On Triangles in Colored Pseudoline Arrangements

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Abstract

We consider the faces in pseudoline arrangements in which the pseudolines are colored with two colors. Björner, Las Vergnas, Sturmfels, White, and Ziegler conjecture the existence of a two-colored triangle in such arrangements. We consider variants of this problem. We show that in any non-trivial two-coloring of a pseudoline arrangement there exists a two-colored triangle or quadrangle. We also investigate the existence of a bichromatic triangle assuming certain structures on the coloring.

We turn our attention to the hypergraph whose vertices correspond to the pseudolines of an arrangement and its hyperedges to the triangular faces. Previously, several authors investigated the chromatic number and independence number of hypergraphs whose vertices correspond to the pseudolines of an arrangement and the hyperedges correspond to the faces of any size of the arrangement. We prove that the maximum of the independence numbers of the line-triangle hypergraphs is $n - \Theta(\log n)$.

1 Introduction

An *Euclidean pseudoline arrangement* is a finite collection of bi-infinite, simple curves called *pseudolines* in the Euclidean plane, such that they pairwise cross in exactly one point, which we will call *crossing*. If they exist, we call pseudolines with no crossings on one side *extremal*. An arrangement is *simple*, if no three pseudolines intersect in a common point. We only consider simple arrangements.

A pseudoline arrangement gives rise to a collection of *vertices*, *edges* and *faces*, see Figure 1. We call a bounded face a *triangle* resp. *quadrangle*, if it is supported by exactly three resp. four pseudolines. Sometimes we consider the area between three pseudolines and call it a *non-empty triangle*.

The minimum and maximum number of triangles in pseudoline arrangements are known[8, 6]. Other questions about triangles remain open. We will consider arrangements whose pseudolines are colored blue and red so that at least one pseudoline of each color exists. We call such arrangements *bicolored*. In 1993, Björner, Las Vergnas, Sturmfels, White, and Ziegler asked about the existence of bichromatic triangles in bicolored arrangements.

▶ **Conjecture 1.1** ([5, p. 280]). *Every bicolored arrangement has a bichromatic triangle.*

Another way to look at this problem is to consider the *triangle-pseudoline incidence graph*. As its vertices, take the triangles and pseudolines and add an edge between a pseudoline and a triangle if the pseudoline supports the triangle. It is intuitive to ask about the connectivity of this graph.

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▶ **Conjecture 1.2** ([5, p. 278]). *The triangle-pseudoline incidence graph of a pseudoline arrangement is connected.* [1]

Note that Conjecture 1.1 and Conjecture 1.2 are equivalent. The existence of bichromatic triangles in a bicolored straight line arrangement has a simple proof: Shift the blue sub-arrangement up and down. Consider the first moment the combinatorics of the arrangement changes. At this point, a blue-blue crossing moved over a red line or a red-red crossing moved over a blue line. The crossing and the line form a bichromatic triangle in the original arrangement.

A generalisation of this result was given in [9]: An arrangement $\mathcal{A} = \{\ell_1, \ldots, \ell_n\}$ is *approaching*, if every pseudoline ℓ_i is the graph of a function f_i, such that $f_i - f_j$ is strictly monotone increasing, for $i < j$. These pseudolines can be shifted and the collection of curves remains an arrangements of pseudolines. This implies that Conjecture 1.1 holds for all approaching arrangements.

The number of approaching arrangements has asymptotics similar to the class of all arrangements, but examples of small arrangements that are not isomorphic to an approaching arrangement are known.

Although the proof in the case of approaching arrangements is very elegant and intuitive, not much progress has been made towards the general case so far. We present results about easier variants of the question.

Also towards the goal of understanding triangles in pseudoline arrangements, we consider the *line-triangle hypergraph*, which is the 3-uniform hypergraph whose vertices correspond to the pseudolines and a triple forms a hyperedge if the corresponding lines form a triangle. Then Conjecture 1.1 claims that every true 2-coloring of this hypergraph has a non-monochromatic hyperedge. We present a theorem about the independence number of this hypergraph, i.e. we consider how large the discrepancy of the cardinalities of the two color classes in an arrangement can be while not including any monochromatic triangles.

