**Università degli Studi di Pisa**

FACOLTÀ DI MATEMATICA
Corso di Laurea magistrale in Matematica

Tesi di laurea magistrale

# Hyperplane Arrangement and Discrete Morse Theory

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**Anno Accademico 2017–2018**

# Contents

# Introduction

The aim of this thesis is to study the complement of a hyperplane arrangement using the techniques of Discrete Morse theory.

A hyperplane arrangement is simply a set $\mathcal{A} = \{H_1, H_2, \ldots\}$ of hyperplanes in a vector space. This object has been widely studied especially to find correlations between its topological properties and the combinatorics of the intersections of the hyperplanes. One of the most studied topological objects is the complement, i.e. the vector space minus the hyperplanes and its homology and homotopy groups. One of the question that we are going to answer is if this complement is a minimal space, meaning that it is homotopy equivalent to a $CW-$complex with as many $i-$cells as the $i-$th Betti number. We will see that the answer is positive in various settings.

Having a minimal complex, if it is given explicitly, can also help in studying various properties of the complement, we will focus in particular on abelian local homology. Local homology is an important tool for the study of hyperplane arrangements because it gives us informations on a special fibration on the complement, called Milnor fibration as well as informations about the characteristic varieties. Even if there is plenty of research on the subject there is still a lot unknown about local homology even for the most famous arrangements, like the Braid ones.

In the first chapter we talk about Discrete Morse theory, first introduced by Forman in [For98],[For02]. The aim is to reduce a $CW-$complex or in general various type of topological and combinatorial object to smaller ones, called *Morse complex* with a series of elementary collapsments such that the properties of the complex are still the same. We will study explore the correlation of this theory with shellability [Del08] and focus on different aspect, all of which will became useful in the following chapters.

In the second chapter, we first give a brief introduction to the theory of hyperplane arrangement, presenting some of the most important known results, following [OT13], concerning in particular their combinatorial properties and homology groups. In the special case of complexified real arrangement we introduce the Salvetti complex [Sal87], a $CW-$complex homotopy equivalent to the complementary of the arrangement. We then review three different articles that with similar techniques have reduced this complex to a minimal one (with as many cells as the Betti numbers).

The following chapter introduce the concept of local homology and its correlations with Discrete Morse theory, in particular how we could compute the local homology of the Morse complex. We focus then our attention to a special kind of hyperplane arrangement, called the Braid arrangement and we explicitly write a program in Sage to compute the boundary in local homology.

In the last chapter we try to give our contribute to the subject. It is a joint work with Giovanni Paolini in which in a similar way to what has been done in [Del08] in the case of oriented matroids, we reduce the Salvetti complex in the case of affine, locally finite hyperplane arrangment to a minimal Morse complex giving a special characterization to the critical cells.

# Chapter 1

# Discrete Morse Theory

In this chapter we will introduce Discrete Morse Theory, an important tool to study the homology and cohomology of $CW-$complexes collapsing some cells. Our approach will be similar to the one presented in [Koz08] but sometimes we will also reference to the originals Forman's articles [For02],[For98]. We will also give briefly an introduction to shellability and how it compares to the Discrete Morse Theory since it will be useful to understand [Del08].

## 1.1 Collapsibility of posets

We start here with the case of posets and then in the following section we will talk with more details about the case of $CW-$complex. The idea behind the Discrete Morse Theory is that of collapsibility, in particular of elementary collapse.

**Definition 1.1.1.** Given a generalized simplicial complex $X$, an *elementary collapse* is the removal of the interiors of two simplices $\sigma$ and $\tau$ such that:

- $\dim\sigma = \dim\tau + 1$,
- the only simplex containing $\sigma$ is $\sigma$ itself,
- the only simplices containing $\tau$ are $\sigma$ and $\tau$.

The idea is then to have a set of collapses and to encode them in a matching, consisting of collection of pairs. Of course not every matching arise from a series of elementary collapsment, so what we want to do is to characterize said matchings.

**Definition 1.1.2.** 1. A *partial matching* of a poset $(P, \prec)$ is a subset $\mathcal{M} \subseteq P \times P$ such that

- $(a, b) \in \mathcal{M}$ implies $b \succ a$ and $\nexists\, c$ such that $b \succ c \succ a$, we call $b = u(a)$,

- each $a \in P$ belongs to at most one element in $\mathcal{M}$.

2. Given the Hasse diagram of $P$ we can orient all the edges so that they point from the larger element to the smaller one and then we change the orientation of the edges in a partial matching $\mathcal{M}$. $\mathcal{M}$ is then called *acyclic* if this oriented graph has no cycles.

The idea now is to remove all the matched elements in some appropriate order without changing the homotopy of the underlying space. We call the unmatched elements *critical*.

The next theorem is the crucial step in being able to prove so in the following section; the idea is to look for linear extensions of our poset that in a certain way respect the matching.

**Theorem 1.1.3.** *A partial matching on $P$ is acyclic if and only if there exists a linear extension $L$ of $P$ such that the elements $a$ and $u(a)$ follow consecutively in $L$ for every pair $(a, u(a))$ in the matching.*

*Proof.* We prove first the second arrow since it is easier. We have a partial matching $\mathcal{M}$ and a linear extension $L$. We want to show that given a short sequence of the form $a_1 \nearrow u(a_1) \searrow a_2$ we have that $a_1 >_L a_2$. From this follow immediately that the matching is acyclic. But the proof is clear because we have that $u(a_1) >_L a_2$ since it is true in $P$ and $L$ is an extension. Moreover we also now that $a_1$ is the biggest element smaller than $u(a_1)$ and that the extension is linear so $a_1 >_L a_2$.

To prove the contrary now we have an acyclic matching $\mathcal{M}$ and we want to define inductively the linear extension $L$. We denote with $Q$ the set of elements that are already ordered in $L$ and we start with $Q = \emptyset$. Let $W$ be the set of minimal elements in $P \backslash Q$. At each step we have two possibilities:

- One of the elements $c$ in $W$ is critical.

  We add $c$ to $L$ as the largest element and proceed with $Q \cup \{c\}$.

- All elements in $W$ are matched.

  Let's consider the subgraph of the Hasse diagram of $P \backslash Q$ induced by $W \cup u(W)$ with the edges oriented as in the definition above and call this graph $G$.

  If there is an element $a \in W$ such that there is only one edge towards or outwards from $u(a)$ (the one starting in $a$), then we can add $a$ and $u(a)$ on top of $L$ and proceed with $Q \cup \{a, u(a)\}$. If this is not true we want to say that $G$ has a cycle that contradicts our hypothesis. This follow from the fact that every vertex has an edge that point outwards.

□

**Theorem 1.1.4** (Patchwork lemma). *Assume that $\phi : P \to Q$ is an order-preserving map, and assume that we have acyclic matchings on subposets $\phi^{-1}(q)$, for all $q \in Q$. Then the union of these matching is itself an acyclic matching on $P$.*

*Proof.* The proof of this lemma is straightforward. Indeed the union is surely a partial matching so we only need to prove that it has no cycles.

Let's suppose we have a cycle of the form

$$a_1 \nearrow b_1 \searrow a_2 \ldots \searrow a_n = a_1$$

Since the matching on each fiber is acyclic in at least one point we have to change fiber, let $i$ be that point, meaning that $\phi(a_i) = \phi(b_i) \neq \phi(a_{i+1})$. But $b_i \succ a_{i+1}$ implies that $\phi(b_i) \succ \phi(a_{i+1})$ since the map is order preserving. This is true every times we change fiber so we can't come back to the starting fiber if we have change at least once. This implies that the matching is acyclic.

□

## 1.2 Discrete Morse Theory for CW-complex

We want now to focus our study to the special case of $CW-$complexes. We recall here that we can associate to a $CW-$complex $X$ the poset of its faces $\mathcal{F}(X)$ (with a minimal element $\hat{0}$ representing the empty face). Then we can construct an acyclic matching on $\mathcal{F}(X)$ and study what this tell us about the complex.

**Definition 1.2.1** (cellular elementary collapse). Let $Y$ be a subcomplex of a $CW-$complex $X$. We say that $Y$ is obtained from $X$ by a *cellular elementary collapse* if $X$ can be obtained from $Y$ by attaching two cells: $B^{n-1}_+$ and $B^n$, where $B^{n-1}_+$ is one of the closed hemisphere on the boundary of $B^n$ and $B^n$ is attached to $Y$ by a map $\phi$ on the other hemisphere.

We are now ready to state and prove the main theorem in this chapter that explains in detail how, given an acyclic matching, we are able to "reduce" a $CW-$complex to a smaller one preserving homotopy and homology.

**Theorem 1.2.2.** *Let $X$ be a regular $CW-$complex, and let $\mathcal{M}$ be an acyclic matching on $\mathcal{F}(X) \backslash \{\hat{0}\}$. Let $c_i$ denote the number of critical $i-$dimensional cells of $X$.*

*(i) If the critical cells form a subcomplex $X_c$ of $X$, then there exists a sequence of cellular collapses leading from $X$ to $X_c$.*

(ii) *In general, the space $X$ is homotopy equivalent to $X_c$, where $X_c$ is a $CW-$complex with $c_i$ cells in dimension $i$ called the* Morse complex *of $\mathcal{M}$.*

(iii) *There is a natural indexing of cells of $X_c$ with the critical cell of $X$ such that for any two cells $\sigma$ and $\tau$ of $X_c$ satisfying $\dim \sigma = \dim \tau + 1$, the incidence number is given by*

$$[\tau : \sigma] = \sum_c \omega(c)$$

*Where the sum is taken over all alternating paths $c$ connecting $\sigma$ with $\tau$, i.e. sequences $c = (\sigma, a_1, u(a_1), \dots, a_t, u(a_t), \tau)$ such that $\sigma \succ a_1$, $u(a_t) \succ \tau$, and $u(a_i) \succ a_{i+1}$, for $I = 1, \dots, t-1$. The quantity $\omega(c)$ is defined by:*

$$\omega(c) = (-1)^t [a_1 : \sigma][\tau : u(a_t)] \prod_{i=1}^{t} [a_i : u(a_i)] \prod_{i=1}^{t-1} [a_{i+1} : u(a_i)]$$

*where all the incidence numbers are taken in $X$.*

*Proof.* (i) Since the critical cells form a subcomplex, watching closely at the proof of 1.1.3 we note that we can choose a linear extension $L$ such that all the critical cells come first. Hence, $L$ gives a sequence of cellular collapses from $X$ to $X_c$ if we read in decreasing order.

(ii) The proof here is by induction on the cardinality of $\mathcal{F}(X)$, $L$ is again a linear extension given by 1.1.3. The base step when $|\mathcal{F}(X)| = 1$ is clear so we can do the induction step. Let then $\sigma \in X$ be the bigger cell with respect to $L$. There are two possibilities.

- $\sigma$ is critical.
  Let $\tilde{X} = X \backslash Int\sigma$ and $\varphi : \delta\sigma \to \tilde{X}$ the attaching map of $\sigma$.
  $\tilde{X}$ has one less cell than $X$ and the restriction of $\mathcal{M}$ is clearly still an acyclic matching so for the inductive hypothesis we have that there exists an homotopy equivalence $h$ between $\tilde{X}$ and a $CW-$complex $\tilde{X}_c$ with as many cells as the critical cells of $\tilde{X}$ (that are the same as the critical cells of $X$ minus $\sigma$.
  We are now able to attach $\sigma$ to $\tilde{X}_c$ toward the map $h \circ \varphi : \delta\sigma \to \tilde{X}_c$. The theorem now follows if we set $X_c = \tilde{X}_c \cup_{h \circ \varphi} \sigma$.
- $\sigma$ is not critical.
  There exist $\tau$ such that $(\tau, \sigma) \in \mathcal{M}$ and since they are the highest element in $L$ removing this pair from $X$ is a cellular collapse, in particular $X$ and $\tilde{X} = X \backslash (Int\sigma \cup Int\tau)$ are homotopy equivalent. Again as before the matching $\mathcal{M} \backslash \{(\tau, \sigma)\}$ is an acyclic matching on $\tilde{X}$ and by induction

$\tilde{X}$ is homotopy equivalent to a $CW-$complex $\tilde{X}_c$ with $c_i$ $i-$dimensional cells since the critical cells of $\tilde{X}$ are the same of $X$. But composition of homotopy equivalences is still an homotopy equivalence so if we set $X_c = \tilde{X}_c$ we obtain the thesis.

(iii) Let $\sigma$ be a critical cell of $X$ of dimension, we want to study the attaching map of $\sigma$ after all the collapses.

Recall from the proof of the previous point and the construction of $L$ in 1.1.3 that the collapses can be performed in a order such that the dimension of the collapsing pair does not increase.

Let's at first study what happens after one collapse of a pair $(a, u(a))$ with $a$ of dimension $n-1$. If $a$ was not in the image of the attaching map of $\sigma$ then this collapse does not alter the attaching map. Otherwise $a$ gets replaced with $\delta u(a) \backslash Int(a)$. This process will continue in sequence until all the collapses of pair of dimension $(n, n-1)$ are done.

Once all this pairs have been collapsed the cells in the image of the attaching map of $\sigma$ are the critical ones and those that are matched to the cells of dimension $n-2$. But the latters after their collapse leave no contribution to the incidence number. Then the only thing that we are interested in is how often and with which orientations the critical cells of dimension $n-1$ will appear on $\delta\sigma$.

From our procedure above follows that the appearance of a critical cell $\tau$ is in one-to-one with the alternating paths between $\sigma$ and $\tau$, moreover when $a$ gets replaced by $\delta u(a) \backslash Int(a)$ each cell $b$ in the latter gets the incidence number $-\epsilon[a : u(a)][b : u(a)]$ where $\epsilon$ is the incidence number of $a$. Putting together this two facts concludes the proof.

□

## 1.3 Out-j Collapsibility

This section is devoted to the study of a particular use of the Discrete Morse theory that will be useful to review the article [Adi14] in the following chapter.

We begin with some definitions.

**Definition 1.3.1.** Let $X$ be a regular $CW-$complex and $Y$ a non-empty subcomplex. Given an acyclic matching on $X$ we say that a face $\tau$ of $Y$ is *outwardly matched* with respect to the pair $(X, Y)$ if it is matched with a face that does not belong to $Y$.

**Definition 1.3.2.** Let $X$ and $Y$ as above. The pair $(X, Y)$ *out-j collapses* to the pair $(X', Y \cap X')$, and we write

$$(X, Y) \searrow_{out-j} (X', Y \cap X')$$

if there exists a collapsing sequence that reduces $X$ to $X'$ and every outwardly matched face with respect to the pair $(X, Y)$ has dimension $j$.

We say that the pair $(X, Y)$ is *out-j collapsible* if there is a vertex $v$ of $Y$ such that

$$(X, Y) \searrow_{out-j} (v, v).$$

Even if the notion of out-$j$ collapses may seem at a first sight not very useful we want now to show that it is actually quite natural. In particular, the number of outwardly matches faces is independent of the collapsing sequence.

**Proposition 1.3.3.** *Let $(X, Y)$ be an out-j collapsible pair, with $Y \neq \emptyset$. The number of outwardly matched $j-$faces of $Y$ is independent of the out-j collapsing sequence chosen, and equal to $(-1)^j(\chi(Y)-1)$ (where $\chi$ is the Euler characteristic). Moreover, the following facts are equivalent:*

1. *$Y$ is contractible,*
2. *There exists a collapsing sequence that has no outwardly matched faces with respect to $(X, Y)$,*
3. *$Y$ is collapsible.*

*Proof.* The second part is simply a consequence of the first one, indeed:

- If $Y$ is contractible then $\chi(Y) = 1$ but then if we take any collapsing sequence we have that the number of outwardly matched $j-$faces is equal to zero for each $j$. This proves 1) $\Rightarrow$ 2).
- If the sequence has no outwardly matched faces, then we can restrict it to $Y$ and obtain a collapsing sequence for $Y$. This proves 2) $\Rightarrow$ 3).
- It is always true that collapsible implies contractible, so 3) $\Rightarrow$ 1).

We need now to prove the first statement. Let's then fix an out-$j$ collapsing sequence. The proof is an easy use of double counting. To do it, we need to define some sets:

- $F = \{\text{faces of } Y\}$ and $f_i = \#\{i-\text{faces of } F\}$,
- $O = \{\text{outwardly matched faces of } Y\}$ and $o_i = \#\{i-\text{faces in } O\}$,

- $Q = \{\text{faces of } Y \text{ matched with faces of } Y\}$ and $q_i = \#\{i\text{−faces in } Q\}$.

Moreover, let $v$ be the vertex onto which $X$ collapses. We then have that $f_i = o_i + q_i$ if $i \geq 1$ and that $f_0 = o_0 + q_0 + 1$. From this follow the chain of equalities:

$$\chi(Y) = \sum(-1)^i f_i = 1 + \sum(-1)^i o_i + \sum(-1)^i q_i = 1 + (-1)^j o_j$$

and this prove the proposition.

□

In the following we will also need this lemma that can be proved pretty easily.

**Lemma 1.3.4.** *Let $X$ be a regular $CW-$complex, $Y$ a subcomplex and $v$ any vertex of $X$. Assume that $v \notin Y$ or $Lk(v, Y)$ is nonempty, and that $(Lk(v, X), Lk(v, Y))$ is out-$j$ collapsible. Then $(X, Y)$ out-$j$ collapses to $(X - v, Y - v)$ (where $X - v$ is the subcomplex of $X$ without all the faces that contained $v$). On the other hand if $v \in Y$ but $Lk(v, Y) = \emptyset$, then $(X, Y)$ out-0 collapses to $(X - v, Y - v)$ if and oly if $Lk(v, X)$ is collapsible.*

The last thing that we want to state in this section is a generalization of 1.2.2 that include the outwardly matches introduce above, the prove is similar to the one written in 1.2.2. For more detail we send the reader to [Adi14].

**Theorem 1.3.5.** *Let $X$ be a regular $CW-$complex and $Y$ a subcomplex. Let $\mathcal{M}$ be an acyclic matching on $X$ that does not have any outwardly matched faces with respect to the pair $(X, Y)$. Then $X$ is up to homotopy equivalence obtained from $Y$ by attaching one cell of dimension $k$ for every critical $k-$cell of $\mathcal{M}$ not in $Y$.*

## 1.4 Shellability

As a first thing we need to define what shellable means and see why it is useful. Then we will explore the connection with the Discrete Morse Theory.

We will first study the case of simplicial complex, again following [Koz08].

**Definition 1.4.1.** A simplicial complex $X$ is called *shellable* if its maximal simplices can be arranged in linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} F_i) \cap F_k$ is pure (meaning that all its maximal simplex have the same dimension) and $(\dim F_k - 1)-$ dimensional for all $k = 2, \ldots, t$.

The idea behind shellability is that we want to use this order to build the complex gluing together the maximal simplices starting from the first one and going on.

The most important topological fact that we need to highlight is that for any $n-$dimensional simplex, if we take the union of some of the $(n-1)-$dimensional simplices in its boundary then or we obtain something homeomorphic to a sphere if we have taken the entire boundary or in all the other cases is contractible. While gluing our simplices then if we glue along the entire boundary we are adding a generator in the right homology group, in the other cases we are simply adding something contractible. To make this clearer we need a definition and a theorem.

**Definition 1.4.2.** A maximal simplex $\sigma$ is called *spanning* with respect to a given shelling order if it is glued along its entire boundary.

**Theorem 1.4.3.** *[Koz08, Theorem 12.3] Assume that $X$ is a shellable simplicial complex with $F_1, F_2, \dots, F_t$ being the corresponding shelling order of the maximal simplices, and $\Sigma$ being the set of spanning simplices. Then the following facts hold:*

*(i) The generalized simplicial complex obtained by the removal of the interiors of the spanning simplices, that is, the complex $\tilde{X} = X \setminus \bigcup_{\sigma \in \Sigma} Int\sigma$, is collapsible.*

*(ii) The generalized simplicial complex $X$ is homotopy equivalent to a wedge of spheres that are indexed by the spanning simplices and have corresponding dimensions.*

*(iii) The cohomology groups of $X$ with integer coefficients are free, and the set of elementary cochains $\{\sigma^*\}_{\sigma \in \Sigma}$ can be taken as a basis.*

In reality we are interested to work with $CW-$complexes and in this case the definition is a bit more complex but it is immediate to see that it reduces to the above definition in case of simplicial complexes.

**Definition 1.4.4.** [BW97, Definition 13.1] A regular $CW-$complex $X$ is called *shellable* if its maximal cells can be arranged in linear order $F_1, F_2, \dots, F_t$ in such a way that if $\dim X \geq 1$ the following conditions are satisfied ($\delta(F)$ is the subcomplex consisting of all proper faces of $F$):

- $\delta(F_1)$ is shellable,
- $(\bigcup_{i=1}^{k-1} F_i) \cap F_k$ is pure and $(dimF_k - 1)-$ dimensional for all $k = 2, \dots, t$,
- $\delta(F_k)$ admits a shelling in which the cells of $(\bigcup_{i=1}^{k-1} F_i) \cap F_k$ come first, for all $k = 2, \dots, t$.

Even for $CW-$complexes can be proved a theorem analogous to 1.4.3 as done in [BW97]. As a consequence of this we can immediately see the following proposition that will be useful.

**Proposition 1.4.5.** *Let $K$ be a regular $CW-$decomposition of a sphere. Then in every shelling order of $K$ the only spanning facet is the last one.*

*Proof.* We prove it by contraposition. Let $K$ be a $CW-$complex with a homeomorphism $\phi: \ K \to S^d$ and a shelling order such that the last facet $F$ of the order is not the only spanning facet. If now we call $K'$ the union of all the facets except $F$ we obtain that $K'$ is not contractible since it contains at least one spanning facet by Theorem 1.4.3. But $K'$ is homeomorphic to $S^d \backslash \phi(F \backslash K')$ and this is contractible since $F \backslash K'$ is. This gives us a contradiction. □

We now want to see the correlation between shellability and Discrete Morse Theory. It is clear that there is one since both are about collapsibility of complexes but we want to make clear how we can make an acyclic matching starting from a shelling order following [Del08].

The first thing we need is a new definition and a new way to see the shellability.

**Notation.** Given a poset $(P, <)$ we will denote

- $coat(p) = \{q < p \mid \nexists x \in P : \ q < x < p \ \}$,
- $P_{\leq q} = \{p \in P \mid p \leq q\}$,
- A totally ordered subset $\omega \subset P$ is called *chain* and its length is $l(\omega) = |\omega| - 1$,
- $l(P)$ is the maximum length of a chain contained in $P$,
- $P$ is said *bounded* if it has a maximal and a minimal element.

**Observation 1.4.6.** *The poset $\mathcal{F}$ of facets of a $CW-$complex is not bounded but we can add to it a maximal element if necessary to make it bounded, we will call this element $\hat{1}$ and the corresponding poset $\hat{\mathcal{F}}$.*

**Definition 1.4.7.** A bounded poset $(P, <)$ is said to admit a *recursive coatom ordering* $\prec$ if $l(P) = 1$ or if $l(P) > 1$ and there is a total ordering on the set $coat(\hat{1})$ such that:

- for all $p \in coat(\hat{1})$, the interval $[\hat{0}, p]$ admits a recursive coatom ordering $\prec_p$ in which the coatoms of the intervals $[hat0, q]$ for $q \prec p$ come first.
- for all $p \prec q$, if $p, q > y$ then there is $p' \prec q$ and $z \in coat(q)$ such that $p' > z \geq y$.

**Theorem 1.4.8.** *[BW97, Theorem 13.2] If $\mathcal{F}$ is the poset of faces of a regular $CW-$complex $X$, then a total ordering of the maximal faces of $X$ is a shelling order if and only if it is a recursive coatom ordering of $\hat{\mathcal{F}}$.*

We want now to describe, using this order, an acyclic matching of the poset of cells of every shellable $CW-$complex. We will continue to follow [Del08] but specialize to the case of poset of facets.

The first thing we need is a technical lemma of which we omit the proof.

**Lemma 1.4.9.** *Let $\mathcal{F}$ be the poset of facets of a $d-$dimensional $CW-$complex $X$ and let $\mathcal{F}^k$ be the $(d-k)-$dimensional facets. Given a recursive coatom ordering $\prec$ on $\hat{\mathcal{F}}$ it is then possible to define a family of total orders $\{(\mathcal{F}^k, \sqsubset_k)\}$ with the following properties: Given $\sigma \in \mathcal{F}^i$, and writing $Q_\sigma = \bigcup_{\sigma' \sqsubset_i p} coat(\sigma'$,*

1. *The order induced by $\sqsubset_{i+1}$ on $D_\sigma = coat(\sigma) \backslash Q_\sigma$ can be extended to a recursive coatom ordering $\prec_\sigma$ of $coat(p)$ in which the elements of $Q_\sigma$ come first.*

2. *For all $\sigma' \sqsubset_i \sigma$, if $\sigma', \sigma > \tau$, then there exist $\sigma'' \sqsubset_i \sigma$ and $\rho \in coat(\sigma)$ such that $\sigma'' > \rho \geq \tau$.*

**Definition 1.4.10.** Let $\mathcal{F}$ as before. $\pi_i : \mathcal{F}^i \to \mathcal{F}^{i+1}$ is defined by:

$$\pi_i(\sigma) = \max_{\sqsubset_{i+1}} \{\tau \in \mathcal{F}^{i+1} \mid \sigma > \tau\}.$$

A *shelling-type ordering* of $\mathcal{F}$ is a linear extension $\lhd$ of $\mathcal{F}$ given by $\sigma \lhd \tau$ if and only if $\sigma \sqsubset_i \tau$ in case $\sigma, \tau$ are cells of the same dimension or $\sigma \sqsubseteq_i \pi_i \pi_{i+1} \dots \pi_{j-1}(\tau)$ in case $\sigma \in \mathcal{F}^i$, $\tau \in \mathcal{F}^j$ and $i > j$.

