{p_-} & I_{p_-} \end{bmatrix}$$

and noting that

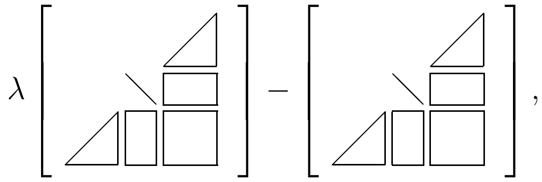
$$\begin{aligned} &P^* \left( \lambda \begin{bmatrix} I_{p_+-p_-} & 0 & 0 \\ 0 & I_{p_-} & 0 \\ 0 & 0 & -I_{p_-} \end{bmatrix} - \begin{bmatrix} \lambda_0 I_{p_+-p_-} & 0 & 0 \\ 0 & \lambda_0 I_{p_-} & 0 \\ 0 & 0 & -\lambda_0 I_{p_-} \end{bmatrix} \right) P \\ = \ &\lambda \begin{bmatrix} I_{p_+-p_-} & 0 & 0 \\ 0 & 0 & I_{p_-} \\ 0 & I_{p_-} & 0 \end{bmatrix} - \begin{bmatrix} \lambda_0 I_{p_+-p_-} & 0 & 0 \\ 0 & 0 & \lambda_0 I_{p_-} \\ 0 & \lambda_0 I_{p_-} & 0 \end{bmatrix}, \end{aligned}$$

by applying Remark 10, we obtain that

$$\lambda G - H \sim \lambda \begin{bmatrix} 0 & 0 & 0 & \check{G}_{14} \\ 0 & I_{p_+-p_-} & 0 & 0 \\ 0 & 0 & G_2 & 0 \\ \check{G}_{14}^* & 0 & 0 & \check{G}_{44} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \check{H}_{14} \\ 0 & \lambda_0 I_{p_+-p_-} & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ \check{H}_{14}^* & 0 & 0 & \check{H}_{44} \end{bmatrix},$$

where $\check{G}_{14}$ and $\check{H}_{14}$ are anti-triangular and the block $\lambda I_{p_+-p_-} - \lambda_0 I_{p_+-p_-}$ displays the sign sum of $\lambda_0$. Using the induction hypothesis on $\lambda G_2 - H_2$, the result follows by one more application of Remark 10. $\square$

**Remark 12** The pencil $P^*(\lambda G - H)P$ has the pattern



and the sign sum of each real eigenvalue or the eigenvalue $\infty$ of $\lambda G - H$ can be easily read off the subpencil $\lambda G_{22} - H_{22}$, since obviously we have

$$\mathrm{Signsum}(\lambda_\alpha, G_{22}, H_{22}) = \varepsilon_\alpha p_\alpha \quad \text{for } \alpha = 1, \dots, k+1.$$

**Remark 13** In [16], it was shown how to obtain an analogue of form (10) for skew-Hamiltonian/Hamiltonian pencils. This method can be easily adapted to Hermitian pencils. Doing so, one can see that in a step-wise reduction, *the reduction to the blocks $G_{13}$ and $H_{13}$ can be executed via unitary transformations.*

In the following we will deduce necessary and sufficient conditions for the existence of anti-triangular forms and anti-$m$-Hessenberg forms for Hermitian pencils. Given a Hermitian pencil $\lambda G - H$, we note that for every $t \in \mathbb{R}$, we have a Hermitian matrix $tG - H$. It is clear that if the pencil $\lambda G - H$ is in anti-triangular form then so is the Hermitian matrix $tG - H$. It will turn out that also the converse is true - at least in the case that the size of the pencil is even. Therefore, the results of section 2 imply that the existence of anti-triangular forms for the Hermitian pencil $\lambda G - H$ is linked to conditions on the indices of the matrices $tG - H$, where $t$ is real.

Moreover, we will see that these conditions on indices can be interpreted as conditions on the sign sums of the real eigenvalues and the eigenvalue $\infty$ of the pencil $\lambda G - H$. Since we may assume that the pencil is in sign condensed form and since the blocks $G_{13}$ and $H_{13}$ in (10) are already in anti-triangular form, it remains to consider the block (11) that inherits all information on the sign sums. The following lemma examines this block and will be applied repeatedly.

**Lemma 14** *Consider the pencil $\lambda G_{22} - H_{22}$ in form (11). Furthermore, let $t_1, t_2 \in \mathbb{R}$ such that*

$$(\lambda_1 \le \ldots \le \lambda_{\alpha-1} <)\ t_1 < \lambda_\alpha \le \ldots \le \lambda_{\alpha+\beta} < t_2\ (< \lambda_{\alpha+\beta+1} \le \ldots \le \lambda_k).$$

*(Here, we allow $\alpha, \beta = 0, \dots, k$, where $\alpha + \beta \le k$, and we ignore terms if they are not defined.) Then setting* $\operatorname{Ind}(tG_{22} - H_{22}) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$, *we obtain that*

$$\Big(\nu_+(t_2) - \nu_-(t_2)\Big) - \Big(\nu_+(t_1) - \nu_-(t_1)\Big) = 2\sum_{j=\alpha}^{\alpha+\beta} \varepsilon_j p_j. \qquad (12)$$

*and*

$$\begin{aligned} &\Big(\nu_+(t_2) - \nu_-(t_2)\Big) + \Big(\nu_+(t_1) - \nu_-(t_1)\Big) \\ = \quad & 2\left(\sum_{j=1}^{\alpha-1} \varepsilon_j p_j\right) - 2\left(\sum_{j=\alpha+\beta+1}^{k+1} \varepsilon_j p_j\right). \end{aligned} \qquad (13)$$

**Proof**. We obtain that

$$\nu_+(t_1) - \nu_-(t_1) = \left(\sum_{j=1}^{\alpha-1} \varepsilon_j p_j\right) - \left(\sum_{j=\alpha}^{\alpha+\beta} \varepsilon_j p_j\right) - \left(\sum_{j=\alpha+\beta+1}^{k+1} \varepsilon_j p_j\right), \qquad (14)$$

$$\nu_+(t_2) - \nu_-(t_2) = \left(\sum_{j=1}^{\alpha-1} \varepsilon_j p_j\right) + \left(\sum_{j=\alpha}^{\alpha+\beta} \varepsilon_j p_j\right) - \left(\sum_{j=\alpha+\beta+1}^{k+1} \varepsilon_j p_j\right). \qquad (15)$$

This implies the assertion. $\square$

We are now able to discuss necessary and sufficient conditions for the existence of anti-triangular forms for regular Hermitian pencils. We start with a result for the case that the size of the pencil is even.

**Theorem 15** *Let $\lambda G - H \in \mathbb{C}^{2n \times 2n}$ be a regular Hermitian pencil and for $t \in \mathbb{R}$ let Ind$(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-triangular form.*
2. *$\lambda G - H$ is unitarily congruent to a pencil in anti-triangular form.*
3. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t)$.*
4. *For almost all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t)$.*
5. *If $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ is an eigenvalue of $\lambda G - H$ then* $\operatorname{Signsum}(\lambda_0) = 0$.

**Proof**. 1) $\Rightarrow$ 2): This follows directly from Lemma 2.

2) $\Rightarrow$ 3): Let $P \in \mathbb{C}^{2n \times 2n}$ be nonsingular such that $P^*(\lambda G - H)P$ is in anti-triangular form. Then clearly $P^*(tG - H)P$ is Hermitian anti-triangular for all $t \in \mathbb{R}$. Thus, 2) follows from Corollary 5.

3) $\Rightarrow$ 4): This implication is trivial.

4) $\Rightarrow$ 5): W.l.o.g. we may assume that $\lambda G - H$ is in sign condensed form (10). If $\lambda_0$ is not an eigenvalue of $\lambda G_{22} - H_{22}$ then trivially $\mathrm{Signsum}(\lambda_0) = 0$. Thus, let us consider an eigenvalue $\lambda_\alpha$ of $\lambda G_{22} - H_{22}$. There are two possible cases.

Case (1) Assume that $\lambda_\alpha \in \mathbb{R}$, that is $\alpha \in \{1, \dots, k\}$, where $\lambda_1, \dots, \lambda_k$ are as in (11).

Choose $t_1, t_2 \in \mathbb{R}$ such that

$$\lambda_1 < \cdots < \lambda_{\alpha-1} < t_1 < \lambda_\alpha < t_2 < \lambda_{\alpha+1} < \cdots < \lambda_k,$$

and furthermore such that $|\nu_+(t_j) - \nu_-(t_j)| \leq \nu_0(t_j)$ holds for $j = 1, 2$ and that $t_1 G - H$ and $t_2 G - H$ are nonsingular. This is possible, since the pencil $\lambda G - H$ is regular, i.e., $tG - H$ is nonsingular for almost all $t \in \mathbb{R}$, and, in addition, condition 3) holds. Then, we obtain from (10) and Lemma 6 that

$$\Big(\nu_+(t_j), \nu_-(t_j), \nu_0(t_j)\Big) = (m, m, 0) + \mathrm{Ind}(t_j G_{22} - H_{22}) \quad \text{for } j = 1, 2.$$

Since $t_1 G - H$ and $t_2 G - H$ are nonsingular, we have $\nu_0(t_1) = \nu_0(t_2) = 0$. Therefore, we obtain from Lemma 14 that

$$\begin{aligned} 0 &= \nu_0(t_2) + \nu_0(t_1) \geq |\nu_+(t_2) - \nu_-(t_2)| + |\nu_+(t_1) - \nu_-(t_1)| \\ &\geq \Big|\Big(\nu_+(t_2) - \nu_-(t_2)\Big) - \Big(\nu_+(t_1) - \nu_-(t_1)\Big)\Big| = 2 \cdot |\mathrm{Signsum}(\lambda_\alpha)|. \end{aligned}$$

This implies $\mathrm{Signsum}(\lambda_\alpha) = 0$.

