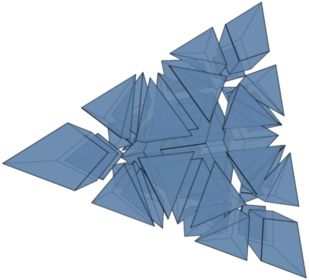
Triangulations, Discriminants, and Teichmüller Theory

Robert Löwe

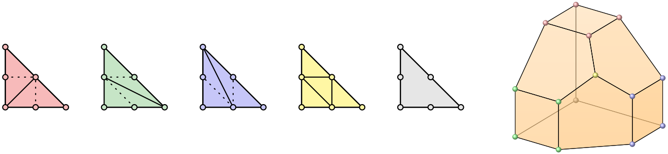
Technische Universität Berlin



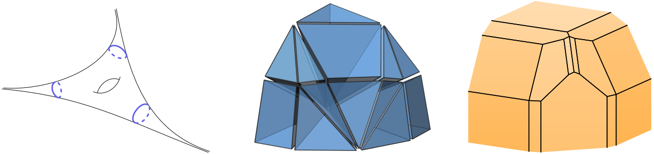
Overview

Use polyhedral geometry to study:

1. fundamental questions in algebraic geometry
   - regular triangulations and discriminants



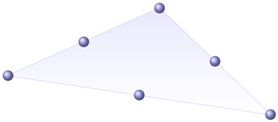
2. decorated Teichmüller space
   - secondary fans and secondary polyhedra of punctured Riemann surfaces



Regular Subdivisions of Point Configurations

A **regular** subdivision of a point configuration $A \subset \mathbb{R}^d$ is obtained by:

1. lifting points by weight function $\omega \in \mathbb{R}^A$,
2. projecting upper faces.



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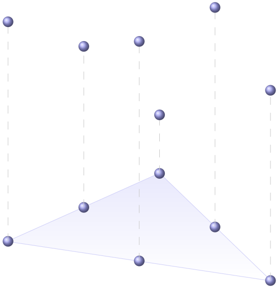

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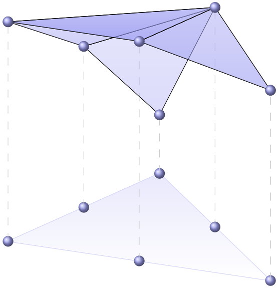

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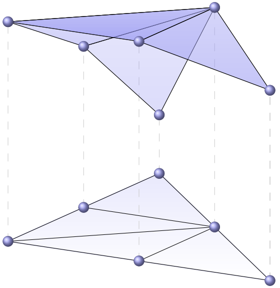

Secondary Fan of a Point Configuration

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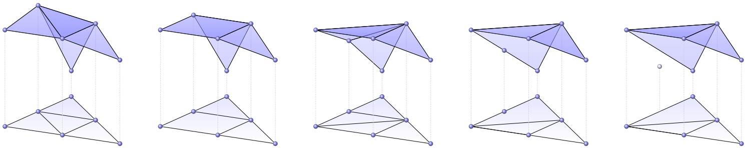

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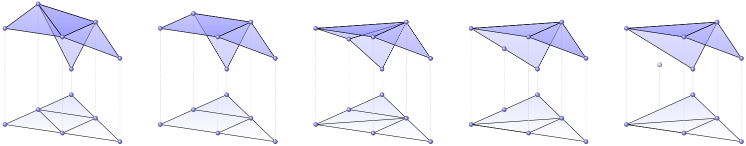
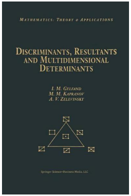


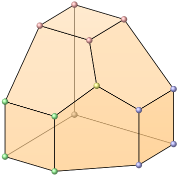
The **secondary fan** of A stratifies the weight space $\mathbb{R}^A$ according to the induced regular subdivisions.

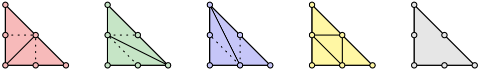
Secondary Polytope of a Point Configuration



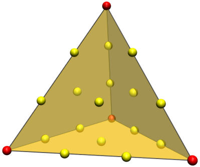
Theorem (Gelfand, Kapranov, Zelevinsky)

Let $A \subset \mathbb{R}^d$ be a point configuration. There is a convex polytope, the **secondary polytope** *of A, whose normal fan is the secondary fan of A.*





Towards the Discriminant of a Quaternary Cubic

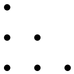


Theorem (Kastner, L 2019)

The point configuration $A = \{z \in \mathbb{Z}^4_{\geq 0} \mid z_1 + z_2 + z_3 + z_4 = 3\}$ *admits* 910 974 879 *regular triangulations up to* S_4*-symmetry.*