It is an easy check to see that this is a well defined linear order. Using this we are now finally able to explicitly write the matching.

**Lemma 1.4.11.** *Every shelling type ordering $\lhd$ of $\mathcal{F}$ induces an acyclic matching $\mathcal{M}$ on $\mathcal{F}$.*

*Proof.* We will write $\mathcal{F}^i = \{\sigma^i_1, \dots, \sigma^i_{k_i}\}$, where $\sigma^i_j \sqsubset_i \sigma^i_{j+1}$ for all $j$. We want now to define the matching.

We start with $\mathcal{M} = \{(\sigma^0_1, \pi_0(\sigma^0_1))\}$. For every $j = 2, \dots, k_0$ we add $\sigma^0_j, \pi_0(\sigma^0_j$ to $\mathcal{M}$ if the second element is not already matched.

We then further expand $\mathcal{M}$ is a similar way i. e. for $i = 1, \dots, d-1$ and for $j = 1, \dots, k_i$ if $\sigma^i_j$ is not alreasy matched and $\pi_i(\sigma^i_j) \neq \pi_i(\sigma^i_l)$ for all $l < j$ then we add $(\sigma^i_j, \pi_i(\sigma^i_j))$ to $\mathcal{M}$.

The shelling-type ordering assures us that what we have defined is acyclic. □

Now that we have construct the matching we want to compare its critical cells with the spanning facets and we obtain the best possible result.

**Theorem 1.4.12.** *Every shelling of a regular $CW-$complex $X$ induces an acyclic matching of the poset of faces of $X$. Moreover, the critical cells of this matching correspond to the spanning faces of the given shelling.*

We want now to briefly recall a couple of notions about polytope and polyhedra that will come useful later.

**Definition 1.4.13.** A *polyhedron* is an intersection of finitely many closed halfspaces in some $\mathbb{R}^d$. A *polytope* is a bounded polyhedron.

Given a polyhedron $P$, denote by $\mathcal{F}(P)$ the set of its faces (considering the polyhedron $P$ itself as a trivial face). The faces of codimension 1 are called *facets*. In addition, denote by $\mathcal{F}(\partial P)$ the *boundary complex* of $P$, i.e. the complex that contains only the proper faces of $P$.

**Definition 1.4.14.** We say that a facet $G \in \mathcal{F}(P)$ is *visible* from a point $p \in \mathbb{R}^d$ if every line segment segment from $p$ to a point of $G$ does not intersect the interior of $P$ (cf. [Zie12, Theorem 8.12]). We say that a face $F \in \mathcal{F}(P)$ is *visible* from $p$ if all the facets $G \supseteq F$ of $P$ are visible from $p$. In particular, notice that the entire polyhedron $P$ is always visible.

The notion of shellability is the same given above for a $CW-$complex, where we say that a polytope is *shellable* if it's boundary complex admits a shelling order.
The following theorems tell us that a polytope is always shellable.

**Lemma 1.4.15** ([Zie12, Lemma 8.10])**.** *If $F_1, F_2, \dots, F_s$ is a shelling order for the boundary of a polytope $P$, then so is the reverse order $F_s, F_{s-1}, \dots, F_1$.*

**Theorem 1.4.16** ([BM72], [Zie12, Theorem 8.12])**.** *Let $P \subseteq \mathbb{R}^d$ be a $d$-polytope, and let $x \in \mathbb{R}^d$ be a point outside $P$. If $x$ lies in general position (that is, not in the affine hull of a facet of $P$), then the boundary complex of the polytope has a shelling in which the facets of $P$ that are visible from $x$ come first.*

## 1.5 Discrete Morse theory in the infinite case

We want here to expand discrete Morse theory to not necessarily finite $CW-$complexes following [BW02].
The first thing needed is the definition of grading and of proper acyclic matching.

**Definition 1.5.1** (Grading [BW02])**.** Let $Q$ be a poset. A poset map $\varphi\colon P \to Q$ is called a $Q$-*grading* of $P$. The $Q$-grading $\varphi$ is *compact* if $\varphi^{-1}(Q_{\leq q}) \subseteq P$ is finite for all $q \in Q$. A matching $M$ on $P$ is *homogeneous* with respect to the $Q$-grading $\varphi$ if $\varphi(p) = \varphi(p')$ for all $(p, p') \in M$. An acyclic matching $M$ is *proper* if it is homogeneous with respect to some compact grading.

The following is a direct consequence of the definition of proper matching (cf. [BW02, Definition 3.2.5 and Remark 3.2.17]).

**Lemma 1.5.2** ([BW02]). *Let $\mathcal{M}$ be a proper acyclic matching on a poset $P$, and let $p \in P$. Then there is a finite number of alternating paths starting from $p$, and each of them has a finite length.*

*Proof.* By definition, $P$ is homogeneous with respect to a compact $Q-$grading $\varphi$. Let $\varphi(p) = q$. It is clear that the restriction on the matching $\mathcal{M}$ to $\varphi^{-1}(Q_{\leq q})$ is still acyclic, we call it $\mathcal{M}_q$ and the corresponding oriented graph $G_{\mathcal{M}_q}$. This is a subgraph of $G_{\mathcal{M}}$ and it is easy to see that every alternatin path starting in $p$ is actually in this subgraph. Indeed every edge, or is in $\mathcal{M}$ and then does not change the fiber with respect of $\varphi$, or is in $\mathcal{E} \setminus \mathcal{M}$ and then goes from a fiber to a smaller one.

The Lemma is now obvious, since $G_{\mathcal{M}_q}$ has no cycles and a finite number of edges. □

We are now ready to state the main theorem of discrete Morse theory in the infinite case. We will see that it is really similar to our previous version 1.2.2 but with one more hypothesis on the matching. This particular formulation follows from [BW02, Theorem 3.2.14 and Remark 3.2.17]

**Theorem 1.5.3.** *Let $X$ be a regular CW complex, and let $P$ be its poset of cells. If $\mathcal{M}$ is a proper acyclic matching on $P$, then $X$ is homotopy equivalent to a CW complex $X_c$ (called the* Morse complex *of $M$) with cells in dimension-preserving bijection with the critical cells of $X$.*

The construction of the Morse complex is explicit in terms of the CW complex $X$ and the matching $M$ and the proof is similar to 1.2.2 (see for example [BW02]), and in a similar way we are able to compute the incidence numbers of the Morse complex studying the alternating paths.

# Chapter 2

# Hyperplane arrangements

The aim of this chapter is to introduce the reader to the hyperplanes arrangements and their structure. In the first part we will partially follow [OT13] for the general definitions, several facts will be listed without proof which can however be found in the previous cited reference.

Then we will focus on the use of Discrete Morse Theory to find perfect acyclic matching and review article from Salvetti [SS07], Delucchi [Del08] and Adiprasito [Adi14].

## 2.1 First definitions

An arrangement of hyperplanes is a finite (sometimes even infinite) collection $\mathcal{A} = \{H_1, \cdots, H_r\}$ of codimension one affine subspace in a $n-$dimensional $\mathbb{K}-$vector space $V$.

**Definition 2.1.1.** $L = L(\mathcal{A})$ is the poset of nonempty intersections of elements of $\mathcal{A}$ with order given by the reverse inclusion; meaning that

$$X \leq Y \Leftrightarrow Y \subseteq X$$

.

**Definition 2.1.2.**
- $\mathcal{A}$ is *central* if $\bigcap \mathcal{A} \neq \emptyset$,
- $\mathcal{A}$ is *linear* if $0 \in \bigcap \mathcal{A}$, is called *affine* otherwise,
- $\mathcal{A}$ is *essential* if $rk(\mathcal{A}) = n$, where the rank of an arrangement is the rank (the codimension) of a maximal element of $L(\mathcal{A})$.

It is an easy proof that the rank is well defined, i.e. that maximal elements of $L(\mathcal{A})$ have all the same rank.

Moreover we can define two operations on $L(\mathcal{A})$ called the join and the meet as follow:

**Definition 2.1.3.** Given $X, Y \in L(\mathcal{A})$ we define their *meet* by

$$X \wedge Y = \cap\{Z \in L \mid X \cup Y \subseteq Z\}$$

If $X \cap Y \neq \emptyset$ we define their *join* by

$$X \vee Y = X \cap Y$$

It is easy to prove that if $\mathcal{A}$ is a central arrangement than $L(\mathcal{A})$ is a geometric lattice, in particular all joins exist.

It is now useful to have some definitions that let us in a certain way consider only a part of an arrangement. Of course, if we take $\mathcal{B} \subset \mathcal{A}$ we can consider $\mathcal{B}$ as an arrangement in the same $V$ and we will call it a subarrangement, but the more important constructions are the following:

**Definition 2.1.4.** For $X \in L(\mathcal{A})$ we define the subarrangement $\mathcal{A}_X$ as

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}$$

and the restriction of $\mathcal{A}$ to $X$ as an arrangement $\mathcal{A}^X$ in the vector space $X$ defined by

$$\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \backslash \mathcal{A}_X \textit{ and } X \cap H \neq \emptyset\}.$$

Another important construction is the coning, a method that allows to compare affine and central arrangements.

First, for each hyperplane $H_i$ there exists $\alpha_i \in (\mathbb{K}^*)^n, a_i \in \mathbb{K}$ (not unique) such that $H_i = \{v \mid \alpha_i v = a_i\}$. We can then define the arrangement's polynomial $Q_{\mathcal{A}}(x) \in \mathbb{K}[x_1, \cdots, x_n]$ as $\prod_i(\alpha_i(x) - a_i)$. We should note that this polynomial defines completely the arrangement, meaning that we are able from $Q_{\mathcal{A}}$ to extract all the hyperplanes.

Then, given an affine arrangement $\mathcal{A}$ we define its cone, called $c\mathcal{A}$ as the arrangement with polynomial $Q(c\mathcal{A}) = x_0 Q'$ where $Q'$ is the homogenization of $Q(\mathcal{A}) \in \mathbb{K}[x_o, \cdots, x_n]$.

The last construction that we want to define is the complexification. Let suppose that we have a field extension $\mathbb{L}$ over $\mathbb{K}$ and an arrangement $\mathcal{A}_{\mathbb{K}}$ in a $\mathbb{K}-$vector space that we will call $V_{\mathbb{K}}$. This gives rise naturally to an arrangement over $\mathbb{L}$ where the vector space will be $V = V_{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L}$ and the collection of hyperplanes $\mathcal{A}_{\mathbb{L}} = \{H \otimes_{\mathbb{K}} \mathbb{L} \mid H \in \mathcal{A}_{\mathbb{K}}\}$.

The most important example of this, and the only one that we will study is that of complexified real arrangements, i.e. when $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathcal{C}$.

### 2.1.1 Poincaré polynomial

The Poincaré polynomial is a very important tool in the study of arrangements of hyperplanes but before being able to define it we need to study the Möbius function.

**Definition 2.1.5.** Let $\mathcal{A}$ be an arrangement and $L = L(\mathcal{A})$, the *Möbius function* $\mu : L \times L \to \mathbb{Z}$ is defined as follows:

$$\mu(X, X) = 1 \;\; if \;\; X \in L$$
$$\sum_{X \leq Z \leq Y} \mu(X, Z) = 0 \;\; if X, Y, Z \in L \; and \; X < Y$$
$$\mu(X, Y) = 0 \;\; otherwise.$$

From this function we are then able to define $\mu(X)$:

**Definition 2.1.6.** For $X \in L$ $\mu(X) = \mu(V, X)$.

At this point we are able to define the object we are interested in:

**Definition 2.1.7.** The *Poincaré polynomial* of $\mathcal{A}$ is defined by:

$$\pi(\mathcal{A}, t) = \sum_{X \in L} \mu(X)(-t)^{rk(X)}$$

**Example 2.1.8** (Braid arrangement and its Poincaré polynomial)**.** The braid arrangement is probably one of the most important family of reflection arrangements since they arise naturally from the symmetric groups. But since we are not interested in reflection arrangement at the moment we will simply define the $n-$th braid arrangement $\mathcal{A}_n$ as the set of hyperplanes $H_{ij} = ker(x_i - x_j)$ for $1 \leq i < j \leq n+1$ where the $x_i$ are the coordinates in $\mathbb{R}^{n+1}$.

The Poincaré polynomial of $\mathcal{A}_n$ is

$$\pi(\mathcal{A}_n, t) = (1 + t)(1 + 2t) \dots (1 + nt)$$

a proof of which can be found in [OT13].

The reason we are so interested in this polynomial is because it is equal to the Poincaré polynomial of several graded algebras and, as we will see in the next sections, it has a close relation with the cohomology of the complement of $\mathcal{A}$.

### 2.1.2 Oriented Matroids

We want now to give a brief introduction to a very important argument, that is the definition of oriented matroids. These are combinatorial objects that in a certain way can be seen as a generalization of central hyperplane arrangement. We will here only give the basic definitions, following [BLVS+99].

**Definition 2.1.9.** Given a ground set $E$, a collection $\mathcal{V} \in \{+,-,0\}^E$ is the set of *vectors of an oriented matroid* $\mathcal{M}$ if and only if the following properties are satisfied:

1. $(0,\dots,0) \in \mathcal{V}$,
2. if $X \in \mathcal{V}$, then $-X \in \mathcal{V}$,
3. for all $X, Y \in \mathcal{V}$, $X \circ Y \in \mathcal{V}$,
4. for all $X, Y \in \mathcal{V}$, given $e, f \in E$ such that $X_e = -Y_e$ and not both $X_f, Y_f$ equal 0, there is $Z \in \mathcal{V}$ such that $Z_e = 0$, $Z_f \neq 0$, and if $Z_i \neq 0$ then $Z_i$ equals $X_i$ or $Y_i$.

Let us now take a real linear arrangement of hyperplanes $\mathcal{A} = \{H_1,\dots,H_n\}$. Its combinatorial data is encoded by the associated oriented matroid $\mathcal{M}_{\mathcal{A}}$ construct as follows:

$E$ is the set $\{v_1,\dots,v_n\}$ where, for all $i$, $v_i$ is normal to $H_i$. An element $X$ is in $\mathcal{V}$ if there exists $\lambda_i \in \mathbb{R}^+$ such that:

$$\sum_{i=1}^{n} X_i \lambda_i v_i = 0$$

**Definition 2.1.10.** An oriented matroid is called *realizable* if it is of the form $\mathcal{M}_{\mathcal{A}}$ for some real linear arrangement $\mathcal{A}$.

## 2.2 The complement of a hyperplane arrangement

Given an arrangement $\mathcal{A}$ we will call with $M(\mathcal{A})$ the complement $V \setminus \bigcup \mathcal{A}$ and a chamber will be a connected component of $M(\mathcal{A})$. We will call the set of chamber as $\mathcal{C}(\mathcal{A})$.

The number of chambers if $\mathcal{A}$ is a real arrangement is determined by its Poincaré polynomial as follows.

**Theorem 2.2.1.** *Let $\mathcal{A}$ be a real arrangement, then*

$$|\mathcal{C}(\mathcal{A})| = \pi(\mathcal{A}, 1)$$

We won't prove this or the following theorems in this section since everything can be found with every detail in [OT13]. Here we will just give a sketch of what has been done regarding the cohomology of $M(\mathcal{A})$ and the results that will be used in the following.

Starting from an hyperplane arrangement there are different graded algebras that one can build. One of these is the algebra of differential forms, called $R(\mathcal{A})$.

**Notation.** Given an affine arrangement $\mathcal{A}$ in $V$ a $\mathbb{K}-$vector space we will denote by:

- S the symmetric algebra of $V^*$ and we will write $S = \mathbb{K}[V]$,
- F the quotient field of $S$ and we will write $S = \mathbb{K}(V)$,
- $\Omega(V)$ the graded exterior algebra of $F \otimes V^*$ with the usual differential $d$.

**Definition 2.2.2.** For every $H \in \mathcal{A}$ let $\alpha_H \in S$ be a polynomial of degree 1 such that $H = ker(\alpha_H)$ and let $\omega_H = d\alpha_H/\alpha_H \in \Omega^1(V)$. $R(\mathcal{A})$ is the subalgebra of $\Omega(V)$ generated by 1 and $\omega_H$ for $H \in \mathcal{A}$.

**Theorem 2.2.3.** *The Poincaré polynomial of $R(\mathcal{A})$ is*

$$Poin(R(\mathcal{A}), t) = \pi(\mathcal{A}, t)$$

So we already have another characterization of the Poincaré polynomial of an arrangement and a reason for which is called in this way, but this is not yet the characterization we are interested in.

Let's call $\eta_H = \omega_H/2\pi i$ and with $[\eta_H]$ the cohomology class of $\eta_H$ in $H^1(M(\mathcal{A}))$. We then have the following theorem,

**Theorem 2.2.4.** *The map $\eta : R(\mathcal{A}) \to H^*(M(\mathcal{A})$ that sends $\omega_H$ to $[\eta_H]$ is an isomorphism of graded algebras.*

Since we have already stated that the Poincaré polynomial of $R(\mathcal{A})$ is the same of $\pi(\mathcal{A}, t)$ we immediately obtain from the previous theorem that if $\mathcal{A}$ is a complex arrangement:

$$Poin(H^*(M(\mathcal{A})), t) = \pi(\mathcal{A}, t).$$

But

$$Poin(H^*(M(\mathcal{A})) = \sum_{p \geq 0} b_p(M) t^p$$

where the $b_p(M)$ are the Betti number.

If we put together what we have just said with 2.2.1 we immediately obtain one of the most used theorem in the following.

**Theorem 2.2.5.** *Let $\mathcal{A}$ be a real arrangement than the number of chambers of $\mathcal{A}$ is equal to the sum of the Betti numbers of the complement of the complexification of $\mathcal{A}$.*

### 2.2.1 The Salvetti complex

The Salvetti complex, first introduced in [Sal87], is a CW-complex homotopy equivalent to the complement $M(\mathcal{A})$ in the case of complexified real arrangement.

This complex is particular helpful in studying this particular class of arrangement as we will see in the next section. For now let's try to understand how it is built.

A real hyperplane arrangement $\mathcal{A}$ gives rise to a subdivision of $\mathbb{R}^n$ into facets, where the chambers defined above are the codimension 0 facets. We will call the set of all the facets with $\mathcal{F}$. This set has a natural partial order given by $F \prec G$ if and only if the closure of $F$ contains $G$. We will also call with $B(\mathcal{F})$ the union of the bounded facets, that is known to be a compact connected subset of V.

Then the $k-$cells in $\mathbf{S}$ are in one to one correspondence with the pair $\langle C, F^k \rangle$ where $C$ is a chamber,$k$ is the codimension of $F \in \mathcal{F}$ and $C \prec F^k$. Moreover a cell $\langle C, F^k \rangle$ is in the boundary of $\langle D, G^j \rangle$ if and only if:

- $k < j$ and $F^k \prec G^j$,
- the chambers $C$ and $D$ are contained in the same chamber of $\mathcal{A}_{F^k}$.

**Notation.** Given a chamber $C$ and a facet $F$ we will denote by $C.F$ the unique chamber containing $F$ and lying in the same chamber as $C$ in $\mathcal{A}_F$.

## 2.3 Minimality

In this section we will review three different article that have found with similar methods perfect acyclic matching on different classes of arrangements.

### 2.3.1 Salvetti-Settepanella and the polar order

The first article, [SS07], is also the first to appear in 2007. In it the two authors have used Discrete Morse theory on the Salvetti complex to obtain a minimal complex i.e. with exactly as many $i-$cells as the $i-$th Betti number for the complement of a complexified real arrangement. The idea is that of giving a total order $\lhd$ on $\mathcal{F}$ which they call polar ordering. Given the order the gradient field can then be recursively defined as we will see; but let's start from the beginning.

First, we need some notation. Let $e_1, \ldots, e_n$ be an orthonormal basis of a $n-$vector space $V$.

**Notation.**

- $V_i =< e_1, \dots, e_i >$ for $i = 0, \dots, n$ ($V_0 = 0$),
- $W_i =< e_i, \dots, e_n >$ for $i = 1, \dots, n$,
- $pr_W : V \to W$ is the orthogonal projection onto a subspace,
- Given a point $P \in V$ $P_i := pr_{W_i}(P)$.

**Definition 2.3.1.** The *polar coordinates* of $P$ will be given by the module $\rho = ||P||$ also called $\theta_0$ and the angles $\theta_i$ that $OP_i$ makes with $e_i$ for $i = 1, \dots, n$.

**Definition 2.3.2.** Given a sequence $\bar{\theta}$ of angles ($\bar{\theta} = \bar{\theta}_i, \dots, \bar{\theta}_{n-1}$) we indicate by $V_i(\bar{\theta})$ the set:

$$V_i(\bar{\theta}) = \{P \mid \theta_i(P) = \bar{\theta}_i, \dots, \theta_{n-1}(P) = \bar{\theta}_{n-1}\}$$

that is an $i-$dimensional open half-subspace.

Given a codimensional$-k$ facet $F \in \mathcal{F}$ we denote by

$$F(\bar{\theta}) = F \cap V_i(\bar{\theta})$$

When $\bar{\theta} = \emptyset$ then $F(\bar{\theta}) = F$.

Moreover for $\delta \in (0, \pi/2)$ we define the open cone:

$$\tilde{B}(\delta) = \{P \mid \theta_i(P) \in (0, \delta) \ for \ i = 1, \dots, n-1, \ \rho(P) > 0\}$$

Let now $\mathcal{A}$ be an essential finite arrangement of hyperplanes in $\mathbb{R}^n$ and let $M_{\mathcal{C}}(\mathcal{A})$ be the complement of the associated complexification.

The first important concept that we introduce is that of genericity of a system with respect to an arrangement. This definition is crucial for the following steps and it is a bit different from the usual one and also from the ones that will be used in the following subsection where we will review Adiprasito's article.

**Definition 2.3.3** (Generic system)**.** We say that a system of polar coordinates in $\mathbb{R}^n$, defined by an origin $O$ and a base $e_1, \dots, e_n$, is generic with respect to the arrangement $\mathcal{A}$ if it satisfies the following conditions:

(a) the origin $O$ is contained in a chamber $C_0$ of $\mathcal{A}$,

(b) there exist $\delta \in (0, \pi/2)$ such that

$$B(\mathcal{F}) \subset \tilde{B}(\delta)$$

(c) subspaces $V_i(\bar{\theta})$ which intersect $clos(\tilde{B})$ are generic with respect to $\mathcal{A}$, in the sense that, for each codim$-k$ subspace $L \in L(\mathcal{A})$,

$$i \geq k \Rightarrow V_i(\bar{\theta}) \cap L \cap clos(\tilde{B}) \neq \emptyset \ and \ \dim(|V_i(\bar{\theta})| \cap L) = i - k.$$

The first thing we want to do is to be sure that for every arrangement there exists a generic system of polar coordinates and this follow from the following theorem, whose proof can be found in [SS07, p. 7-9].

**Theorem 2.3.4.** *For each unbounded chamber $C$ such that $C \cap H_\infty$ is relatively open, the set of points $O \in C$ such that exists a polar coordinate system centered in $O$ and generic with respect to $\mathcal{A}$ forms an open subset of $C$.*

Having fixed a generic system with center $O$ and frame $e_1, \ldots, e_n$ we are now ready to define a total order on $\mathcal{F}$. Let $\delta$ be the one from the definition of genericity.

Let's now fix a $\theta = \theta_1, \ldots, \theta_{n-1}$ with $\theta_j \in [0, \delta], j = i, \ldots, n-1$ and a codimensional$-k$ facet $F$ with $i \geq k$. Suppose that $F(\theta)$ is not empty and set

$$i_{F(\theta)} = min\{j \geq 0 \mid V_j \cap clos(F(\theta) \neq \emptyset\}$$

For the third condition of a generic system, taking $L = |F(\theta)|$ (the subspace spanned by $F(\theta)$) we immediately see that

$$i_{F(\theta)} \geq codim(F(\theta)). \tag{2.3.1}$$

There are now two possibilities:

- $i_{F(\theta)} \geq i$:

  This means that $V_{i-1} \cap clos(F(\theta)) = \emptyset$. Then there exists a $0-$dimensional facet $P_{F(\theta)} \in clos(F(\theta))$ such that:

  $$\theta_{i-1}(P_{F(\theta)}) = min\{\theta_{i-1}(Q) \mid Q \in clos(F(\theta))\}$$

  the minimum exists because otherwise there is a facet in $V_i(\theta)$ completely contained in $V_{i-1}(\theta_{i-1}(P_{F(\theta)})$ since there are at least two points, but this is impossible for genericity.

