Maximum of a Rough Path from Its Signature

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Definition

Let $1 \leq p < 2$ a real number, and V a Banach space. Let $X : J \longrightarrow V$ be a continuous path. Then the $p-$variation of X on $[0, T]$ is defined by

$$||X||_{p,[0,T]} = \left[\sup_{D \subset [0,T]} \sum_{j=0}^{r-1} ||X_{t_j} - X_{t_{j+1}}||^p \right]^{\frac{1}{p}}$$

where $D = \{t_0, t_1, \ldots, t_r\}$ is a finite partition of $[0, T]$, and $0 = t_0 < t_1 < \ldots < t_{r-1} < t_r = T$.

Definition

$X : [0, T] \to V$ is a path with finite $p-$variation, $1 \leq p < 2$. The $p-$Length of X, which we denote by $l_p : \triangle_T \to R$ is a real valued function, defined as

$$l_p(s, t) = ||X||^p_{p,[s,t]}$$

where $\triangle_T := \{(s, t) | 0 \leq s \leq t \leq T\}$

Definition

Suppose $X : [0, T] \to V$ is a path of finite $p-$variation, $1 \leq p < 2$. Then the Signature of X: $S(X) : [0, T] \to T((V))$, $(T((V)) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots)$ is defined as

$$S(X_{0,t}) = (1, X^1_{0,t}, X^2_{0,t}, \ldots, X^k_{0,t}, \ldots)$$

Here $X^k_{0,t} \in V^{\otimes k}$, and $X^k_{0,t}$ is called the Iterated Integral of X:

$$X^k_{0,t} = \int \cdots \int_{0<u_1<\cdots<u_k<t} dX_{u_1} \otimes \ldots \otimes dX_{u_k}$$

Definition

If $e = (e^*_{i_1}, e^*_{i_2}, \ldots, e^*_{i_k}) \in (V^*)^{\otimes k}$, $X : [0, T] \to V$ is a path of finite $p-$variation, $1 \leq p < 2$. Define kth Coordinate Iterated Integral of X with respect to e as

$$\varphi^X_e(t) = \int \cdots \int_{s<u_1<\cdots<u_k<t} e^*_{i_1}(dX_{u_1}) \ldots e^*_{i_k}(dX_{u_k})$$

Reparametrization

Definition

Suppose X is a path of finite $p-$variation, $1 \leq p < 2$. Reparametrize X by its $p-$Length, i.e. let $\widehat{X_{l_p(0,t)}} = X_t$.

$$||\widehat{X_{l_p(0,t_2)}} - \widehat{X_{l_p(0,t_1)}}||^p \leq l_p(t_1, t_2) \leq l_p(0, t_2) - l_p(0, t_1)$$

$$||\widehat{X_{l_p^1}} - \widehat{X_{l_p^2}}|| \leq (l_p^1 - l_p^2)^{\frac{1}{p}} \text{ with } l_p^1 = l_p(0, t_1), \; l_p^2 = l_p(0, t_2)$$

- $\widehat{X_{l_p}}$ is non-constant on any subinterval.
- Reparametrization does not change the path, nor does it change the Signature of the path.

$$\begin{aligned}(\widehat{X_{l_p(0,t)}})^k &= \int \cdots \int_{0<l_p^k<\cdots<l_p^k<l_p(0,t)} d\widehat{X_{l_p^1}} \otimes \ldots \otimes d\widehat{X_{l_p^k}} \\ &= \int \cdots \int_{0<u_1<\cdots<u_k<t} dX_{u_1} \otimes \ldots \otimes dX_{u_k}\end{aligned}$$

2n-Norm Method

Theorem

If f is $\alpha-$Hőlder continous on interval J, with $0 < \alpha \leq 1$, $c > 0$: $|f(u) - f(v)| \leq c|u - v|^{\alpha}$, for any $u, v \in J$. Denote $M := \max |f|_J$, then

$$|M - (\int_J f^{2n}(s)ds)^{\frac{1}{2n}}| \leq M[\frac{\ln 2n}{2\alpha n} + o(\frac{\ln 2n}{2\alpha n})] \quad as\ n \to \infty$$

And, there exists f, such that $|M - (\int_J f^{2n}(s)ds)^{\frac{1}{2n}}| = M[\frac{\ln 2n}{2\alpha n} + o(\frac{\ln n}{n})]$.

