Scaled Limit and Rate of Convergence for the Largest Eigenvalue from the Generalized Cauchy Random Matrix Ensemble

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Outline

Introduction

- Interest: Study eigenvalues of (large) random matrices.
  - In particular the largest eigenvalue.
  - Source of interesting distributions, occurring in different fields of mathematics and physics.
- Started with Physicists (Wigner) in the 50's.
  - Model to understand statistical behavior of slow neutron resonances.
  - Applications in statistical mechanics (eg. limit laws of random growth models, tilting problems).
- 70's: Applications to number theory (Montgomery).

Two Types of Models

Essentially two types of matrix models:

- Choose each entry of a $N \times N$ matrix (independently) according to some given distribution.
  - eg. GUE, GOE, GSE.
- Choose a compact group and endow it with its normalized Haar measure.
  - eg. CUE (Dyson Ensemble).

The GUE

Definition (see eg. Mehta, 1991)

A random $N \times N$ Hermitian matrix belongs to the *GUE*, if the diagonal elements x_{jj} and the upper triangular elements $x_{jk} = u_{jk} + iv_{jk}$ $(j < k)$ are chosen independently with normal densities of the form:

$$\frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \quad \sim \mathcal{N}(0, \frac{1}{2}) \text{ (diagonal)},$$
$$\frac{2}{\pi} e^{-2(u_{jk}^2 + v_{jk}^2)} \quad \sim \mathcal{N}(0, \frac{1}{4}) + i\mathcal{N}(0, \frac{1}{4}) \text{ (upper triangular)}$$

Joint density function:

$$p(X) = \prod_{j=1}^{N} \frac{1}{\sqrt{\pi}} e^{-x_{jj}^2} \prod_{1 \le j < k \le N} \frac{2}{\pi} e^{-2|x_{jk}|^2} = \frac{1}{Z_N} \exp\{-\mathrm{Tr}(X^2)\}.$$

Eigenvalues and Point Processes I

- Eigenvalue distribution?
- Apply basis transformation and integrate out elements independent of the eigenvalues:

Eigenvalue measure on $\mathbb{R}^N$: If $x_1, \ldots, x_N \in \mathbb{R}$,

$$\begin{aligned} u_N(x_1, \ldots, x_N) &= \frac{1}{Z_N N!} \prod_{1 \le j < k \le N} |x_j - x_k|^2 \exp\left(-\sum_{j=1}^{N} x_j^2 \right) \\ &= \frac{1}{Z_N N!} \left(\det(p_{j-1}(x_i) e^{(-x_i^2)/2})_{1 \le i,j \le N} \right)^2 \\ &= \frac{1}{N!} \det(K_N(x_i, x_j))_{i,j=1}^{N}, \end{aligned}$$

Eigenvalues and Point Processes II

where

$$K_N(x,y) = \sum_{j=0}^{N-1} p_j^H(x) p_j^H(y) e^{-\frac{x^2+y^2}{2}}$$

$$= \mathrm{cst} \cdot e^{-(x^2+y^2)/2} \frac{p_N^H(x)p_{N-1}^H(y) - p_N^H(y)p_{N-1}^H(x)}{x-y}.$$

p_i^H: normalized Hermite polynomial of degree i.

Definition

The *n-th correlation function* ρ_n is defined by:

$$\rho_n(x_1,\ldots,x_n) = \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} u_N(x_1,\ldots,x_N) dx_{n+1} \cdots dx_N$$

$$= \det(K_N(x_i,x_j))_{i,j=1}^n, \text{ for } n \leq N.$$

Eigenvalues and Point Processes III

The eigenvalue distribution can be viewed as a point process on $\mathbb{R}$ via the application $(x_1, \ldots, x_N) \mapsto \sum_{i=1}^{N} \delta_{x_i}$. Point processes with a correlation function of this determinantal form are called *determinantal point processes*.

