Tutorial on hyperbolic surfaces in `polymake`

February 20, 2018

A *hyperbolic surface with cusps* is a topological surface together with a hyperbolic structure of finite area. `polymake` can deal with hyperbolic surfaces in view of Penners coordinates of the decorated Teichmüller space (lambda lengths). These allow to pick a hyperbolic surface by choosing a triangulation of the surface along with one positive parameter for each edge.

The secondary fan of a hyperbolic surface stratifies the space of weight vectors (horocyclic decorations) according to which Delaunay triangulations are induced by the Epstein-Penner convex hull construction. For each point on the surface, there is a secondary polyhedron whose normal fan is the secondary fan.

This tutorial shows how to deal with secondary fans and secondary polyhedra of hyperbolic surfaces.

Construction of hyperbolic surfaces

To define a hyperbolic surface we need to specify (a) a triangulation and (b) Penner coordinates.

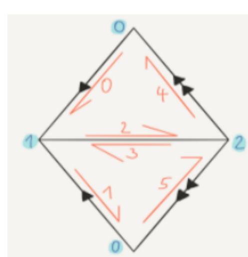
(a) The triangulation is obtained by specifying the `DCEL_DATA` as an `Array<Array<Int>>`. This constructs a doubly connected edge list as follows: Each row of `DCEL_DATA` reads { (*2i*).head , (*2i+1*).head , (*2i*).next , (*2i+1*).next }. In general, for each edge *i* of the triangulation there are two half edges *2i* and *2i+1*, one for each orientation.

(b) The `PENNER_COORDINATES` assign a positive rational number to each edge of the triangulation, ordered in the same sense as prescribed by the `DCEL_DATA`.

Example 1: hyperbolic sphere with three cusps

```
In [1]: application 'fan'; application 'topaz';
```

Let us initialize the following triangulation of a sphere with three punctures. We use this triangulation to obtain a hyperbolic surface by setting all lambda lengths to 1.



```
In [2]: $S3 = new Array<Array<Int>>([[1,0,2,5],[2,1,4,1],[0,2,0,3]]);
        $s = new HyperbolicSurface(DCEL_DATA=>$S3,PENNER_COORDINATES=>[1,1,1]);
```

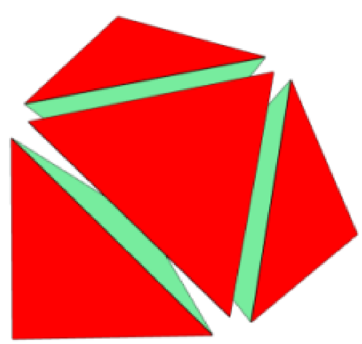
The secondary fan

The secondary fan of the hyperbolic sphere from above can now be computed as follows.

```
In [3]: $f = $s->SECONDARY_FAN;
        $f->properties;
```

```
Out[3]: type: PolyhedralFan<Rational>

        RAYS
        0 1 1
        1 0 1
        1 1 0
        0 0 1
        1 0 0
        0 1 0


        MAXIMAL_CONES
        {0 1 2}
        {0 1 3}
        {1 2 4}
        {0 2 5}
```

```
In [4]: $f->VISUAL;
```



```
In [5]: $s->properties;
```

```
Out[5]: name: s
        type: HyperbolicSurface

        DCEL_DATA
        1 0 2 5
        2 1 4 1
        0 2 0 3
```

```
PENNER_COORDINATES
1 1 1

SECONDARY_FAN
type: PolyhedralFan<Rational>

FLIP_WORDS
{}
{0}
{1}
{2}
```

The FLIP_WORDS indicate how to obtain the Delaunay triangulations. The k-th flip word is a list of integers (the indices of the edges) that describe which edge flips produce the k-th Delaunay triangulation. Note that the k-th Delaunay triangulation also corresponds to the k-th maximal cone of the SECONDARY_FAN.

GKZ vectors & secondary polyhedra

In order to compute GKZ_VECTORS or a secondary_polyhedron of a hyperbolic surface one needs to additionally specify a SPECIAL_POINT on the surface. This is done by choosing two rational numbers.

Continuing with the above example, lets look at the following.

```
In [6]: $s = new HyperbolicSurface(DCEL_DATA=>$S3,
                                  PENNER_COORDINATES=>[1,1,1],
                                  SPECIAL_POINT=>[1,0]);
```

Now we may compute an approximation of the GKZ_VECTORS of the surface. The approximation depends on a parameter *depth* that restricts the depth of the (covering) triangles that are summed over in the definition of the GKZ vectors.