Case (2) If the assumption of Case (1) does not hold, then $\lambda_\alpha = \infty$.

In this case, we choose $t_1, t_2 \in \mathbb{R}$ such that

$$t_1 < \lambda_1 < \ldots < \lambda_k < t_2,$$

and furthermore such that $|\nu_+(t_j) - \nu_-(t_j)| \leq \nu_0(t)$ holds for $j = 1, 2$ and that $t_1 G - H$ and $t_2 G - H$ are nonsingular. Then we obtain from Lemma 14 that

$$0 \geq \Big|\Big(\nu_+(t_2) - \nu_-(t_2)\Big) + \Big(\nu_+(t_1) - \nu_-(t_1)\Big)\Big| = 2\,|\mathrm{Signsum}(\lambda_\infty)|.$$

5) $\Rightarrow$ 1): This follows directly from Theorem 11, since 5) implies that the subpencil $\lambda G_{22} - H_{22}$ does not appear. $\square$

**Remark 16** The condition $\mathrm{Signsum}(\lambda_0) = 0$ means that in the canonical form (9) the odd-sized blocks associated with $\lambda_0$ occur in pairs with opposite signs $+1$ and $-1$, respectively. (The pairing applies only to the signs, but not to the sizes of the blocks!) This condition can also be interpreted in the following way. If the columns of $V_0$ form a basis of the deflating subspace associated with $\lambda_0 \in \mathbb{R}$, then $\mathrm{Ind}(V_0^* G V_0) = (k, k, 0)$ for an integer $k \in \mathbb{N}$. Analogously, if the columns of $V_0$ form a basis of the deflating subspace associated with $\infty$, then $\mathrm{Ind}(V_0^* H V_0) = (k, k, 0)$ for an integer $k \in \mathbb{N}$. (For a proof see [16] on related work for skew-Hamiltonian/Hamiltonian pencils.)

Our next result gives necessary and sufficient conditions for the existence of anti-Hessenberg forms for a Hermitian pencil $\lambda G - H$. Again, we will consider the indices of the Hermitian matrices $tG - H$, where $t \in \mathbb{R}$, and then interpret these conditions in terms of the sign sums of the real eigenvalues and the eigenvalue $\infty$. First, we consider the case that the size of the pencil is odd.

**Theorem 17** *Let $\lambda G - H \in \mathbb{C}^{(2n+1)\times(2n+1)}$ be a regular Hermitian pencil and for $t \in \mathbb{R}$ let $\mathrm{Ind}(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-Hessenberg form.*
2. *$\lambda G - H$ is unitarily congruent to a pencil in anti-Hessenberg form.*
3. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t) + 1$.*
4. *For almost all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t) + 1$.*
5. *For every real eigenvalue $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ we have that $|\mathrm{Signsum}(\lambda_0)| \le 1$ and if $\lambda_1 < \ldots < \lambda_r \le \infty$ denote the real eigenvalues (including $\infty$) with nonzero sign sum, then $\lambda_1, \ldots, \lambda_r$ satisfy the property*

$$\mathrm{Signsum}(\lambda_\alpha) = -\mathrm{Signsum}(\lambda_{\alpha+1}), \quad \alpha = 1, \ldots, r-1. \tag{16}$$

**Proof**. 1) $\Rightarrow$ 2): This follows directly from Lemma 2.

2) $\Rightarrow$ 3): Let $P \in \mathbb{C}^{(2n+1)\times(2n+1)}$ be nonsingular such that the pencil $P^*(\lambda G - H)P$ is in anti-Hessenberg form. Then $P^*(tG - H)P$ is Hermitian anti-Hessenberg for all $t \in \mathbb{R}$. Thus, 2) follows from Corollary 4.

3) $\Rightarrow$ 4): This implication is trivial.

4) $\Rightarrow$ 5): W.l.o.g. we may assume that $\lambda G - H$ is in sign condensed form (10). Again, it is sufficient to consider the subpencil $\lambda G_{22} - H_{22}$ that has the form (11). Let us consider an eigenvalue $\lambda_\alpha$ of $\lambda G_{22} - H_{22}$.

Case (1) Assume that $\lambda_\alpha \in \mathbb{R}$, that is $\lambda_\alpha \in \{\lambda_1, \ldots, \lambda_k\}$. Choose $t_1, t_2 \in \mathbb{R}$ such that

$$\lambda_1 < \cdots < \lambda_{\alpha-1} < t_1 < \lambda_\alpha < t_2 < \lambda_{\alpha+1} < \cdots < \lambda_k,$$

and such that $t_jG - H$ is nonsingular and $|\nu_+(t_j) - \nu_-(t)| \le \nu_0(t_j) + 1$ for $j = 1, 2$. Then we obtain from Lemma 14 and $\nu_0(t_1) = \nu_0(t_2) = 0$ that

$$\begin{array}{rcl} 2 & \ge & |\nu_+(t_1) - \nu_-(t_1)| + |\nu_+(t_2) - \nu_-(t_2)| \\ & \ge & \left|\Big(\nu_+(t_1) - \nu_-(t_1)\Big) - \Big(\nu_+(t_2) - \nu_-(t_2)\Big)\right| = |2\mathrm{Signsum}(\lambda_\alpha)|. \end{array}$$

This implies $|\mathrm{Signsum}(\lambda_\alpha)| \le 1$.

Case (2) If the assumption of Case (1) does not hold, then $\lambda_\alpha = \infty$.

In this case, we choose $t_1, t_2 \in \mathbb{R}$ such that

$$t_1 < \lambda_1 < \cdots < \lambda_k < t_2,$$

and such that $t_jG - H$ is nonsingular and $|\nu_+(t_j) - \nu_-(t_j)| \leq \nu_0(t_j) + 1$ for $j = 1, 2$. Applying Lemma 14 once more, we conclude that

$$2 \geq 2|\mathrm{Signsum}(\lambda_\infty)|.$$

For the second part of 3) we first note that $|\mathrm{Signsum}(\lambda_\beta)| = 1$ for all the eigenvalues $\lambda_\beta$ of $\lambda G_{22} - H_{22}$, since this subpencil does not contain eigenvalues with sign sum zero. We pick an $\alpha \in \{1, \ldots, k\}$ and distinguish two cases.

Case (a) Assume $\alpha < k$. Then choose $t_1, t_2 \in \mathbb{R}$ such that $t_jG - H$ is nonsingular, $|\nu_+(t_j) - \nu_-(t)| \leq \nu_0(t_j) + 1$ for $j = 1, 2$, and such that

$$\lambda_1 < \cdots < \lambda_{\alpha-1} < t_1 < \lambda_\alpha < \lambda_{\alpha+1} < t_2 < \lambda_{\alpha+2} < \cdots < \lambda_k.$$

Applying Lemma 14 again, we obtain that

$$2 \geq 2|\mathrm{Signsum}(\lambda_\alpha) + \mathrm{Signsum}(\lambda_{\alpha+1})|.$$

This implies $\mathrm{Signsum}(\lambda_\alpha) = -\mathrm{Signsum}(\lambda_{\alpha+1})$, since both terms do not vanish.

Case (b) If the assumption of Case (a) does not hold, then $\alpha = k$. If $\lambda G_{22} - H_{22}$ does not have the eigenvalue $\infty$, then $\lambda_\alpha$ is already the eigenvalue of maximal modulus and nothing must be proved. Otherwise, choose $t_1, t_2 \in \mathbb{R}$ such that $t_jG - H$ is nonsingular, $|\nu_+(t_j) - \nu_-(t)| \leq \nu_0(t_j) + 1$ for $j = 1, 2$, and such that

$$t_1 < \lambda_1 < \cdots < \lambda_{k-1} < t_2 < \lambda_k.$$

Then we obtain from Lemma 14 that

$$\begin{aligned} 2 &\geq |\nu_+(t_2) - \nu_-(t_2)| + |\nu_+(t_1) - \nu_-(t_1)| \\ &\geq |\nu_+(t_2) - \nu_-(t_2) + \nu_+(t_1) - \nu_-(t_1)| \\ &= 2|\mathrm{Signsum}(\lambda_k) + \mathrm{Signsum}(\lambda_\infty)|. \end{aligned}$$

This implies $\mathrm{Signsum}(\lambda_k) = -\mathrm{Signsum}(\lambda_\infty)$.

5) $\Rightarrow$ 1): Again, we may assume that the pencil is in sign condensed form (10). It remains to show that the subpencil $\lambda G_{22} - H_{22}$ of the form (11) is congruent to anti-Hessenberg form. From 5) we find in particular that all the eigenvalues of $\lambda G_{22} - H_{22}$ are simple. Again, we consider two different cases.