Fredholm Determinant and Law of the Largest Eigenvalue

Problem: Describe probabilities of the form

$$E(k,J) = P[\text{there are exactly k eigenvalues in J}].$$

- $K_N(x,y)$ is the kernel of a trace class integral operator K_N on $L^2(\mathbb{R})$:

$$K_N f(x) = \int_{\mathbb{R}} K_N(x,y) f(y) dy, \quad \text{for } f \in L^2(\mathbb{R}).$$

- *Fredholm determinant* ($\phi \in L^\infty(J)$):

$$\det(I + K_N \phi)|_J = \sum_{n \geq 0} \frac{1}{n!} \int_{J^n} \prod_{j=1}^{n} \phi(x_j) \rho_n(x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

Now:

- $E(k,J) = \frac{(-1)^k}{k!} \frac{d^k}{d\lambda^k} \det(I - \lambda K_N|_J)|_{\lambda=1}$.
- Law of the largest eigenvalue $\lambda_1(N)$:

$$P[\lambda_1(N) \leq t] = E(0,(t,\infty)) = \det(I - K_N)|_{(t,\infty)}.$$

GUE: Airy Kernel and Painlevé-II

Scale around the largest eigenvalue $\lambda_1(N)$ of the GUE, to obtain for $N \to \infty$:

$$P\left[\lambda_1(N) \leq \sqrt{2N} + \frac{s}{\sqrt{2}N^{1/6}}\right] \longrightarrow F_{TW}(s) = \det(Id - K_{Airy})|_{L_2(s,\infty)},$$

$$\text{where} \qquad F_{TW}(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)dx\right).$$

q is the solution of a Painlevé-II equation $q'' = sq + 2q^3$ with boundary condition $q(s) \sim Ai(s)$ for $s \to \infty$.

(see Tracy, Widom 1994)

The GCyE I

Setting: Unitary group $U(N)$ with *Haar measure* μ_N, complex parameter s, $\Re(s) \geq -1/2$.

- Deformed Haar measure: $\mathrm{cst} \cdot \det((I-U)^{\bar{s}}) \det((I-U^*)^s) \mu_N(dU)$.
- Eigenvalue measure on $\mathbb{S}^N / S(N)$:

$$\mathrm{cst} \cdot \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} (1 - e^{i\theta_j})^{\bar{s}} (1 - e^{-i\theta_j})^s.$$

Cayley transform: $U(N) \longleftrightarrow H(N)$, via $X \in H(N) \mapsto \frac{X+i}{X-i} \in U(N)$.

- *Generalized Cauchy measure* on $H(N)$:

$$\mathrm{cst} \cdot \det((1+iX)^{-s-N}) \det((1-iX)^{-\bar{s}-N}) \prod_{1 \leq j < k \leq N} dX_{jk} \prod_{i=1}^{N} dX_{ii}.$$

The GCyE II

- Eigenvalue measure on $\mathbb{R}^N/S(N)$:

$$m_N^s = \mathrm{cst} \cdot \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} w_H(x_j),$$

where $w_H(x_j) = (1 + ix_j)^{-s-N}(1 - ix_j)^{-\bar{s}-N}$.

$\Rightarrow$ As in the GUE case, this eigenvalue process is determinantal.