- $i_{F(\theta)} < i$:

  In this case we put $P_{F(\theta)} = O$ if $F$ is the base chamber, or the unique point such that:

  $$\theta_{i_{F(\theta)}-1}(P_{F(\theta)}) = min\{\theta_{i_{F(\theta)}-1}(Q) \mid Q \in clos(F(\theta)) \cap V_{i_{F(\theta)}}\}$$

  where the minimum exists for an argument similar to the previous one.

We will denote with $\Theta(F(\theta)) = (\theta_0(P_{F(\theta)}), \ldots, \theta_{i_{F(\theta)}-1}(P_{F(\theta)}), 0, \ldots, 0)$.

**Definition 2.3.5** (Polar Ordering)**.** Given $F, G \in \mathcal{F}$ and $\theta$ as above and such that $F(\theta), G(\theta) \neq \emptyset$, we set

$$F(\theta) \lhd G(\theta)$$

if one of the following occur:

(i) $P_{F(\theta)} \neq P_{G(\theta)}$ and $\Theta(F(\theta)) < \Theta(G(\theta))$ according to the anti-lexicographic ordering of the coordinates.

(ii) $P_{F(\theta)} = P_{G(\theta)}$ (so $\Theta = \Theta(F(\theta)) = \Theta(G(\theta))$) and either:

(a) $dim(F(\theta)) = 0$ and $F(\theta) \neq G(\theta)$,

(b) $dim(F(\theta)) > 0$, $dim(G(\theta)) > 0$, let $i_0 = i_{F(\theta)} = i_{G(\theta)}$

- if $i_0 \geq i$ then $\forall \varepsilon > 0$, $\varepsilon \ll \delta$, it must happen:

$$F(\Theta_{i-1} + \varepsilon, \theta_i, \ldots, \theta_{i_0-1}, 0, \ldots 0) \lhd G(\Theta_{i-1} + \varepsilon, \theta_i, \ldots, \theta_{i_0-1}, 0, \ldots 0)$$

- if $i_0 < i$ then $\forall \varepsilon > 0$, $\varepsilon \ll \delta$, it must happen:

$$F(\Theta_{i_0-1} + \varepsilon, 0, \ldots, 0) \lhd G(\Theta_{i_0-1} + \varepsilon, 0, \ldots, 0)$$

It is easy to see that the polar order gives a total ordering of the facets of $V_i(\theta)$ for any given $\theta$, in particular it gives a total ordering on $\mathcal{F}$.

We now need a theorem that compare the polar ordering with the $\prec$ partial ordering that will be used in the following to prove that the polar gradient is effectively an acyclic matching.

**Theorem 2.3.6.** *Each codimensional–$k$ facet $F^k \in \mathcal{F}$ ($k < n$) such that $F^k \cap V_k = \emptyset$ has the following property: among all codimensional–$(k+1)$ facets $G^{k+1}$ with $F^k \prec G^{k+1}$ there exists a unique one such that*

$$F^{k+1} \lhd F^k.$$

*If $F^k \cap V_k \neq \emptyset$ then*

$$F^k \lhd G^{k+1}, \forall G^{k+1} \text{ with } F^k \prec G^{k+1}.$$

*Proof.* Let's $G$ be a facet in the closure of $F^k$ if $P(G) \neq P(F^k)$ from the definition of polar order 2.3.5 we have that $F^k \lhd G$. Instead if $P(G) = P(F^k)$ again from 2.3.5 we see that to which one between $F^k$ and $G$ is bigger we can reduce to the case where $F^k$ is of dimension one (after $\epsilon-$ deforming) and in this case the assertion is obvious.

If $F^k \cap V_k \neq \emptyset$ then from the definitions we have that $F^k \cap V_k = P(F^k)$. Let's take $G$ a $k+1-$facet with $F^k \prec G$ i.e. in the closure of $F^k$. From 2.3.1 we have that $i_G \geq k+1$, in particular $P(G) \notin V_k$, so $F^k \lhd G$. □

We are finally able to define a gradient field over **S** and then prove that we obtain a perfect matching.

**Definition 2.3.7** (Polar Gradient)**.** Let **S** be the Salvetti complex, the $(j+1)-$th component $\phi_{j+1}$ of the *polar gradient field* $\phi$ is given by the pairs

$$(\langle C, F^j\rangle, \langle C, F^{j+1}\rangle),\ \ F^j \prec F^{j+1}$$

such that $F^{i+1} \lhd F^j$ and $\forall F^{j-1} \prec F^j$ the pair

$$(\langle C, F^{j-1}\rangle, \langle C, F^j\rangle) \notin \phi_j$$

**Theorem 2.3.8.** *The following are true:*

*(i)* $\phi$ *is an acyclic matching on* $\boldsymbol{S}$.

*(ii) The pair*

$$(\langle C, F^j\rangle, \langle C, F^{j+1}\rangle),\ \ F^j \prec F^{j+1}$$

*belong to* $\phi$ *if the following conditions hold:*

*(a)* $F^{j+1} \lhd F^j$,

*(b)* $\forall F^{j-1}$ *such that* $C \prec F^{j-1} \prec F^j$, *one has* $F^{j-1} \lhd F^j$.

*(iii) For each chamber* $C$ *such that exists* $F^{j-1}$ *with*

$$C \prec F^{j-1} \prec F^j,\ \ F^j \lhd F^{j-1}$$

*the pair* $(\langle C, \bar{F}^{j-1}\rangle, \langle C, F^j\rangle) \in \phi$, *where* $\bar{F}^{j-1}$ *is the maximum* $(j-1)-$*facet satisfying the previous condition.*

*(iv) The set of* $k-$*dimensional singular cells is given by*

$$Sing_k(\boldsymbol{S}) = \{\langle C, F^k\rangle \mid F^k \cap V_k \neq \emptyset,\ \ F^j \lhd F^k,\ \forall C \prec F^j \precneqq F^k\}$$

*Equivalently,* $F^k \cap V_k$ *is the maximum among all facets of* $C \cap V_k$.

*Proof.* iii

Let $\bar{F}^{j-1}, C, F^j$ satisfying the conditions. Let's also suppose that exist $F^{j-2}$ such that $C \prec F^{j-2} \prec \bar{F}^{j-1}$ and $\bar{F}^{j-1} \lhd F^{j-2}$. But then there exists another facet $G^{j-1}$, with $F^{j-2} \prec G^{j-1} \prec F^j$, moreover, by theorem 2.3.6 it follows that $F^{j-2} \lhd G^{j-1}$, which implies that $\bar{F}^{j-1} \lhd G^{j-1}$, contradicting the maximality of $\bar{F}^{j-1}$. Then, by the definitions of polar gradient we have that

$$(\langle C, \bar{F}^{j-1}\rangle, \langle C, F^j\rangle) \in \phi \tag{2.3.2}$$

ii

Indeed, if conditions iia and iib hold for a triplet $C \prec F^j \prec F^{j+1}$, then surely

$$(\langle C, F^j\rangle, \langle C, F^{j+1}\rangle) \in \phi$$

because we are asking more restrict conditions. On the other side if a pair as above, belongs to $\phi$ then iia is clearly verified. Assuming iib does not hold, then there exist $F^{j-1}$ such that $C \prec F^{j-1} \prec F^j$ and $F^j \lhd F^{j-1}$. But we are in the conditions of iii and we have already proved that there exist $\bar{F}^{j-1}$ such that 2.3.2, which contradicts our assumption that the pair

$$(\langle C, F^j\rangle, \langle C, F^{j+1}\rangle) \in \phi$$

iv

A cell $\langle C, F^k\rangle$ does not belong to $\phi$, according to ii and iii if and only if

$$F^k \lhd F^{k+1},\ \forall F^k \prec F^{k+1} \tag{2.3.3}$$

and

$$F^{k-1} \lhd F^k,\ \ \forall C \prec F^{k-1} \prec F^k \tag{2.3.4}$$

By theorem 2.3.6 conditions 2.3.3 holds if and only if $P = F^k \cap V_k \neq \emptyset$. By genericity $P$ is a $0-$dimensional facet in $V_k$ and 2.3.4 holds iff $P$ is the maximum facet of the chamber $C \cap V_k$ that is exactly what we want to prove.

i

First we show that no cell $\langle C, F^{j+1}\rangle$ belongs to two different pairs of $\phi$. The only non obvious part is that it is not the end of two different pairs. This however can be seen $\epsilon-$deforming till we reduce $F^{j+1}$ to a $0-$dimensional facet and here the uniqueness follows easily from the convexity of the chamber $C$. What we then need to prove, according to discrete Morse theory is that $\phi$ has no closed loop. To do this let us consider a path

$$(\langle C_1, F_1^j\rangle, \langle C_1, F_1^{j+1}\rangle, \ldots, \langle C_{m+1}, F_{m+1}^j\rangle$$

First of all we remember the conditions of the boundary and of the polar gradient and we have the following inequalities at the $k-$th step:

$$F_{k+1}^j \prec F_k^{j+1},\quad F_{k+1}^j \prec F_{k+1}^{j+1},\quad F_{k+1}^{j+1} \lhd F_{k+1}^j$$

Moreover by theorem 2.3.6, for the uniqueness, if $F_k^{j+1} \lhd F_{k+1}^j$ then $F_{k+1}^{j+1} = F_k^{j+1}$. If this is false we have necessary that $F_{k+1}^{j+1} \lhd F_k^{j+1}$. Then if the path is closed than all the $j+1-$facets are equal to a unique $F^{j+1}$.

What we want now to show is that at every step we necessarily have that $F_k^j \lhd F_{k+1}^j$ which will conclude the theorem.

At this point, up to $\epsilon-$deforming, we can assume that the path is in some $V_{i-1}(\theta)$. By iii we have that $F^j_{k+1}$ is the maximum vertex of $C_{k+1}$. But, by the definitions of boundary we have that $C_k$ and $C_{k+1}$ belong to the same chamber of $\mathcal{A}_{F^j_{k+1}}$. Then all the facets of $C_k$ are lower than $F^j_{k+1}$ as we want. □

**Theorem 2.3.9.** *The singular cells of the polar gradient are in one-to-one correspondence with the set of all the chambers of $\mathcal{A}$, so the matching is perfect.*

*Proof.* The second part follow from what we have said in the previous section because we already know that the number of chambers is equal to the sum of the Betti numbers. Since we have as many cells as chambers we have then as many cells as the Betti numbers, so every cell must be a generator in the right homology group and all the boundary map must be zero, otherwise the dimension of the homology group will be strictly less that the number of cells.

So we need only to prove the correspondence. From iv of theorem 2.3.8 we have that, watching the arrangement $\mathcal{A}_k = \mathcal{A} \cap V_k$, the singular $k-$cells of $\mathbf{S}$ corresponds to pair $(C, v)$ in $\mathcal{A}_k$ where $C$ is a chamber and $v$ its maximum vertex. Of course $v$ can also be seen as the minimum vertex of the chamber $C^{opp}$ of $\mathcal{A}_k$ that is opposite to $C$ with respect to $v$. Since $v$ is the minimum of $C^{opp} \cap V_{k-1} = \emptyset$ and for the genericity condition every chamber in $\mathcal{A}_k$ comes from a chamber in $\mathcal{A}$, we have then proved that there is a one-to-one correspondence between $Sing_k(\mathbf{S})$ and the chamber of $\mathcal{A}$ that intersect $V_k$ but not $V_{k-1}$,in particular this gives a correspondence between all the chambers and all the singular cells. □

### 2.3.2 Adiprasito and 2-arrangements

In the article [Adi14] minimality is proved in a more general case.

We are going to work with arrangement in $S^d$ and treat then the case of $\mathbb{R}^d$ as a special one. Here for a $i-$dimensional subspace in $S^d$ we mean the intersection of some $(i+1)-$ dimensional linear subspace in $\mathbb{R}^{d+1}$ with the sphere $S^d$.

**Definition 2.3.10.** If $H$ is a hyperplane in $S^d$, then $H$ is in general position with respect to a polyhedron if $H$ intersects the span of any face of the polyhedron transversally. This naturally extends to collection of polyhedra. A hemisphere is in general position if its boundary is.

**Definition 2.3.11.** A $2-$arrangement in $S^d$ (resp. in $\mathbb{R}^d$) is a finite collection of distinct affine subspace of codimension 2 such that the codimension of every non-empty intersection is even.

Clearly any collection of hyperplanes in $\mathbb{C}^d$ can be viewed as a $2-$arrangement in $\mathbb{R}^d$ but this family is actually bigger.

What we want now to do is associate to any 2−arrangement a complex in a similar way as what we have done for the complexified case with the Salvetti complex.

**Definition 2.3.12.** A *sign extension* $\mathcal{A}_\sigma$ of a 2− arrangement $\mathcal{A} = \{h_i : i \in [1, n]\}$ in $S^d$ is any collection of hyperplanes $\{H_i \subset S^d : i \in [1, n]\}$ such that for each $i$, we have that $h_i \subset H_i$.

A *hyperplane extension* $\mathcal{A}_e$ of $\mathcal{A}$ in $S^d$ is a sign extension of a together with an arbitrary finite collection of hyperplanes in $S^d$.

Any hyperplane extension of $S^d$, and in general any 1−arrangement gives a stratification **s** of $S^d$.

**Definition 2.3.13.** Let $\mathcal{A}$ be a 2−arrangement. An extension $\mathcal{A}_e$ of $\mathcal{A}$ is *fine* if it gives rise to a stratification **s**, called *combinatorial*, that together with the canonical attaching maps is a regular $CW$−complex.

We need some notations to go on

**Notation.** Given a $CW$−complex $X$ in , a subcomplex $Y$ and a subset $M$ we denote by

- $R(X, M)$ the maximal subcomplex of $X$ all whose faces are contained in $M$,
- $X^*$ the dual block complex, meaning the complex with same set of faces but opposite inclusion,
- $R^*(Y^*, M)$ is the minimal subcomplex of $Y^*$ containing all those faces of $Y^*$ that are dual to faces of $X$ intersecting $M$.

With the above notations we are able to define the space that we are going to study.

**Definition 2.3.14.** Let $\mathcal{A}$ be a 2−arrangement in $S^d$ and $s$ a combinatorial stratification induced by $\mathcal{A}$. The *complement complex* of $\mathcal{A}$ with respect to $s$ is the regular $CW$- complex

$$K(\mathcal{A}, s) = R^*(s^*, S^d \backslash \mathcal{A})$$

If instead $\mathcal{A}$ is a 2−arrangement in $R^d$, given a radial projection $\rho$ of $R^d$ into an open hemisphere $O$ in $S^d$, calling $\mathcal{A}'$ the extension to a 2−arrangement in $S^d$ of the images $\rho(h)$ for $h \in \mathcal{A}$ and $s$ a combinatorial stratification of $\mathcal{A}'$. The *complement complex* of $\mathcal{A}$ is $R^*(s^*, O \backslash \mathcal{A}')$.

**Lemma 2.3.15.** *[BZ92, Proposition 3.1] Let $\mathcal{A}$ be a 2−arrangement in $\mathbb{R}^d$ (resp. $S^d$). Then every complement complex $K(\mathcal{A}, \boldsymbol{s})$ of $\mathcal{A}$ is a model for the complement $M(\mathcal{A})$.*

Let from now on $\mathbf{s}$ denote a combinatorial stratification of the sphere $S^d$. Our goal is to obtain a perfect acyclic matching on $K(\mathcal{A}, \mathbf{s})$. The idea is to study instead acyclic matching on $\mathbf{s}$ and see how they relate.

**Notation.** Let $\phi$ be an acyclic matching on a regular $CW-$complex $C$ and $D$ a subcomplex.

We will denote with $\phi_D$ the restriction of $\phi$ to $D$, that is, the collection of matching pairs in $\phi$ involving two faces of $D$.

We will denote with $\phi^*$ the matching on the dual complex $C*$ that is exactly the same matching pairs of $\phi$.

Let now $\phi$ be a matching on $\mathbf{s}$. We will denote with $\phi*_{K(\mathcal{A},\mathbf{s})}$ the *complement matching* induced by $\phi$ on $K(\mathcal{A}, \mathbf{s})$. That is the restriction to $K(\mathcal{A}, \mathbf{s})$ of the dual matching.

The following theorem, whose proof is easy, gives us a clear correlation between the two matchings on $\mathbf{s}$ and $K$.

**Theorem 2.3.16.** *Consider an acyclic matching $\phi$ on $\boldsymbol{s}$. Then the critical $i-$faces of $\phi*_K$ are in one to one correspondence with the union of*

- *the critical $(d-1)-$faces of $\phi$ that are not faces of $R(\boldsymbol{s}, \mathcal{A})$,*
- *the outwardly matched $(d-i-1)-$faces with respect to the pair $(\boldsymbol{s}, R(\boldsymbol{s}, \mathcal{A}))$.*

*If $M$ is furthermore an open subset of $S^d$ such that all noncritical faces of $\phi$ intersect $M$, then the critical $i-$faces of $\phi*_{R*(K,M)}$ are in bijection with the union of*

- *the critical $(d-1)-$ faces of $\phi$ that are not faces of $R(\boldsymbol{s}, \mathcal{A})$ and that intersect $M$,*
- *the outwardly matched $(d-i-1)-$faces with respect to the pair $(\boldsymbol{s}, R(\boldsymbol{s}, \mathcal{A})$.*

We will now focus to the study of acyclic matching on $\mathbf{s}$.

Let first consider the special case of the empty arrangement and then the general case.

**Lemma 2.3.17.** *Let $\boldsymbol{s}$ be a combinatorial stratification of a fine extension of the empty arrangement in $S^d$. Let $F$ be a closed hemisphere that is in general position with respect to $\boldsymbol{s}$. Then $R(s, F)$ is collapsible.*

**Theorem 2.3.18.** *Let $\mathcal{A}$ be a non-empty $2-$arrangement in $S^d$. Let $\boldsymbol{s}$ be a combinatorial stratification of a fine extension $\mathcal{A}_e$ of $\mathcal{A}$, and $F$ a closed hemisphere in general position with respect to $\boldsymbol{s}$. Then, for any $k-$dimensional subspace $H$ of $\mathcal{F}(A_\sigma)$ extending an element of $\mathcal{A}$, we have the following:*

1. *The pair* $(R(\boldsymbol{s}, F \cap H), (R(\boldsymbol{s}, F \cap \mathcal{A} \cap H))$ *is out-*$\iota(d)$ *collapsible.*

2. *If* $\mathcal{A}$ *is non-essential, then* $(R(\boldsymbol{s}, F \cap H), (R(\boldsymbol{s}, F \cap \mathcal{A} \cap H))$ *is a collapsible pair.*

The previous theorem is the crucial step to prove the following that will be one of the two results needed to construct the matching.

**Theorem 2.3.19.** *[Adi14, Corollary 4.2] Let* $F$ *denote a closed hemisphere in* $S^d$*, let* $O$ *denote its open complement. Let* $\mathcal{A}$ *be a* $2-$*arrangement w.r.t.* $O$*, and* $s$ *the associated combinatorial stratification. If* $H$ *is a hyperplane in* $S^d$ *in general position with respect to* $\mathcal{A}_e$ *and* $\delta F$ *then:*

$$(R(s, S^d \backslash (O \cap H)), (R(s, \mathcal{A} \cap S^d \backslash (O \cap H))) \searrow_{out-\iota(d)} (R(s, F), R(s, F \cap \mathcal{A}))$$

Before being able to construct the matching on $s$ we just need one more lemma, that is an easy consequence of the Goresky-MacPherson formula.

**Lemma 2.3.20.** *Let* $\mathcal{A}$ *denote a subspace arrangement in* $S^d$*,* $O$ *an open hemisphere, and let* $H$ *be a hyperplane. Then we have the following*

1. *If* $O$ *is in general position w.r.t.* $\mathcal{A}$*, then for all* $i$*,* $\beta_i(S^d \backslash \mathcal{A}) \geq \beta_i(O \backslash \mathcal{A})$*. Where the* $\beta_i$ *are the Betti numbers.*

2. *If* $H$ *is in general position w.r.t.* $\mathcal{A}$ *and* $O$*, then for all* $i$*,* $\beta_i(O \backslash \mathcal{A}) \geq \beta_i((O \cap H) \backslash \mathcal{A})$*.*

**Theorem 2.3.21.** *Let* $F$ *be a closed hemisphere of* $S^d$*,* $O = F^c$*,* $\mathcal{A}$ *a* $2-$ *arrangement with respect to* $O$ *and* $s$ *a combinatorial stratification induced by* $\mathcal{A}$*, and* $K$ *the associated complement complex.*

*Then, there exists an acyclic matching* $\phi$ *on* $s$ *whose critical faces are the subcomplex* $R(s, F)$ *and some additional facet such that the restriction of the complement matching to* $R * (K, O)$ *is perfect.*

*Proof.* We prove it by induction on the dimension, where the case $d = 0$ is clearly true. Let us assume now that $d \geq 1$ and let $H$ be a generic hyperplane in $S^d$. Calling $s^H$ the restriction of the combinatorial stratification to $H$ we have by induction hypothesis a matching $\varphi$ on it that respects all the hypothesis. We can lift this matching to a matching $\phi$ on $s$ of the faces intersecting $H$, this will still be an acyclic matching, moreover, using Theorem 2.3.19 we have that:

$$(R(s, S^d \backslash (O \cap H)), (R(s, \mathcal{A} \cap S^d \backslash (O \cap H))) \searrow_{out-\iota(d)} (R(s, F), R(s, F \cap \mathcal{A})) \quad (2.3.5)$$

We now define $\psi$ as the union of the matching $\phi$ and the out-$\iota(d)$ sequence associated to the equation above and claim that $\psi$ is the acyclic matching that we want.

By construction $\psi$ has the desired critical faces so we need only to show that it is perfect.

The equation 2.3.5 also tells us that if we consider the matching $\psi^*_{R^*[K,O]}$ all the critical faces that are not in $R^*[O \cap H]$ have dimension $e = d - \iota(d) - 1$. Then Theorem 1.3.5 tells us that $R^*[K, O]$ is obtained from the complex $R^*[K, O \cap H]$ by attaching cells of dimension $e$. We want to show that each of these cells add a generator in homology, but if that's not the case one of them should delete a generator in the $(e-1)-$ homology and then

$$\beta_{e-1}(O \backslash \mathcal{A}) = \beta_{e-1}(R^*[K, O]) < \beta_{e-1}(R^*[K, O \cap H]) = \beta_{e-1}((O \cap H) \backslash \mathcal{A}),$$

in contradiction with Lemma 2.3.20. We have then proved that the matching is perfect.

□

As an easy consequence of the previous theorem we then see that we are able to construct perfect acyclic matching on every $2-$arrangement.

**Corollary 2.3.22.** *[Adi14, Theorem 5.4] Any complement complex of any $2-$arrangement $\mathcal{A}$ in $S^d$ or $R^d$ admits a perfect acyclic matching.*

*Proof.* Let us first solve the case of $2-$arrangement in $\mathbb{R}^d$. Let then $\mathcal{A}$ be a $2-$arrangement in $\mathbb{R}^d$, $\rho$ a radial projection of $\mathbb{R}^d$ into an open hemisphere $O$ in $S^d$ and $\mathcal{A}'$ the corresponding arrangement in $S^d$. Then the acyclic matching construct in the previous theorem with respect to $R^*[K(\mathcal{A}', s), O]$ is perfect.

Let then study now the case of a $2-$arrangement $\mathcal{A}$ in $S^d$. Let $F$ be a generic close hemisphere, $O = F^c$ and $s$ a combinatorial stratification. Again using the previous theorem we are able to construct an acyclic matching $\psi$ on $s$ such that the complement matching restricted to $R^*[K(\mathcal{A}, s), O]$ is perfect.

By theorem 2.3.18 $R^*(s, F)$ is out-$\iota(d)$ collapsible. If we now consider the matching $\psi$ together with this collapsment we can see using again Lemma 2.3.20 that the obtained acyclic matching is perfect on $K(\mathcal{A}, s)$. □

### 2.3.3 Delucchi and the central case

The article [Del08] main aim is to answer a question asked in [SS07] about a completely combinatorial formulation of the polar ordering. The approach used here is that of constructing an acyclic matching in the context of oriented matroids, so the results won't hold for affine arrangement but it has the big advantage that it does not require the choice of a generic flag in the ambient space.

From now on we will used, unless explicitly stated, the notation for hyperplane arrangement, since it's more geometrically intuitive and closer to the one used in the previous subsection. However everything written is true for oriented matroids, even if not realizable.

Let $\mathcal{A}$ be a finite central arrangement of hyperplanes in $\mathbb{R}^n$.

The first thing we need is the definition of an order of the chambers.

**Definition 2.3.23.** Let $B \in \mathcal{C}(\mathcal{A})$ a base chamber, a *valid order* $\dashv$ on $\mathcal{C}(\mathcal{A})$ is a linear extension of the partial order $<_B$. Given $C, C' \in \mathcal{C}(\mathcal{A})$ we will denote by $s(C, C')$ the set of hyperplanes that separates the two chambers, then:

$$C <_B C' \Leftrightarrow s(B, C) \subset s(B, C')$$

We will denote by $C_B(\mathcal{A})$ the set of chambers endowed with this partial order.

In the following we will suppose that we have a fixed base chamber $B$ and a valid order $\dashv$ and we can define one of the most important object of our study.