The A-discriminant

$$f \;=\; a_{00}+a_{01}\cdot y+a_{02}\cdot y^2+a_{10}\cdot x+a_{11}\cdot xy+a_{20}\cdot x^2 \qquad A=\operatorname{supp}(f)=$$



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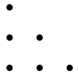
$A = \mathrm{supp}(f) =$ 

$$\boxed{V(f) \text{ singular hypersurface } \Leftrightarrow D_A = 0}$$

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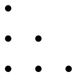

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$$D_A \;=\; a_{00}{a_{11}}^2 + {a_{01}}^2 a_{20} + a_{02}{a_{10}}^2 - a_{01}a_{10}a_{11} - 4a_{00}a_{02}a_{20}$$

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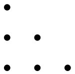
$A = \mathrm{supp}(f) =$ 

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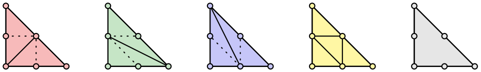

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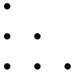

$V(f)$ singular hypersurface $\Leftrightarrow D_A = 0$

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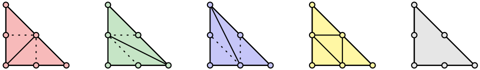
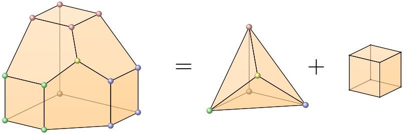
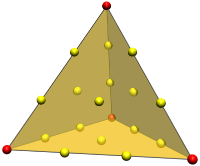


Theorem (Gelfand, Kapranov, Zelevinsky)

The Newton polytope of the A-discriminant is a Minkowski summand of the secondary polytope of A.



Discriminant of a Quaternary Cubic



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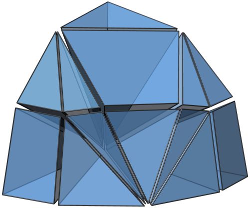
Theorem (Kastner, L 2019)

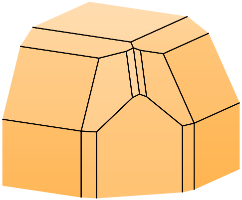
The Newton polytope of the A*-discriminant has* 166 104 *vertices that split into* 7 132 S_4*-orbits. Equivalently,* A *admits* 166 104 D*-equivalence classes of regular triangulations.*

Secondary Polyhedra of Punctured Riemann Surfaces

Theorem (Joswig, L, Springborn 2019)

*For each punctured hyperbolic Riemann surface its **secondary fan** is the normal fan of an unbounded **secondary polyhedron**.*



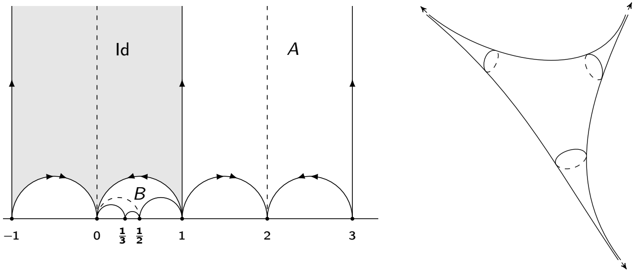


Hyperbolic Surfaces with Cusps

- $\mathbb{H}$ hyperbolic plane
- $\Gamma < \mathrm{Isom}^+(\mathbb{H})$ discrete, non-cocompact subgroup
- $\mathscr{R} = \mathbb{H}/\Gamma$ hyperbolic surface with cusps

Example (Sphere with three cusps)

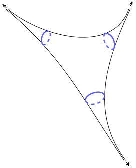
$$\Gamma = \langle A, B \rangle < \mathrm{PSL}_2(\mathbb{R}) \quad \text{where} \quad A(z) = z + 2, \quad B(z) = \frac{z}{2z+1}$$

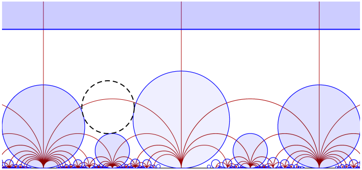


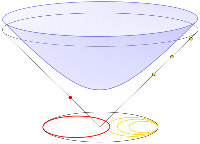
Epstein–Penner Convex Hull Construction

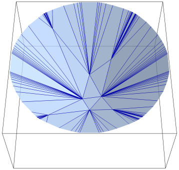
Fix surface $\mathscr{R} = \mathbb{H}/\Gamma$:

1. Decorate each cusp with a horocycle.
2. Take convex hull of corresponding Γ-orbits in positive light-cone.
3. Project facets to $\mathscr{R}$ to obtain subdivision into ideal polygons.