The Kernel and its Scaling Limit

- *The kernel*: $K_N(x,y) = \frac{\phi(x)\psi(y)-\phi(y)\psi(x)}{x-y}$,
  - where $\phi(x) = \mathrm{cst}\cdot\sqrt{w_H(x)}p_N(x)$, and $\psi(x) = \mathrm{cst}\cdot\sqrt{w_H(x)}p_{N-1}(x)$, and
  - $p_m(x) = (x-i)^m {}_2F_1[-m, s+N-m, 2\Re(s)+2N-2m; 2/(1+iX)]$ are the *pseudo Jacobi polynomials*.
  - The *Gauss Hypergeometric function*: ${}_2F_1[a,b,c;x] = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n n!}x^n$, and $(a)_n = a\cdot\ldots\cdot(a+n-1)$.
- *The scaling limit*: $NK_N(Nx,Ny) \longrightarrow K_\infty(x,y)$, when $N\to\infty$,
  - where $K_\infty(x,y) = \mathrm{cst}\cdot\frac{\tilde{P}(x)Q(y)-Q(x)\tilde{P}(y)}{x-y}$, and
  - $\tilde{P}(x) = |2/x|^{\Re(s)}e^{-i/x+\pi\Im\mathrm{Sgn}(x)/2}{}_1F_1[s, 2\Re(s)+1; 2i/x]$, and $Q(x) = (2/x)|2/x|^{\Re(s)}e^{-i/x+\pi\Im\mathrm{Sgn}(x)/2}{}_1F_1[s+1, 2\Re(s)+2; 2i/x]$.
  - The *Confluent Hypergeometric function*: ${}_1F_1[r,q;x] = \sum_{n\geq 0}\frac{(r)_n}{(q)_n n!}x^n$.

(see Borodin, Olshanski 2001)

U-invariant Measures on H, Ergodicity I

Definition

Assume there is a group acting on a Borel space. A probability measure on the Borel space which is invariant under this action is called *ergodic*, if any invariant set has either measure 0 or 1.

- If the group is compact and the action continuous, the ergodic measures are exactly the orbital measures. Ie. invariant measures supported by orbits.
- Consider the group $U = \bigcup_N U(N)$ and the space H of infinite Hermitian matrices. U acts on H by conjugation. m_N^s are consistent with natural projections and define a two parameter family of U-invariant probability measures m^s on H.

U-invariant Measures on H, Ergodicity II

- U-invariant probability measures can be written as continual convex combinations of ergodic measures (*spectral decomposition*). This decomposition is determined by a *spectral measure* on the space of point configurations $Conf(\mathbb{R}\backslash\{0\})$.

$\Rightarrow$ The spectral measure of m^s defines a determinantal point process on $\mathbb{R}\backslash\{0\}$ with kernel K_∞.

(see Borodin, Olshanski 2001)

Irreducible Spherical Representation of $U \ltimes H$

- Set $G(N) = U(N) \ltimes H(N)$, $U(N)$ acting on $H(N)$ (additive group) by conjugation. Also, $G = \varinjlim G(N)$.
- A unitary representation T of G is *spherical* if it posesses a cyclic unit vector ξ which is invariant wrt the subgroup $U \subset G$.
- One to one correspondence of equivalence classes of spherical representation (T, ξ) and U-invariant probability measures on H. Ie. irreducible spherical representations are parametrized by ergodic measures on H.

(see Borodin, Olshanski 2001)

Convergence of the Rescaled Largest Eigenvalue I

Set $F_N(t) = \det(I - K_N)|_{(Nt,\infty)}$ the Fredholm determinant of the rescaled kernel $NK_N(Nx, Ny)$, respectively $F_\infty(t) = \det(I - K_\infty)_{(t,\infty)}$, where $t > 0$.

Theorem (NNR, 2009)

For s such that $\Re(s) > -1/2$, *and* $t > 0$, F_N *and* F_∞ *are in* $\mathcal{C}^3(\mathbb{R}_+^*, \mathbb{R})$ *and for* $p \in \{0, 1, 2, 3\}$, *the p-th derivative (wrt t) of* F_N *converges pointwise to the p-th derivative of* F_∞.

Convergence of the Rescaled Largest Eigenvalue II

Corollary (NNR, 2009)

The law of $\lambda_1(N)/N$ converges to the distribution of the largest point of the determinantal process on $\mathbb{R}^*$ described by the limiting kernel $K_\infty(x, y)$:

$$P\left[\frac{\lambda_1(N)}{N} \le x_0\right] = \det(I - K_N)|_{(Nx_0,\infty)} \longrightarrow \det(I - K_\infty)|_{(x_0,\infty)}, \;\; as \;\; N \to \infty,$$

for any $x_0 > 0$.