Theorem

If f is a Lipschitz function on interval J : $|f(u) - f(v)| \leq c|u - v|$, for any $u, v \in J$. $M := \max |f|_J$, and $\frac{c}{M} \geq 1$. Then

$$|(\int_J f^{2n}(s)ds)^{\frac{1}{2n}} - M| \leq \frac{M}{2N}|\ln(2n+1) + \ln \frac{c}{M}|$$

Proposition: If $M := \max |f|_J$ is fixed, the error of approximation increases when c increases. That is, the speed of convergence is slower when the value of f changes dramatically in a small interval.

Example

Let $f_k(t) = e^{-kt}$, $k \geq 1$. Then $\frac{c}{M} = k$ for $f_k(t)$ on $t \in [0,1]$. For some fixed $\epsilon > 0$, if we want

$$|\,(\int_J f_k(s)^{2n} ds)^{\frac{1}{2n}} - 1\,| = |(\frac{1 - e^{-2nk}}{2nk})^{\frac{1}{2n}} - 1\,| < \epsilon$$

It is equivalent to

$$2nk(1-\epsilon)^{2n} + e^{-2nk} < 1$$

if it holds, we must have

$$2n(1-\epsilon)^{2n} < \frac{1}{k}$$

So, as $k \longrightarrow \infty$, $n \longrightarrow \infty$.

Proposition: The Convergence is quicker when the mass of f is more evenly distributed.

- If f is a real-valued function on J, and $J_1 \subset J$

$$M - (\int_{J_1} f(s)^{2n} ds)^{\frac{1}{2n}} \geq M - (\int_J f^{2n}(s) ds)^{\frac{1}{2n}}$$

- If f is a Non-negative real-valued function on J, then for any $a \geq 0$

$$M + a - (\int_J [f(s) + a]^{2n} ds)^{\frac{1}{2n}} \geq M - (\int_J f^{2n}(s) ds)^{\frac{1}{2n}}$$

Theorem

If f is a real-valued function on J. Then for any given n

$$\min_{a \in R} \theta_n(a) = \theta_n\left(-\frac{ess \sup f_J + ess \inf f_J}{2}\right)$$

where $\theta_n(a) := ess \sup(f(s) + a)_J - (\int_J [f(s) + a]^{2n} ds)^{\frac{1}{2n}}$

Definition

$$\theta_n(a) := ess\sup(f(s)+a)_J - \left(\int_J [f(s)+a]^{2n} ds\right)^{\frac{1}{2n}}$$

Corollary

For any fixed n, $-\frac{ess\sup f_J + ess\ \inf f_J}{2}$ *is a globally (strict) minimum point of* $\theta_n(a)$*, i.e.*

$$\theta_n(a) > \theta_n\left(-\frac{ess\sup f_J + ess\ \inf f_J}{2}\right) \quad \text{if } a \neq -\frac{ess\sup f_J + ess\ \inf f_J}{2}$$

- If this minimum point is found, $ess\sup f_J + ess\ \inf f_J$ is known. Moreover

$$ess\sup\left(f - \frac{ess\sup f_J + ess\ \inf f_J}{2}\right) = \frac{ess\sup f_J - ess\ \inf f_J}{2}$$

Therefore, we could extract $ess\sup f_J$ and $ess\ \inf f_J$ from its $2n-$norm, especially the diameter of f on J: $ess\sup f_J - ess\ \inf f_J$, which, however, is not obvious from the definition of $2n-$norm.

Approximation Using Signature

Theorem

Suppose f is a continuous real-valued function, and $|f| \leq M$. If

$$\int_J \left(\frac{(M+|K|)S_N(\frac{f-K}{M+|K|}) + f + K}{2} - K\right)^2 dt > \frac{(M+|K|)^4 m(J)}{4(2N+1)^2}$$

then $\max f_J > K$, *where* $S_N(x) := \frac{2}{\pi} + \frac{4}{\pi}\sum_{r=1}^{N} \frac{(-1)^{r+1}}{4r^2-1} T_{2r}(x)$ *is the partial sum of Chebyshev series of* $|x|$.

- Suppose $X : J \to R^n$ is a path of finite $p-$variation $1 \leq p < 2$, and $\varphi_e^X(s,t)$ is a Coordinate Iterated Integral function of X, then $\varphi_e^X(s,t)$ is continuous on $\triangle_T := \{(s,t) | 0 \leq s \leq t \leq T\}$, thus bounded on compact interval J.
- Polynomial of φ_e^X is a linear combination of higher ordered Coordinate Iterated Integrals of X.