**Definition 2.3.24.** For every $C \in \mathcal{C}(\mathcal{A})$ we let

$$\mathcal{J}(C) = \{X \in L(\mathcal{A}) \mid \operatorname{supp}(X) \cap s(C, K) \neq \emptyset \ for\ every\ K \dashv C\}$$

It is easily seen that $\mathcal{J}(C)$ is an upper ideal in $L(\mathcal{A})$ but what we are really interested in is that it is principal. This is not obvious and follow from a series of proposition that can be found in [Del08, p. 16-18]. We will here just lay down the last one, together with some notation.

**Notation.** Given $H \in \mathcal{A}$, let $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ and similarly, given a chamber $C$ we will denote with $C'$ the unique chamber in $\mathcal{A}'$ that contains $C$. This inclusion induce an order preserving map:

$$\varphi : C(\mathcal{A}') \to C(\mathcal{A})$$

where we sent $C'$ to the minimum chamber $C$, with respect to a valid order $\dashv$ on $C(\mathcal{A})$, that is contained in $C'$. This let us build a valid order on $C(\mathcal{A}')$ that is the pullback of $\dashv$ along $\varphi$ and that we will call $\dashv'$. Similarly we are able to build $\mathcal{J}'(C')$ for any chamber $C'$.

The inclusion $\mathcal{A}' \hookrightarrow \mathcal{A}$ induces an order preserving injection

$$\iota : L(\mathcal{A}') \to L(\mathcal{A}), \quad X \to \bigcap \operatorname{supp}(X).$$

We will identify $\mathcal{J}'(C')$ with its image under this map.

**Lemma 2.3.25.** *Given a chamber $C$ different from $-B$ (the chamber opposite to $B$), choose $H \in \mathcal{A} \backslash s(B, C)$ and let $\mathcal{A}' = \mathcal{A} \backslash \{H\}$. For every $Y \in \mathcal{J}(C)$ we have*

$$\bigcap (\operatorname{supp}(Y) \backslash \{H\}) \in \mathcal{J}'(C')$$

Given this lemma, that we are not going to prove, we are then able to prove the theorem.

**Theorem 2.3.26.** *Given a valid order $\dashv$ with respect to a chamber $B$, for every $C \in \mathcal{C}(\mathcal{A})$, $\mathcal{J}(C) \subset L(\mathcal{A})$ is a principal upper ideal.*

*Proof.* We will argue by induction on the size of $\mathcal{A}$. The base step, when $\mathcal{A}$ contains only one hyperplane, is trivial, so we can prove the inductive step, assuming that $|\mathcal{A}| > 1$. We will prove that $\mathcal{J}(C)$ is closed under the meet operator, which will immediately imply the claim.

If $C = -B$, then clearly $\mathcal{J}(C)$ contains only the intersection of all the hyperplanes, so is principal. We can then suppose that $C \neq -B$, in particular there is an hyperplane $H \in \mathcal{A} \backslash s(B, C)$. We call $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ and by induction hypothesis $\mathcal{A}'$ satisfies the theorem.

We introduce now the order preserving map

$$\lambda : L(\mathcal{A}) \to L(\mathcal{A}'), \quad Y \to \bigcap (\mathrm{supp}(Y) \backslash \{H\})$$

that by lemma 2.3.25 satisfies $\lambda(\mathcal{J}(C)) \subseteq \mathcal{J}'(C')$. Moreover the inclusion $\iota$ of $\mathcal{J}'(C')$ in $\mathcal{J}(C)$ is well defined. Now that we have introduced the instruments let's consider two elements $Y_1, Y_2 \in \mathcal{J}(C)$, as already said we want to prove that $Y_1 \wedge Y_2$ exists and $\in \mathcal{J}(C)$. The induction tells us that $\lambda(Y_1) \wedge \lambda(Y_2) \in \mathcal{J}'(C')$. Applying $\iota$ to this element we have then an element $\iota(\lambda(Y_1) \wedge \lambda(Y_2)) \in \mathcal{J}(C)$ and $\leq Y_1 \wedge Y_2$ because $\iota\lambda(Y) \leq Y$ for every $Y \in \mathcal{J}(C)$. But now we already now that $L(\mathcal{A})$ is a lattice, meaning that $Y_1 \wedge Y_2$ surely exists in $L(\mathcal{A})$, The fact that $\mathcal{J}(C)$ is an upper ideal concludes the proof.

□

Thanks to the theorem, we are now able to define the following object.

**Definition 2.3.27.** For every $C \in \mathcal{C}(\mathcal{A})$ let

$$X_C = min \mathcal{J}(C)$$

**Corollary 2.3.28.** *If we define $F_C = X_C \cap C$, we have $|F_C| = X_C$.*

*Proof.* As in the proof of the theorem before we can assume that $C \neq -B$ because otherwise the claim is trivial and proceed by induction. Again the base step when $|\mathcal{A}| = 1$ is easy so we can assume that $|\mathcal{A}| > 1$. We will call with $\mathcal{W}_C$ the set of hyperplanes adjacent to $C$. Since $C \neq -B$ there exist $H \in \mathcal{W}_C \cap s(C, -B)$ and call $\mathcal{A}' = \mathcal{A} \backslash \{H\}$. If we call $H^+$ the closed half-space bounded by $H$ and containing $B$ we have that

$$C = C' \cap H^+.$$

Moreover by induction hypothesis we have that $|F'_{C'}| = X'_{C'}$ which implies that

$$dim(X'_{C'} \cap C') = dim(X'_{C'}).$$

$$\lambda(X_C) = X'_{C'}.$$

By definition of $\lambda$ we have that

$$\lambda(X_C) = \bigcap(\text{supp}(X_C)\backslash\{H\}$$

then there are two possible cases:

- $X_C = X'_{C'}$

  $$dim(C \cap X_C) = dim(C' \cap H^+ \cap X'_{C'}) = dim(X'_{C'} \cap H^+) = dim(X_C)$$

  where the last equality is true because $X_C \subset H^+$.

- $X_C = X'_{C'} \cap H$

  $$\begin{aligned} dim(C \cap X_C) &= dim(C' \cap H^+ \cap X'_{C'} \cap H) = \\ &= dim(C' \cap X'_{C'} \cap H) = dim(X'_{C'} \cap H) = dim(X_C) \end{aligned}$$

In both cases we have then prove that $dim(X_C \cap C) = dim(X_C)$ that is equivalent to our thesis.

□

From everything written above is then clear the following lemma

**Lemma 2.3.29.** *$X_C$ is uniquely determined by the following properties:*

1. *$s(K, C) \cap \text{supp}(X_C) \neq \emptyset$ for all $K \dashv C$,*
2. *For every $Y \in L(\mathcal{A})$ such that $Y$ does not contain $X_C$ there is a chamber $K \dashv C$ such that $s(K, C) \cap \text{supp}(Y) = \emptyset$.*

At this point we have all the instruments and we should understand how to apply them to the Salvetti complex $\mathbf{S}$ of $\mathcal{A}$ to construct a perfect acyclic matching. The idea is to use the patchwork lemma and, instead of constructing directly the matching for all the complex, subdivide the complex in subcomplexes $N(C)$, one per chamber $C \in C(\mathcal{A})$ and construct for each of this subcomplexes an acyclic matching with exactly one singular cell. Let's then see what this subcomplexes are.

**Definition 2.3.30.** Let $\mathbf{S}$ be the Salvetti complex of a linear arrangement $\mathcal{A}$ with the partial order given by "be a face of". Then, for every chamber $C$, we will denote by

$$S(C) = \mathbf{S}_{\leq \langle C,P \rangle}$$

where $P$ is the unique minimal element of $\mathcal{F}(\mathcal{A})$. In other words $S(C)$ is the smaller subcomplex of $\mathbf{S}$ that contains the cell $\langle C, P \rangle$. Moreover we define

$$N(C) = S(C) - \bigcup_{C' \dashv C} S(C')$$

The definition of $S(C)$ is slightly different from the one given in [Del08] but it is easy to see that the definitions of $N(C)$ are the same.

**Lemma 2.3.31.** *Under the above definitions, we have for every $C \in \mathcal{C}(\mathcal{A})$*

$$N(C) \simeq \mathcal{F}(\mathcal{A}^{X_C})$$

*Proof.* First we note that from the definitions follows that

$$N(C) = \{\langle D, F \rangle \mid D = C.F \textit{ and } C.F \neq K.F \textit{ for all } K \dashv C\}$$

but then $D$ is uniquely determined by $F$ so we have a correspondence between $N(C)$ and $\mathcal{F}(\mathcal{A})$, we now want to say that this correspondence is actually one-to-one with $\mathcal{F}(\mathcal{A}^{X_C})$.

Let us then suppose that $F \in \mathcal{F}(\mathcal{A}^{X_C})$. Then, for all $K \in C(\mathcal{A})$ we have that $s(C.F, K) \cap \operatorname{supp}(F) = s(C, K) \cap \operatorname{supp}(F)$. But then from the lemm 2.3.29 we have that for all $K \dashv C$ $s(C, K) \cap \operatorname{supp}(F) \neq \emptyset$, which implies $K.F \neq C.F$ since obviously $s(K.F, K) \cap \operatorname{supp}(F) = \emptyset$.

For the other inclusion, suppose $\langle C.F, F \rangle \in N(C)$ and $F \notin \mathcal{F}(\mathcal{A}^{X_C})$. But then, again by lemma 2.3.29, there is $K \dashv C$ with $s(K, C) \cap \operatorname{supp}(F) = \emptyset$ which implies $K.F = C.F$ and thus a contradiction. □

Now, we need to study $\mathcal{F}(\mathcal{A}^{X_C})$ and in general $\mathcal{F}(\mathcal{A})$ for a linear arrangement $\mathcal{A}$. We recall that $\mathcal{F}(\mathcal{A})$ is a stratification of a certain $\mathbb{R}^d$, the stratification can also be seen as a $CW-$decomposition of $\mathbb{R}^d$. Using the shellability can then be proved

**Theorem 2.3.32.** *Every valid order $\dashv$ on $\mathcal{C}(\mathcal{A})$ defines an acyclic matching of the face poset $\mathcal{F}(\mathcal{A})$ such that the only critical element is the chamber opposite to $B$.*

*Proof.* Every valid order on $\mathcal{C}(\mathcal{A})$ induces a recursive coatom ordering of $\mathcal{F}(\mathcal{A})$ ([BLVS+99, Proposition 4.3.2]). In theorem 1.4.12 we have seen that this induces an acyclic matching with the critical faces corresponding to the spanning faces of the given shelling. It is now easy to see that the definition of valid order implies that the only spanning chamber is the last one. □

This is the last piece needed to be able to construct a perfect acyclic matching on all the Salvetti complex of $\mathcal{A}$.

**Theorem 2.3.33.** *Let $\mathcal{A}$ be an arrangement of real hyperplanes in a real space and fix any $B \in \mathcal{C}(\mathcal{A})$. To any valid order $\dashv$ corresponds a family of perfect acyclic matchings of the associated Salvetti complex $\boldsymbol{S}$ which critical cells are in natural bijection with the chambers of $\mathcal{A}$.*

*Proof.* Given a valid order $\dashv$ with respect to a chamber $B$ and thanks to the previous theorem and lemma we are able to construct on $N(C)$ an acyclic matching with exactly one critical cell. In particular we also know that this cell is the one in correspondence with the chamber $-B$ in $\mathcal{A}^{X_C}$.

Since the $N(C)$ are a partitions of $\mathbf{S}$ we are then able, attaching this matching, to construct a matching on $\mathbf{S}$. Let $f$ be the function that sends a cell $\langle D, F\rangle$ to the chamber $C$ if and only if $\langle D, F\rangle \in N(C)$. $f$ is a map of posets that preserves the order, since by construction given $\sigma, \tau \in \mathbf{S}$, $\sigma < \tau$ and $\tau \in N(C)$ then surely $\sigma \in S(C)$ which implies that $f(\sigma) \leq C$.

We are then on the hypothesis of the patchwork lemma that assures us that the induced matching on $\mathbf{S}$ is still acyclic.

The critical cells are the union of the critical cells of the $N(C)$s which immediately implies that we have a bijection between critical cells and chambers.

$\square$

# Chapter 3

# Local Abelian Homology

The aim of this chapter is to use what done in the previous pages to study the abelian local homology of the hyperplane arrangements. In the first part we will see what local homology is and its principal property, as well as how it behaves with Discrete Morse theory. In the second section instead we will see how the acyclic matching, in particular the polar matching proposed by [SS07], can help us in compute abelian local homology.

In the last section, eventually, we will talk about the Braid arrangements. We will see a particular matching on them and use it to define an algorithmic way to compute the boundary of the reduced complexes.

## 3.1 Homology with local coefficients

Homology with local coefficients, or local homology, is a homology theory first introduced by Steenrod in [Ste43].

In the first part of this section we will then follow the article to see the basic definitions before then moving to its correlations with Discrete Morse theory.

Let $R$ be an arcwise connected topological space, $x$ a point of $R$ and $\pi_1(R, x)$ the fundamental group with base point $x$.

The fundamental idea of this theory is that we are goint to compute the homology where the coefficients will be a group $G$ with an action of $\pi_1(R, x)$.

**Definition 3.1.1.** A *system of local groups* in the space $R$ is a family of groups $\{G_y\}_{y\in R}$ such that for each class of path $\alpha_{yz}$ from $y$ to $z$ there is a group isomorphism between $G_y$ and $G_z$ and the composition of isomorphism $\alpha_{yz}\beta_{zw}$ is the isomorphism corresponding to the path from $y$ to $w$.

**Theorem 3.1.2.** *If $G$ is a group with an action of $\pi_1(R, x)$ for some $x \in R$, then there is a system $\{G_y\}$ of local groups in $R$ such that the operations are determined by $\pi_1(R, x)$.*

*Proof.* For each point $y \in R$ we choose a class of path $\lambda_{xy}$ from $x$ to $y$, with $\lambda_{xx}$ =identity. Let now $G_y$ be a group isomorphic to $G$ for each $y \in R$ and we associate this isomorphism with $\lambda_{yx} = \lambda_{xy}^{-1}$.

If we have a path $\alpha_{yz}$, we obtain a isomorphism from $G_y$ to $G_z$ in the following way

$$\alpha_{yz}(g) = \lambda_{xz}[\lambda_{xy}\alpha_{yz}\lambda_{zx})(\lambda_{yx}(g))]$$

where the interiors $\lambda$ are paths, while the exteriors group isomorphisms.

It is now easy to control that what defined above is a system of local groups. □

We now suppose that our space $R$ has a $CW-$decomposition $\Delta$ that we suppose finite and see how, from a system of local groups, we can create our homology theory.

For each cell $\sigma$ we choose a representative point $x(\sigma)$ and we call $G_{x(\sigma)} = G_\sigma$.

**Definition 3.1.3.** A *q-chain* of $\Delta$ is a function $f$ attaching to each oriented $q-$cell $\sigma$ of $\Delta$ an element $f(\sigma) \in G_\sigma$ such that $f(-\sigma) = -f(\sigma)$. Chains are added by adding functional values, so they form a group isomorphic to the direct sum of the groups $G_\sigma$ for all $q-$cells $\sigma$.

We want now to define the boundary maps between chains.

If $\sigma' < \sigma$ we choose a path in the closure of $\sigma$ joining $x(\sigma)$ to $x(\sigma')$. We obtain an isomorphism $G_\sigma \to G_{\sigma'}$ which is denoted by $h_{\sigma'\sigma}$. We postulate that the closure of each cell is simply connected in order for $h_{\sigma'\sigma}$ to be independent of the path.

Using $h$ we are now able to define the boundary $\partial$ and co-boundary $\delta$ of a $q-$chain $f$.

$$\partial f(\sigma^{q-1}) = \sum_{\sigma^q} [\sigma^{q-1} : \sigma^q] h_{\sigma^{q-1}\sigma^q}(f(\sigma^q)),$$
$$\delta f(\sigma^{q+1}) = \sum_{\sigma^q} [\sigma^q : \sigma^{q+1}] h^{-1}_{\sigma^q\sigma^{q+1}}(f(\sigma^q)).$$

It is a simple check that $\delta\delta f = 0$ and $\partial\partial f = 0$ so we are able to define homology and cohomology groups as usual.

We have now to talk a bit about algebraic Discrete Morse theory, in order to study this and in general all kind of homology theory, again following [Koz08].

We will denote in the following with $C_*$ a chain complex of modules over some commutative ring. We will call it free if each $C_n$ is a finitely generated free module. A basis $\Omega$ of $C_*$ is simply a set of free generators $\Omega_n$ for each $C_n$. Given $b \in \Omega_n$ and $a \in \Omega_{n-1}$ we denote by $[b : a]$ the coefficients of $\partial(b)$ with respect to $a$, if $a \notin \Omega_{n-1}$ by convention we set $[b : a] = 0$.

*Remark* 3.1.4. Since the following theory is valid only for modules over commutative ring from now on we will work with a group $G$ with an action of the abelianization of $\pi_1(R, x)$ that is, an action of $H_1(R)$.

Clearly a free chain complex with a basis $(C_*, \Omega)$ can be represented as a ranked poset, called $P(C_*, \Omega)$, where the weights are the incidence numbers defined above.

**Definition 3.1.5.** Let $C_*, \Omega)$ be a free chain complex with a basis. A *partial matching* $\mathcal{M} \subseteq \Omega \times \Omega$ on $(C_*, \Omega)$ is a partial matching on $P(C_*, \Omega)$ such that, if $(a, b) \in \mathcal{M}$ then $[b : a]$ is invertible.

The notion of acyclic matching is the same as before.

We want to make clear that we are exactly in this situation since, given a $CW-$complex and a local homology chain on in, we have that the chain is free and we can take as a basis the cells of $\Delta$. The above definition then asks us that every incidence number is invertible but this is always true because our modules are over a group. Then in our particular case the above definition of partial matching is actually the same as the one given at the beginning of the thesis.

Given an acyclic matching $\mathcal{M}$ we let $C_n(\Omega)$ be the set of critical elements of $\Omega_n$, meaning the elements that are not in any pair.

Given two elements $b \in \Omega_n$ and $a \in \Omega_{n-1}$, an alternating path between them is a path of the form:

$$p = b \searrow a_1 \nearrow b_1 \searrow \cdots \nearrow b_n \searrow a$$

where $(a_i, b_i) \in \mathcal{M}$ for each $i = 1, \ldots, n$. Its weight is

$$\omega(p) = (-1)^n \frac{[b : a_1] \cdot [b_1 : a_2] \cdots [b_n : a]}{[b_1 : a_1] \cdot [b_2 : a_2] \cdots [b_n : a_n]}$$

Comparing the formula with the one in 1.2.2 we see that they are really similar, but now we are taking into account the possibility that the incidence numbers are different from $\pm 1$.

**Definition 3.1.6.** Given a free chain complex with a basis $(C_*, \Omega)$ and an acyclic matching $\mathcal{M}$. The *Morse complex* $C_*^{\mathcal{M}}$ is defined as follows. $C_n^{\mathcal{M}}$ is freely generated by the elements of $C_n(\Omega)$ and the boundary operator is defined by

$$\partial^{\mathcal{M}}(s) = \sum_p \omega(p) \cdot p^\bullet$$

where the sum is taken over all alternating paths $p$ starting in $s$ and $p^\bullet$ is the ending point of the path.

We are nearly ready to set out the main theorem of algebraic Morse theory, we just need one more definition.

**Definition 3.1.7.** The chain complex where the only nontrivial modules are in the dimensions $d$ and $d-1$, both free of dimension one and the boundary map is the identity is called *atom chain complex* and denoted by $\mathrm{Atom}(d)$.

**Theorem 3.1.8** ([Koz08, Theorem 11.24])**.** *Assume that we have a free chain complex with a basis $(C_*, \Omega)$, and an acyclic matching $\mathcal{M}$. Then $C_*$ decomposes as a direct sum of chain complexes $C_*^{\mathcal{M}} \bigoplus T_*$ where $T_* \simeq \bigoplus_{(a,b)\in\mathcal{M}} \mathrm{Atom}(\dim b)$.*

In particular, since the atom chain complexes are all acyclic by construction, we obtain immediately the corollary

**Corollary 3.1.9.** *Assume that we have a free chain complex with a basis $(C_*, \Omega)$, and an acyclic matching $\mathcal{M}$. Then the homology of the chain complex is the same as that of the Morse complex.*

To conclude this section we simply want to underline that we are exactly in this situations with the abelian local homology. Theorem 3.1.8 can be applied to the abelian local homology defined above, so, studying the alternating path in an acyclic matching, in principle we are then able to use them to study the local homology.

## 3.2 Local homology of hyperplane arrangement

In this section we will focus on the study of local homology of hyperplane arrangement.

Let $\mathcal{A}$ be a complexified real arrangement and choosing a base chamber $C_0$ and a base point $O \in C_0$ we define, for each hyperplane $H_i$, a positive oriented loop around this hyperplane as an element of $\pi_1(M(\mathcal{A}), O)$, calling it $t_i$.

Given now an abelian local system $\mathbb{L}$ over $M(\mathcal{A})$ we have a homomorphism

$$\mathbb{Z}[\pi_1(M(\mathcal{A}, O)] \to \mathbb{Z}[H_1(M(\mathcal{A})] \to \mathbb{Z}[t_i^{\pm 1}]_{H_i \in \mathcal{A}} \subseteq End(\mathbb{L})$$

The basepoint $O \in C_0$ can be taken as the unique $0-$cell of $\mathbf{S}$ contained in $C_0$, namely equal to $\langle C_0, C_0 \rangle$. We can do the same for each cell of $\mathbf{S}$, and associate to $\langle C, F \rangle$ the point $\langle C, C \rangle$. This will be the point that we consider in the construction of our chain complex, moreover since all the points are in the $0-$skeleton of $\mathbf{S}$ up to homotopy we can consider only paths in the $1-$skeleton of $\mathbf{S}$.

A sequence, or galleries, of adjacent chambers uniquely correspond to a special kind of combinatorial paths in the $1-$skeleton of $\mathbf{S}$, which we call *positive path.*

Two galleries with the same ends and of minimal length determine two homotopic paths [Sal87]. This implies that when we are going from a cell to another we are only interested in the hyperplanes that separate the corresponding chambers.

**Example 3.2.1.** To make things a bit clearer we want to see explicitly how we can calculate the boundary. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$, $C, D \in \mathcal{C}$ such that $s(C, C_0) = \{H_1, \ldots, H_i\}$, $s(D, C_0) = \{H_1, \ldots H_{i-1}\}$, $s(C, D) = \{H_i\}$.

The incidence coefficients between two cells $\langle C, F\rangle$ and $\langle D, G\rangle$ is given by $t_i$ times the incidence number with integral coefficients.

Let us take the polar order and the corresponding acyclic matching defined in [SS07] Studying the alternating paths between critical cells we are then able, thanks to 3.1.8, to study the local homology of the arrangement. This has been done in the second part of [SS07] for the general case and in [GS09] for a more specific approach to the case of linear arrangement.

## 3.3 Local homology of the Braid arrangements

In this section we want to talk about a new matching on the Braid arrangements that help us find the alternating path in a somewhat easier way. In the first part we then present our matching and in the second we will see how to compute the boundary in local homology and a program in Python written to do so.

The idea is to use the description of the Braid arrangement given at the end of [SS07] that uses the concept of tableau.

We recall that the braid arrangement of dimension $n$ is the arrangement $\mathcal{A}_n = \{H_{ij} = \{x_i = x_j\}, 1 \leq i < j \leq n+1\}$ in $\mathbb{R}^{n+1}$ where the $x_i$ is a system of coordinates.

**Proposition 3.3.1.** *A $k-$cell $\langle C, F\rangle \in \boldsymbol{S}(\mathcal{A}_n)$ is represented by a* tableau *with $n+1$ boxes and $n+1-k$ rows filled with all the integers in $\{1, \ldots, n+1\}$ such that $(x_1, \ldots, x_{n+1}$ is a point in $F$ if and only if:*

- *$i$ and $j$ belong to the same row if and only if $x_i = x_j$,*
- *$i$ belongs to a row less than the one containing $j$ if and only if $x_i < X_j$.*

*The chamber $C$ belong to the half-space $x_i < x_j$ if and only if*

- *The row that contains $i$ is less than the one containing $j$ or*
- *$i$ and $j$ belong to the same row and the column which contains $i$ is less than the one containing $j$.*

**Example 3.3.2.** The tableau in $\mathcal{A}_2$

| | |
|---|---|
| 2 | 1 |
| 3 | |

represents the cell $\langle C, F\rangle$ with $C = \{x_2 < x_1 < x_3\}$ and $F = \{x_2 = x_1 \wedge x_2 < x_3\}$.

Using the tableaux there is an easy way to see when a cell is in the boundary of another.

**Lemma 3.3.3.** *Given a tableau $T$ all the tableaux in its boundary can be obtained by $T$ by taking a row and spitting it in two rows while preserving the order (if $i, j$ where in the row and $i$ was before $j$ than or $i$ and $j$ now belong to different rows or still $i$ is before $j$).*

The matching that we are now going to discuss was first proposed by Giovanni Paolini. Later on we found out that was already been studied in [Dja09] but without our focus on the boundary in local homology.

Given a tableau $T$ we will call with $T_{ij}$ the element in row $i$ and column $j$ of $T$, we give to the elements of a tableau $T$ an order, saying that $T_{ij}$ is before $T_{i'j'}$ if $i < i'$ or $i = i'$ and $j < j'$.