Case (1) Assume that $\lambda G_{22} - H_{22}$ does not have the eigenvalue $\infty$.

This implies in particular that $k = 2q+1$ is odd, since the size of $\lambda G_{22} - H_{22}$ is necessarily odd and all its eigenvalues are simple. Let us assume, w.l.o.g., that the sign $\varepsilon_1$ of $\lambda_1$ is equal to one. Otherwise, we may consider the pencil $-(\lambda G - H)$. Then, property (16) implies that the eigenvalues with sign $+1$ interlace the eigenvalues with sign $-1$. We visualize that by the following formula.

$$\begin{array}{cccccccccc} \lambda_1 < & & \lambda_3 < & & \cdots & < \lambda_{2q-1} & & < \lambda_{2q+1} & \text{sign } 1 \\ & \lambda_2 < & & \lambda_4 < & \cdots & & < \lambda_{2q} & & \text{sign } -1 \end{array} \tag{17}$$

By row and column permutations we find that

$$\lambda G_{22} - H_{22} \sim \lambda \begin{bmatrix} -I_q & 0 \\ 0 & I_{q+1} \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix},$$

where $\operatorname{spec}(\tilde{H}_1) = \{\lambda_2, \lambda_4, \ldots, \lambda_{2q}\}$ and $\operatorname{spec}(\tilde{H}_2) = \{\lambda_1, \lambda_3, \ldots, \lambda_{2q+1}\}$.

The interlacing property (17) allows us to solve an inverse eigenvalue problem (see [2] or [8]). There, it is shown that (17) is sufficient for the existence of a unitary matrix $Q \in \mathbb{C}^{(q+1)\times(q+1)}$ such that

$$Q^* \tilde{H}_2 Q = \begin{bmatrix} \tilde{H}_{21} & \tilde{H}_{22} \\ \tilde{H}_{22}^* & \tilde{H}_{23} \end{bmatrix},$$

where $\tilde{H}_{23} \in \mathbb{R}$ and $\operatorname{spec}(\tilde{H}_{21}) = \operatorname{spec}(\tilde{H}_1)$. From this, we see that

$$\lambda G_{22} - H_{22} \; \sim \; \lambda \begin{bmatrix} -I_q & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 & 0 \\ 0 & \tilde{H}_{21} & \tilde{H}_{22} \\ 0 & \tilde{H}_{22}^* & \tilde{H}_{23} \end{bmatrix}.$$

Note that we obtain from $\operatorname{spec}(\tilde{H}_{21}) = \operatorname{spec}(\tilde{H}_1)$ that every eigenvalue of the upper principal subpencil

$$\lambda \begin{bmatrix} -I_q & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 \\ 0 & \tilde{H}_{21} \end{bmatrix}$$

occurs with algebraic multiplicity 2 and opposite signs. Hence, the pencil satisfies condition 4) of Theorem 15 and there exists a nonsingular $P \in \mathbb{C}^{2q\times 2q}$ such that

$$P^* \left( \lambda \begin{bmatrix} -I_q & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 \\ 0 & \tilde{H}_{21} \end{bmatrix} \right) P$$

is in anti-triangular form. This implies that

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}^* \left( \lambda \begin{bmatrix} -I_q & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -\tilde{H}_1 & 0 & 0 \\ 0 & \tilde{H}_{21} & \tilde{H}_{22} \\ 0 & \tilde{H}_{22}^* & \tilde{H}_{23} \end{bmatrix} \right) \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$$

is in anti-Hessenberg form.

Case (2) If the assumption of Case (1) does not hold, then $\lambda G_{22} - H_{22}$ has the eigenvalue $\infty$.

This implies that $k = 2q$ is even. Again, property (16) implies that the eigenvalues with sign $+1$ interlace the eigenvalues with sign $-1$, where we assume again that $\varepsilon_1 = 1$. Thus, we have the following situation.

$$\begin{array}{cccccccccl} \lambda_1 < & & \lambda_3 < & & \cdots & < \lambda_{2q-1} & & & \text{with sign } +1 \\ & \lambda_2 < & & \lambda_4 < & \cdots & & < \lambda_{2q} & & \text{with sign } -1 \end{array} \tag{18}$$

Furthermore, the eigenvalue $\infty$ has the sign $+1$. By row and column permutations we find that

$$\lambda G_{22} - H_{22} \sim \lambda \begin{bmatrix} I_q & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{H}_1 & 0 & 0 \\ 0 & -\tilde{H}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\mathrm{spec}(\tilde{H}_1) = \{\lambda_1, \lambda_3, \ldots, \lambda_{2q-1}\}$ and $\mathrm{spec}(\tilde{H}_2) = \{\lambda_2, \lambda_4, \ldots, \lambda_{2q}\}$.

The interlacing property (18) allows us to solve another inverse eigenvalue problem. In [26], it is shown that (18) is sufficient for the existence of a rank-one updating with a vector $x \in \mathbb{R}^q$ such that $\mathrm{spec}(\tilde{H}_1 + xx^*) = \mathrm{spec}(\tilde{H}_2)$. From this, we see that

$$\begin{aligned} & \begin{bmatrix} I_q & 0 & x \\ 0 & I_q & 0 \\ 0 & 0 & 1 \end{bmatrix} \left( \lambda \begin{bmatrix} I_q & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{H}_1 & 0 & 0 \\ 0 & -\tilde{H}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} I_q & 0 & 0 \\ 0 & I_q & 0 \\ x^* & 0 & 1 \end{bmatrix} \\ = \; & \lambda \begin{bmatrix} I_q & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{H}_1 + xx^* & 0 & x \\ 0 & -\tilde{H}_2 & 0 \\ x^* & 0 & 1 \end{bmatrix}. \end{aligned}$$

Again, we see from Theorem 15 that the upper principal $2q \times 2q$ subpencil is congruent to a pencil in anti-triangular form, and thus, $\lambda G_{22} - H_{22}$ is congruent to a pencil in anti-Hessenberg form. □

Theorem 15 and Theorem 17 are special cases of a more general result for anti-$m$-Hessenberg forms. This general result can be shown by induction on $m$. For the induction step, we need the following lemma.

**Lemma 18** *Let $\lambda G_{22} - H_{22} \in \mathbb{C}^{n \times n}$ be a pencil in form (11). Furthermore, let us denote* $\mathrm{Ind}(tG_{22} - H_{22}) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$, *and assume that*

$$|\nu_+(t) - \nu_-(t)| \le \nu_0(t) + m + 1 \quad \textit{for almost all } t \in \mathbb{R}.$$

*Then there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that*

$$P^*(\lambda G_{22} - H_{22})P = \lambda \begin{bmatrix} G' & 0 \\ 0 & G'' \end{bmatrix} - \begin{bmatrix} H' & 0 \\ 0 & H'' \end{bmatrix},$$

*where the size of $\lambda G'' - H''$ is odd and such that the following conditions are satisfied.*

1. *Setting* $\operatorname{Ind}(tG' - H') = \Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big)$, *we have that*

$$|\mu_+(t) - \mu_-(t)| \leq \mu_0(t) + m \quad \textit{for almost all } t \in \mathbb{R}.$$

2. *Setting* $\operatorname{Ind}(tG'' - H'') = \Big(\pi_+(t), \pi_-(t), \pi_0(t)\Big)$, *we have that*

$$|\pi_+(t) - \pi_-(t)| \leq \pi_0(t) + 1 \quad \textit{for almost all } t \in \mathbb{R}.$$

**Proof**. Let $s_1, \ldots, s_{k+1} \in \mathbb{R}$ be arbitrary with the condition that we have for $j = 1, \ldots, k+1$ that $|\nu_+(s_j) - \nu_-(s_j)| \leq \nu_0(s_j) + m + 1$, and such that

$$s_1 < \lambda_1 < s_2 < \ldots < s_k < \lambda_k < s_{k+1}.$$

This implies in particular that $\nu_0(s_j) = 0$. Applying Lemma 14, we find the recursive formula

$$\Big(\nu_+(s_{\alpha+1}) - \nu_-(s_{\alpha+1})\Big) - \Big(\nu_+(s_\alpha) - \nu_-(s_\alpha)\Big) = 2p_\alpha\varepsilon_\alpha. \tag{19}$$

Thus, the map $\alpha \mapsto \Big(\nu_+(s_\alpha) - \nu_-(s_\alpha)\Big)$ is increasing whenever $\varepsilon_\alpha$ is positive and decreasing whenever $\varepsilon_\alpha$ is negative. Hence, 'extremal points' such that $|\nu_+(s_\alpha) - \nu_-(s_\alpha)| = m + 1$, can only be reached for an $\alpha$ such that $\varepsilon_\alpha \neq \varepsilon_{\alpha-1}$.