We will consider Euclidean arrangements as *marked arrangements*, which have a fixed unbounded face, which we will call the *north face*. This gives use a canonical numbering of the pseudolines, after choosing the north face, see Figure 1.This induces an orientation on every triples of lines: we say that a triple of lines has "-" orientation, if the middle line goes above the crossing of the other two, otherwise it is $+$, see Figure 1.

▶ **Definition 1.3** ([10]). The function $\sigma : \binom{[n]}{k} \to \{-,+\}$ is a rank k signotope on n elements, if for all

$$S := \{x_1 < x_2 < \cdots < x_{k+1}\} \subseteq [n],$$

the sequence

$$\sigma(S \setminus x_1), \sigma(S \setminus x_2), \ldots, \sigma(S \setminus x_{k+1})$$

has at most one sign change.

The orientation of triples of pseudolines of $\mathcal{A}$ give rise to the rank 3 signotope $\sigma_{\mathcal{A}}$. Conversely, every rank 3 signotope can be seen as triangle orientations of a pseudoline arrangement [10]. It is possible to locally mutate a pseudoline arrangement $\mathcal{A}$ by *flipping* a triangle T to obtain $\mathcal{A}'$, i.e. changing the orientation of the pseudolines supporting T. This is equivalent to $\sigma_{\mathcal{A}}$ and $\sigma_{\mathcal{A}'}$ differing on a single value, corresponding to T. See Figure 1. The most elementary tool to locate triangles is the so called *sweeping lemma for pseudolines*.

[1] The original (weaker) conjecture regarded projective arrangements.

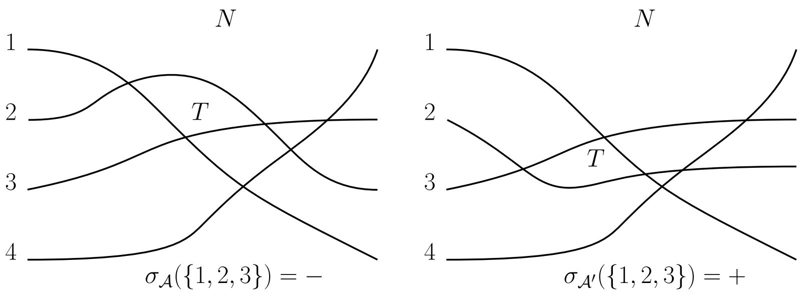

Figure 1 The north face is designated by N. The face T is a triangle. The crossings (1,2) and (1,3) are vertices, which are connected by an edge. The left arrangement corresponds to the all "$-$" signotope. The right arrangement is obtained by flipping T.

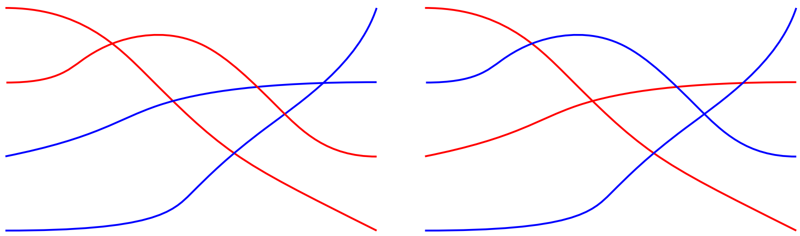

Figure 2 The arrangement on the left is block-bicolored. The arrangement on the right is not.

▶ **Lemma 1.4** ([10]). *Let ℓ be a pseudoline in a marked arrangement. If there is a crossing above ℓ, then there is a crossing above ℓ that forms a triangle supported by ℓ. The same holds for crossings below ℓ.*

There are two a-priori different orders on the set of signotopes with the same parameters: the *inclusion order*, where $\sigma_1 \leq \sigma_2$, when $\sigma_1^{-1}\{+\} \subseteq \sigma_2^{-1}\{+\}$ and the *single-step inclusion order*, where $\sigma_1 \leq \sigma_2$, if there is a sequence of $-$ to $+$ flips that transform σ_1 into σ_2. The following was proven by Felsner and Weil.

