# On "Discovering and Proving that $\pi$ Is Irrational"

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## 1 A needle and a haystack.

Once upon a time, there was a village with a huge haystack. Some villagers found the challenge of retrieving needles from the haystack rewarding, but also frustrating at times. Instead of looking for needles directly, one talented villager had the great idea and gift of looking for threads, and found some sharp and useful needles, by their threads, from the haystack. Decades passed. A particularly beautiful golden needle he found was polished, with its thread removed, and displayed in the village temple. Many more decades passed. The golden needle has been admired in the temple, but mentioned rarely together with its gifted founder and the haystack. Some villagers of a new generation start to claim and believe that the needle could be found simply by its golden color and elongated shape.

## 2 The golden needle.

Let me first show you the golden needle, with a different polish from what you may be used to see. You are no doubt aware of the name(s) of its polisher(s). But you will soon learn the name of its original founder.

**Theorem 1.** *$\pi^2$ is irrational.*

*Proof.* Let $f_n(x) = x^n(\pi - x)^n/n!$ and $I_n = \int_0^\pi f_n(x)\sin x\,dx$ for $n \geq 0$. Then $I_0 = 2$ and $I_1 = 4$. For $n \geq 2$, it is easy to verify that

$$f_n''(x) = -(4n-2)f_{n-1}(x) + \pi^2 f_{n-2}(x). \tag{1}$$

Using (1) and integration by parts, we get $I_n = (4n-2)I_{n-1} - \pi^2 I_{n-2}$. Inducting on $n$, we see that each $I_n$ is a polynomial in $\pi^2$ with integer coefficients and degree at most $\lfloor n/2 \rfloor$. Now assume that $\pi^2 = b/a$ with $a, b \in \mathbb{N}$. Then each $a^{\lfloor n/2 \rfloor} I_n$ is an integer, but $0 < a^{\lfloor n/2 \rfloor} I_n \to 0$ as $n \to \infty$, a contradiction. □

## 3 Hay and needles that are not golden.

To motivate $I_n$, the author of [6] claims in the abstract:

Using the concept that a quadratic function with the same symmetric properties as sine should when multiplied by sine and integrated obey upper and lower bounds for the integral, a contradiction is generated for rational candidate values of $\pi$. This simplifying concept yields a more motivated proof of the irrationality of $\pi$ and $\pi^2$.

Really? Throughout [6], the author emphasizes repeatedly the significance of the common graphical symmetry shared by $x^n(\pi - x)^n$ and $\sin x$ over $[0, \pi]$, yet fails to explain how this graphical similarity is used. So let me first try to counter this claim. Consider two new integrals

$$H_n = \int_0^{2\pi/3} \frac{x^n(2\pi - 3x)^n}{n!} \left(2 - \sec\left(x - \frac{\pi}{3}\right)\right) dx$$

and

$$K_n = \int_0^{\pi/2} \frac{x^n(\pi - 2x)^n}{n!} \cos x \, dx.$$

Clearly, $x^n(2\pi - 3x)^n$ and $2 - \sec(x - \pi/3)$ share a common symmetry, while $x^n(\pi - 2x)^n$ and $\cos x$ do not. According to the simplifying concept, $H_n$ would be a more motivated choice than $K_n$ to use in proving the irrationality of $\pi$. However, does $H_n$ work? On the other hand, we can show that $K_n$ works. Indeed, we have similarly that $K_0 = 1$, $K_1 = 4 - \pi$, and $K_n = (8n - 4)K_{n-1} - \pi^2 K_{n-2}$ for $n \geq 2$. Hence each $K_n$ is a polynomial in $\pi$ with integer coefficients and degree at most $n$. Now assume that $\pi = p/q$ with $p, q \in \mathbb{N}$. Then each $q^n K_n$ is an integer, but $0 < q^n K_n \to 0$ as $n \to \infty$, a contradiction.

From the similarity in the proofs using $I_n$ and $K_n$, the reader can see that it is much more significant that $(\sin x)'' = -\sin x$ and $f_n(x)$ satisfies (1), which render their product $f_n(x) \sin x$ perfect for integration by parts to generate the crucial recurrence relation for $I_n$. This also explains why the integral

$$\int_0^r \frac{x^n(r - x)^n}{n!} e^x \, dx$$

can be used to prove the irrationality of $e^r$ for nonzero rational $r$ [12], because $(e^x)'' = e^x$, even though $x^n(r - x)^n$ and $e^x$ share no graphical symmetry either.