Next, assume that there exists an index $l \in \{1, \ldots, k+1\}$ such that

$$|\nu_+(s_l) - \nu_-(s_l)| = m + 1.$$

(We may always start with the largest $\tilde{m}$ such that there exists an index $l \in \{1, \ldots, k+1\}$ with $|\nu_+(s_l) - \nu_-(s_l)| = \tilde{m}+1$. The statement of the lemma is then correct for any $m \geq \tilde{m}$.) Then, we obtain from the recursive formula (19) that the possible values for $\nu_+(s_j) - \nu_-(s_j)$, $j = 1, \ldots, k+1$, include $m + 1$ and $m - 1$, but neither $m$ nor $-m$. Moreover, we may assume w.l.o.g. that $\varepsilon_1 = +1$. Then, the recursive formula (19) implies in particular

$$-(m + 1) \leq \nu_+(s_1) - \nu_-(s_1) < m. \tag{20}$$

Define

$$G_0' = \begin{bmatrix} I_{p_1-1} & & & & \\ & \varepsilon_2 I_{p_2} & & & \\ & & \ddots & & \\ & & & \varepsilon_k I_{p_k} & \\ & & & & 0 \end{bmatrix}, \quad G_0'' = [1], \quad \text{and}$$

$$H_0' = \begin{bmatrix} \lambda_1 I_{p_1-1} & & & & \\ & \varepsilon_2\lambda_2 I_{p_2} & & & \\ & & \ddots & & \\ & & & \varepsilon_k\lambda_k I_{p_k} & \\ & & & & \varepsilon_{k+1} I_{p_{k+1}} \end{bmatrix}, \quad H_0'' = [\lambda_1].$$

Then $G = G_0'' \oplus G_0'$ and $H = H_0'' \oplus H_0'$. Moreover, setting

$$\Big(\varrho_+(t), \varrho_-(t), \varrho_0(t)\Big) := \mathrm{Ind}(\mathrm{tG_0'} - \mathrm{H_0'}),$$

we obtain using formula (14) that

$$\begin{array}{ll} \varrho_+(s_1) - \varrho_-(s_1) = \nu_+(s_1) - \nu_-(s_1) + 1 & , \\ \varrho_+(s_j) - \varrho_-(s_j) = \nu_+(s_j) - \nu_-(s_j) - 1 & \text{for } j > 1. \end{array}$$

This implies

$$\begin{array}{rl} -m \le \varrho_+(s_1) - \varrho_-(s_1) \le m & \text{and,} \\ -m - 2 \le \varrho_+(s_j) - \varrho_-(s_j) \le m & \text{for } j > 1. \end{array}$$

Next, let $l < k+1$ be the smallest index such that $\varrho_+(s_l) - \varrho_-(s_l) = -m-2$ if there exists such an index. From our discussion of 'extremal points' of the map $\alpha \mapsto \Big(\nu_+(s_\alpha) - \nu_-(s_\alpha)\Big)$, we find that this is only possible if $\varepsilon_{l-1} = -1$ and $\varepsilon_l = +1$. Let $G_1'$ and $H_1'$ be the matrices that are obtained from $G_0'$ and $H_0'$, respectively, by changing the $(l-1)$th and $l$th diagonal blocks in the following way:

$$G_1' = \begin{bmatrix} \ddots & & & \\ & -I_{p_{l-1}-1} & & \\ & & I_{p_l-1} & \\ & & & \ddots \end{bmatrix}, \quad H_1' = \begin{bmatrix} \ddots & & & \\ & -\lambda_{l-1} I_{p_{l-1}-1} & & \\ & & \lambda_l I_{p_l-1} & \\ & & & \ddots \end{bmatrix}.$$

Here, the dotted parts stand for the blocks that have remained unchanged. Furthermore, set

$$G_1'' = \begin{bmatrix} G_0'' & & \\ & -1 & \\ & & 1 \end{bmatrix} \quad \text{and} \quad H_1'' = \begin{bmatrix} H_0'' & & \\ & -\lambda_{l-1} & \\ & & \lambda_l \end{bmatrix}$$

and redefine $\Big(\varrho_+(t), \varrho_-(t), \varrho_0(t)\Big) = \mathrm{Ind}(\mathrm{tG_1'} - \mathrm{H_1'})$. Then $\varrho_+(s_l) - \varrho_-(s_l) = -m$, i.e.,

$$\begin{array}{rcll} -m \le \varrho_+(s_j) - \varrho_-(s_j) & \le & m & \text{for } j \le l \\ \varrho_+(s_j) - \varrho_-(s_j) & = & \nu_+(s_j) - \nu_-(s_j) - 1 & \text{for } j > l. \end{array}$$

After a finite number of steps, analogously constructing $G_r'$, $H_r'$, $G_r''$, and $H_r''$, respectively, from given matrices $G_{r-1}'$, $H_{r-1}'$, $G_{r-1}''$, and $H_{r-1}''$, and redefining

$$\Big(\varrho_+(t), \varrho_-(t), \varrho_0(t)\Big) = \mathrm{Ind}(\mathrm{tG_r'} - \mathrm{H_r'}),$$

we finally obtain

$$\begin{array}{rcll} -m \le \varrho_+(s_j) - \varrho_-(s_j) & \le & m & \text{for } j < k+1, \\ \varrho_+(s_{k+1}) - \varrho_-(s_{k+1}) & = & \nu_+(s_{k+1}) - \nu_-(s_{k+1}) - 1. & \end{array}$$

We now distinguish two cases.

Case (1) Assume $\varrho_+(s_{k+1}) - \varrho_-(s_{k+1}) > -m-2$.

In this case, we have in particular that

$$-m \le \varrho_+(s_{k+1}) - \varrho_-(s_{k+1}) \le m$$

taking into account the possible values of the map $\alpha \mapsto \Big(\nu_+(s_\alpha) - \nu_-(s_\alpha)\Big)$. Next, set $\lambda G' - H' = \lambda G'_r - H'_r$ and $\lambda G'' - H'' = \lambda H''_r - H''_r$. Then $\lambda G' - H'$ satisfies condition 1) of the lemma. On the other hand, note that the eigenvalues of $\lambda G'' - H''$ by construction have sign sum with modulus equal to one and satisfy the interlacing property (16). Hence, Theorem 17 implies that $\lambda G'' - H''$ satisfies condition 2) of the lemma. Moreover, it is clear that the size of $\lambda G'' - H''$ is odd. This concludes the proof of case (1).

Case (2) Assume $\varrho_+(s_{k+1}) - \varrho_-(s_{k+1}) = -m-2$.

This implies $\nu_+(s_{k+1}) - \nu_-(s_{k+1}) = -m-1$ and from Lemma 14 and (20) we obtain that

$$\begin{aligned}
-2\varepsilon_{k+1}p_{k+1} &= \nu_+(s_{k+1}) - \nu_-(s_{k+1}) + \nu_+(s_1) - \nu_-(s_1) \\
&< -m-1+m = -1.
\end{aligned}$$

This implies $\varepsilon_{k+1} = +1$ and $p_{k+1} > 0$, i.e., the pencil $\lambda G - H$ has the eigenvalue $\infty$. Furthermore, $\nu_+(s_{k+1}) - \nu_-(s_{k+1})$ is minimal and therefore, we must have $\varepsilon_k = -1$. Let $\lambda G' - H'$ be obtained from $\lambda G'_r - H'_r$ by changing the $k$th and $(k+1)$th diagonal blocks only, in detail

$$\lambda G' - H' = \lambda \begin{bmatrix} \ddots & & \\ & -I_{p_k-1} & \\ & & 0 \end{bmatrix} - \begin{bmatrix} \ddots & & \\ & -\lambda_k I_{p_k-1} & \\ & & I_{p_{k+1}-1} \end{bmatrix},$$

where the dotted parts stand again for the blocks that have remained unchanged. Moreover, set

$$\lambda G'' - H'' = \lambda \begin{bmatrix} G''_r & & \\ & -1 & \\ & & 0 \end{bmatrix} - \begin{bmatrix} H''_r & & \\ & -\lambda_k & \\ & & 1 \end{bmatrix}.$$

Redefining $\Big(\varrho_+(t), \varrho_-(t), \varrho_0(t)\Big) = \mathrm{Ind}(\mathrm{tG'_r - H'_r})$, we obtain that

$$|\varrho_+(s_j) - \varrho_-(s_j)| \le m$$

for all $j = 1, \ldots, k+1$. The rest of case (2) is analogous to case (1). This concludes the proof. $\square$

**Theorem 19** *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a regular Hermitian pencil and let $m \leq n$ be such that $n-m$ is even. Furthermore, let Ind$(tG-H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$ for $t \in \mathbb{R}$. Then the following statements are equivalent:*

1. *$\lambda G - H$ is congruent to a pencil in anti-$m$-Hessenberg form.*
2. *$\lambda G - H$ is unitarily congruent to a pencil in anti-$m$-Hessenberg form.*
3. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \leq \nu_0(t) + m$.*
4. *For almost all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \leq \nu_0(t) + m$.*

**Proof**. 1) $\Rightarrow$ 2): This follows directly from Lemma 2.

2) $\Rightarrow$ 3): Let $P \in \mathbb{C}^{n\times n}$ be nonsingular such that $P^*(\lambda G - H)P$ is in anti-$m$-Hessenberg form. Then 2) follows from Corollary 4.

3) $\Rightarrow$ 4): This implication is trivial.

4) $\Rightarrow$ 1): We proceed by induction on $m$.

$m = 0$ and $m = 1$: These have already been proved, see Theorems 15 and 17.