▶ **Theorem 1.5** ([11]). *The single-step inclusion order and the inclusion order on rank three signotopes coincide.*

1.1 Our results about bichromatic faces

We consider multiple weaker variants of Conjecture 1.1. We first assume certain structures on the coloring of the arrangements. The following is already implicit in [3].

▶ **Theorem 1.6.** *A bicolored arrangement with at most 5 red pseudolines has a bichromatic triangle.*

This implies that if we color an arrangement with $n/5$ colors and use all of them, then there is a non-monochromatic triangle. It also implies the conjecture holds for all arrangements with $n \leq 11$. We call an (unmarked) bicolored pseudoline arrangment *block-bicolored*, if we can choose a north cell, such that the first pseudolines in the numbering induced by the north cell are red and the rest are blue, see Figure 2. The following result was first proven by Felsner [personal communication]. Here we provide a simpler proof.

▶ **Theorem 1.7** ([2]). *If a pseudoline arrangement is block-bicolored it has a bichromatic triangle.*

We then broadened the question and considered the existence of bichromatic faces with low complexity.

▶ **Theorem 1.8.** *Every bicolored arrangement contains a bichromatic triangle or a bichromatic quadrangle.*

1.2 The line-triangle hypergraph

Let $H_{face}(L)$ denote the line-face hypergraph of the arrangement defined by a set of lines L, i.e., the hypergraph whose vertices correspond to the lines and a subset forms a hyperedge if the corresponding lines form a face in the arrangement (including the unbounded faces). Properties of this hypergraph were regarded earlier by Bose et al.[7] and then their initial results were improved by Ackerman et al. [1] and Balogh and Solymosi [4].

Let α be the independence number of a hypergraph, that is, the largest size of an independent set. The following theorem summarizes previous knowledge about the relevant parameters of line-face hypergraphs:

▶ **Theorem 1.9** ([7, 1, 4])**.**

- $\Omega(\sqrt{n \log n}) = \min_{|L|=n} \alpha(H_{face}(L)) \leq n^{5/6+o(1)}$,
- $n/2 \leq \max_{|L|=n} \alpha(H_{face}(L)) < \frac{2}{3}n$,
- $n^{1/6-o(1)} \leq \max_{|L|=n} \chi(H_{face}(L)) = O(\sqrt{n/\log n})$.

We note that the proofs of Theorem 1.9 work also if we replace lines by pseudolines. From now on, we only consider pseudoline arrangements. Also, the first and the third result hold if we consider the sub-hypergraph containing only the bounded faces as hyperedges.

Instead of the line-face hypergraph, we can consider the line-triangle hypergraph, which we denote by $H_\Delta(L)$. The next is a direct corollary of Theorem 1.9, using that the line-triangle hypergraph is a subhypergraph of the line-face hypergraph. We also need that in the construction in [4] it is enough to consider only triangular faces to obtain a small independence number.

▶ Corollary 1.

- $\Omega(\sqrt{n \log n}) = \min_{|L|=n} \alpha(H_\Delta(L)) \leq n^{5/6+o(1)}$,
- $n^{1/6-o(1)} \leq \max_{|L|=n} \chi(H_\Delta(L)) = O(\sqrt{n/\log n})$.

Our contribution in this area is the following theorem.

▶ **Theorem 1.10.** *The maximum of $\alpha(H_\Delta(\mathcal{A}))$ over every arrangement of pseudolines $\mathcal{A}$ of size n is $\max_{|\mathcal{A}|=n} \alpha(H_\Delta(\mathcal{A})) = n - \Theta(\log n)$.*

This implies the same upper bound for families of lines. We do not provide a proof that our construction works with straight line arrangement.

2 Proof sketches

We present proof sketches for our main results. See the soon to appear full-version for detailed proofs.