## 4 More hay.

As the second component of his simplifying concept, the author of [6] also argues for the significance of the upper and lower bounds of $I_n$ in eliminating some rational candidate values of $\pi$, and uses this elimination as evidence that he is discovering $\pi$ is irrational. Let me summarize his argument first.

Clearly, $0 < I_n < \pi^{2n+1}/(4^n n!)$ for $n \geq 1$. For $n = 2, 3, 4$, direct integrations yield [1]

$$0 < I_2 = 24 - 2\pi^2 < \frac{\pi^5}{32}, \tag{2}$$

$$0 < I_3 = 240 - 24\pi^2 < \frac{\pi^7}{384}, \tag{3}$$

$$0 < I_4 = 3360 - 360\pi^2 + 2\pi^4 < \frac{\pi^9}{6144}. \tag{4}$$

The candidate 7/2 for $\pi$ contradicts the lower bound of (2). The upper or lower bound of (3) is contradicted by the candidates 3, 13/4, 16/5, and 19/6. He discovers that 22/7 is not $\pi$ using (4), and concludes:

> We have evidence that our method can be used to prove $\pi$ is irrational.

Is this conclusion convincing or non sequitur? Let's scrutinize the argument for a particular $n$, say $n = 3$. Notice that (3) implies $3.113\cdots < \pi < \sqrt{10}$, which obviously eliminates the rational candidates 3, 13/4, 16/5, and 19/6. But this is a magic trick of misdirection, because all other candidate values outside the interval $(3.113\cdots, \sqrt{10})$, rational or irrational, are eliminated as well. In other words, these rational candidate values violate the lower or upper bound of $I_3$ because they are either too small or too big; not because they are rational.

We can also illustrate this logical fallacy further with the integral [7]

$$L_n = \int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2}\,dx,$$

which satisfies $0 < L_n < (1/4)^{4n}$ and yields

$$L_1 = \frac{22}{7} - \pi,\ L_2 = 4\pi - \frac{188684}{15015},\ L_3 = \frac{431302721}{8580495} - 16\pi,\ \ldots.$$

Evidently, many extremely good rational candidate values of $\pi$ contradict the upper or lower bounds of $L_1$, $L_2$, and $L_3$. Can we then conclude that we have evidence that $L_n$ can be used to prove $\pi$ is irrational?

In fact, we can penetrate the heart of the issue even without an elaborate integral. Following Archimedes, we can use inscribed and circumscribed regular $6 \cdot 2^n$-gons to get $b_n < \pi < a_n$, where [2, pp. 15–19]

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \qquad b_{n+1} = \sqrt{a_{n+1} b_n},$$

with $a_0 = 2\sqrt{3}$ and $b_0 = 3$. Then the candidate 7/2 or 3 contradicts $a_0 = 3.4641\cdots$ or $b_0 = 3$. The value of $a_1 = 3.2153\cdots$ eliminates the candidate 13/4. We discover that 16/5, 19/6, or 22/7 is not $\pi$ using $a_2 = 3.1596\cdots$ or

[1] Instead of direct integrations, it is of course easier to use the recursive formula $I_n = (4n-2)I_{n-1} - \pi^2 I_{n-2}$ with $I_0 = 2$ and $I_1 = 4$. However, this formula is not noticed in [6].

$a_4 = 3.1427\cdots$. Moreover, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, any candidate value of $\pi$, too big or too small, will be eliminated by $a_n$ or $b_n$ for a sufficiently large $n$. But we certainly cannot conclude that Archimedes had evidence that his method could be used to prove the irrationality of $\pi$, because we are unable to eliminate the possibility of the last candidate standing, the common limit of $a_n$ and $b_n$, being rational, which is what really matters.

This comparison with Archimedes's squeeze reveals that we should have paid no attention whatsoever to the red herring that $I_1$, $I_2$, $I_3$, or $I_4$ can eliminate some rational candidate values. Instead, our attention should have been directed to the truly significant and remarkable fact that $I_2$, $I_3$, and $I_4$ are polynomials in $\pi$, actually in $\pi^2$, with integer coefficients and degrees at most 2, 3, and 4, respectively. So why is $I_n$ so remarkable and seems so perfectly crafted for the task? How can we motivate it convincingly and, above all, sincerely? Perhaps it is time to recover the missing thread.