**Proposition 3.3.4.** *The following matching $\mathcal{M}$ on $\mathcal{A}_n$ is acyclic for all $n$. Given a tableau $T$ let $i$ be the minimum such that*

- $T_{i1} \neq \max_j T_{ij}$ *In this case the tableau $T$ is matched with the tableau obtained by $T$ by splitting the row $i$ taking $\max_j T_{ij}$ and all the following elements in the row $i$ and moving them up.*
- $T_{i1} = \max_j T_{ij}$ *but $T_{i1} > \max_j T_{(i+1)j}$ then $T$ is matched with the tableau obtained by gluing together the rows $i$ and $i+1$ putting the row $i$ after the row $i+1$.*

*The critical tableaux are in correspondence with the permutations of the numbers between 1 and $n+1$, where given a permutation we associate a tableau such that the elements are in the same order of the permutation, the first column is increasing and on each row the maximum is on the first column.*

**Example 3.3.5.** The tableau

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 3 | 5 | 4 |

is matched with the tableau

| | | |
|---|---|---|
| 5 | 4 | |
| 2 | 1 | 3 |

To the permutation $(4, 3, 5, 6, 2, 1)$ we associate the critical tableau

| | | |
|---|---|---|
| 4 | 3 | |
| 5 | | |
| 6 | 2 | 1 |

*Proof.* The fact that the one described above is a partial matching and that the critical tableaux are the one written is clear, so we only have to show that the matching is acyclic.

Let us suppose that there is a cyclic alternating path, namely

$$T = T_0 \searrow T_1' \nearrow T_1 \searrow \dots T_n' \nearrow T_n = T$$

where, for each $k$, $(T_k', T_k) \in \mathcal{M}$ and $T_k' < T_{k-1}$.

Let suppose that $T_1$ is obtained by $T_1'$ by gluing together the rows $i$ and $i+1$, this implies that $T_2'$ is obtained by $T_1$ by splitting the row $i$. To prove this we need to divide in two cases:

- If $T_2'$ is obtained by $T_1$ by splitting a row $j > i$ then our matching will split the row $i$ in two and so the path is not alternating.
- If $T_2'$ is obtained by $T_1$ by splitting a row $j < i$ we can repeat the same argument and see that at all the following step we will work only in a row smaller or equal to $j$ and then at the end we cannot return to $T$.

The same argument used above can be repeated for each boundary operation, even for the one between $T_0$ and $T_1'$ so in our path the only things that we can do is to split a certain row $i$ in two and then glue them together. Let $T_{il}$ be the maximum element in the row $i$ of $T$.

Let us now consider $T_1'$. $T_{il}$ must become the first element of the row $i$ because otherwise the matching does not glue the rows $i$ and $i+1$ together. If we recall when a tableau is in the boundary of another then in the $i-$row of $T_1'$ there are only elements of the row $i$ of $T$ and after $T_{il}$. If there are all said elements then $T_1 = T$, otherwise in $T_1$ the maximum in the row $i$ is in a column bigger than $l$. It is now immediate to see that there is no way to move the maximum in a smaller column so to obtain again $T$ in the path and this concludes the proof.

□

We need now to give a definition to some special elements of $T$. An element $T_{ij}$ of $T$ is a *block maximum* if it is bigger than all the elements in the row before it ($T_{ij} > T_{ik}\ \forall 0 < k < j$). A *block* is a sequence of elements in a row of $T$ starting with a block maximum and ending just before the following block maximum.

Let now consider the following map:

$$\beta\colon \mathbf{S}(\mathcal{A}_n) \to \mathcal{P}\{1, \dots, n+1\}$$

Where with $\mathcal{P}\{1, \dots, n+1\}$ we mean the set of ordered partitions of $\{1, \dots, n+1\}$, that sends a tableau $T$ to the sequence of its blocks. The first block of the sequence is the one starting with $T_{11}$, and we go on from left to right, from the first row to the last.

**Example 3.3.6.** The tableau

| | | | | | |
|---|---|---|---|---|---|
| 7 | 4 | | | | |
| 5 | 2 | 8 | 9 | 1 | 3 |

is sent to $((7,4),(5,2)),(8),(9,1,3)$.

It is immediate to see that if $(T,T') \in \mathcal{M}$ then $\beta(T) = \beta(T')$ and that if a tableau is critical then the blocks correspond to the rows.

We can now try to describe the alternating paths between critical cells.

**Proposition 3.3.7.** *Given two critical tableaux $T$ and $T'$, there exist an alternating path between them if and only if:*

- *The block maximums of $T'$ are the block maximums of $T$ plus one,*
- *All the blocks of $T$ included one block of $T'$ (the one with same block maximum) preserving the order of the elements.*
- *Let $i$ be the row such that $T'_{i1}$ is not a block maximum of $T$ and let $k$ be the length of the row $i$ of $T'$. Then for every $1 \leq l < m \leq k$ the element $T'_{il}$ in $T$ is in a row bigger or equal to that that contains the element $T'_{im}$. Moreover any elements of this row must be in $T$ in a row with block maximum bigger that $T'_{i1}$.*

*Proof.* Let us suppose then that we have an alternating path between critical tableaux

$$T = T_0 \searrow T'_1 \nearrow T_1 \searrow \dots T'_n = T'$$

Since the number of blocks of $T'$ is one plus the number of blocks of $T$ and we cannot remove a block with a boundary operation then the only time we are adding a block is between $T'_1$ and $T$ (here we are obliged). After this, every other times we do a boundary operation we have to split the only row with two blocks otherwise we will create an additional block. This row during the path can only decrease and at any time after the first passage the block that is not in $T$ is always the first of the row. Then with a boundary operation if we don't want to create new blocks we can only add elements at the end of this block. This implies that all the three conditions are necessary.

We show now a way to construct an acyclic path for every pair of tableaux that respect all the conditions.

$T'_1$ is obtained by $T$ by splitting the row that contains $T'_{i1}$ in two parts, where the part that goes below has the single elements $T'_{i1}$. Then $T_1$ is equal to $T$ with the exception that in the $i-$row it contains as first element $T'_{i1}$. After this we proceed by induction, let us suppose that we have done $j$ step. In the following step we check if the row of $T_j$ that contains two blocks contains in the second block some element that should go in the first one in $T'$, in this case we split the row by putting below the first block together with the smaller of this elements. Otherwise we split the row but putting above the first block. It is now easy to see that this construction always arrives at $T'$ and we stop there. As we have already noticed the row that contains two blocks can only decrease then whichever path we consider between two critical cells the first time that we have two blocks in the $i-$rows the intermediate tableau is exactly the same because we have taken all the elements in higher rows in the right order even if maybe in different ways. □

*Remark* 3.3.8. Watching closely the proof above we see that in the construction of the alternating path we do not have a variety of choice in what to do. First of all, we recall that we have to work row per row, so here we can only study the case of a single row, or even better of a single inductive step. This is because when we change the row with two blocks we cannot go back. At a certain step a certain row must have two blocks and if the second block does not contains any element that should go in the first one we are obliged to split as in the proof. If it contains at least one, we should check if they are already ordered. Here two things could happen, first of all if there are at least two consecutive elements in the right order we can take instead of only one at times any subsequence of them and add them together to the first block. Moreover if all the elements are ordered (for example if there is only one remaining) we can add it to the first block and put the block above instead of below.

This description of the alternating paths is the idea behind our algorithm. Before arriving to it we need yet another tool, an explicit description of the local homology boundary of two tableaux.

**Lemma 3.3.9.** *Given two tableaux $T$ and $T'$, we define the set $Q$ as the set of pairs $(a, b)$ of elements in $1, \ldots, n+1$ such that $a < b$ and in $T$ $a$ is before $b$ and after in $T'$. Then, the boundary in local homology is given by:*

$$[T : T'] \prod_{(a,b)\in Q} t_{ab}$$

*Where $t_{ab}$ is the loop around the hyperplane $x_a = x_b$ and $[T : T']$ is the incidence number with integer coefficients, in particular can be taken equal to*

$$[T : T'] = sgn(n_m - n_{m+1})(-1)^k$$

*where $n_j$ is the maximum in the row $j$ of $T'$, $m, m+1$ are the row of $T'$ that are joined in $T$ and $k = \sharp\{n_j \mid n_j < \min\{n_m, n_{m+1}\}$.*

*Proof.* The first part of the lemma follows from the definition of abelian local homology and of the tableau. For the second part it is an easy calculation to verify that the incidence numbers given derives from an appropriate orientation of the cells of the Salvetti complex, see [Sal87] for more details of how the complex is actually built. □

Thanks to the previous Lemma and Theorem 3.1.8, using the previous description of the alternating paths between critical tableaux we are able to write an algorithm that explicitly do the calculations. The program, written in Sage, is divided in two subfunctions to make everything more readable.

The first function that we are listing takes the following as input:

- a critical tableau (*tab*), written as a permutation of $1, \dots, n+1$.
- The block maximums of *tab* (*heads*) as an array whose elements are the positions of the block maximums in the permutation. In particular since the tableau is critical *heads[i]* gives us the position of the first element in the $i-$row of *tab*.
- A certain row of *tab*, called *line*, the number actually identifies the first element of the row in the list.
- the number of block maximums of *tab*.

The output are all the critical tableau $T$ for which there exist an alternating path between $T$ and *tab* and whose block maximums are the ones of *tab* minus the one in the row specified. The idea of the algorithm, is pretty easy, we follow Proposition 3.3.7 and we do a recursion after having moved the first element of the row and modified every input accordingly.

```
def faces(tab, heads, line, finpos):
  t=[]
  if heads[line]==heads[line+1]:
    return [tab]
  else :
    for k in range(line+1,finpos+1):
      headsK=heads[:]
      for l in range(line+1,k+1):
        headsK[l]=headsK[l]-1
      for h in range(heads[k],heads[k+1]):
        tabbb=tab[:heads[line]]+tab[heads[line]+1:]
        tabbb.insert(h,tab[heads[line]])
        for p in faces(tabbb,headsK,line,k):
          t.append(p)
    return t
```

The second function is the one that actually compute the boundary, given two critical tableaux (*tab1* and *tab2*), their block maximums (*heads1* and *heads2*) and the row (*line*) in the first tableau that is not contained in a block of the second; return us the local boundary.

The idea behind the algorithm is the same described above, i.e. to work row per row. Putting together 3.3.8 and 3.3.9 we start with the row that contains the block maximums, we compute the partial boundary of each paths that sends all the elements in the special block of this row in the row below and then we continue till we arrive at our critical cell. The idea is that even if there may be more than one path between two critical cells in every path the first times that we have two blocks in a certain row the tableau is always the same so we can actually divide the computations of the boundary row per row. On a specific row the algorithm study the order in which are the element that we want to move and act accordingly, separating the case when they are already ordered and when they are not. More details are given as comment in the algorithm.

*Remark* 3.3.10. In the algorithm is also used the following observation: let us call with $a_{i1}, \ldots, a_{ik}$ the elements of *line* in *tab1* contained in the row $i$ of *tab2* with the order given by *tab1* (meaning that $a_{ij}$ is before $a_{il}$ in *line* whenever $j < l$). Then the boundary is not zero if and only if there exist $m \in [1, k]$ such that in the row $i$ of *tab2* the elements $a_{i1}, \ldots, a_{im}$ are in the reversed order while the elements $a_{im}, \ldots, a_{ik}$ are ordered.

The idea of the proof is simply to do the calculation and see that there are a lot of paths with same incidence number apart from the sign.

```
def bord(tab1, tab2, heads1, heads2, line):
  #f=position in tab2 of the elements in line of tab1
  f=[]
  for i in range(heads1[line],heads1[line+1]):
    j=heads1[line]
    while tab1[i]!=tab2[j]:
      j=j+1
    f.append(j)

  R = PolynomialRing(GF(5),(n+2)**2,"z").gens()

  #finding the row that contains f[0]
  g=len(heads2)-2
  while heads2[g]>f[0]:
    g=g-1

  #initializing the boundary delta
```

```
delta=(-1)**(line+1)
i=0

#Starting to work row per row till we arrive at line
while g>=line:

 #We consider the elements of f in the row g
 #that are in the wrong order
 while (i<len(f)-1)and(f[i+1]>heads2[g])and(f[i]>f[i+1]):
    delta=-delta
    for z in range(heads2[line],f[i]):
      if tab2[f[i]]>tab2[z]:
        delta=delta/R[(n+2)*(tab2[z]-1)+tab2[f[i]]]

    i=i+1

  #The first i for which f[i] and the following elements
  #are in the right order so we have two different paths
  if f[i]>heads2[g]:
    p2=-1
    for z in range(heads2[g],f[i]):
      if tab2[f[i]]>tab2[z]:
        p2=p2/R[(n+2)*(tab2[z]-1)+tab2[f[i]]]

    p1=1
    for z in range(heads2[g],f[i]):
      k=0
      while z!=f[k] and k<i:
        k=k+1
        if (k==i) and tab2[f[i]]<tab2[z]:
          p1=p1*R[(n+2)*(tab2[f[i]]-1)+tab2[z]]
    delta=delta*(p1+p2)

    for z in range(heads2[line],heads2[g]):
      if tab2[f[i]]>tab2[z]:
        delta=delta/R[(n+2)*(tab2[z]-1)+tab2[f[i]]]

    i=i+1

  #We check the last elements in the row g,
```

```
  #for the remark they must be ordered
  while (i<len(f))and(f[i]>heads2[g])and(f[i-1]<f[i]):
    for z in range(heads2[g],f[i]):
      k=0
      while z!=f[k] and k<i:
        k=k+1
      if (k==i) and tab2[f[i]]<tab2[z]:
        delta=delta*R[(n+2)*(tab2[f[i]]-1)+tab2[z]]

    for z in range(heads2[line],heads2[g]):
      if tab2[f[i]]>tab2[z]:
        delta=delta/R[(n+2)*(tab2[z]-1)+tab2[f[i]]]
    i=i+1


  if (i>len(f)-1):
    g=line-1
  elif (f[i]<heads2[g]):
    g=g-1
  else:
    delta=0
    g=line-1

return delta
```

Finally, we use the two previous function to compute the boundary of an $\mathcal{A}_n$ with $n$ given as input.

```
n=input("Choose the value for n: ")

tab=[]
for x in range(1,n+2):
  tab.append(x)

for A in itertools.permutations(tab):

  #definition of headsA
  A = list(A)
  headsA=[0]
  for i in range(1,n+1):
    if A[i]>A[headsA[len(headsA)-1]]:
      headsA.append(i)
```

| | 2 1<br>3<br>4 | 2<br>3 1<br>4 | 2<br>3<br>4 1 | 1<br>3 2<br>4 | 1<br>3<br>4 2 | 1<br>2<br>4 3 |
|---|---|---|---|---|---|---|
| 1<br>2<br>3<br>4 | $1-t_{12}$ | $1-t_{13}$ | $1-t_{14}$ | $-1+t_{23}$ | $-1+t_{24}$ | $1-t_{34}$ |

Table 3.1: The boundary $\partial_1$ of $\mathcal{A}_3$

```
    headsA.append(n+1)

lineA=0
for lineA in range(0,len(headsA)-1):
  if len(headsA)>2:
    for B in faces(A,headsA,lineA,len(headsA)-2):

#definition of headsB
headsB=[0]
for i in range(1,n+1):
  if B[i]>B[headsB[len(headsB)-1]]:
    headsB.append(i)
  headsB.append(n+1)

C=bord(A,B,headsA,headsB,lineA)
print("{0} is a face of {1} with delta={2}").format(A,B,C)
```

As an example we see the boundary $\partial_2$ of $\mathcal{A}_3$ in 3.1,3.2,3.3.

Using this tables one can then compute the local homology of $\mathcal{A}_3$ and in general of $\mathcal{A}_n$ for small $n$. The same calculations have already been done in literature in the case all the $t_i$ equal to $t$ for example in [DL16] or in [Set09] for $n \leq 7$. Doing the calculations in our example again with a single $t$ using Sage to compute the

Smith forms of the matrices we obtain the known result:

$$
\begin{aligned}
H_0(\mathcal{M}(\mathcal{A}_3), \mathbb{Q}[t^{\pm 1}]) &\cong \frac{\mathbb{Q}[t^{\pm 1}]}{(t-1)} \cong \mathbb{Q} \\
H_1(\mathcal{M}(\mathcal{A}_3), \mathbb{Q}[t^{\pm 1}]) &\cong \left(\frac{\mathbb{Q}[t^{\pm 1}]}{(t-1)}\right)^4 \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{(t^3-1)} \\
H_2(\mathcal{M}(\mathcal{A}_3), \mathbb{Q}[t^{\pm 1}]) & \left(\frac{\mathbb{Q}[t^{\pm 1}]}{(t-1)}\right)^3 \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{(t^3-1)} \oplus \left(\frac{\mathbb{Q}[t^{\pm 1}]}{(t^6-1)}\right)^2 \\
H_3(\mathcal{M}(\mathcal{A}_3), \mathbb{Q}[t^{\pm 1}]) &\cong 0
\end{aligned}
$$

This method seems slightly different from the previous approaches but it is still not fast enough to be used for large $n$.

| | 2 1 / 4 3 | 2 / 4 1 3 | 2 / 4 3 1 | 1 / 4 2 3 | 1 / 4 3 2 | 3 1 2 / 4 | 3 2 1 / 4 | 3 2 / 4 1 | 3 1 / 4 2 | 3 / 4 1 2 | 3 / 4 2 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 / 2 / 4 3 | $1 - t_{12}$ | $1 - t_{14}$ | $1 - t_{13}t_{14}$ | $-1 + t_{24}$ | $-1 + t_{23}t_{24}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| 1 / 3 2 / 4 | $0$ | $0$ | $0$ | $t_{23}^{-1}(1 - t_{24})$ | $t_{24}(t_{34} - 1)$ | $1 - t_{13}$ | $1 - t_{12}t_{13}$ | $1 - t_{14}$ | $0$ | $0$ | $0$ |
| 1 / 3 / 4 2 | $0$ | $0$ | $0$ | $t_{34} - t_{23}^{-1}$ | $t_{34} - 1$ | $0$ | $0$ | $0$ | $1 - t_{13}$ | $1 - t_{14}$ | $1 - t_{12}t_{14}$ |
| 2 1 / 3 / 4 | $t_{34} - 1$ | $0$ | $0$ | $0$ | $0$ | $t_{12}^{-1}(t_{13} - 1)$ | $t_{13}(1 - t_{23})$ | $0$ | $t_{12}^{-1}(1 - t_{13})(t_{24} - 1)$ | $t_{12}^{-1}(t_{14} - 1)$ | $t_{14}(1 - t_{24})$ |
| 2 / 3 1 / 4 | $0$ | $t_{13}^{-1}(1 - t_{14})$ | $t_{14}(t_{34} - 1)$ | $0$ | $0$ | $t_{12}^{-1} - t_{23}$ | $1 - t_{23}$ | $0$ | $t_{12}^{-1}(1 - t_{24})$ | $0$ | $0$ |
| 2 / 3 / 4 1 | $0$ | $t_{34} - t_{13}^{-1}$ | $-1 + t_{34}$ | $0$ | $0$ | $0$ | $0$ | $1 - t_{23}$ | $0$ | $t_{12}^{-1} - t_{24}$ | $1 - t_{24}$ |

Table 3.2: The boundary $\partial_2$ of $\mathcal{A}_3$

| | 4 1 2 3 | 4 2 1 3 | 4 2 3 1 | 4 1 3 2 | 4 3 1 2 | 4 3 2 1 |
|---|---|---|---|---|---|---|
| 1 / 4 2 3 | $1-t_{14}$ | $1-t_{12}t_{14}$ | $1-t_{12}t_{13}t_{14}$ | $0$ | $0$ | $0$ |
| 1 / 4 3 2 | $0$ | $0$ | $0$ | $1-t_{14}$ | $1-t_{13}t_{14}$ | $1-t_{12}t_{13}t_{14}$ |
| 2 1 / 4 3 | $t_{12}^{-1}(t_{14}-1)$ | $t_{14}(1-t_{24})$ | $t_{13}t_{14}(1-t_{24})$ | $t_{12}^{-1}(t_{14}-1)$ | $t_{12}^{-1}(t_{13}t_{14}-1)$ | $t_{13}t_{14}(1-t_{23}t_{24})$ |
| 2 / 4 1 3 | $t_{12}^{-1}-t_{24}$ | $1-t_{24}$ | $0$ | $t_{12}^{-1}-t_{23}t_{24}$ | $0$ | $0$ |
| 2 / 4 3 1 | $0$ | $0$ | $1-t_{24}$ | $0$ | $t_{12}^{-1}-t_{23}t_{24}$ | $1-t_{23}t_{24}$ |
| 3 1 2 / 4 | $t_{24}(t_{13}t_{23})^{-1}(t_{14}-1)$ | $(t_{13}t_{23})^{-1}(1-t_{24})$ | $0$ | $t_{13}^{-1}t_{24}(t_{14}-1)$ | $t_{14}t_{24}(1-t_{34})$ | $0$ |
| 3 2 1 / 4 | $(t_{12}t_{13}t_{23})^{-1}(1-t_{14})$ | $t_{14}(t_{13}t_{23})^{-1}(t_{24}-1)$ | $t_{23}^{-1}t_{14}(t_{24}-1)$ | $0$ | $0$ | $t_{14}t_{24}(1-t_{34})$ |
| 3 2 / 4 1 | $(t_{13}t_{23})^{-1}(t_{24}-t_{12}^{-1})$ | $(t_{13}t_{23})^{-1}(t_{24}-1)$ | $t_{23}^{-1}(t_{24}-1)$ | $t_{24}(t_{13}^{-1}-t_{34})$ | $t_{24}(1-t_{34})$ | $t_{24}(1-t_{34})$ |
| 3 1 / 4 2 | $(t_{13}t_{23})^{-1}(t_{14}-1)$ | $(t_{13}t_{23})^{-1}(t_{12}t_{14}-1)$ | $t_{12}t_{14}(t_{23}^{-1}-t_{34})$ | $t_{13}^{-1}(t_{14}-1)$ | $t_{14}(1-t_{34})$ | $t_{12}t_{14}(1-t_{34})$ |
| 3 / 4 1 2 | $t_{13}^{-1}t_{23}^{-1}-t_{34}$ | $0$ | $0$ | $t_{13}^{-1}-t_{34}$ | $1-t_{34}$ | $0$ |
| 3 / 4 2 1 | $0$ | $t_{13}^{-1}t_{23}^{-1}-t_{34}$ | $t_{23}^{-1}-t_{34}$ | $0$ | $0$ | $1-t_{34}$ |

Table 3.3: The boundary $\partial_3$ of $\mathcal{A}_3$

# Chapter 4

# Minimality of infinite affine arrangement

As described in the previous chapter, in [Del08] Delucchi constructs a perfect acyclic matching in the central case using an order of the chambers and the shellability. This chapter is a joint work with Giovanni Paolini in which we will try to understand how much is still valid in the affine case, even with infinite hyperplane. In the first section we will see how to modify some definitions to make them still work and in particular that we must ask a more rigid requirement for an order to be valid. Following this we will prove that if we have a valid order we are able to build an acyclic matching with as many singular cells as chambers in the arrangements in the case of locally finite affine hyperplane arrangements. Finally, in the last section we will show a valid ordering of the chamber that we will call euclidean order and prove that even in the infinite case all the boundary maps are zero.

## 4.1 Decomposition of the Salvetti complex

We are going to construct an acyclic matching on the Salvetti complex of a locally finite affine arrangement $\mathcal{A}$, with critical cells in explicit bijection with the chambers of $\mathcal{A}$. Here with locally finite we mean that each compact set intersect a finite number of hyperplanes and also that each chamber has a finite number of walls. Following the ideas of Delucchi [Del08], we want to decompose the Salvetti complex into "pieces" (one piece for every chamber) and construct an acyclic matching on each of these pieces with exactly one critical cell. More formally, we are going to decompose the poset of cells $\mathbf{S}(\mathcal{A})$ as a disjoint union

$$\mathbf{S}(\mathcal{A}) = \bigsqcup_{C \in \mathcal{C}} N(C),$$

so that every subposet $N(C) \subseteq \mathbf{S}(\mathcal{A})$ admits an acyclic matching with one critical cell.

**Definition 4.1.1.** Given a chamber $C \in \mathcal{C}$, let $S(C) \subseteq \mathbf{S}(\mathcal{A})$ be the set of all the cells $\langle C', F \rangle \in \mathbf{S}(\mathcal{A})$ such that $C' = C.F$. In other words, a cell is in $S(C)$ if all the hyperplanes in $\operatorname{supp}(F)$ do not separate $C$ and $C'$.

Notice that the cells in $S(C)$ form a subcomplex of the Salvetti complex (using poset terminology, $S(C)$ is a lower ideal in $\mathbf{S}(\mathcal{A})$). This subcomplex is dual to the stratification of $\mathbb{R}^n$ induced by $\mathcal{A}$. Also, the natural map $S(C) \to \mathcal{F}$ which sends $\langle C', F \rangle$ to $F$ is an order-preserving bijection. This yields a poset isomorphism $S(C) \cong \mathcal{F}$.