$m \Rightarrow (m+1)$: Once again we may assume that $\lambda G - H$ is in sign condensed form (10) and it is sufficient to consider the subpencil $\lambda G_{22} - H_{22}$ that has the form (11). By Lemma 18, we find that there exists a nonsingular matrix $\tilde{P} \in \mathbb{C}^{n\times n}$ such that

$$\tilde{P}^*(\lambda G_{22} - H_{22})\tilde{P} = \lambda \begin{bmatrix} G' & 0 \\ 0 & G'' \end{bmatrix} - \begin{bmatrix} H' & 0 \\ 0 & H'' \end{bmatrix},$$

where $\lambda G'' - H''$ has odd size, and setting $\mathrm{Ind}(tG' - H') = \Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big)$ and $\mathrm{Ind}(tG'' - H'') = \Big(\pi_+(t), \pi_-(t), \pi_0(t)\Big)$, the following conditions are satisfied for almost all $t \in \mathbb{R}$:

$$\begin{array}{rcl} |\mu_+(t) - \mu_-(t)| & \leq & \mu_0(t) + m, \\ |\pi_+(t) - \pi_-(t)| & \leq & \pi_0(t) + 1. \end{array}$$

Let $n'$ and $n''$ denote the sizes of $\lambda G' - H'$ and $\lambda G'' - H''$, respectively. By assumption, $n - (m+1)$ is even and thus, so is $n' - m$, since $n - n' = n''$ is odd. Therefore, by the induction hypothesis and by Theorem 17, the pencil $\lambda G' - H'$ is congruent to a pencil in anti-$m$-Hessenberg form and $\lambda G'' - H''$ is congruent to a pencil in anti-Hessenberg form, i.e.,

$$\lambda G_{22} - H_{22}$$

$$\sim \quad \lambda \begin{bmatrix} 0 & \hat{G}_{12} & \hat{G}_{13} & 0 & 0 & 0 \\ \hat{G}_{12}^* & \hat{G}_{22} & \hat{G}_{23} & 0 & 0 & 0 \\ \hat{G}_{13}^* & \hat{G}_{23}^* & \hat{G}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \check{G}_{12} & \check{G}_{13} \\ 0 & 0 & 0 & \check{G}_{12}^* & \check{G}_{22} & \check{G}_{23} \\ 0 & 0 & 0 & \check{G}_{13}^* & \check{G}_{23}^* & \check{G}_{33} \end{bmatrix} - \begin{bmatrix} 0 & \hat{H}_{12} & \hat{H}_{13} & 0 & 0 & 0 \\ \hat{H}_{12}^* & \hat{H}_{22} & \hat{H}_{23} & 0 & 0 & 0 \\ \hat{H}_{13}^* & \hat{H}_{23}^* & \hat{H}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \check{H}_{12} & \check{H}_{13} \\ 0 & 0 & 0 & \check{H}_{12}^* & \check{H}_{22} & \check{H}_{23} \\ 0 & 0 & 0 & \check{H}_{13}^* & \check{H}_{23}^* & \check{H}_{33} \end{bmatrix}$$

$$\sim \quad \lambda \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{G}_{12} & \hat{G}_{13} \\ 0 & 0 & \check{G}_{12} & \check{G}_{13} & 0 & 0 \\ 0 & \check{G}_{12}^* & \check{G}_{22} & \check{G}_{23} & 0 & 0 \\ 0 & \check{G}_{13}^* & \check{G}_{23}^* & \check{G}_{33} & 0 & 0 \\ \hat{G}_{12}^* & 0 & 0 & 0 & \hat{G}_{22} & \hat{G}_{23} \\ \hat{G}_{13}^* & 0 & 0 & 0 & \hat{G}_{23}^* & \hat{G}_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & \hat{H}_{12} & \hat{H}_{13} \\ 0 & 0 & \check{H}_{12} & \check{H}_{13} & 0 & 0 \\ 0 & \check{H}_{12}^* & \check{H}_{22} & \check{H}_{23} & 0 & 0 \\ 0 & \check{H}_{13}^* & \check{H}_{23}^* & \check{H}_{33} & 0 & 0 \\ \hat{H}_{12}^* & 0 & 0 & 0 & \hat{H}_{22} & \hat{H}_{23} \\ \hat{H}_{13}^* & 0 & 0 & 0 & \hat{H}_{23}^* & \hat{H}_{33} \end{bmatrix} \quad (21)$$

where the submatrices have the following forms.

$$\begin{aligned} \hat{G}_{12}, \hat{H}_{12} \quad &\in \quad \mathbb{C}^{(\frac{n'-m}{2})\times(\frac{n'-m}{2})} \quad \text{are anti-triangular, } \hat{G}_{13}, \hat{H}_{13} \in \mathbb{C}^{(\frac{n'-m}{2})\times m}, \\ \check{G}_{12}, \check{H}_{12} \quad &\in \quad \mathbb{C}^{(\frac{n''-1}{2})\times(\frac{n''-1}{2})} \quad \text{are anti-triangular, } \check{G}_{13}, \check{H}_{13} \in \mathbb{C}^{(\frac{n''-1}{2})\times 1}, \end{aligned}$$

and the other blocks have corresponding sizes. Hence, the pencil (21) is in anti-(m+1)-Hessenberg form. □

In Theorem 19, we did not give conditions on the sign sums as in the Theorems 15 and 17. In principle, this is also possible for the case $m > 1$. But then the conditions become very complicated, since we have to consider many subcases. Therefore, we prefer the conditions given in Theorem 19.

Clearly, Theorem 19 does not hold in the case that $n - m$ is odd. For example, let us consider the case $m = 0$ and $n = 3$. The Hermitian pencil

$$\lambda \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is in anti-triangular form, but we immediately obtain Signsum(2) = 1. We see from this example that the eigenvalue that is displayed in the middle of the anti-diagonal plays an exceptional role and has to be treated differently from the rest of the eigenvalues. In fact, we may omit the eigenvalue that is displayed in the middle of the anti-diagonal, and its tribute to the sign sum may also be omitted, such that we can use the fact that $n - 1 - m$ is even and apply Theorem 19. This is done in the proof of the next theorem.

**Theorem 20** *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a regular Hermitian pencil and let $m \leq n$ be such that $n - m$ is odd. Furthermore, let Ind$(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$ for $t \in \mathbb{R}$. Then the following statements are equivalent.*

1. *$\lambda G - H$ is congruent to a pencil in anti-$m$-Hessenberg form.*

2. *$\lambda G - H$ is unitarily congruent to a pencil in anti-$m$-Hessenberg form.*

3. *There exists $t_0 \in \mathbb{R} \cup \{\infty\}$ and $\varepsilon \in \{1, -1\}$ such that*

$$\begin{array}{rlll} & |\nu_+(t) - \nu_-(t) + \varepsilon| \leq \nu_0(t) + m & \text{for all} & t < t_0 \\ \text{and} & |\nu_+(t) - \nu_-(t) - \varepsilon| \leq \nu_0(t) + m & \text{for all} & t > t_0. \end{array}$$

4. *There exists $t_0 \in \mathbb{R} \cup \{\infty\}$ and $\varepsilon \in \{1, -1\}$ such that*

$$\begin{array}{rlll} & |\nu_+(t) - \nu_-(t) + \varepsilon| \leq \nu_0(t) + m & \text{for almost all} & t < t_0 \\ \text{and} & |\nu_+(t) - \nu_-(t) - \varepsilon| \leq \nu_0(t) + m & \text{for almost all} & t > t_0. \end{array}$$

**Proof**. 1) $\Rightarrow$ 2): This follows directly from Lemma 2.

2) $\Rightarrow$ 3): Let $P \in \mathbb{C}^{n \times n}$ be nonsingular such that $P^*(\lambda G - H)P$ is in anti-$m$-Hessenberg form. Thus, $P^*(tG - H)P$ is Hermitian anti-$m$-Hessenberg for all $t \in \mathbb{R}$. This means in particular that

$$P^*(tG - H)P = \begin{bmatrix} 0 & 0 & tG_{13} - H_{13} \\ 0 & tg_{22} - h_{22} & tG_{23} - H_{23} \\ tG_{13}^* - H_{13}^* & tG_{23}^* - H_{23}^* & tG_{33}^* - H_{33}^* \end{bmatrix},$$

where $tG_{13} - H_{13} \in \mathbb{C}^{(\frac{n-m-1}{2}) \times (\frac{n+m-1}{2})}$, and $tg_{22} - h_{22} \in \mathbb{C}$, and where the other blocks have corresponding sizes. If $g_{22} \neq 0$, then let $t_0 = \frac{h_{22}}{g_{22}}$, otherwise set $t_0 = \infty$. Then Lemma 6 for $t \neq t_0$ implies that

$$\operatorname{Ind}(tG - H) = \operatorname{Ind}(t\tilde{G} - \tilde{H}) + \operatorname{Ind}(tg_{22} - h_{22}), \tag{22}$$

where

$$t\tilde{G} - \tilde{H} = \begin{bmatrix} 0 & tG_{13} - H_{13} \\ tG_{13}^* - H_{13}^* & * \end{bmatrix}.$$

Let $\Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big) = \operatorname{Ind}(t\tilde{G} - \tilde{H})$. Set $\varepsilon = -\sigma(\tilde{t}g_{22} - h_{22})$ for some $\tilde{t} < t_0$, and note that $t\tilde{G} - \tilde{H}$ is in anti-$m$-Hessenberg form with size $n - 1$. Thus, since $n - 1 - m$ is even, we can apply Theorem 19 and we obtain from (22) for $t > t_0$ that

$$|\nu_+(t) - \nu_-(t) + \varepsilon| = |\mu_+(t) - \mu_-(t)| \leq \mu_0(t) + m = \nu_0(t) + m,$$

since $\mu_0(t) = \nu_0(t)$ for $t \neq t_0$. Analogously we obtain for $t > t_0$ that

$$|\nu_+(t) - \nu_-(t) - \varepsilon| = |\mu_+(t) - \mu_-(t)| \leq \mu_0(t) + m = \nu_0(t) + m.$$

3) $\Rightarrow$ 4): This implication is trivial.