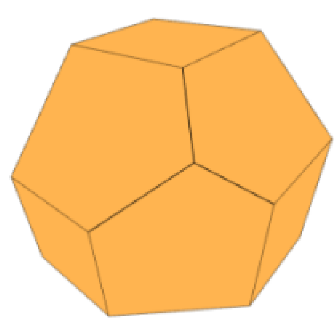
```
In [7]: print $s->GKZ_VECTORS(3);

Out[7]: 1 33346854621/25672050625 33346854621/25672050625 19782163/27238250
        1 2361/3250 3955357/5447650 33346854621/25672050625
        1 10549213550005124385885122/6365327663846199230365625 11433978/13287625
        30327974429709/105771923977850
        1 11433978/13287625 10549213550005124385885122/6365327663846199230365625
        30327974429709/105771923977850
```

The secondary polyhedron can be computed similarly using the function `secondary_polyhedron`.

In [8]:
```
$p = secondary_polyhedron($s,10);
$p->properties;
```

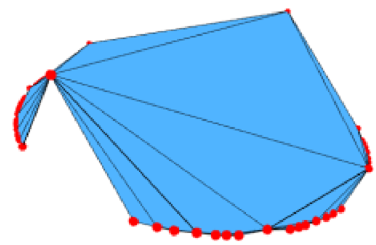
Out[8]:
```
name: p
type: Polytope<Float>

VERTICES
1 1.315301353 1.315301353 0.7316378744
1 0.7316489581 0.7316267908 1.315301353
1 1.752046187 0.8750928112 0.2910011302
1 0.8750928112 1.752046187 0.2910011302
0 -1 0 0
0 0 -1 0
0 0 0 -1


VERTICES_IN_FACETS
{0 1 3 4}
{0 1 2 5}
{0 2 3 6}
{1 4 5}
{2 5 6}
{3 4 6}


CONE_AMBIENT_DIM
4
```

In [9]:
```
$p->VISUAL(FacetColor=>'255 180 80');
```

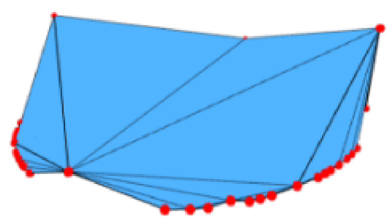


We may look at the GKZ domes of the individual Delaunay triangulations.

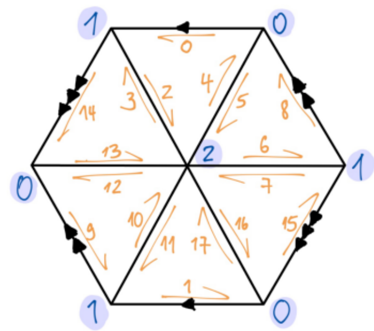
```
In [10]: $d0 = $s->gkz_dome(0,5);
         $d0->VISUAL(FacetColor=>'80 180 255');
```



```
In [11]: $d1 = $s->gkz_dome(1,5);
         $d1->VISUAL(FacetColor=>'80 180 255');
```



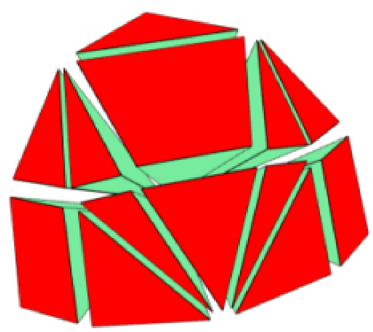
Example 2: a hyperbolic torus with three cusps



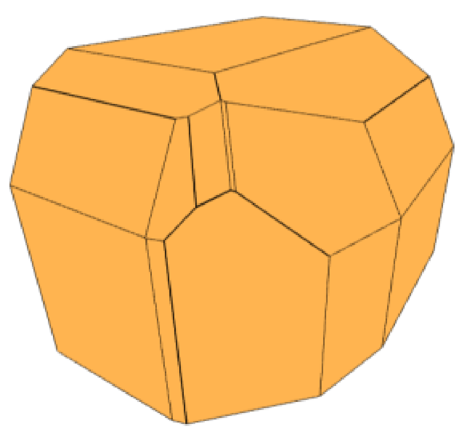

```
In [12]: $T3 = new Array<Array<Int>>([[1,0,2,17],[2,1,4,14],[0,2,0,6],[1,2,8,16],
                           [0,1,5,10],[2,1,12,1],[0,2,9,3],[0,1,13,7],[0,2,15,11]]);
         $s = new HyperbolicSurface(DCEL_DATA=>$T3,
                                    PENNER_COORDINATES=>[2,1,1,1,1,1,1,1,1],
                                    SPECIAL_POINT=>[1,0]);

In [14]: $f = $s->SECONDARY_FAN;
         $s->properties;

Out[14]: name: s
         type: HyperbolicSurface

         DCEL_DATA
         1 0 2 17
         2 1 4 14
         0 2 0 6
         1 2 8 16
         0 1 5 10
         2 1 12 1
         0 2 9 3
         0 1 13 7
         0 2 15 11


         PENNER_COORDINATES
         2 1 1 1 1 1 1 1 1

         SPECIAL_POINT
         1 0

         SECONDARY_FAN
         type: PolyhedralFan<Rational>

         FLIP_WORDS
         {0}
```

```
{}
{0 3}
{0 4 7}
{0 6}
{3}
{6}
{0 3 1 5}
{0 6 2 8}
{3 1 5}
{6 2 8}
{0 3 1 5 0 1}
{0 6 2 8 0 2}
```