Now fix a total order $\dashv$ of the chambers:

$$\mathcal{C} = \{C_0 \dashv C_1 \dashv C_2 \dashv \dots\}$$

(when $\mathcal{C}$ is infinite, the order type is that of natural numbers).

**Definition 4.1.2.** Let $N(C) \subseteq S(C)$ be the subset consisting of all the cells not included in any $S(C')$ with $C' \dashv C$.

The subsets $N(C)$, for $C \in \mathcal{C}$, form a partition of $\mathbf{S}(C)$. All the 0-cells are contained in $N(C_0) = S(C_0)$. Therefore, for $C \neq C_0$, the cells of $N(C)$ do not form a subcomplex of the Salvetti complex. If $\mathcal{A}$ is a (finite) central arrangement, this definition of $N(C)$ coincides with the one given in [Del08, Section 4].

We want to choose the total order $\dashv$ of the chambers so that each $N(C)$ admits an acyclic matching with exactly one critical cell. In [Del08] this is done taking any linear extension of the partial order $\leq_{C_0}$, for any base chamber $C_0$. Such a total order works well for central arrangements but not for general affine arrangements, as we see in the following two examples.

**Example 4.1.3.** Consider a non-central arrangement of three lines in the plane, as in Figure 4.1 on the left. Choose $C_0$ to be one of the three simplicial unbounded chambers. In any linear extension of $\leq_{C_0}$, the last chamber $C_6$ must be the non-simplicial unbounded chamber opposite to $C_0$. However $S(C_6) \subseteq \bigcup_{C \neq C_6} S(C)$, so $N(C_6)$ is empty, and therefore it does not admit an acyclic matching with one critical cell. Figure 4.2 shows the decomposition of the Salvetti complex for one of the possible linear extensions of $\leq_{C_0}$.

**Example 4.1.4.** Consider the arrangement of five lines depicted on the right of Figure 4.1. For every choice of a base chamber $C_0$ and for every linear extension of $\leq_{C_0}$, there is some chamber $C$ such that $N(C)$ is empty.

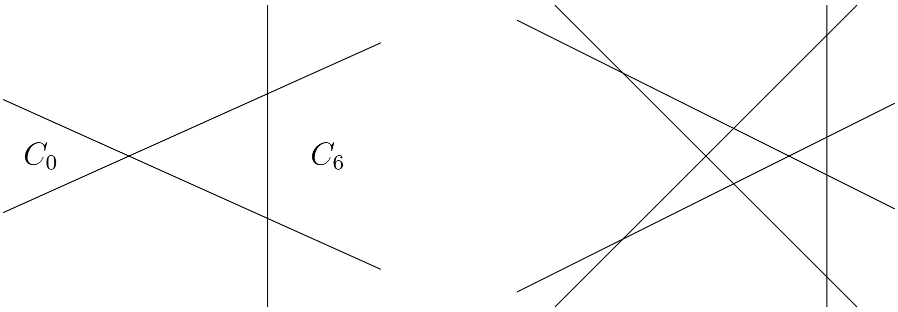



Figure 4.1: Two line arrangements.

Now we are going to state the condition on the total order $\dashv$ on $\mathcal{C}$ that produces a decomposition of the Salvetti complex with the desired properties. First recall the following definition from [Del08].

**Definition 4.1.5.** Given a chamber $C$ and a total order $\leq$ on $\mathcal{C}$, let

$$\mathcal{J}(C) = \{X \in \mathcal{L} \mid \operatorname{supp}(X) \cap s(C, C') \neq \emptyset \;\; \forall\, C' \dashv C\}.$$

Notice that $\mathcal{J}(C)$ is an upper ideal of $\mathcal{L}$, and coincides with $\mathcal{L}$ for $C = C_0$. In [Del08, Theorem 4.15] it is proved that, if $\mathcal{A}$ is a (finite) central arrangement and $\dashv$ is a linear extension of $\leq_{C_0}$ (for any choice of $C_0 \in \mathcal{C}$), then $\mathcal{J}(C)$ is a principal upper ideal for every chamber $C \in \mathcal{C}$. This is the condition we need.

**Definition 4.1.6** (Valid order)**.** A total order $\dashv$ on $\mathcal{C}$ is *valid* if, for every chamber $C \in \mathcal{C}$, $\mathcal{J}(C)$ is a principal upper ideal generated by some flat $X_C = |F_C| \in \mathcal{L}$ where $F_C$ is a face of $C$.

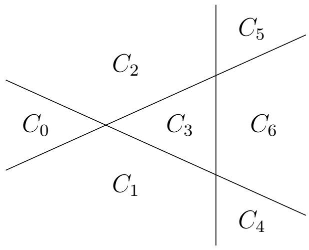
The total orders of Example 4.1.3 are not valid, because $\mathcal{J}(C_6)$ is empty. A valid order that begins with the same chamber $C_0$ is showed in Figure 4.3.

## 4.2 Construction of the acyclic matching

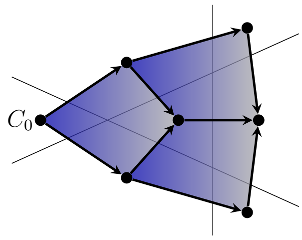
Throughout this section we assume to have an arrangement $\mathcal{A}$ together with a valid order $\dashv$ of $\mathcal{C}$ (as in Definition 4.1.6). Using the decomposition

$$\mathbf{S}(\mathcal{A}) = \bigsqcup_{C \in \mathcal{C}} N(C)$$

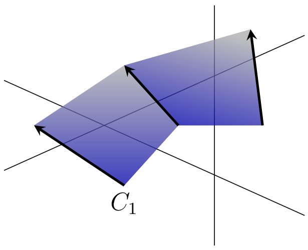
of Section 4.1 (associated to the valid order $\dashv$), we are going to construct a proper acyclic matching on $\mathbf{S}(A)$ with critical cells in bijection with the chambers. In more detail, we are going to construct an acyclic matching on every $N(C)$ with exactly



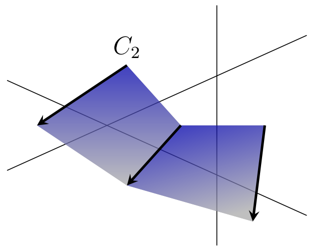

(a) Total order of the chambers



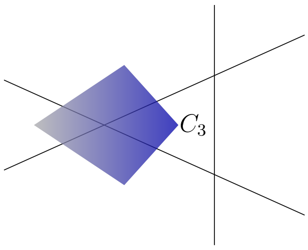

(b) $N(C_0)$



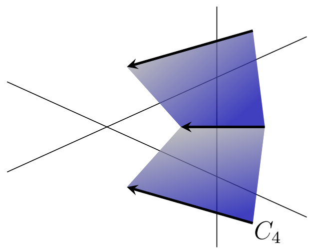

(c) $N(C_1)$



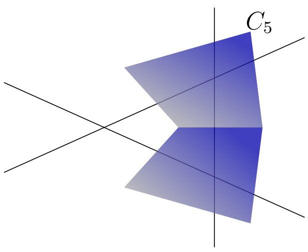

(d) $N(C_2)$



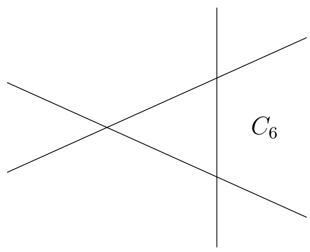

(e) $N(C_3)$




(f) $N(C_4)$

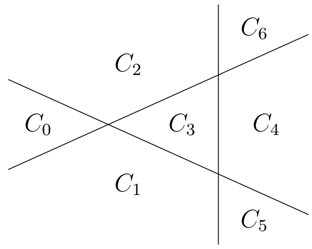



(g) $N(C_5)$

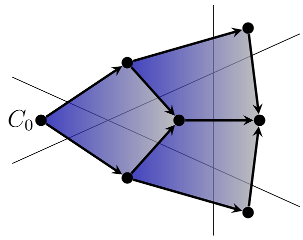



(h) $N(C_6)$ is empty

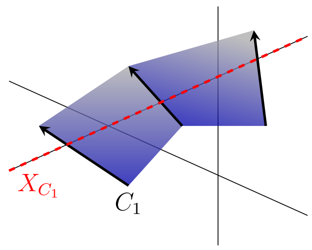
Figure 4.2: A non-central arrangement of three lines in the plane, with a linear extension of $\leq_{C_0}$. Here $N(C_5)$ and $N(C_6)$ do not admit an acyclic matching with one critical cell.



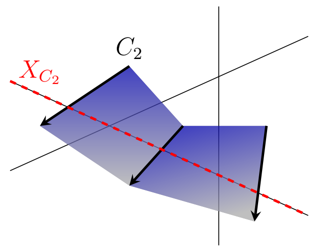

(a) Total order of the chambers



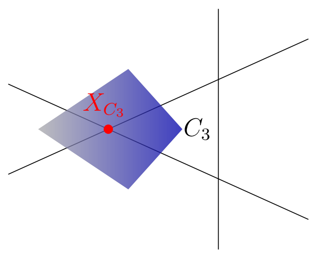

(b) $N(C_0)$



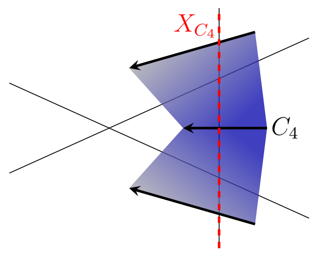

(c) $N(C_1)$



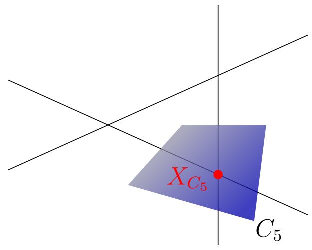

(d) $N(C_2)$



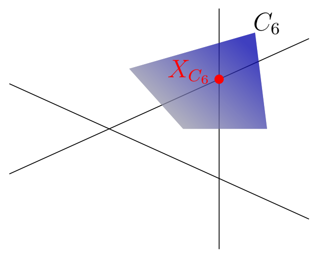

(e) $N(C_3)$




(f) $N(C_4)$




(g) $N(C_5)$




(h) $N(C_6)$

Figure 4.3: A non-central arrangement of three lines in the plane, with a valid order of the chambers. For every chamber $C$ except $C_0$, the generator $X_C$ of $\mathcal{J}(C)$ is highlighted.

one critical cell, and then attach these matchings together using the Patchwork Theorem (Theorem 1.1.4). This strategy is the same as the one employed in [Del08], but our proofs are different since we deal with affine and possibly infinite arrangements.

**Lemma 4.2.1.** *Suppose that $\dashv$ is a valid order of $\mathcal{C}$, in the sense of Definition 4.1.6. Then*

$$N(C) = \{\langle D, F\rangle \in S(C) \mid F \subseteq X_C\}.$$

*Proof.* To prove the inclusion $\subseteq$, assume by contradiction that there exists some cell $\langle D, F\rangle \in N(C)$ with $F \nsubseteq X_C$. By minimality of $X_C$ in $\mathcal{J}(C)$ we have that $|F| \notin \mathcal{J}(C)$. This means that there exists a chamber $C' \dashv C$ such that $\operatorname{supp}(F) \cap s(C, C') = \emptyset$. Then $C$ and $C'$ belong to the same chamber of $\mathcal{A}_{|F|}$, which implies $C'.F = C.F$. By definition of $S(C)$, we have that $C.F = D$. Then $C'.F = D$, so $\langle D, F\rangle \in S(C')$. This is a contradiction since $\langle D, F\rangle \in N(C)$ and $C' \dashv C$.

For the opposite inclusion, consider a cell $\langle D, F\rangle \in S(C)$ with $F \subseteq X_C$. Then $|F| \in \mathcal{J}(C)$, so for every chamber $C' \dashv C$ there exists an hyperplane in $\operatorname{supp}(F) \cap s(C, C')$. By the same argument as before we can deduce that $D = C.F \neq C'.F$ for all $C' \dashv C$, which means that $\langle D, F\rangle \notin S(C')$ for all $C' \dashv C$. Therefore $\langle D, F\rangle \in N(C)$. □

Now that we have a good description of every $N(C)$, we can begin the construction of the matching. For every chamber $C \in \mathcal{C}$ consider the map

$$\tilde{\eta}_C \colon S(C) \to \mathcal{C}$$

that sends a cell $\langle D, F\rangle$ to the chamber opposite to $D$ with respect to $F$. If we endow $\mathcal{C}$ with the partial order $\leq_C$, then $\eta_C$ becomes a poset map.

**Lemma 4.2.2.** *The map $\tilde{\eta}_C \colon S(C) \to (\mathcal{C}, \leq_C)$ is order-preserving.*

*Proof.* Let $\langle D, F\rangle, \langle D', F'\rangle \in S(C)$ and suppose that $\langle D', F'\rangle \leq \langle D, F\rangle$. Then $F' \preceq F$ and therefore $\operatorname{supp}(F') \subseteq \operatorname{supp}(F)$. Call $E$ and $E'$ the images of these two cells under the map $\eta_C$. By definition of $\eta_C$ and $S(C)$ we have that $s(C, E) = s(C, D) \cup \operatorname{supp}(F)$ and $s(C, E') = s(C, D') \cup \operatorname{supp}(F')$. In addition $F' \preceq F$ implies that $s(D, D') \subseteq \operatorname{supp}(F) \setminus \operatorname{supp}(F')$. Since $s(C, D') \subseteq s(C, D) \cup s(D, D')$, we conclude that

$$\begin{aligned} s(C, E') = s(C, D') \cup \operatorname{supp}(F') &\subseteq s(C, D) \cup s(D, D') \cup \operatorname{supp}(F') \\ &\subseteq s(C, D) \cup \operatorname{supp}(F) = s(C, E). \end{aligned}$$

Therefore $E' \leq_C E$. □

Consider the restriction $\eta_C = \tilde{\eta}_C|_{N(C)} \colon N(C) \to \mathcal{C}$. The matching on $N(C)$ will be obtained as a union of acyclic matchings on each fiber $\eta_C^{-1}(E)$ of $\eta_C$. Lemma 4.2.2, together with the Patchwork Theorem, will ensure that the matching on $N(C)$ is acyclic. We now fix two chambers $C$ and $E$, and study the fiber $\eta_C^{-1}(E)$.

**Lemma 4.2.3.** *Let $\dashv$ be a valid order of $\mathcal{C}$, and let $C, E$ be two chambers. A cell $\langle D, F \rangle \in \boldsymbol{S}(\mathcal{A})$ is in the fiber $\eta_C^{-1}(E)$ if and only if $D$ is opposite to $E$ with respect to $F$, $F \subseteq X_C$ and* $\operatorname{supp}(F) \subseteq s(C, E)$.

*Proof.* Suppose that $\langle D, F \rangle \in \eta_C^{-1}(E)$. In particular $\langle D, F \rangle \in N(C)$, thus by Lemma 4.2.1 we have that $F \subseteq X_C$. By definition of $\eta_C$, $D$ is the unique chamber opposite to $E$ with respect to $F$. Finally $\operatorname{supp}(F) \subseteq s(D, E)$ by definition of $\eta_C$, and $\operatorname{supp}(F) \cap s(C, D) = \emptyset$ by definition of $S(C)$, so $\operatorname{supp}(F) \subseteq s(D, E) \setminus s(C, D) \subseteq s(C, E)$.

We want now to prove that a cell $\langle D, F \rangle$ that satisfies the given conditions is in the fiber $\eta_C^{-1}(E)$. Since $D$ is opposite to $E$ with respect to $F$, we deduce that $\operatorname{supp}(F) \subseteq s(D, E)$. Then, using the hypothesis $\operatorname{supp}(F) \subseteq s(C, E)$, we obtain $\operatorname{supp}(F) \cap s(C, D) = \emptyset$. This means that $C.F = D$, i.e. $\langle D, F \rangle \in S(C)$. By Lemma 4.2.1, we conclude that $\langle D, F \rangle \in N(C)$. The fact that $\eta_C(\langle D, F \rangle) = E$ follows directly from the definition of $\eta_C$. □

A cell $\langle D, F \rangle$ in the fiber $\eta_C^{-1}(E)$ is determined by $F$, because $D$ is the unique chamber opposite to $E$ with respect to $F$. Then we immediately have the following corollary.

**Corollary 4.2.4.** *The fiber $\eta_C^{-1}(E)$ is in order-preserving (and rank-preserving) bijection with the set of faces $F \succeq E$ such that $F \subseteq X_C$ and* $\operatorname{supp}(F) \subseteq s(C, E)$.

Assume from now on that the fiber $\eta_C^{-1}(E)$ is non-empty. By Lemma 4.2.3, this means that $E \cap X_C \neq \emptyset$. Consider the restricted arrangement $\mathcal{A}^{X_C}$, and let $E' = E \cap X_C$ and $C' = C \cap X_C$ be the restrictions of our chambers to $\mathcal{A}^{X_C}$. Notice that $C' \neq \emptyset$ by definition of a valid order, so both $C'$ and $E'$ are chambers of $\mathcal{A}^{X_C}$. With this notation we can restate the above corollary as follows.

**Corollary 4.2.5.** *Suppose that the fiber $\eta_C^{-1}(E)$ is non-empty. Then $C' = C \cap X_C$ and $E' = E \cap X_C$ are chambers of the arrangement $\mathcal{A}^{X_C}$, and $\eta_C^{-1}(E)$ is in order-preserving bijection with the set of faces $F \succeq E'$ such that* $\operatorname{supp}(F) \subseteq s(C', E')$ *in $\mathcal{A}^{X_C}$.*

*Proof.* By Definition 4.1.6, $X_C = |F_C|$ for some face $F_C$ of $C$. Then $C' = C \cap X_C = F_C$ is a chamber of $\mathcal{A}^{X_C}$.

Consider now any cell $\langle D, F \rangle \in \eta_C^{-1}(E)$, and let $D' = D \cap X_C$. If we prove that $D'$ is a chamber of $\mathcal{A}^{X_C}$ then the same is true for $E'$ since they are opposite with respect to $F$ and $F \subseteq X_C$ (by Lemma 4.2.1). Let $F'_C = F_C.F$ in the arrangement $\mathcal{A}^{X_C}$ (so $F'_C$ is a chamber of $\mathcal{A}^{X_C}$), and consider the chamber $\tilde{D} = C.F'_C$ in $\mathcal{A}$. Then $\tilde{D} = C.F = D$ (the first equality holds because $F'_C \preceq F$, and the second equality because $D \in S(C)$). Therefore $D' = D \cap X_C = \tilde{D} \cap X_C = F'_C$ is a chamber of $\mathcal{A}^{X_C}$.

The second part is mostly a rewriting of Corollary 4.2.4, but some care should be taken since we are passing from the arrangement $\mathcal{A}$ to the arrangement $\mathcal{A}^{X_C}$. To avoid confusion, in $\mathcal{A}^{X_C}$ write $\operatorname{supp}'$ and $s'$ in place of supp and $s$. Given a face $F \subseteq X_C$, we need to prove that $\operatorname{supp}(F) \subseteq s(C,E)$ in $\mathcal{A}$ if and only if $\operatorname{supp}'(F) \subseteq s'(C',E')$ in $\mathcal{A}^{X_C}$. This is true because

$$\operatorname{supp}'(F) = \{H \cap X_C \mid H \in \operatorname{supp}(F) \text{ and } H \not\supseteq X_C\};$$
$$s'(C',E') = \{H \cap X_C \mid H \in s(C,E) \text{ and } H \not\supseteq X_C\}. \qquad \square$$

Constructing an acyclic matching on $\eta_C^{-1}(E)$ is then the same as constructing one on the set of faces of $E'$ given by Corollary 4.2.5. We start by considering the special case $E' = C'$.

**Lemma 4.2.6.** *Suppose that the fiber $\eta_C^{-1}(E)$ is non-empty. Then $E' = C'$ if and only if $E$ is the chamber opposite to $C$ with respect to $X_C$. In this case, $\eta_C^{-1}(E)$ contains the single cell $\langle C, F_C\rangle$.*

*Proof.* If $E$ is opposite to $C$ with respect to $X_C$, then clearly $E' = C'$. Conversely, suppose that $E' = C' = F_C$. Let $\langle D, F\rangle$ be any cell in $\eta_C^{-1}(E)$. As in the proof of Corollary 4.2.5, we have that $D \cap X_C = F'_C$ where $F'_C = F_C.F$ in $\mathcal{A}^{X_C}$. Notice that $F \subseteq E \cap X_C = E' = F_C$, so $F'_C = F_C.F = F_C$. In other words, the chambers $C$, $D$ and $E$ all contain the face $F_C$. Since $F \subseteq F_C \subseteq C \cap D$, we have that $s(C,D) \subseteq \operatorname{supp}(F)$. But $D \in S(C)$ implies that $D = C.F$, i.e. $s(C,D) \cap \operatorname{supp}(F) = \emptyset$. Therefore $s(C,D) = \emptyset$, so $C = D$. Now $E$ is the opposite of $D$ with respect to $F$, and $E \cap X_C = D \cap X_C = F_C$, so $F = F_C$. This means that $E$ is the opposite of $C$ with respect to $X_C$. The previous argument also shows that $\eta_C^{-1}(E)$ contains the single cell $\langle C, F_C\rangle$. $\square$

In particular, for every chamber $C$ there is exactly one fiber $\eta_C^{-1}(E)$ for which $E' = C'$. This *special fiber* contains exactly one cell, which is going to be critical with respect to our matching.

Consider now the case $E' \neq C'$. In view of Corollary 4.2.5, we work with the restricted arrangement $\mathcal{A}^{X_C}$ in $X_C$. Until Lemma 4.2.8, all our notations (for example, $\operatorname{supp}(F)$ and $s(C',E')$) are intended with respect to the arrangement $\mathcal{A}^{X_C}$. In what follows we make use of the definitions and facts of Section 1.4.

**Lemma 4.2.7.** *Let $y_{C'}$ be a point in the interior of $C'$. The faces $F \succeq E'$ such that $\operatorname{supp}(F) \subseteq s(C',E')$ are exactly the faces of $E'$ that are visible from $y_{C'}$.*

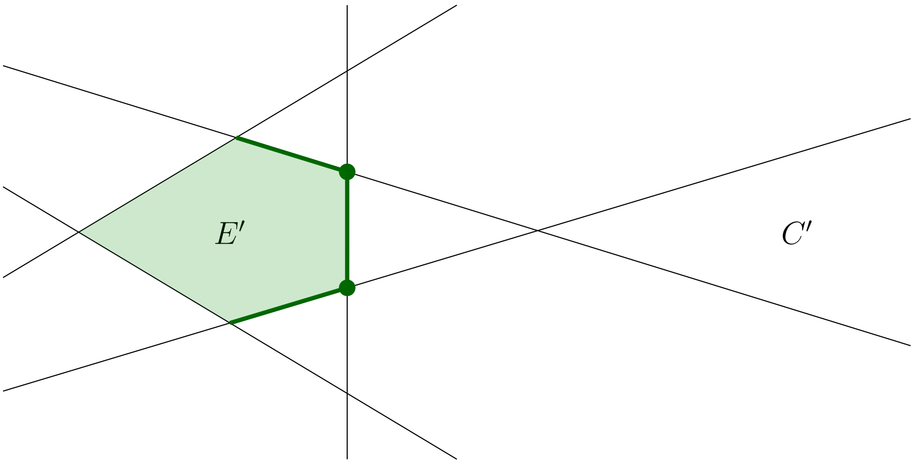



Figure 4.4: The faces of $E'$ that are visible from a point in the interior of $C'$.

*Proof.* Suppose that $\text{supp}(F) \subseteq s(C', E')$. In particular, for every facet $G \supseteq F$ of $E'$, the hyperplane $|G|$ separates $C'$ and $E'$ and so $G$ is visible from $y_{C'}$. Then $F$ is visible from $y_{C'}$.

Conversely, suppose that $F$ is visible from $y_{C'}$. Denote by $\mathcal{B} \subseteq \text{supp}(F)$ the set of hyperplanes $|G|$ where $G \supseteq F$ is a facet of $E'$. All the facets $G \supseteq F$ of $E'$ are visible from $y_{C'}$, so the hyperplanes $|G|$ separate $C'$ and $E'$. In other words, $\mathcal{B} \subseteq s(C', E')$. In the central arrangement $\mathcal{A}_{|F|}^{X_C} = \text{supp}(F)$, the chambers $\pi_{|F|}(C')$ and $\pi_{|F|}(E')$ are therefore opposite to each other, and $\mathcal{B}$ is the set of their walls. Then every hyperplane in $\text{supp}(F)$ separates $C'$ and $E'$. □

Fix an arbitrary point $y_{C'}$ in the interior of $C'$. By the previous lemma, the faces $F$ given by Corollary 4.2.5 are exactly the faces of $E'$ that are visible from $y_{C'}$. See Figure 4.4 for an example.

The idea now is that, if $E'$ is bounded, the boundary of $E'$ is shellable and we can use a shelling to construct an acyclic matching on the set of visible faces. We first need to reduce to the case of a bounded chamber (i.e. a polytope).