4) $\Rightarrow$ 1): W.l.o.g. we may assume that $\varepsilon = 1$. Otherwise, we may consider the pencil $-(\lambda G - H)$. Repeating our proof strategy once more, we assume that $\lambda G - H$ is in sign condensed form (10) and we consider the subpencil $\lambda G_{22} - H_{22}$. By 2) there exists $t_0 \in \mathbb{R} \cup \{\infty\}$ such that

$$\begin{aligned} |\nu_+(t) - \nu_-(t) + 1| &\leq \nu_0(t) + m \quad \text{for almost all} \quad t < t_0 \\ |\nu_+(t) - \nu_-(t) - 1| &\leq \nu_0(t) + m \quad \text{for almost all} \quad t > t_0. \end{aligned} \tag{23}$$

Case (1) Assume that $t_0$ can be chosen to be finite, i.e., $t_0 \in \mathbb{R}$.

We show next that we may assume that $t_0$ is an eigenvalue of $\lambda G_{22} - H_{22}$. For this, let $\lambda_\alpha$ be the largest eigenvalue $\lambda_{\tilde{\alpha}} \leq t_0$ of $\lambda G_{22} - H_{22}$. Clearly, we have

$$\begin{aligned} |\nu_+(t) - \nu_-(t) + 1| &\leq \nu_0(t) + m \quad \text{for almost all} \quad t < \lambda_\alpha \quad (\text{since } \lambda_\alpha \leq t_0), \\ |\nu_+(t) - \nu_-(t) - 1| &\leq \nu_0(t) + m \quad \text{for almost all} \quad t > t_0. \end{aligned}$$

Thus, it remains to show that $|\nu_+(t) - \nu_-(t) - 1| \leq \nu_0(t) + m$ for almost all $t \in (\lambda_\alpha, t_0]$ if this interval is nonempty. But this follows from the fact that $t \mapsto \Big(\nu_+(t) - \nu_-(t) - 1\Big)$ and $\nu_0(t)$ are constant on $(\lambda_\alpha, \lambda_{\alpha+1})$ (or $(\lambda, \infty)$ if there exists no finite eigenvalue $\lambda_{\alpha+1} > \lambda_\alpha$), and by the choice of $\lambda_\alpha$ we have $t_0 \in (\lambda_\alpha, \lambda_{\alpha+1})$ (or $t_0 \in (\lambda_\alpha, \infty)$, respectively).

Hence, we may assume that $t_0 = \lambda_\alpha$ is an eigenvalue of $\lambda G_{22} - H_{22}$. Let $\alpha$ be chosen minimal with the property that (23) is satisfied for all $t_0 = \lambda_\beta$, where $\beta \geq \alpha$, i.e.,

$$\begin{aligned} |\nu_+(t) - \nu_-(t) + 1| &\leq \nu_0(t) + m \quad \text{for almost all} \quad t < \lambda_\beta, \\ |\nu_+(t) - \nu_-(t) - 1| &\leq \nu_0(t) + m \quad \text{for almost all} \quad t > \lambda_\beta, \end{aligned} \tag{24}$$

if $\beta \geq \alpha$, but

$$|\nu_+(t) - \nu_-(t) - 1| \leq \nu_0(t) + m \quad \text{for almost all } t > \lambda_\gamma \tag{25}$$

is not true if $\gamma < \alpha$. For the rest of Case (1), we distinguish two different subcases.

Subcase (1a) Assume that $\alpha > 1$. Then (25) is not true for $\gamma = \alpha - 1$, i.e., there exist infinitely many $t_1$ such that $\lambda_{\alpha-1} < t_1 < \lambda_\alpha$ and such that

$$|\nu_+(t_1) - \nu_-(t_1) - 1| > \nu_0(t_1) + m.$$

On the other hand, we know from (24) for $\beta = \alpha$ that $t_1$ can be chosen such that

$$|\nu_+(t_1) - \nu_-(t_1) + 1| \leq \nu_0(t_1) + m.$$

Both inequalities hold simultaneously only if

$$\nu_+(t_1) - \nu_-(t_1) - 1 < -\Big(\nu_0(t_1) + m\Big).$$

Next, we show that $\varepsilon_\alpha = +1$. Choose $t_2$ such that $\lambda_\alpha < t_2$ ($< \lambda_{\alpha+1}$ if $\lambda_{\alpha+1}$ exists) and $\Big|\nu_+(t_2) - \nu_-(t_2) - 1\Big| \le \nu_0(t_2) + m$. Then Lemma 14 implies that

$$\Big(\nu_+(t_2) - \nu_-(t_2)\Big) - \Big(\nu_+(t_1) - \nu_-(t_1)\Big) = 2\varepsilon_\alpha p_\alpha.$$

If $\varepsilon_\alpha$ is equal to $-1$, then $\nu_0(t_1) = \nu_0(t_2) = 0$ implies that

$$\nu_+(t_2) - \nu_-(t_2) < \nu_+(t_1) - \nu_-(t_1) \le -\Big(\nu_0(t_1) + m\Big) = -\Big(\nu_0(t_2) + m\Big),$$

which is a contradiction to $|\nu_+(t_2) - \nu_-(t_2) - 1| \le \nu_0(t_2) + m$. Thus, $\varepsilon_\alpha = 1$. By permuting some rows and columns, we obtain that

$$\lambda G_{22} - H_{22} \;\sim\; \lambda \begin{bmatrix} g & 0 \\ 0 & \tilde{G} \end{bmatrix} - \lambda \begin{bmatrix} h & 0 \\ 0 & \tilde{H} \end{bmatrix},$$

where $\lambda g - h \in \mathbb{C}$ is a $1 \times 1$ pencil with eigenvalue $\lambda_\alpha$. Setting

$$\Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big) = \operatorname{Ind}(t\tilde{G} - \tilde{H}),$$

we find that

$$|\mu_+(t) - \mu_-(t)| = \begin{cases} |\nu_+(t) - \nu_-(t) + 1| & \text{for all} \quad t < \lambda_\alpha \\ |\nu_+(t) - \nu_-(t) - 1| & \text{for all} \quad t > \lambda_\alpha. \end{cases}$$

This implies that $|\mu_+(t) - \mu_-(t)| \le \nu_0(t) + m = \mu_0(t) + m$ for almost all $t \in \mathbb{R}$. Hence, by Theorem 19 the pencil $\lambda G_{22} - H_{22}$ is congruent to a pencil

$$\lambda \begin{bmatrix} g & 0 & 0 \\ 0 & 0 & \hat{G}_{23} \\ 0 & \hat{G}_{23}^* & \hat{G}_{33} \end{bmatrix} - \begin{bmatrix} h & 0 & 0 \\ 0 & 0 & \hat{H}_{23} \\ 0 & \hat{H}_{23}^* & \hat{H}_{33} \end{bmatrix},$$

where the subpencil

$$\lambda \begin{bmatrix} 0 & \hat{G}_{23} \\ \hat{G}_{23}^* & \hat{G}_{33} \end{bmatrix} - \lambda \begin{bmatrix} 0 & \hat{H}_{23} \\ \hat{H}_{23}^* & \hat{H}_{33} \end{bmatrix}$$

is in anti-$m$-Hessenberg form. Thus, we finally obtain

$$\lambda G_{22} - H_{22} \sim \lambda \begin{bmatrix} 0 & 0 & \hat{G}_{23} \\ 0 & g & 0 \\ \hat{G}_{23}^* & 0 & \hat{G}_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \hat{H}_{23} \\ 0 & h & 0 \\ \hat{H}_{23}^* & 0 & \hat{H}_{33} \end{bmatrix},$$

and this pencil is in anti-$m$-Hessenberg form.