Definition

$S_N(x) := \frac{2}{\pi} + \frac{4}{\pi}\sum_{r=1}^{N} \frac{(-1)^{r+1}}{4r^2-1} T_{2r}(x)$, where T_{2r}, $r \geq 1$, are the first kind Chebyshev Polynomials.

Theorem

Suppose X: $J \rightarrow R^n$ is an $n-$dimensional path of finite $p-$variation, $1 \leq p < 2$. e is an unit constant $n-$dimensional vector. If $0 < m \leq |X \cdot e| \leq M$, then there exists a series of polynomials S_N^{m+M},s.t.

$$\left|\left|S_N^{m+M}(X \cdot e) - |X \cdot e|\right|\right|_{p,J} \leq \frac{C(\frac{\pi}{2} - \arccos\frac{m}{m+M})}{N} ||X||_{p,J}$$

where $S_N^{m+M}(f) = (m+M)S_N(\frac{f}{m+M})(x)$

$$C(\theta) := \frac{1}{\pi\sin(\frac{1}{2}\theta)} + \frac{1}{2\pi}\int_\theta^\pi \left|\frac{\cos\frac{1}{2}t\cos[(N+\frac{1}{2})t]}{\sin^2\frac{1}{2}t}\right| dt$$

Extract Abstract Value of x from Power Series

Definition

Let $T_N(x) := \sum_{i=0}^{N} \frac{x^{2i}}{i!}$.

Lemma

Suppose f is a real-valued function, then

$$\left| |f| - \sqrt{\ln[T_N(f)]} \right| \leq \max\{\delta_N^1(f), \delta_N^2(f)\} := \delta_N(f)$$

$$\delta_N^1(f) \ : \ = \sqrt{\ln[T_N(f)] - \ln[1 - e(\frac{ef^2}{N+1})^{N+1}]} - \sqrt{\ln[T_N(f)]}$$

$$\delta_N^2(f) \ : \ = \sqrt{\ln[T_N(f)]} - \sqrt{\ln[T_N(f)] - \ln[1 + e(\frac{ef^2}{N+1})^{N+1}]}$$

Lemma

For any fixed $N \geq 7$, $\delta_N(x)$ *is increasing on* $[0, \sqrt{\frac{N+1}{e}}]$.

Theorem

Suppose f *is a real-valued function, and* $N \geq 7$. *If* $|f(J)| \leq a \leq \sqrt{\frac{N+1}{e}}$, *then on* J

$$0 \leq |f| - \sqrt{\ln[T_N(f)]} \leq \delta_N(a)$$

Corollary

Suppose f *is a real-valued function,* $N \geq 7$, *if for some* $K \in R$, $|f(J) - K| \leq a \leq \sqrt{\frac{N+1}{e}}$, *and*

$$\int_J (\sqrt{\ln[T_N(f-K)]} + f - K)^2 dt > m(J)[\delta_N(a)]^2$$

then $\max f_J > K$.

Definition

$$\lambda_N(x) := \max\{\frac{\delta_N^2(x)}{\sqrt{\ln[T_N(x)]}}, \frac{\delta_N^1(x)}{\sqrt{\ln[T_N(x)]}}\}$$

Lemma

There exists $x_\lambda^N \geq 0$*, s.t.* $\lambda_N(x)$ *is increasing on* $[x_\lambda^N, \sqrt{\frac{N+1}{2e}}]$*, where* $\lim_{N\to\infty} x_\lambda^N = 0$*.*

Theorem

$X{:}J \to R^n$ *is a* $n-$*dimensional path of finite* $p-$*variation,* $1 \leq p < 2$*,* e *is a* $n-$*dimensional constant unit vector, then for any fixed* $N \geq 2$*, if* $x_\lambda^N \leq |X \cdot e| \leq \sqrt{\frac{N+1}{2e}}$

$$\left|\left| |X\cdot e| - \sqrt{\ln[T_N(X\cdot e)]} \right|\right|_{p,J} \leq \lambda_N(\sqrt{\frac{N+1}{2e}})\,||X||_{p,J}$$

Integration Method

For all $f : J \to R$, is a continuous function, $V_f : R \to \overline{R^+}$ is defined as

$$V_f(a) := \int_{\{f \geq a\}} (f(t) - a)dt$$

Then V_f has the following properties:

- non-negative.
- non-increasing.
- differentiable with the derivative $-m(f \geq a)$, which is bounded by $m(J)$.
- convex.
- $ess \sup f_J$ is the smallest point where $V_f = 0$.