**Lemma 4.2.8.** *There exists a finite set $\mathcal{A}'$ of hyperplanes in $X_C$, and a bounded chamber $\tilde{E} \subseteq E'$ of the hyperplane arrangement $\mathcal{A}' \cup \mathcal{A}^{X_C}$, such that the poset of faces of $\tilde{E}$ that are visible from $y_{C'}$ is isomorphic to the poset of faces of $E'$ that are visible from $y_{C'}$.*

*Proof.* Let $X_C \cong \mathbb{R}^k$. Let $Q$ be a finite set of points which contains $y_{C'}$ and a point in the (relative) interior of each visible face of $E'$. For $i = 1, \ldots, k$, define $q_i \in \mathbb{R}$ as the minimum of all the $i$-th coordinates of the points in $Q$, and $q^i$ as the maximum.

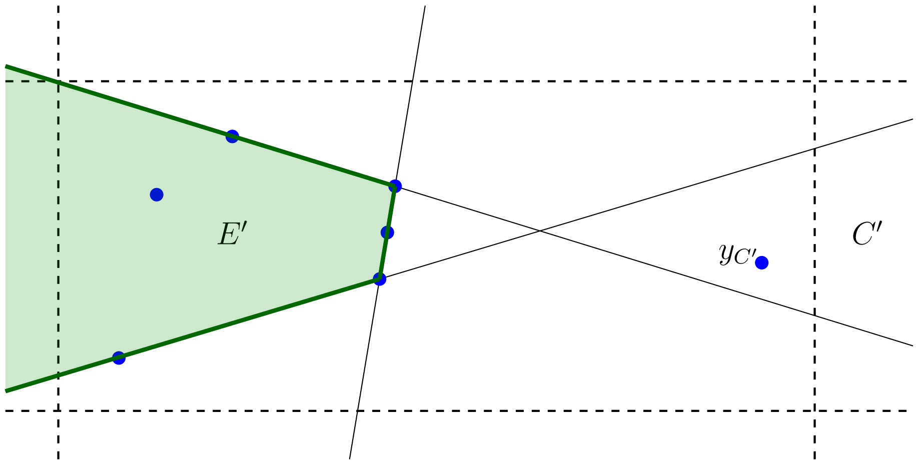



Figure 4.5: Construction of the bounded chamber $\tilde{E} \subseteq E'$ in the proof of Lemma 4.2.8. The points of $Q$ are colored in blue, the hyperplanes of $\mathcal{A}'$ are the dashed lines.

Choose $\mathcal{A}'$ as the set of the $2k$ hyperplanes of the form $x_i = q_i - 1$ and $x_i = q^i + 1$, for $i = 1, \ldots, k$. See Figure 4.5 for an example.

Let $\tilde{E}$ be the chamber of $\mathcal{A}^{X_C} \cup \mathcal{A}'$ that contains $Q \setminus \{y_{C'}\}$. By construction, $\tilde{E}$ is bounded.

The walls of $E'$ and of $\tilde{E}$ are related as follows: $\mathcal{W}_{\tilde{E}} = \mathcal{W}_{E'} \cup \mathcal{A}''$ for some $\mathcal{A}'' \subseteq \mathcal{A}'$. The hyperplanes in $\mathcal{W}_{E'}$ separate $y_{C'}$ and $\tilde{E}$, whereas the hyperplanes in $\mathcal{A}''$ do not. This means that a facet $\tilde{G}$ of $\tilde{E}$ is visible if and only if $|\tilde{G}| \in \mathcal{W}_{E'}$.

There is a natural order-preserving (and rank-preserving) injection $\varphi$ from the set $\mathcal{V}$ of the visible faces $F$ of $E'$ to the set of faces of $\tilde{E}$, which maps a face $F$ to the unique face $\tilde{F}$ of $\tilde{E}$ such that $F \cap Q \subseteq \tilde{F} \subseteq F$. We want to show that the image of $\varphi$ coincides with the set of visible faces of $\tilde{E}$.

Consider a facet $\tilde{G}$ of $\tilde{E}$. Then $\tilde{G}$ is in the image of $\varphi$ if and only if $|\tilde{G}| \notin \mathcal{A}''$, which happens if and only if $\tilde{G}$ is visible.

Consider now a generic face $\tilde{F}$ of $\tilde{E}$. If $\tilde{F} = \varphi(F)$ for some $F \in \mathcal{V}$, then $Q \cap F \subseteq \tilde{F}$ and so $\tilde{F}$ is not contained in any hyperplane of $\mathcal{A}''$. Then all the facets $\tilde{G} \supseteq \tilde{F}$ of $\tilde{E}$ are visible, and so $\tilde{F}$ is visible. Conversely, if $\tilde{F}$ is not in the image of $\varphi$, then $\tilde{F}$ is contained in some hyperplane of $\mathcal{A}''$ and therefore also in some non-visible facet $\tilde{G}$. Then $\tilde{F}$ is not visible. □

We now show that the poset of visible faces of a polytope admits an acyclic matching such that no face is critical. We will use this result on the polytope $\tilde{E}$, in order to obtain a matching on the fiber $\eta_C^{-1}(E)$.

**Theorem 4.2.9.** *Let $X$ be a $k$-dimensional polytope in $\mathbb{R}^k$, and let $y \in \mathbb{R}^k$ be a point outside $X$ that does not lie in the affine hull of any facet of $X$. Then there* *exists an acyclic matching on the poset of faces of $X$ visible from $y$, such that no face is critical.*

*Proof.* By [Zie12, Theorem 8.12] and [Zie12, Lemma 8.10], there is a shelling $G_1, \dots, G_s$ of $\partial X$ such that the facets visible from $y$ are the last ones. Suppose that $G_t, G_{t+1}, \dots, G_s$ are the visible facets. Notice that there is at least one visible facet and at least one non-visible facet. In particular, the first facet $G_1$ is not visible and the last facet $G_s$ is visible. In other words, we have $2 \leq t \leq s$.

In [Del08, Proposition 1] it is proved that a shelling of a regular CW complex $Y$ induces an acyclic matching on the poset of cells $(P, <)$ of $Y$ (augmented with the empty face $\emptyset$), with critical cells corresponding to the spanning facets of the shelling. In our case, $Y = \partial X$ is a regular CW decomposition of a sphere, so the only spanning facet of a shelling is the last one (see for example [Del08, Lemma 2.13]).

Let $\mathcal{M}$ be an acyclic matching on $\partial X$ induced by the shelling $G_1, \dots, G_s$, as in [Del08]. We claim that the construction of [Del08] produces a matching which is homogeneous with respect to the grading $\varphi\colon (P, <) \to \{1, \dots, s\}$ given by

$$\varphi(F) = \min\{i \in \{1, \dots, s\} \mid F \leq G_i\}.$$

To prove this, we need to briefly go through the construction of $\mathcal{M}$. The first step [Del08, Lemma 2.10] is to construct a total order $\sqsubset_i$ on each $P_i$ (the set of faces of codimension $i$). The order $\sqsubset_0$ is simply the shelling order of the facets. It follows from the recursive construction of $\sqsubset_i$ that each $\varphi|_{P_i}\colon (P_i, \sqsubset_i) \to \{1, \dots, s\}$ is order-preserving. Then the linear extension $\lhd$ of $P$ constructed in [Del08, Definition 2.11] is such that $\varphi\colon (P, \lhd) \to \{1, \dots, s\}$ is also order-preserving. By construction of the matching [Del08, Lemma 2.12], if $(p, q) \in \mathcal{M}$ (with $p \geq q$) then $p \lhd q$. From this we obtain $\varphi(p) \geq \varphi(q)$ and $\varphi(p) \leq \varphi(q)$, so $\varphi(p) = \varphi(q)$. Therefore the matching is homogeneous with respect to $\varphi$.

The set of visible faces of $X$ is $\varphi^{-1}(\{t, \dots, s\}) \cup \{X\}$. Notice that the empty face $\emptyset$ belongs to $\varphi^{-1}(1)$, so it does not appear in $\varphi^{-1}(\{t, \dots, s\})$ because $t \geq 2$.

Let $\mathcal{M}'$ be the restriction of $\mathcal{M}$ to $\varphi^{-1}(\{t, \dots, s\})$. This is an acyclic matching on $\varphi^{-1}(\{t, \dots, s\})$ with exactly one critical face, the facet $G_s$. Then $\mathcal{M}' \cup \{(X, G_s)\}$ is an acyclic matching on the poset of visible faces of $X$ such that no face is critical. □

We are finally able to attach the matchings on the fibers $\eta_C^{-1}(E)$, putting all the results of this section together.

**Theorem 4.2.10.** *Let $\mathcal{A}$ be a locally finite hyperplane arrangement, and let $\dashv$ be a valid order of the set of chambers $\mathcal{C}$. For every chamber $C \in \mathcal{C}$, there exists a proper acyclic matching on $N(C)$ such that the only critical cell is $\langle C, F_C \rangle$. The* *union of these matchings forms a proper acyclic matching on* $\boldsymbol{S}(\mathcal{A})$ *with critical cells in bijection with the chambers.*

*Proof.* Consider the map

$$\eta\colon \mathbf{S}(\mathcal{A}) \to \mathcal{C} \times \mathcal{C}$$

defined as

$$\langle D, F\rangle \mapsto \big(C,\, \eta_C(\langle D, F\rangle)\big),$$

where $C \in \mathcal{C}$ is the chamber such that $\langle D, F\rangle \in N(C)$.

Corollary 4.2.5 provides a description of the non-empty fibers $\eta^{-1}(C, E)$, since by definition $\eta^{-1}(C, E) = \eta_C^{-1}(E)$. By Lemma 4.2.6 we know that for every $C \in \mathcal{C}$ there is exactly one non-empty fiber such that $E \cap X_C = C \cap X_C$, and this fiber contains the single cell $\langle C, F_C\rangle$. By Lemma 4.2.7 and Lemma 4.2.8, every other non-empty fiber $\eta^{-1}(C, E)$ is isomorphic to the poset of visible faces of some polytope in $X_C$ (with respect to some external point not lying on the affine hull of the facets). Finally, by Theorem 4.2.9 this poset admits an acyclic matching with no critical faces.

We want to use the Patchwork Theorem (Theorem 1.1.4) to attach these matchings together. To do so, we first need to define a partial order on $\mathcal{C} \times \mathcal{C}$ that makes $\eta$ a poset map. The order $\leq$ on $\mathcal{C} \times \mathcal{C}$ is the transitive closure of:

$$(C', E') \leq (C, E) \quad \text{if and only if} \quad C' \trianglelefteq C \text{ and } E' \leq_C E$$

(we denote by $\trianglelefteq$ the "less than or equal to" with respect to the total order $\triangleleft$).

To prove that $\eta$ is a poset map, suppose to have $\langle D', F'\rangle \leq \langle D, F\rangle$ in $\mathbf{S}(\mathcal{A})$. Let $\eta(\langle D', F'\rangle) = (C', E')$ and $\eta(\langle D, F\rangle) = (C, E)$. Since $S(C)$ is a lower ideal of $\mathbf{S}(\mathcal{A})$, we immediately obtain that $\langle D', F'\rangle \in S(C)$ and $C' \triangleleft C$. Then Lemma 4.2.2 implies that $E' \leq_C E$. Therefore $(C', E') \leq (C, E)$.

By the Patchwork Theorem, the union of the matchings on the fibers of $\eta$ forms an acyclic matching on $\mathbf{S}(\mathcal{A})$, with critical cells in bijection with the chambers.

We now have to prove that the obtained matching is proper. To do so, we prove that the $(\mathcal{C} \times \mathcal{C})$-grading $\eta$ is compact. Since every fiber $\eta^{-1}(C, E)$ is finite by Lemma 4.2.3, we only need to show that the poset $(\mathcal{C} \times \mathcal{C})_{\leq (C,E)}$ is finite for every pair of chambers $(C, E)$.

We prove this by double induction, first on the chamber $C$ (with respect to the order $\triangleleft$) and then on $n = |s(C, E)|$. The base case, $C = C_0$ and $n = 0$, is trivial since $E = C_0$.

We want now to prove the inductive step. Given a pair $(C, n) \in \mathcal{C} \times \mathbb{N}$, suppose that the claim is true for every pair $(C', n')$ such that either $C' \triangleleft C$, or $C' = C$ and $n' < n$. For every chamber $E$ with $|s(C, E)| = n$ we have that

$$(\mathcal{C} \times \mathcal{C})_{\leq (C,E)} = \bigcup_{\substack{C' \trianglelefteq C \\ E' \leq_C E \\ (C', E') \neq (C, E)}} (\mathcal{C} \times \mathcal{C})_{\leq (C', E')} \;\cup\; \{(C, E)\}$$

This is a union over a finite number of sets, and by induction hypothesis every set $(\mathcal{C} \times \mathcal{C})_{\leq (C',E')}$ is finite. Therefore the set $(\mathcal{C} \times \mathcal{C})_{\leq (C,E)}$ is finite.

By the Patchwork Theorem, the matchings on the fibers $\eta^{-1}(C,E)$ can be attached together to form a proper acyclic matching on $\mathbf{S}(\mathcal{A})$. By construction this matching is a union of proper acyclic matchings on the subposets $N(C)$ for $C \in \mathcal{C}$, each of them having exactly one critical cell $\langle C, F_C \rangle$. □

We end this section with a few remarks. We are not going to use them in the following, but they are interesting by themselves (especially in relation with [Del08]).

The first remark is that, without the need of a valid order, the results of this section allow to obtain a proper acyclic matching on $S(C_0)$ (for any chamber $C_0 \in \mathcal{C}$) with the single critical cell $\langle C_0, C_0 \rangle$. This is because $N(C_0) = S(C_0)$, and in the construction of the matching on $N(C_0)$ we do not use the existence of a valid order that begins with $C_0$. As noted in Section 4.1, there is a natural poset isomorphism $S(C_0) \cong \mathcal{F}$ for every chamber $C_0 \in \mathcal{C}$. Then the existence of an acyclic matching on $S(C_0)$ can be stated purely in terms of $\mathcal{F}$, without speaking of the Salvetti complex. This result is stated in [Del08, Theorem 3.6] in the case of the face poset of an oriented matroid.

**Theorem 4.2.11.** *Let $\mathcal{A}$ be a locally finite hyperplane arrangement. For every chamber $C \in \mathcal{C}(\mathcal{A})$ there is a proper acyclic matching on the poset of faces $\mathcal{F}(\mathcal{A})$ such that $C$ is the only critical face.*

The second remark is that, given a valid order $\dashv$ of $\mathcal{C}$ and a chamber $C \in \mathcal{C}$, the poset $N(C)$ is isomorphic to $\mathcal{F}(\mathcal{A}^{X_C})$.

**Lemma 4.2.12** (cf. [Del08, Lemma 4.20])**.** *Suppose that $\dashv$ is a valid order of $\mathcal{C}$. For every chamber $C \in \mathcal{C}$ there is a poset isomorphism*

$$N(C) \cong \mathcal{F}\big(\mathcal{A}^{X_C}\big).$$

*Proof.* The isomorphism in the left-to-right direction sends a cell $\langle D, F \rangle \in N(C)$ to the face $F$, which is in $\mathcal{F}(\mathcal{A}^{X_C})$ by Lemma 4.2.1. The inverse map sends a face $F \in \mathcal{F}(\mathcal{A}^{X_C})$ to the cell $\langle C.F, F \rangle$, which is in $N(C)$ by definition of $S(C)$ and by Lemma 4.2.1. These maps are order-preserving. □

Together, Lemma 4.2.12 and Theorem 4.2.11 give an alternative (but equivalent) construction of our matching on $\mathbf{S}(\mathcal{A})$, closer to the approach of [Del08].

## 4.3 Euclidean orders

In this section we are going to construct a valid order $\dashv_{\mathrm{eu}}$ of the set of chambers $\mathcal{C}$, for any locally finite arrangement $\mathcal{A}$, using the Euclidean distance in $\mathbb{R}^n$. Then we are going to prove that the matching induced by this order (given by Theorem 4.2.10) yields a minimal Morse complex.

Denote by $d$ the Euclidean distance in $\mathbb{R}^n$. Also, if $K$ is a closed convex subset of $\mathbb{R}^n$, denote by $\rho_K(x)$ the projection of a point $x \in \mathbb{R}^n$ onto $K$. The point $\rho_K(x)$ is the unique point $y \in K$ such that $d(x, y) = d(x, K)$.

The first step is to prove that there exist a lot of *generic points* with respect to the arrangement $\mathcal{A}$. For this, we need the following technical lemma. By *measure* we always mean the Lebesgue measure in $\mathbb{R}^n$.

**Lemma 4.3.1.** *Let $K_1$ and $K_2$ be two closed convex subsets of $\mathbb{R}^n$. Let*

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid d(x, K_1) = d(x, K_2) \text{ and } \rho_{K_1}(x) \neq \rho_{K_2}(x)\}.$$

*Then $\mathcal{S}$ has measure zero.*

*Proof.* This proof was suggested by Federico Glaudo. Let $d_i(x) = d(x, K_i)$ for $i = 1, 2$. Each function $d_i \colon \mathbb{R}^n \to \mathbb{R}$ is differentiable on $\mathbb{R}^n \setminus K_i$ by [GM12, Lemma 2.19], and its gradient in a point $x \notin K_i$ is the versor with direction $x - \rho_{K_i}(x)$.

Let $f(x) = d_1(x) - d_2(x)$. Denote by $A$ the open set of points $x \in \mathbb{R}^n \setminus (K_1 \cup K_2)$ such that $\rho_{K_1}(x) \neq \rho_{K_2}(x)$. On this set, the function $f$ is differentiable and its gradient does not vanish. It is known that the gradient of $f$ must vanish almost everywhere on $A \cap f^{-1}(0)$ [EG92, Corollary 1 of Section 3.1], hence $A \cap f^{-1}(0)$ has measure zero.

It is easy to check that the points in $K_1 \cup K_2$ cannot belong to $\mathcal{S}$. Then $\mathcal{S} = A \cap f^{-1}(0)$ has measure zero. □

**Lemma 4.3.2** (Generic points)**.** *Given a locally finite hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$, let $\mathcal{G} \subseteq \mathbb{R}^n$ be the set of points $x \in \mathbb{R}^n$ such that:*

*(i) for every $C, C' \in \mathcal{C}$ with $d(x, C) = d(x, C')$, we have $\rho_C(x) = \rho_{C'}(x) \in C \cap C'$;*

*(ii) for every $L, L' \in \mathcal{L}$ with $L' \subsetneq L$, we have $d(x, L') > d(x, L)$.*

*Then the complement of $\mathcal{G}$ has measure zero. In particular, $\mathcal{G}$ is dense in $\mathbb{R}^n$.*

*Proof.* Given $C, C' \in \mathcal{C}$, let $\mathcal{S}_{C,C'}$ be the set of points $x \in \mathbb{R}^n$ such that $d(x, C_1) = d(x, C_2)$ and $\rho_{C_1}(x) \neq \rho_{C_2}(x)$. By Lemma 4.3.1, every $\mathcal{S}_{C,C'}$ has measure zero.

Similarly, for every $L, L' \in \mathcal{L}$ with $L' \subsetneq L$, denote by $\mathcal{T}_{L,L'}$ the set of points $x \in \mathbb{R}^n$ such that $d(x, L') = d(x, L)$. We have that $\mathcal{T}_{L,L'}$ is an affine subspace of $\mathbb{R}^n$ of codimension at least 1, and in particular it has measure zero.

The complement of $\mathcal{G}$ is the union of all the sets $\mathcal{S}_{C,C'}$ for $C, C' \in \mathcal{C}$ and $\mathcal{T}_{L,L'}$ for $L, L' \in \mathcal{L}$ with $L' \subsetneq L$. This is a finite or countable union of sets of measure zero, hence it has measure zero. □

We call *generic points* the elements of $\mathcal{G}$, as defined in Lemma 4.3.2. Notice that, by condition (ii) with $L = \mathbb{R}^n$, a generic point must lie in the complement of $\mathcal{A}$.

We are now able to define Euclidean orders.

**Definition 4.3.3** (Euclidean orders)**.** A total order $\dashv_{\text{eu}}$ of the set of chambers $\mathcal{C}$ is *Euclidean* if there exists a generic point $x_0$ such that $C \dashv_{\text{eu}} C'$ implies that $d(x_0, C) \leq d(x_0, C')$. The point $x_0$ is called a *base point* of the Euclidean order $\dashv_{\text{eu}}$.

In other words, a Euclidean order is a linear extension of the partial order on $\mathcal{C}$ given by $C < C'$ if $d(x_0, C) < d(x_0, C')$, for some fixed generic point $x_0 \in \mathbb{R}^n$. In particular, for every generic point $x_0$ there exists at least one Euclidean order with $x_0$ as a base point. Since the set of generic points is dense, we immediately get the following corollary.

**Corollary 4.3.4.** *For every chamber $C_0 \in \mathcal{C}$, there exists a Euclidean order $\dashv_{\text{eu}}$ that starts with $C_0$.*

*Proof.* It is enough to take the base point $x_0$ in the interior of the chamber $C_0$. □

**Theorem 4.3.5.** *Let $\dashv_{\text{eu}}$ be a Euclidean order with base point $x_0$. For every chamber $C$, let $x_C = \rho_C(x_0)$ and let $F_C$ be the smallest face of $C$ that contains $x_C$. Then $\mathcal{J}(C)$ is the principal upper ideal generated by $X_C = |F_C|$. Therefore $\dashv_{\text{eu}}$ is a valid order.*

*Proof.* First we want to prove that $X_C \in \mathcal{J}(C)$. This is equivalent to proving that for every chamber $C' \dashv_{\text{eu}} C$ there exists a hyperplane $H \in \operatorname{supp}(X_C) \cap s(C, C')$. We have that $\rho_{X_C}(x_0) = x_C$ because $F_C$ is the smallest face that contains $x_C$. Then it is also true that $\rho_{\pi_{X_C}(C)}(x_0) = x_C$. Given a chamber $C' \dashv_{\text{eu}} C$, we have two possibilities.

- $d(x_0, C') < d(x_0, C)$. Then $C' \nsubseteq \pi_{X_C}(C)$, because all the points of $\pi_{X_C}(C)$ have distance at least $d(x_0, C)$ from $x_0$. This means that there exists a hyperplane $H \in \operatorname{supp}(X_C) = \mathcal{A}_{X_C}$ which separates $C$ and $C'$.

- $d(x_0, C') = d(x_0, C)$. Since $x_0$ is a generic point, we have that $x_C = x_{C'} \in C \cap C'$. Then $F_C$ is a common face of $C$ and $C'$, and every hyperplane in $s(C, C')$ contains $F_C$.

Now we want to prove that $X \subseteq X_C$ for every $X \in \mathcal{J}(C)$. Suppose by contradiction that $X \nsubseteq X_C$ for some $X \in \mathcal{J}(C)$. In particular, $X_C \neq \mathbb{R}^n$ and thus $x_0 \neq x_C$. We first prove that $\operatorname{supp}(X_C \cup X)$ is non-empty.

Let $C'$ be the chamber of $\mathcal{A}$ such that $x_0 \in \pi_{X_C}(C')$ and $C' \prec F_C$. Since $x_C \in X_C \subseteq \pi_{X_C}(C')$, the entire line segment $\ell$ from $x_0$ to $x_C$ is contained in $\pi_{X_C}(C')$. Then there is a neighbourhood of $x_C$ in $\ell$ which is contained in $C'$, hence $d(x_0, C') < d(x_0, x_C)$ and therefore $C' \dashv_{\mathrm{eu}} C$. Since $X \in \mathcal{J}(C)$, there exists a hyperplane $H \in \operatorname{supp}(X) \cap s(C, C')$. We also have that $F_C \subseteq C \cap C'$, and thus $X_C \subseteq H$.

Consider now the flat $X' = \cap\{Z \in \mathcal{L} \mid X_C \cup X \subseteq Z\}$, i.e. the meet of $X_C$ and $X$ in $\mathcal{L}$. The flat $X'$ is contained in the hyperplane $H$ constructed above, so in particular $X' \neq \mathbb{R}^n$. In addition, since $X \nsubseteq X_C$, $X'$ is different from $X_C$. Then the point $y_0 = \rho_{X'}(x_0)$ is different from $x_C$, and we have $d(x_0, y_0) < d(x_0, x_C)$, because $x_0$ is generic (see condition (ii) of Lemma 4.3.2). Let $F$ be the smallest face that contains the line segment $[x_C, x_C + \epsilon(y_0 - x_C)]$ for some $\epsilon > 0$. By construction, for every chamber $C''$ such that $C'' \preceq F$ we have that $C'' \dashv_{\mathrm{eu}} C$. This holds in particular for $C'' = C.F$. Then we have $\operatorname{supp}(F) \cap s(C, C'') = \emptyset$.