Subcase (1b) If $\alpha \not> 1$, then $\alpha = 1$. Thus, (23) holds for all $t_0 = \lambda_\beta$. This means in particular that both

$$|\nu_+(t) - \nu_-(t) + 1| \le \nu_0(t) + m \qquad , \tag{26}$$
$$|\nu_+(t) - \nu_-(t) - 1| \le \nu_0(t) + m \tag{27}$$

hold for almost all $t \in \mathbb{R}$. Permuting some rows and columns, we obtain that

$$\lambda G_{22} - H_{22} \sim \lambda \begin{bmatrix} g & 0 \\ 0 & \tilde{G} \end{bmatrix} - \lambda \begin{bmatrix} h & 0 \\ 0 & \tilde{H} \end{bmatrix},$$

where $\lambda g - h \in \mathbb{C}$ is a $1 \times 1$ pencil with eigenvalue $\lambda_1$ . Setting

$$\Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big) = \operatorname{Ind}(t\tilde{G} - \tilde{H}),$$

we find that

$$|\mu_+(t) - \mu_-(t)| = \begin{cases} |\nu_+(t) - \nu_-(t) + \varepsilon_1| & \text{for all} \quad t < \lambda_1 \\ |\nu_+(t) - \nu_-(t) - \varepsilon_1| & \text{for all} \quad t > \lambda_1. \end{cases}$$

Then (26) and (27) imply that

$$|\mu_+(t) - \mu_-(t)| \leq \nu_0(t) + m = \mu_0(t) + m,$$

for almost all $t \in \mathbb{R}$ and hence we may proceed as in Case (1a).

Case (2) Assume that $t_0$ cannot be chosen to be finite.

In this case we have

$$|\nu_+(t) - \nu_-(t) + 1| \leq \nu_0(t) + m \quad \text{for all } t < \infty, \tag{28}$$

but for any $c \in \mathbb{R}$, there exist infinitely many $t > c$ such that

$$|\nu_+(t) - \nu_-(t) - 1| > \nu_0(t) + m.$$

Choose $t_2 > \lambda_k$ such that $|\nu_+(t_2) - \nu_-(t_2) - 1| > \nu_0(t_2) + m = m$ and $t_1 < \lambda_1$. Then we have in particular that $\nu_+(t_2) - \nu_-(t_2) - 1 < -m$ using the same argumentation as in Case (1a) and moreover $\nu_+(t_1) - \nu_-(t_1) + 1 \leq m$ by (28). We obtain from Lemma 14 that

$$\begin{aligned} -2\varepsilon_{k+1}p_{k+1} &= \Big(\nu_+(t_2) - \nu_-(t_2)\Big) + \Big(\nu_+(t_1) - \nu_-(t_1)\Big) \\ &< (-m+1) + (m-1) = 0. \end{aligned}$$

This implies $\varepsilon_{k+1} = +1$ and then, we may proceed as in Case (1a). This concludes the proof. □

Analogous to the proof of Theorem 15, we obtain conditions on the sign sum for the real eigenvalues and the eigenvalue $\infty$. We only state this for the anti-triangular case.

**Corollary 21** *Let $\lambda G - H \in \mathbb{C}^{(2n+1)\times(2n+1)}$ be a regular Hermitian pencil. Then the following statements are equivalent:*

1. *$\lambda G - H$ is congruent to a pencil in anti-triangular form.*
2. *$\lambda G - H$ is unitarily congruent to a pencil in anti-triangular form.*
3. *There exists exactly one eigenvalue $\lambda_0 \in \mathbb{R}$ with $\operatorname{Signsum}(\lambda_0) = \pm 1$ and for every eigenvalue $\lambda_\alpha \in \mathbb{R} \cup \{\infty\}$ with $\lambda_\alpha \neq \lambda_0$ we have that $\operatorname{Signsum}(\lambda_\alpha) = 0$.*

# 4 Condensed forms for singular Hermitian pencils

In this section we include the case of singular Hermitian pencils. Although in this case an anti-triangular form does not necessarily display the roots of the elementary divisors, it still displays a nested set of invariant subspaces and therefore, the consideration of condensed forms of singular Hermitian pencils does still make sense.

Analogous to the regular case, we derive a sign condensed form and then discuss the existence of anti-triangular and anti-$m$-Hessenberg forms. Let us first consider the canonical form (see [22]).

**Theorem 22** *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a Hermitian pencil. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^*(\lambda G - H)P = \lambda \begin{bmatrix} G' & 0 \\ 0 & G'' \end{bmatrix} - \begin{bmatrix} H' & 0 \\ 0 & H'' \end{bmatrix}, \tag{29}$$

*where the following conditions are satisfied.*

1. *The subpencil $\lambda G' - H'$ is block diagonal with diagonal blocks of the form*

$$\lambda \begin{bmatrix} 0 & 0 & Z_r \\ 0 & 0 & 0 \\ Z_r & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \mathcal{J}_r(0)^* Z_r \\ 0 & 0 & e_1^* \\ Z_r \mathcal{J}_r(0) & e_1 & 0 \end{bmatrix} \in \mathbb{C}^{(r+1)\times(r+1)}, \tag{30}$$

   *where $r \geq 0$.*

2. *The subpencil $\lambda G'' - H''$ is regular and in canonical form (9).*

**Proof**. The proof follows directly from [22], Lemma 3. □

In the following, if we speak of the sign characteristic or the sign sum of $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ with respect to $\lambda G - H$, we mean the sign characteristic or sign sum, respectively, of $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ with respect to the regular subpencil $\lambda G'' - H''$ in the canonical form (29) of $\lambda G - H$. Next, we generalize Theorem 11 to the case of singular pencils.

**Theorem 23 (Sign condensed form)** *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a Hermitian pencil. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^*(\lambda G - H)P = \lambda \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & G_{23} \\ G_{13}^* & G_{23}^* & G_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & H_{13} \\ 0 & H_{22} & H_{23} \\ H_{13}^* & H_{23}^* & H_{33} \end{bmatrix}, \tag{31}$$

*where the subpencil*

$$\lambda \begin{bmatrix} 0 & G_{13} \\ G_{13}^* & G_{33} \end{bmatrix} - \begin{bmatrix} 0 & H_{13} \\ H_{13}^* & H_{33} \end{bmatrix}$$

*is regular and* $G_{13}, H_{13} \in \mathbb{C}^{m\times m}$ *are lower anti-triangular. Furthermore,*

$$\begin{aligned}
&\lambda G_{22} - H_{22} \\
= \; &\lambda \begin{bmatrix} O_l & & & & 0 \\ & \varepsilon_1 I_{p_1} & & & \\ & & \ddots & & \\ & & & \varepsilon_k I_{p_k} & \\ 0 & & & & 0 \end{bmatrix} - \begin{bmatrix} O_l & & & & 0 \\ & \varepsilon_1 \lambda_1 I_{p_1} & & & \\ & & \ddots & & \\ & & & \varepsilon_k \lambda_k I_{p_k} & \\ 0 & & & & \varepsilon_{k+1} I_{p_{k+1}} \end{bmatrix},
\end{aligned} \tag{32}$$

*where* $\lambda_1 < \cdots < \lambda_k$*. In addition, we have for all* $\lambda_0 \in \mathbb{R} \cup \{\infty\}$ *that*

$$\operatorname{Signsum}(\lambda_0, G, H) = \operatorname{Signsum}(\lambda_0, G_{22}, H_{22}).$$

**Proof**. Let $\lambda G - H$ be in canonical form (29) and let $l$ denote the number of singular blocks of type (30). We prove the result by induction on $l$.

$l = 0$: This is Theorem 11.

$l \Rightarrow (l+1)$: It follows from Remark 10 that

$$\begin{aligned}
\lambda G - H &= \lambda \begin{bmatrix} 0 & 0 & Z_r & 0 \\ 0 & 0 & 0 & 0 \\ Z_r & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{G} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \mathcal{J}_r(0)^* Z_r & 0 \\ 0 & 0 & e_1^* & 0 \\ Z_r \mathcal{J}_r(0) & e_1 & 0 & 0 \\ 0 & 0 & 0 & \tilde{H} \end{bmatrix} \\
&\sim \lambda \begin{bmatrix} 0 & 0 & 0 & Z_r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{G} & 0 \\ Z_r & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \mathcal{J}_r(0)^* Z_r \\ 0 & 0 & 0 & e_1^* \\ 0 & 0 & \tilde{H} & 0 \\ Z_r \mathcal{J}_r(0) & e_1 & 0 & 0 \end{bmatrix},
\end{aligned}$$

where the number of blocks of type (30) of the subpencil $\lambda \tilde{G} - \tilde{H}$ is equal to $l$. By the induction hypothesis we find that $\lambda \tilde{G} - \tilde{H}$ is congruent to a pencil that is in sign condensed form (31). Thus, the result follows by again applying Remark 10. □

We are now able to discuss necessary and sufficient conditions for the existence of anti-triangular and anti-$m$-Hessenberg forms for the singular case. A condition on sign sums of real eigenvalues (including $\infty$) of the regular subpencil that is analogous to the condition in Theorem 15 or Corollary 21 does not hold as we can see from the following example. The Hermitian pencil

$$\lambda \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is already in anti-triangular form, but $\operatorname{Signsum}(\infty) = 1$. The background is that the problem of reducing a singular Hermitian pencil to anti-$m$-Hessenberg form is basically the problem of reducing a regular subpencil to anti-$(m+l)$-Hessenberg form, where $l$ denotes the number of singular blocks of the pencil.