Proof Sketch of Theorem 1.6. If the red sub-arrangment has an extremal line, either a simple sweeping argument shows the existence of a bichromatic triangle or we can delete the extremal line and continue with a smaller red sub-arrangement. The only arrangement without extremal line with $n \leq 5$ is the 5-star, where an explicit argument can be given. ◀

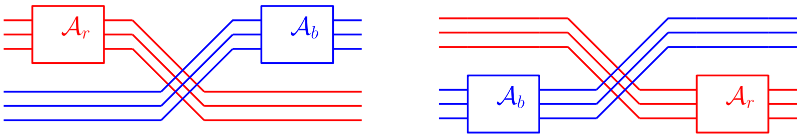

Figure 3 Both extremal arrangements with prescribed sub-arrangements

Proof of Theorem 1.7. Given a block-bicolored arrangement $\mathcal{A}$, we call the arrangement induced by the red pseudolines $\mathcal{A}_r$ and the arrangement induced by the blue pseudolines $\mathcal{A}_b$. Consider the arrangements shown in Figure 3. In the arrangement to the left, call it $\mathcal{A}_{min}$, for every triple of lines, not all of them of the same color, the pseudoline with the highest index is going below the crossing of the other two. This implies that every bichromatic triple has $-$ orientation. A similar argument implies that every bichromatic triple in the right arrangement, call it $\mathcal{A}_{max}$, has orientation $+$. The remaining, monochromatic, triples in $\mathcal{A}, \mathcal{A}_{max}$ and $\mathcal{A}_{min}$ have the same orientation. By the observation above clearly $\mathcal{A}_{min} \leq \mathcal{A} \leq \mathcal{A}_{max}$ holds in the inclusion order. By Theorem 1.5, there is a sequence (possibly of length zero!) of "$-$ to $+$" triangle flips, transforming $\mathcal{A}$ into $\mathcal{A}_{max}$. Since both arrangements agree on the monochromatic triples, the sequence only consists of bichromatic triangle flips. A similar statement holds for $\mathcal{A}_{min}$. Clearly $\mathcal{A}_{min} \neq \mathcal{A}_{max}$, so not both $\mathcal{A} = \mathcal{A}_{min}$ and $\mathcal{A} = \mathcal{A}_{max}$ can hold, so not both flip sequences can be empty, so there exists at least one bichromatic triangle in $\mathcal{A}$. Moreover if both $\mathcal{A} \neq \mathcal{A}_{min}$ and $\mathcal{A} \neq \mathcal{A}_{max}$, $\mathcal{A}$ has two bichromatic triangles, of opposite orientation. ◀

Note that this proof gives us a sequence of arrangements which look like we shift a sub-arrangement to the left, similar to the proof for line arrangements.

We will use a sweeping lemma for lenses, which was stated in [3].

▶ **Lemma 2.1** (Lens Sweeping Lemma). *Let Q be a lens bounded by two curves L and R. Assume there is a collection of curves inside Q which pairwise intersect at most once and where every curve intersects L and R exactly once. If there is a crossing inside Q, there is a triangle that is supported by L.*

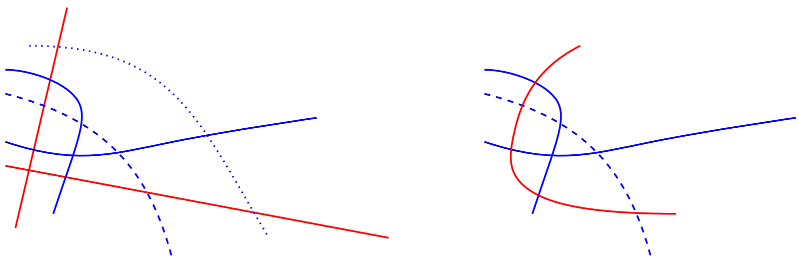

Figure 4 Illustration for the proof of Theorem 1.8. The dashed blue and both red lines form a lens.