Since $X \in \mathcal{J}(C)$ and $C'' \dashv_{\mathrm{eu}} C$, there exists a hyperplane $H \in \operatorname{supp}(X) \cap s(C, C'')$. By construction, $x_C \in C \cap C''$ and then $X_C$ is contained in every hyperplane of $s(C, C'')$. In particular, $X_C \subseteq H$. Therefore $X_C \cup X \subseteq H$, which means that $H \in \operatorname{supp}(X_C \cup X) \subseteq \operatorname{supp}(X')$. Both $x_C$ and $y_0$ belong to $X'$, hence $F \subseteq X'$. Putting everything together, we get $H \in \operatorname{supp}(X') \cap s(C, C'') \subseteq \operatorname{supp}(F) \cap s(C, C'') = \emptyset$. This is a contradiction. □

*Remark* 4.3.6. For a given generic point $x_0$, there might be more than one Euclidean order $\dashv_{\mathrm{eu}}$ with base point $x_0$. Nonetheless, all Euclidean orders with a given base point produce the same faces $F_C$ (by Theorem 4.3.5) and the same critical cells (by Theorem 4.2.10). The decomposition

$$\mathbf{S}(\mathcal{A}) = \bigsqcup_{C \in \mathcal{C}} N(C)$$

also depends only on $x_0$ (by Lemma 4.2.1), and therefore the construction of the matching is not influenced by the choice of $\dashv_{\mathrm{eu}}$ (once the base point $x_0$ is given).

Since Euclidean orders are valid, we are able to construct an acyclic matching on the Salvetti complex of any arrangement. We also prove that this matching yields a minimal Morse complex.

**Theorem 4.3.7.** *Let $\mathcal{A}$ be a locally finite hyperplane arrangement in $\mathbb{R}^n$. Let $\mathcal{M}$ be the matching on $\boldsymbol{S}(\mathcal{A})$ given by Theorem 4.2.10, induced by a Euclidean order with base point $x_0$. Then the associated Morse complex $\boldsymbol{S}(\mathcal{A})_{\mathcal{M}}$ is minimal (all the incidence numbers vanish).*

*Proof.* If the arrangement $\mathcal{A}$ is finite, we know that the sum of the Betti numbers of $\mathbf{S}(\mathcal{A})$ is equal to the number of chambers [OS80, Zas97]. By Theorem 4.2.10, the critical cells of $\mathcal{M}$ are in bijection with the chambers. Then the Morse complex is minimal.

Suppose from now on that $\mathcal{A}$ is infinite. Fix a chamber $C \in \mathcal{C}$, and consider the associated critical cell $\langle C, F_C \rangle \in N(C)$. Recall from the proof of Theorem 4.2.10 the poset map $\eta \colon \mathbf{S}(\mathcal{A}) \to \mathcal{C} \times \mathcal{C}$, and let $(C, E) = \eta(\langle C, F_C \rangle)$. Since the matching is proper, the set $\eta^{-1}((\mathcal{C} \times \mathcal{C})_{\leq (C,E)})$ is finite.

Consider now the finite set of chambers

$$\begin{aligned}\mathcal{C}' = \{&D \in \mathcal{C} \mid D \cap F' \neq \emptyset \text{ for some cell } \langle C', F' \rangle \in \eta^{-1}((\mathcal{C} \times \mathcal{C})_{\leq (C,E)})\} \\ &\cup \{D \in \mathcal{C} \mid d(x_0, D) \leq d(x_0, C)\}.\end{aligned}$$

Notice that $C \in \mathcal{C}'$, and every chamber $D \dashv_{\text{eu}} C$ is also contained in $\mathcal{C}'$. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the finite subarrangement of $\mathcal{A}$ consisting of the hyperplanes that intersect at least one chamber in $\mathcal{C}'$. By construction, every chamber $D$ in $\mathcal{C}'$ is also a chamber of the arrangement $\mathcal{A}'$. The point $x_0$ is generic also with respect to $\mathcal{A}'$, thus it induces a Euclidean order $\dashv'_{\text{eu}}$ of the chambers of $\mathcal{A}'$. Choose $\dashv'_{\text{eu}}$ so that it coincides with $\dashv_{\text{eu}}$ until the chamber $C$.

The subcomplex $S = \eta^{-1}((\mathcal{C} \times \mathcal{C})_{\leq (C,E)})$ of $\mathbf{S}(\mathcal{A})$ can be also regarded as a subcomplex of $\mathbf{S}(\mathcal{A}')$, since for every cell $\langle C', F' \rangle \in S$ we have $C' \in \mathcal{C}(\mathcal{A}')$ and $F' \in \mathcal{F}(\mathcal{A}')$. In addition, the restriction of $\mathcal{M}$ to $S$ can be extended to a matching $\mathcal{M}'$ on $\mathbf{S}(\mathcal{A}')$ induced by $\dashv'_{\text{eu}}$ as in Theorem 4.2.10.

Consider now an $\mathcal{M}$-critical cell $\langle D, G \rangle \in \mathbf{S}(\mathcal{A})$ such that there is at least one alternating path from $\langle C, F \rangle$ to $\langle D, G \rangle$. Since $\mathcal{M}$ is homogeneous with respect to $\eta$, every alternating path starting from $\langle C, F \rangle$ is entirely contained in $S$. In particular, $\langle D, G \rangle \in S$. Then the alternating paths from $\langle C, F \rangle$ to $\langle D, G \rangle$ are the same in $\mathbf{S}(\mathcal{A})$ (with respect to the matching $\mathcal{M}$) and in $\mathbf{S}(\mathcal{A}')$ (with respect to the matching $\mathcal{M}'$). In particular, the incidence number between $\langle C, F \rangle$ and $\langle D, G \rangle$ is the same in the two Morse complexes. Since $\mathcal{A}'$ is finite, the Morse complex $\mathbf{S}(\mathcal{A}')_{\mathcal{M}'}$ is minimal and all incidence numbers vanish. Therefore the incidence number between $\langle C, F \rangle$ and $\langle D, G \rangle$ in $\mathbf{S}(\mathcal{A})_{\mathcal{M}}$ also vanishes. □

What we now want to prove is a particular version of the Brieskorn's Lemma, whose first proof can be found in [Bri73].

**Lemma 4.3.8** (Brieskorn)**.** *Let $\mathcal{A}$ be a nonempty complex arrangement and $<$ the Euclidean order with respect to a base point $x_0$. Let $L_k = \{X \in L(\mathcal{A}) \mid codim(X) = k\}$ and $<_X$ the Euclidean order with respect to $x_0$ in the subarrangement $\mathcal{A}_X$. Then there is a one-to-one correspondence between the critical $k-$cells of $\boldsymbol{S}(\mathcal{A})$ and the critical $k-$cells of $\boldsymbol{S}(\mathcal{A}_X)$ for each $X \in L_k$.*

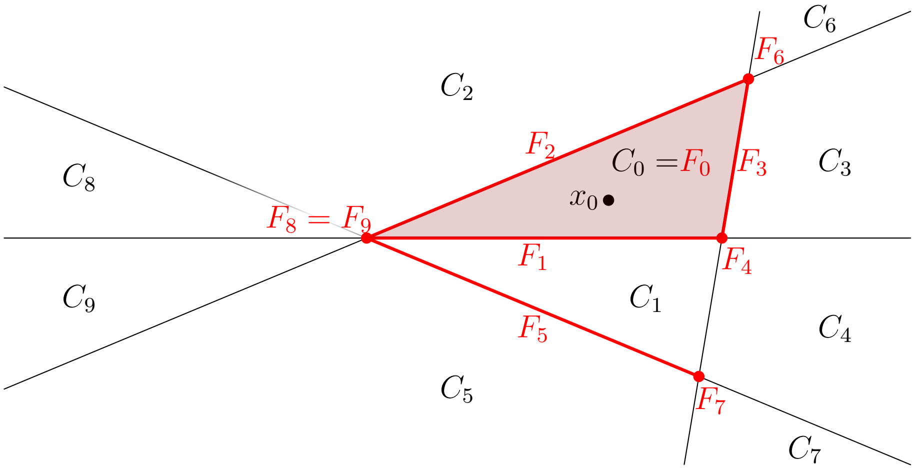



Figure 4.6: Euclidean order with respect to $x_0$

*Proof.* Let $\langle C, F_C\rangle$ be a critical $k-$cell of $\mathbf{S}(\mathcal{A})$, $X_C = |F_C|$. This means that $\text{codim}(X_C) = k$. We want to say that $\langle \pi_{X_C}(C), X_C\rangle$ is a critical $k-$cell of $\mathbf{S}(\mathcal{A}_{X_C})$. This is equivalent to say that, called $x_C$ the point of $C$ of minimum distance from $x_0$, $x_C$ is still the point of minimum distance of $\pi_{X_C}(C)$. This is clearly true because $x_C$ is the point of minimum distance of $X_C$ from $x_0$ otherwise $x_C$ would not have been in the interior of $X_C$.

We have then defined the map, we have now to prove that this map is injective and surjective.

- Injectivity

  Let $\langle C, F_C\rangle$ and $\langle C', F_{C'}\rangle$ two critical $k-$cell of $\mathbf{S}(\mathcal{A})$ with $|F_C| = |F_{C'}| = X_C$. We need to show that $\pi_{X_C}(C) \neq \pi_{X_C}(C')$. This is obvious because by how we have defined the Euclidean order $F_C = C' \cap X_C = F_{C'}$. Then the two chambers $C$ and $C'$ have a face in common that is in $X_C$, so at least one hyperplane of $\text{supp}(X_C)$ separate them.

- Surjectivity

  Let $X \in L_k$ and $\langle D, X\rangle$ a critical $k-$cell of $\mathbf{S}(\mathcal{A}_X)$. Let's call with $x_C$ the point of minimum distance of $X$ from $x_0$ and with $C$ the chamber in $\mathcal{F}(\mathcal{A})$ adjacent to $x_C$ and such that $\pi_X(C) = D$. $x_C$ is also clearly the point of minimum distance of $C$ from $x_0$ since it is for $D$ by hypothesis and this implies that $\langle C, X \cap C\rangle$ is a critical $k-$cell of $\mathbf{S}(\mathcal{A})$.

□

## 4.4 Local homology of line arrangement

The first step to study local homology is that of studying alternating path between critical cells. We will now focus only in the case of line arrangements. In the entire section we suppose that we have fixed a Euclidean order with base point $x_0$.

The alternating paths between a critical 1-cell and the only critical 0-cell are particularly easy, since all the zero cells are in $N(C_0)$ and it is immediate to see that there are always the paths.

We want to see now that there is a correspondence between alternating paths from critical 2-cells to critical 1-cells and a special kind of sequences in $\mathcal{F}_1(\mathcal{A})$, which we call *alternating sequences*, where $F \in \mathcal{F}_1(\mathcal{A})$ if and only if $|F| \in \mathcal{L}_1(\mathcal{A})$.

First of all we notice that, given an alternating path between critical cells of the form

$$\langle D, p\rangle \searrow \langle C_1, F_1\rangle \nearrow \langle D_1, p_1\rangle \searrow \langle C_2, F_2\rangle \cdots \searrow \langle C_n, F_n\rangle, \tag{4.4.1}$$

the starting cell plus the sequence $(F_1, \ldots, F_n)$ completely determines the alternating path. This is because for each $i$ there are only two cells with $F_i$ and one of them is in $N(C_0)$. By construction of the matching, if this cell is in an alternating path then all the following cells in the path are in $N(C_0)$ and so the last one cannot be a critical 1-cell. Then $C_i$ is uniquely determined by $F_i$ for every $i$. Each cell $\langle D_i, p_i\rangle$ is also uniquely determined, since it is matched with $\langle C_i, F_i\rangle$.

We now describe which alternating sequences in $\mathcal{F}_1(\mathcal{A})$ give rise to an alternating path. Given a face $F \in \mathcal{F}_1(\mathcal{A})$, let $H$ be the unique line in $\operatorname{supp}(F)$, if $\rho_H(x_0)$ is not in the interior of $F$ we denote by $p(F)$ the endpoint of $F$ which is closer to $\rho_H(x_0)$, otherwise $p(F) = \emptyset$.

In addition, let $C(F)$ be the unique chamber such that $\langle C(F), F\rangle \notin N(C_0)$.

**Definition 4.4.1.** Given two different faces $F$, $G \in \mathcal{F}_1(\mathcal{A})$, we say that $F \to G$ if

- $F \cap G = p(F)$ and $p(F) \neq \emptyset$;
- Let $H_F = \operatorname{supp}(F)$ and $H_G = \operatorname{supp}(G)$, then $H_F = H_G$ or $F$ and $C_0$ are in the same half-plane with respect to $H_G$.

**Lemma 4.4.2.** *The alternating paths between $\langle D, p\rangle$ and $\langle C, F\rangle$ are in one to one correspondence with the alternating sequences in $\mathcal{F}_1(\mathcal{A})$ of the form $(F_1 \to F_2 \ldots \to F_n = F)$ such that $\langle C(F_1), F_1\rangle < \langle D, p\rangle$.*

*Proof.* We have already said that an alternating path as in 4.4.1 is completely determined by the starting cell and the sequence $(F_1, \ldots, F_n)$. Let us now see that for each $i \in [1, n-1]$ $F_i \to F_{i+1}$ since the condition that $\langle C(F_1), F_1\rangle < \langle D, p\rangle$ is obvious.

Let now $E_i$ be the chamber opposite to $C(F_i)$ with respect to $F_i$, then from how we have construct the matching it is immediate to see that the cell $\langle C(F_i), F_i\rangle$ is matched with $\langle D(F_i), p(F_i)\rangle$ where $D(F_i)$ is the chamber opposite to $E_i$ with respect to $p(F_i)$. By hypothesis $\langle C(F_{i+1}), F_{i+1}\rangle < \langle D(F_i), p(F_i)\rangle$ which implies that $F_i \cap F_{i+1} = p(F_i)$ and that $D(F_i).F_{i+1} = C(F_{i+1})$. If we call with $H_{i+1} = \text{supp}(F_{i+1})$ since $\langle C(F_{i+1}), F_{i+1}\rangle \notin N(C_0)$ $C_0$ and $C(F_{i+1})$ are in opposite semispace with respect to $H_{i+1}$ but the same is true for $F_i$ and $C(F_{i+1})$ because $D(F_i)$ and $F_i$ are in opposite semispace with respect to $H_{i+1}$ (unless of course $F_i \subset H_{i+1}$). Then we have that $F_i \to F_{i+1}$.

We need now to prove that given a sequence as in the hypothesis there exist an alternating paths between $\langle D, p\rangle$ and $\langle C, F\rangle$. This follows easily from what we have said above and a simple induction on the length of the sequence.

The case $n = 1$ is easy since we have by hypothesis that $\langle C(F_1), F_1\rangle < \langle D, p\rangle$. For the induction step we need only to prove that if we have $F, G \in \mathcal{F}_1(\mathcal{A})$, $F \to G$ then $\langle C(G), G\rangle < \langle D(F), p(F)\rangle$. From the first conditions of Definition 4.4.1 we have that $G \prec p(F)$. We need to see that $D(F).G = C(G)$. This is equivalent to prove, by definition of $C(G)$, that $D(F)$ and $C_0$ are in opposite semispace with respect to $H_G$ that again follows from the fact that $F$ and $C_0$ are in the same semispace with respect to $H_G$ and the definition of $D(F)$. □

Now that we have a simple description of the alternating paths, we want to use it to study the boundary of the Morse complex.

**Definition 4.4.3.** Given two different faces $F$, $G \in \mathcal{F}_1(\mathcal{A})$, with $F \to G$, let

$$[F \to G] = \frac{[\langle D(F), p(F)\rangle : \langle C(G), G\rangle]}{[\langle D(F), p(F)\rangle : \langle C(F), F\rangle]},$$

where the incidence numbers on the right are taken in the Salvetti complex $\mathbf{S}(\mathcal{A})$, and $D(F)$ is defined as in the above lemma.

**Theorem 4.4.4.** *Let $\mathcal{A}$ be a line arrangement in $\mathbb{R}^2$, and let $\langle D, p\rangle$ and $\langle C, F\rangle$ be two critical cells. Then their incidence number in the Morse complex is given by*

$$[\langle D, p\rangle : \langle C, F\rangle] = \sum_{s \in \text{Seq}} \omega(s)$$

*where* Seq *is the set of sequence of Lemma 4.4.2 and for each sequence of the form*

$$s = F_1 \to F_2 \to \cdots \to F_n = F$$

*then*

$$\omega(s) = (-1)^n [\langle D, p\rangle : \langle C(F_1), F_1\rangle] \prod_{i=1}^{n-1} [F_i \to F_{i+1}]$$

*Proof.* It follows directly from [Koz08, Definition 11.23] and Lemma 4.4.2. □

**Lemma 4.4.5.** *Given two different faces $F$, $G \in \mathcal{F}_1(\mathcal{A})$, we have*

$$[F \to G] = \begin{cases} \pm 1 & \text{if } p(F) = p(G) \\ \pm \prod t_{H_i}^{-1} & \text{otherwise,} \end{cases}$$

*where $t_{H_i}$ is a positive oriented loop around the line $H_i$ and the product is over the lines through $p(F)$ that separate $F$ and $C_0$.*

*Proof.* We need here some definitions given in [SS07, Chapter 5] about combinatorial and positive paths in the $1-$skeleton of $\mathbf{S}(\mathcal{A})$. By definition of local homology, given two cells $\langle D, p\rangle, \langle C, F\rangle$ we have that

$$[\langle D, p\rangle : \langle C, F\rangle] = [\langle D, p\rangle : \langle C, F\rangle]_{\mathbb{Z}} \bar{u}(D, C)$$

as an element of $\pi_1(\mathcal{M}(\mathcal{A}), x_0)$ where $\bar{u}(D, C) = \Gamma(D)^{-1}u(D, C)\Gamma(C)$ and $\Gamma(C) = u(C, C_0)$ and $u(\cdot, \cdot)$ is a positive path between the first and the second chamber. If we restrict ourselves to the case of abelian local homology, then as an element of $H_1(\mathcal{M}(\mathcal{A}))$ $\bar{u}(D, C)$ is equal to the product of the positive loops around the hyperplanes in $s(C_0, C) \cap s(D, C)$.

Then in our special case we need to study the relation between $\bar{u}(D(F), C(F))$ and $\bar{u}(D(F), C(G))$ that is between $s(C_0, C(F)) \cap s(D(F), C(F))$ and $s(C_0, C(G)) \cap s(D(F), C(G))$. By construction $s(D(F), C(F))$ is equal to the set of hyperplanes through $p(F)$ minus $H_F = \text{supp}(F)$.

Let us now suppose that $p(G) = p(F)$ and let $H \in s(C(F), C(G))$. By hypothesis $H$ does not separate $F$ and $C_0$ (because $H_G$ does not), so $H \notin s(C_0, C(F))$ and $s(C_0, C(G)) = s(C_0, C(F)) \sqcup s(C(F), C(G))$. Moreover, since all the chambers $C(F), C(G)$ and $D(G)$ are adjacent to $p(F)$ then

$$s(D(F), C(G)) = s(D(F), C(F)) \setminus s(C(G), C(F))$$

because $H_F \notin s(D(F), C(G))$. Then

$$s(C_0, C(F)) \cap s(D(F), C(F)) = s(C_0, C(G)) \cap s(D(F), C(G)).$$

Let us now instead suppose that $p(F) \neq p(G)$. We want to prove that $s(C_0, C(G)) \cap s(D(F), C(G)) = \emptyset$ and this will imply the thesis. Let then $H \in s(C_0, C(G)) \cap s(D(F), C(G))$. Since both $D(F), C(G)$ are adjacent to $p(F)$ then $p(F) \in H$. Given $H_G = \text{supp}(G)$, then there exist $G' \subset H_G$ and $p(G') = p(F)$. $s(C(G), C(G'))$ is the set of all hyperplanes trough $p(F)$ minus $H_G$. If $H \neq H_G$ then $H \notin s(C_0, C(G')) \cup s(D(F), C(G')) = s(D(F), C(F)) \cup s(C_0, C(F))$ so all the hyperplanes adjacent to $p(F)$, giving a contradiction. At the same time $H$ cannot be equal to $H_G$ because it does not separate $D(F)$ and $C(G)$ since $D(F).G = C(G)$. □

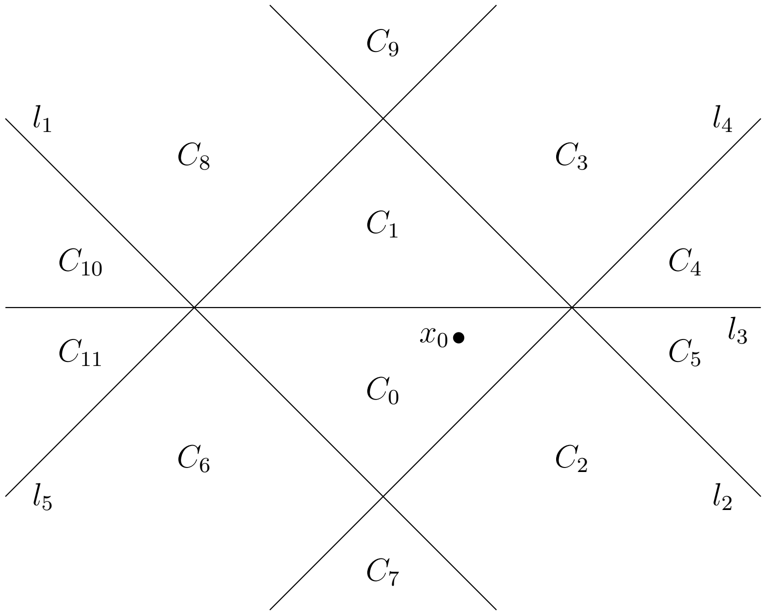



Figure 4.7: Deconing $A_3$

| | $\langle C_4, p_4\rangle$ | $\langle C_5, p_5\rangle$ | $\langle C_7, p_7\rangle$ | $\langle C_9, p_9\rangle$ | $\langle C_{10}, p_{10}\rangle$ | $\langle C_{11}, p_{11}\rangle$ |
|---|---|---|---|---|---|---|
| $\langle C_1, F_1\rangle$ | $q_4 - 1$ | $q_4(1-q_2)$ | 0 | 0 | $q_1 - 1$ | $q_1(1-q_5)$ |
| $\langle C_2, F_2\rangle$ | $q_2q_3 - 1$ | $q_2 - 1$ | $1 - q_1$ | 0 | 0 | 0 |
| $\langle C_3, F_3\rangle$ | $q_4 - 1$ | $q_4 - q_3^{-1}$ | 0 | $1 - q_5$ | 0 | 0 |
| $\langle C_6, F_6\rangle$ | 0 | 0 | $q_4 - 1$ | 0 | $q_3q_5 - 1$ | $q_5 - 1$ |
| $\langle C_8, F_8\rangle$ | 0 | 0 | 0 | $q_2 - 1$ | $q_1 - 1$ | $q_1 - q_3^{-1}$ |

Table 4.1: The boundary $\partial_2$ of the deconing of $\mathcal{A}_3$

**Example 4.4.6** (Deconing $A_3$)**.** In the following example we explicitly compute the matrix associated to $\partial_2$ for the Morse complex of the arrangement in Figure 4.7. Given a chamber $C_i$ we denote by $\langle C_i, F_i\rangle$ the associated critical cell if it is of dimension 1 or $\langle C_i, p_i\rangle$ if it is of dimension 2. ($\langle C_0, C_0\rangle$ is the only critical $0-$cell.

Since in our computations there are only negative oriented loops we will denote with $q_i = t_i^{-1}$ the negative oriented loop around the line $l_i$.

Following theorem 4.4.4 and Lemma 4.4.5 we obtain the matrix 4.1.

Specializing to the case where $q_1 = \ldots = q_5 = q$ we obtain that

$$H_1(\mathcal{M}(\mathcal{A}), \mathbb{Q}[q^{\pm 1}]) \cong \left(\frac{\mathbb{Q}[q^{\pm 1}]}{q-1}\right)^3 \oplus \frac{\mathbb{Q}[q^{\pm 1}]}{q^3 - 1}$$

as already computed for example in [GS09].

# Aknowledgements

La mia avventura quinquennale a Pisa è giunta al termine e questa tesi ne è l'epilogo. Sono stati cinque anni molto densi, in alcuni momenti difficili, ma in molti altri fantastici. Ne esco con un bagaglio non indifferente non solo di nozioni matematiche ma anche di lezioni di vita e per questo devo ringraziare tutti coloro con cui ho condiviso anche solo un attimo di questa avventura.

In particolare ringrazio Giove senza il quale questa tesi sarebbe molto diversa. Lo ringrazio per non avermi mai fatto mancare il suo aiuto e le sue conoscenze e per le molte decine di ore che abbiamo passato insieme a discutere. Ringrazio inoltre il mio relatore, Mario Salvetti, per i molti consigli e tutto il tempo che mi ha dedicato.

Ringrazio tutti i miei amici, di Pisa e non, per tutti i momenti di svago. In particolare ringrazio i miei compagni di corso in Normale con i quali ho condiviso la maggior parte del mio tempo; coloro con cui ho condiviso il collegio e con cui sono andato a mensa insieme, Alice, Benzo, Camilla, Claudio, Fabio e Tess rendendo il viaggio meno lungo; coloro con cui ho giocato a calcetto in queste calde serate pisane e coloro con cui ho giocato ad Hanabi, in particolare Sasha per tutte le ottime cene a casa sua.

Ringrazio la mia dolce metà, Eli, per essermi sempre stata vicina. Per essere sempre stata pronta ad ascoltare tutti i miei dubbi e le mie paure ed avermi sempre incoraggiato a non farmi abbattere dalle difficoltà.

E per ultima, ma certamente non per ultima, ringrazio la mia famiglia per non avermi mai fatto mancare il loro supporto qualunque fossero le mie scelte e per avermi sempre dimostrato il loro affetto in ogni modo possibile.

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