**Theorem 24** *Let $\lambda G - H \in \mathbb{C}^{n\times n}$ be a Hermitian pencil and let $m \le n$ be such that $n-m$ is even. Furthermore, let $\mathrm{Ind}(tG-H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$ for $t \in \mathbb{R}$. Then the following statements are equivalent:*

1. *$\lambda G - H$ is congruent to a pencil in anti-$m$-Hessenberg form.*
2. *$\lambda G - H$ is unitarily congruent to a pencil in anti-$m$-Hessenberg form.*
3. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t) + m$.*
4. *For almost all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t) + m$.*

**Proof**. 1) $\Rightarrow$ 2): This follows directly from Lemma 2.

2) $\Rightarrow$ 3): As in the regular case, this follows from Corollary 4.

3) $\Rightarrow$ 4): This implication is trivial.

4) $\Rightarrow$ 1): Assume that $\lambda G - H$ is in sign condensed form (31), i.e.,

$$\lambda G - H = \lambda \begin{bmatrix} 0 & 0 & 0 & G_{14} \\ 0 & O_l & 0 & G_{24} \\ 0 & 0 & G_{33} & G_{34} \\ G_{14}^* & G_{24}^* & G_{34}^* & G_{44} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & H_{14} \\ 0 & O_l & 0 & H_{24} \\ 0 & 0 & H_{33} & H_{34} \\ H_{14}^* & H_{24}^* & H_{34}^* & H_{44} \end{bmatrix},$$

where $\lambda G_{14} - H_{14} \in \mathbb{C}^{k\times k}$ is regular. For all $t \in \mathbb{R}$ that are not eigenvalues of the regular pencil

$$\lambda \begin{bmatrix} 0 & G_{14} \\ G_{14}^* & G_{44} \end{bmatrix} - \begin{bmatrix} 0 & H_{14} \\ H_{14}^* & H_{44} \end{bmatrix}$$

we have that

$$\mathrm{Ind}(tG - H) = (k, k, 0) + (0, 0, l) + \mathrm{Ind}(tG_{33} - H_{33}).$$

Setting $\Big(\mu_+(t), \mu_-(t), \mu_0(t)\Big) := \mathrm{Ind}(tG_{33} - H_{33})$, we obtain for almost all these $t$ that

$$|\mu_+(t) - \mu_-(t)| = |\nu_+(t) - \nu_-(t)| \le \nu_0(t) + m = \mu_0(t) + m + l.$$

The size of $\lambda G_{33} - H_{33}$ is $n - 2k - l$ such that $n - 2k - l - (m - l) = n - m - 2k - 2l$ is even. Thus, Theorem 19 can be applied and $\lambda G_{33} - H_{33}$ is congruent to a pencil $\lambda \hat{G}_{33} - \hat{H}_{33}$ in anti-$(m + l)$-Hessenberg form. Hence

$$\lambda G - H \sim \lambda \begin{bmatrix} 0 & 0 & 0 & G_{14} \\ 0 & O_l & 0 & G_{24} \\ 0 & 0 & \hat{G}_{33} & * \\ G_{14}^* & G_{24}^* & * & G_{44} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & H_{14} \\ 0 & O_l & 0 & H_{24} \\ 0 & 0 & \hat{H}_{33} & * \\ H_{14}^* & H_{24}^* & * & H_{44} \end{bmatrix},$$

and this pencil is in anti-$m$-Hessenberg form. □

We have a corresponding result for the case that $n-m$ is odd. Analogous to the regular case, the entry on the middel of the leftmost nonzero anti-diagonal plays an exceptional role.

**Theorem 25** *Let* $\lambda G - H \in \mathbb{C}^{n \times n}$ *be a Hermitian pencil and let* $n \geq m \in \mathbb{N}$ *such that* $n - m$ *is odd. Furthermore, let* $\mathrm{Ind}(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$ *for* $t \in \mathbb{R}$. *Then the following statements are equivalent.*

1. $\lambda G - H$ *is congruent to a pencil in anti-*$m$*-Hessenberg form.*
2. $\lambda G - H$ *is unitarily congruent to a pencil in anti-*$m$*-Hessenberg form.*
3. *There exists* $t_0 \in \mathbb{R} \cup \{\infty\}$ *and* $\varepsilon \in \{1, -1\}$, *such that*

$$\begin{array}{rlll} & |\nu_+(t) - \nu_-(t) + \varepsilon| \leq \nu_0(t) + m & \textit{for all} & t < t_0 \\ \textit{and} & |\nu_+(t) - \nu_-(t) - \varepsilon| \leq \nu_0(t) + m & \textit{for all} & t > t_0. \end{array}$$

4. *There exists* $t_0 \in \mathbb{R} \cup \{\infty\}$ *and* $\varepsilon \in \{1, -1\}$, *such that*

$$\begin{array}{rlll} & |\nu_+(t) - \nu_-(t) + \varepsilon| \leq \nu_0(t) + m & \textit{for almost all} & t < t_0 \\ \textit{and} & |\nu_+(t) - \nu_-(t) - \varepsilon| \leq \nu_0(t) + m & \textit{for almost all} & t > t_0. \end{array}$$

**Proof**. 1) $\Rightarrow$ 2): This follows directly from Lemma 2.

2) $\Rightarrow$ 3): Assume, there exists a nonsingular matrix $P \in \mathbb{C}^{2n \times 2n}$ such that $P^*(\lambda G - H)P$ is in anti-$m$-Hessenberg form, thus, $P^*(tG - H)P$ is Hermitian anti-$m$-Hessenberg for all $t \in \mathbb{R}$. This means in particular that

$$P^*(tG - H)P = \begin{bmatrix} 0 & 0 & tG_{13} - H_{13} \\ 0 & tg_{22} - h_{22} & tG_{23} - H_{23} \\ tG_{13}^* - H_{13}^* & tG_{23}^* - H_{23}^* & tG_{33}^* - H_{33}^* \end{bmatrix},$$

where $tG_{13} - H_{13} \in \mathbb{C}^{(\frac{n-m-1}{2}) \times (\frac{n+m-1}{2})}$, $tg_{22} - h_{22} \in \mathbb{C}$, and the other blocks have corresponding sizes. If the subpencil $\lambda g_{22} - h_{22}$ is regular, we may proceed as in the proof of Theorem 20. Otherwise, $\lambda g_{22} - h_{22} \equiv 0$. Then it follows from Lemma 3 that

$$|\nu_+(t) - \nu_-(t)| \leq 2\frac{n + m - 1}{2} + \nu_0(t) - n = \nu_0(t) + m - 1$$

for all $t \in \mathbb{R}$. Hence, 2) is trivially satisfied for any $t_0 \in \mathbb{R} \cup \{\infty\}$.

3) $\Rightarrow$ 4): is trivial.

4) $\Rightarrow$ 1): This implication is proved analogous to the proof of Theorem 24. $\square$

It was our main goal to obtain necessary and sufficient conditions for the existence of anti-triangular forms for general (including singular) Hermitian pencils. This explicit result follows now directly from Theorem 24 and Theorem 25.

**Corollary 26** *Let $\lambda G - H \in \mathbb{C}^{2n\times 2n}$ be a Hermitian pencil. Furthermore, for $t \in \mathbb{R}$ let* $\mathrm{Ind}(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$. *Then the following statements are equivalent:*

1. *$\lambda G - H$ is congruent to a pencil in anti-triangular form.*
2. *$\lambda G - H$ is unitarily congruent to a pencil in anti-triangular form.*
3. *For all $t \in \mathbb{R}$ we have that $|\nu_+(t) - \nu_-(t)| \le \nu_0(t)$.*

**Corollary 27** *Let $\lambda G - H \in \mathbb{C}^{(2n+1)\times(2n+1)}$ be a Hermitian pencil. Furthermore, for $t \in \mathbb{R}$ let* $\mathrm{Ind}(tG - H) = \Big(\nu_+(t), \nu_-(t), \nu_0(t)\Big)$. *Then the following statements are equivalent:*

1. *$\lambda G - H$ is congruent to a pencil in anti-triangular form.*
2. *$\lambda G - H$ is unitarily congruent to a pencil in anti-triangular form.*
3. *There exists $t_0 \in \mathbb{R} \cup \{\infty\}$ and $\varepsilon \in \{1, -1\}$ such that*

$$\begin{array}{lll} |\nu_+(t) - \nu_-(t) + \varepsilon| \le \nu_0(t) & \text{for all} & t < t_0, \\ |\nu_+(t) - \nu_-(t) - \varepsilon| \le \nu_0(t) & \text{for all} & t > t_0. \end{array}$$

# 5 Conclusions

We have obtained the so-called *sign condensed form* for general Hermitian pencils. This form is a mixture of an anti-triangular form and a diagonal form, where the diagonal form displays all the ‘singularity’ and all the sign sums of the real eigenvalues of the pencil (or of the regular subpencil), including the eigenvalue $\infty$. We have furthermore obtained necessary and sufficient conditions for the existence of anti-triangular and anti-$m$-Hessenberg forms for Hermitian pencils in terms of conditions on the sign sum of the real eigenvalues and the eigenvalue $\infty$ and in terms of the inertia indices of certain Hermitian matrices. The latter conditions hold also in the case that the pencil is singular. If a Hermitian pencil can be transformed to anti-$m$-Hessenberg form via congruence, then the transformation matrices can be chosen to be unitary, i.e., in this case both matrices of the pencil are simultaneously unitarily similar to anti-$m$-Hessenberg forms.

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