Convergence of the Rescaled Largest Eigenvalue III

Theorem (NNR, 2009)

For all $x_0 > 0$, and for $x > x_0$,

$$\left| P\left[\frac{\lambda_1(N)}{N} \leq x\right] - \det(I - K_\infty)|_{(x,\infty)} \right| \leq \frac{1}{N} C(x_0, s),$$

where $C(x_0, s)$ is a constant depending only on x_0 and s.

The Painlevé-V equation

Define $\theta_\infty(\tau) = \tau \frac{d \log \det(I - K_\infty)|_{(\tau^{-1},\infty)}}{d\tau}$, $\tau > 0$.

Theorem (NNR, 2009)

Let s be such that $\Re(s) > -1/2$. then, θ_∞ is well defined and is a solution of the Painlevé-V equation on $\mathbb{R}_+^*$:

$$-\tau^2(\theta''(\tau))^2 = [2(\tau\theta'(\tau) - \theta(\tau)) + (\theta'(\tau))^2 + i(\bar{s} - s)\theta'(\tau)]^2 - (\theta'(\tau))^2(\theta'(\tau) - 2is)(\theta'(\tau)) + 2i\bar{s}).$$

This implies the result of Jimbo, Miwa, Môri and Sato (1980) that the sine kernel (case $s = 0$) satisfies the Painlevé-V equation with $s = 0$.

Scaling Limit I

Lemma (NNR, 2009)

Let $x, y \in \mathbb{R}^*$ and let $\Re s > -1/2$. Then $K_{[N]}$ and K_∞ are C^∞ in $(\mathbb{R}^*)^2$ and for all $p, q \in \mathbb{N}_0$,

$$(\mathcal{S}gn(xy))^N \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x,y) \underset{N\to\infty}{\longrightarrow} \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_\infty(x,y).$$

Moreover, for any $x_0 > 0$, and $|x|, |y| \geq x_0 > 0$:

$$\left| \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x,y) \right| \leq \frac{C(x_0, s, p, q)}{|x|^{\Re s+p+1} |y|^{\Re s+q+1}}.$$

- $\rho_{k,N}(x_1, \ldots, x_k) = \det(K_{[N]}(x_i, x_j))_{i,j=1}^n$ and its scaling limit $\rho_{,\infty}$ are in C^3, and $\rho_{k,N}$ and its derivatives converge pointwise to the corresponding derivatives of $\rho_{k,\infty}$.

Scaling Limit II

- For all k, $p \in \{0,1,2,3\}$, $N \in \mathbb{N} \cup \{\infty\}$ and $t \geq x_0$,

$$\sup_{t \geq x_0} \left| \frac{d^p}{dt^p} \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \rho_{k,N}(x_1, \ldots, x_k) dx_1 \cdots dx_k \right| \leq \frac{C(x_0)^k}{(k-1)!}.$$

- F_N and F_∞ are well defined and $\mathcal{C}^3$ (dominated convergence).
- Again by dominated convergence, F_N and its partial derivatives converge pointwise to F_∞ and its partial derivatives.

Painlevé-V

- Show that $F_\infty(t) > 0$, $\forall t > 0$. Then, θ_∞ is well defined.
- $F_N(t) > 0$, $\forall t > 0$ (as $P[\lambda_1(N) \leq Nt] > 0$). Thus,

$$\theta_N(\tau) = \tau \frac{d}{d\tau} \log(F_N(\tau^{-1}))$$

  is well defined.
- The derivatives of θ_N converge pointwise to the derivatives of θ_∞
- By Forrester, Witte (2002), θ_N satisfies a Painlevé-VI equation. This equation is of the form

$$\sum_{k=0}^{m} N^{-k} \frac{P_k(\tau, \theta_N(\tau), \theta_N'(\tau), \theta_N''(\tau))}{\tau^q} = 0,$$

  where m and q are universal integers and the P_k's are polynomials which are independent of N. Moreover, $P_0(\tau, \theta_N(\tau), \theta_N'(\tau), \theta_N''(\tau))\tau^{-q}$ corresponds to the σ-form of the Painlevé-V equation.