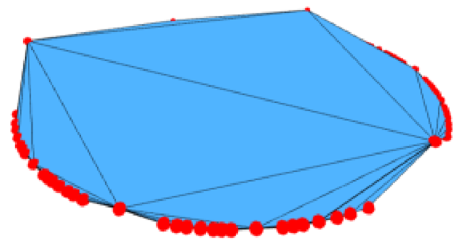
```
In [13]: $f->VISUAL;
```



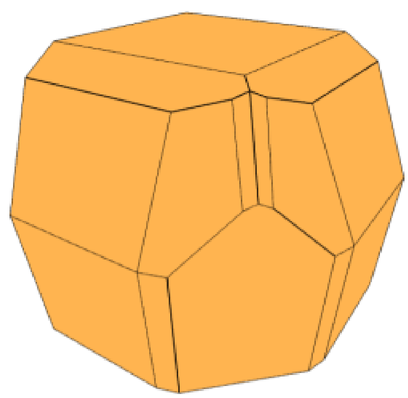
```
In [15]: $p = secondary_polyhedron($s,7);
         $p->VISUAL(FacetColor=>'255 180 80');
```



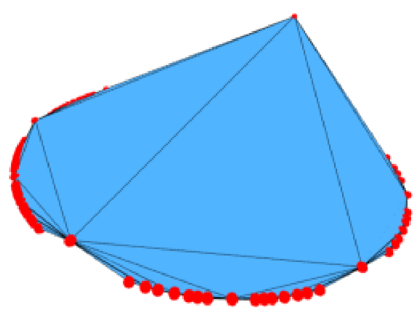
```
In [16]: $d0 = $s->gkz_dome(0,5);
         $d0->VISUAL(FacetColor=>'80 180 255');
```



```
In [17]: $s = new HyperbolicSurface(DCEL_DATA=>$T3, PENNER_COORDINATES=>[2,1,1,1,1,1,1,1,1],
                SPECIAL_POINT=>[new Rational(1.5196714),new Rational(-0.5773503)]);
         $p = secondary_polyhedron($s,7);
         $p->VISUAL(FacetColor=>'255 180 80');
```



```
In [18]: $d0 = $s->gkz_dome(0,5);
         $d0->VISUAL(FacetColor=>'80 180 255');
```



More examples can be studied via the following:

```
In [19]: # a torus with two cusps (6 edges)
         $T2 = new Array<Array<Int>>([[0,0,6,5],[0,0,1,10],[0,0,8,2],[1,0,11,4],
          [1,0,7,3],[1,0,9,0]]);

         # a sphere with four cusps (6 edges)
         $S4 = new Array<Array<Int>>([[1,0,2,6],[2,1,4,9],[0,2,0,11],[3,0,8,5],
          [1,3,1,10],[2,3,3,7]]);

         # a double torus with two cusps (12 edges)
         $DT2 = new Array<Array<Int>>([[0,0,8,10],[0,0,12,14],[0,0,16,18],[0,0,20,22],
          [1,0,23,2],[1,0,13,3],[1,0,9,1],[1,0,11,4],[1,0,15,6],[1,0,21,7],[1,0,17,5],
          [1,0,19,0]]);
```

To study 4-dim. secondary fans the following method is useful. It intersects the secondary fan with the 3-dim. standard simplex.

```
In [20]: sub norm($){
            my $B = new Matrix(shift);
            for (my $i = 0; $i < $B->rows(); ++$i) {
               my $sum = 0;
               for (my $j = 1; $j < $B->cols(); ++$j) {
                  $sum = $sum + $B->elem($i,$j);
               }
               $x = 1/$sum;
               $B->row($i) = $x * $B->row($i);
            }
            return $B;
         }

In [25]: $s = new HyperbolicSurface(DCEL_DATA=>$S4,PENNER_COORDINATES=>[1,1,1,1,1,1],
                                   SPECIAL_POINT=>[1,0]);
         $f = $s->SECONDARY_FAN;
         $v = ones_vector | $f->RAYS;
         $a = norm($v);
         $b = $a->minor(All,~[0]);
         $c = ones_vector | $b;

In [24]: $q = new fan::PolyhedralComplex(POINTS=>$c,INPUT_POLYTOPES=>rows($f->MAXIMAL_CONES));
         $pro = fan::project_full($q);
         $pro->VISUAL(FacetColor=>'255 180 80');
```