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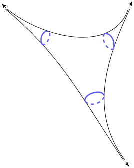
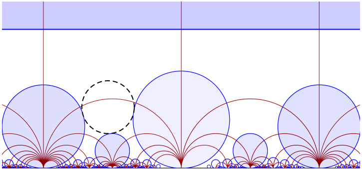
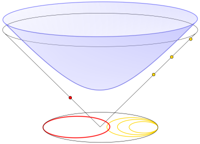
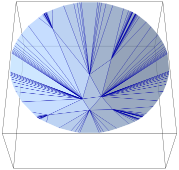


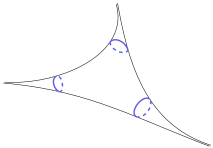


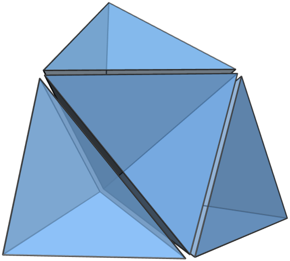




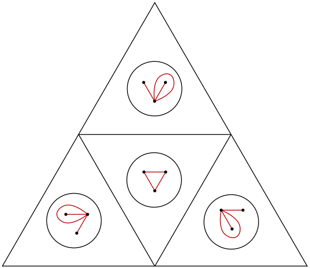
- Parametrize decorating horocycles by their lengths $\omega \in \mathbb{R}^n_{>0}$.
- **Secondary fan** of $\mathscr{R}$ stratifies weight space $\mathbb{R}^n_{>0}$ according to induced ideal subdivision of $\mathscr{R}$.

Example: Thrice Punctured Sphere



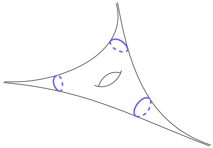


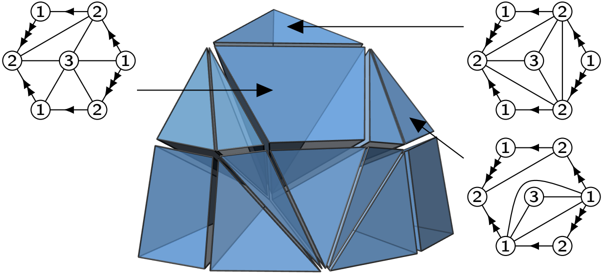
The secondary fan ...



and the corresponding triangulations of the sphere.

Example: Thrice Punctured Torus

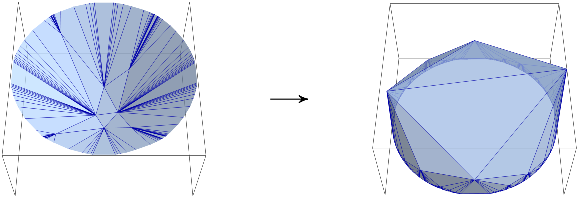




Secondary Polyhedra of Punctured Riemann Surfaces

Theorem (Joswig, L, Springborn 2019)
For each $x \in \mathscr{R}$, there is an unbounded convex polyhedron $\Sigma\text{-poly}(\mathscr{R}, x)$, the ***secondary polyhedron****, whose normal fan is the secondary fan.*

- key idea: switch to hemisphere model and define analogs of GKZ vectors



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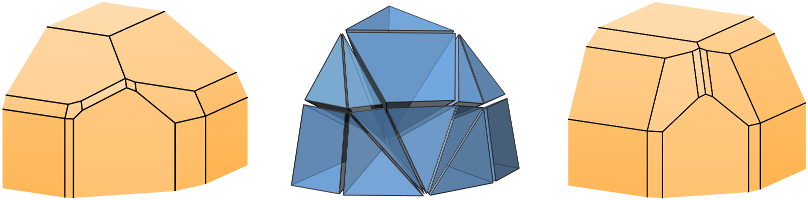
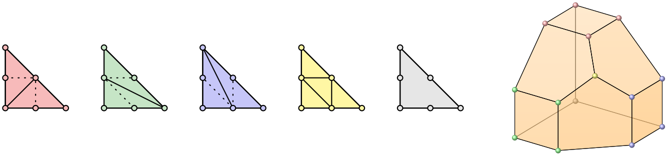
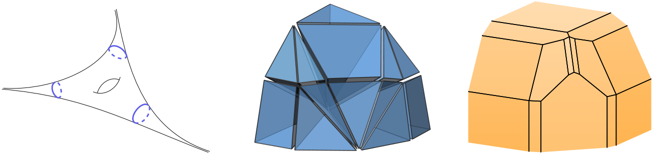


Summary

1. regular triangulations and secondary polytopes of point configurations
   - study cubic surfaces



2. secondary fans and secondary polyhedra of punctured Riemann surfaces
   - study decorated Teichmüller space



Future Research

- direct computation of D-equivalence classes
- Viros patchworking
- dependence of secondary polyhedron Σ-poly$(\mathscr{R}, x)$ on choice of $x \in \mathscr{R}$, canonical secondary polyhedron
- connections to surface cluster algebras
- hyperbolic manifolds
- secondary fans and secondary polyhedra for convex projective structures