# 5 The missing thread.

We see earlier that both $I_n$ and $K_n$ satisfy similar and perfectly-crafted recurrence relations. Can recurrence be the clue? If we vary $I_n$ into

$$I_n(r) = \int_0^r \frac{x^n(r-x)^n}{n!} \sin x \, dx,$$

as done in [12], then $I_0(r) = 1 - \cos r$, $I_1(r) = 2(1-\cos r) - r\sin r$, and

$$I_n(r) = (4n-2)I_{n-1}(r) - r^2 I_{n-2}(r). \tag{5}$$

Notice that (5) can be rewritten as

$$\frac{I_{n-1}(r)}{I_{n-2}(r)} = \cfrac{r^2}{4n-2-\cfrac{I_n(r)}{I_{n-1}(r)}},$$

from which we get a formal continued fraction

$$2 - r\cot\frac{r}{2} = \frac{I_1(r)}{I_0(r)} = \cfrac{r^2}{6-\cfrac{I_2(r)}{I_1(r)}} = \cfrac{r^2}{6-\cfrac{r^2}{10-\cfrac{r^2}{14-\cfrac{r^2}{\ddots}}}},$$

which is equivalent to

$$\tan\frac{r}{2} = \cfrac{r}{2-\cfrac{r^2}{6-\cfrac{r^2}{10-\cfrac{r^2}{14-\cfrac{r^2}{\ddots}}}}}.$$

It is well known that in 1761, J. H. Lambert conceived the first proof of the irrationality of $\tan r$ for nonzero rational $r$, and as a corollary, the irrationality of $\pi$ [2, pp. 129–146]. In this original breakthrough, Lambert exploited the continued fraction for $\tan r$ and the recurrence relation $R_n(r) = (2n-1)R_{n-1}(r) - r^2 R_{n-2}(r)$ associated with the convergents. Is this the missing thread? Is $I_n$ created from Lambert's continued fraction for $\tan x$? If so, who created it?

## 6 The temple records.

Caution is also needed in reading the opening statements in [6]:

> Charles Hermite proved that $e$ is transcendental in 1873 using a polynomial that is the sum of derivatives of another polynomial [5]. Ivan Niven in 1947 found a way to use Hermite's technique to prove that $\pi$ is irrational [8].

The second statement exemplifies a common misperception. Niven did not cite any reference in his famous 1947 paper [8]. Later in 1956, in his equally popular book [9, p. 27], Niven only pointed readers to [5] which is on the transcendence of $e$. Since then, most later authors follow Niven's lead and only give the reference [5]. For example, the encyclopedic source book of $\pi$ [2, pp. 162–193] includes only [5]. Another popular book [1, p. 41] also refers the readers only to [5].

But if we turn a couple of pages back from [5] in Hermite's *Oeuvres III*, we find [4] which contains a simple proof of the irrationality of $\pi^2$, using the integral $R_n(\pi/2) = I_n(\pi)/2^{n+1}$, where

$$R_n(r) = \frac{r^{2n+1}}{2^n n!} \int_0^1 (1-z^2)^n \cos(rz)\, dz. \tag{6}$$

In fact, with the slightest polish, Hermite's proof in [4] is the same as what we give to Theorem 1 at the beginning. If we turn a few pages further back from [4], then we find [3] which reveals that Hermite created (6) and other similar integrals exactly from Lambert's continued fraction for $\tan r$. See [11] for a more detailed discussion.

Moreover, if we read [3], [4], and [5] together, we see that in 1873, Hermite was in a serious hunt first for the transcendence of $\pi$, encountered difficulties, and fell short. But all was not lost. He obtained, as a consoling by-product, a new and elegant proof of the irrationality of $\pi^2$, and then promptly realized that similar ideas and integrals, created from Lambert's continued fraction for $\tanh r$, could make the "easier" transcendence of $e$ transparent. It then did not take long for F. Lindemann, in 1882, to overcome Hermite's difficulties with the transcendence of $\pi$ [2, pp. 194–206]. From [3], we also find that Hermite may have been attracted to $R_n(r)$ through its connection to the Bessel function $J_\nu(r)$. In fact [11],

$$R_n(r) = \sqrt{\frac{\pi}{2}}\, r^{n+\frac{1}{2}} J_{n+\frac{1}{2}}(r).$$

This connection may be directly responsible for C. L. Siegel's 1929 work [10] on the transcendence and algebraic independence of certain values of *E-functions*, which differ from Bessel functions only by simple factors. However, I shall leave the exploration and exposition of these topics to mathematicians who are much more competent in history, transcendental number theory, and a reading knowledge of French and German.

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