Proof sketch of Theorem 1.8. In a bicolored arrangement $\mathcal{A}$, we consider the sub-arrangement induced by the red pseudolines and a single blue pseudoline ℓ_a. There is a blue-red-red triangle T in this subarrangement supported by the blue pseudoline, by Lemma 1.4. Call the red lines ℓ_b and ℓ_c, respectively.

We will only consider the parts of the arrangement $\mathcal{A}$ inside the non-empty triangle T. By infinite descent, we can assume that all curves in T intersect ℓ_a.

We then join ℓ_b and ℓ_c to a single curve ℓ_d and consider the lens formed by ℓ_a and ℓ_d, see Figure 4. By a short case distinction and by using Lemma 2.1 we find a bichromatic triangle in the lens. After reintroducing the red-red crossing this triangle possibly becomes an empty bichromatic quadrangle.

◀

▶ Remark. In the case where we find a quadrangle, this quadrangle is red-red-blue-blue.

Proof Sketch of Theorem 1.10. We first consider the red sub-arrangement. By the Erdős-Szekeres theorem applied in the dual setting to pseudolines, there we find a cyclic sub-arrangement of size $\Theta(\log(n))$. The cyclic arrangement on k pseudolines has $k-2$ triangles. Any non-empty triangle in an arrangement contains a triangle. Since each pseudoline added to the cyclic arrangement can intersect at most two of its triangles, we need to add at least $\Theta(\log(n))$ blue pseudolines to destroy all the red triangles. This yields the upper bound.

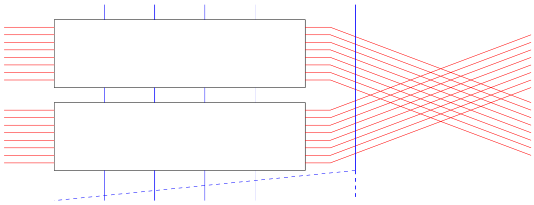

Figure 5 The bottom block has no blue crossings and the top block either has all blue crossings or no blue crossings. The extra blue line either crosses all other blue lines below, or it crosses none.

For the lower bound we inductively construct an arrangement with 2^n red pseudolines and n blue pseudolines without monochromatic triangles, see Fig 5. ◀

3 Discussion

The conjecture about the existence of a bichromatic triangle is solved if there are at most 5 red pseudolines. One particular case of 6 red pseudolines which we could not solve in general (that is, for any addition of blue pseudolines) and find especially interesting is depicted in Figure 6. Note that in any extension with blue pseudolines, as every red pseudoline must have an incident triangle on both sides, either one of them is bichromatic and we are done, or no gray triangles on the figure can be crossed by a blue pseudoline.

The bounds regarding the face-hypergraph and triangle-hypergraph still have large gaps except for the one case in Theorem 1.10. We find the gap between $n/2$ and $2n/3$ for $\max_{|L|=n} \alpha(H_{face}(L))$ a particularly interesting problem.

As an approach to proving Theorem 1.8 we attempted to show the following result which we conjecture to be true.

▶ **Conjecture 3.1.** *Given a blue k-gon P with at least $k-4$ red pseudolines passing through, we find a bichromatic triangle or quadrangle on its inside boundary.*

It is easy to see that $k-4$ is tight if there are no crossings between the red pseudolines.

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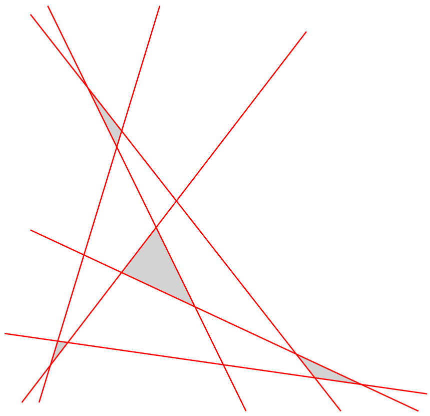

Figure 6 Six red pseudolines to which we need to add several blue pseudolines.

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