Differentiability of martingale driven BSDE and application to hedging in incomplete markets

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Motivation

Risk source: n-dimensional SDE

$$dR_s = \sigma(s, R_s)dM_s + b(s, R_s)dC_s$$

with M d-dim. continuous local martingale, $d\langle M, M\rangle_s = q_s q_s^* dC_s$

Aim: price and hedge a derivative of the form $F(R_T)$

Correlated financial market: $k \leq d$ assets

$$dS_s = S_s(\beta(s, R_s)dM_s + \alpha(s, R_s)dC_s)$$

Preferences:

$$U(x) = -e^{-\eta x}, \quad 0 < \eta = \text{risk aversion}$$

Value function:

$$V^F(x,t,r) = \sup_\lambda \mathbb{E}\left[U(x + \sum_{i=1}^{k} \int_t^T \lambda_s^{(i)} \frac{dS_s^{(i)}}{S_s^{(i)}} - F(R_T^{t,r})) \right]$$

What is a BSDE (M=W)?
A BSDE with terminal condition B and generator f is an equation of the type

$$Y_t = B - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s) ds.$$

$\rightarrow$ A solution is a pair of *adapted* processes (Y, Z).

Utility maximization

Theorem (Hu et al. '05, Morlais '08)

The value function satisfies

$$V^F(x,t,r) = U(x - Y_t^{F,t,r})$$

and the optimal strategy π^F *is given by*

$$\pi_s^F = Z_s^{F,t,r} q_s^* \beta^*(\beta\beta^*)^{-1}(s, R_s^{t,r}) + \frac{1}{\eta}\alpha^*(\beta\beta^*)^{-1}(s, R_s^{t,r}),$$

where $(Y^{F,t,r}, Z^{F,t,r})$ *is the solution of a certain quadratic BSDE with terminal value* $F(R_T)$.

Indifference pricing and delta hedging

Decomposition:

$$\pi^F = \pi^0 + \Delta = \textit{pure investment} + \textit{optimal hedge}$$

Indifference price: $V^F(x - p(t,r), t, r) = V^0(x, t, r)$

Theorem (Ankirchner et al. '07)

Let M be the Brownian motion. The indifference price is given by

$$p(t,r) = Y_t^{0,t,r} - Y_t^{F,t,r}.$$

The optimal hedge Δ can be derived explicitely

$$\begin{aligned}\Delta(t,r) &= \left[Z_t^{0,t,r} - Z_t^{F,t,r}\right] \beta^*(\beta\beta^*)^{-1}(t,r) \\ &= \left[-\partial_2 p(t,r)\sigma(t,r)\right] \beta^*(\beta\beta^*)^{-1}(t,r).\end{aligned}$$

Backround - The Brownian Case (M=W)

Theorem (Ankirchner et al. '07)

Let f be of quadratic growth, i.e. $|f(\cdot, z)| \leq C(1 + |z|^2)$.
Assume that σ, b are Lipschitz, have uniformly bounded partial derivatives, f is differentiable in r, z, ...
Then

- *Markov property:*
$$Y_s^{t,r} = u(s, R_s^{t,r}),$$
- *$Y^{t,r}$ is continuously differentiable in r and Malliavin differentiable:*
$$D_\theta Y_s^{t,r} = \partial_2 u(s, R_s^{t,r}) D_\theta R_s^{t,r}$$
- *Malliavin trace:*
$$Z_s^{t,r} = D_s Y_s^{t,r} = \partial_2 u(s, R_s^{t,r}) \sigma(s, R_s^{t,r}).$$

Our aim

Brownian setting: If the coefficients σ, b and f are nice, then

$$\begin{aligned} Y_s^{t,r} =& u(s, R_s^{t,r}) \\ Z_s^{t,r} =& \partial_2 u(s, R_s^{t,r})\sigma(s, R_s^{t,r}) \end{aligned}$$

Question: Does this relation between Y and Z hold in other settings, f.e. in a continuous martingale setting?

What is a martingale driven BSDE?

- M continuous d-dim. martingale, $(\mathcal{F}_t)$ cont. and complete and thus every martingale is of the form $Z \cdot M + L$
- $d\langle M, M\rangle_t = q_t q_t^* dC_t$
- B $\mathcal{F}_T$-measurable r.v.
- f:$\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ Borel-measurable function

A BSDE with terminal value B and generator f is an equation

$$\begin{aligned} Y_t = & B - \int_t^T Z_s dM_s - \int_t^T dL_s + \frac{\eta}{2} \int_t^T d\langle L, L\rangle_s \\ & + \int_t^T f(s, Y_s, Z_s) dC_s. \end{aligned}$$

A solution is a triple of adapted processes (Y, Z, L) such that the above equation makes sense.

Our Setting - Martingale driven FBSDE

- M continuous d-dim. martingale

For $(x, m) \in \mathbb{R}^n \times \mathbb{R}^d$ and $t \in [0, T]$ we consider

$$X_t^{x,m} = x + \int_0^t \sigma(s, X_s^{x,m}, M_s^m) dM_s + \int_0^t b(s, X_s^{x,m}, M_s^m) dC_s$$

with solution processes $(X^{x,m}, Y^{x,m}, Z^{x,m}, L^{x,m})$.

Our Setting - Martingale driven FBSDE

- M continuous d-dim. martingale
- F is a bounded function
- f is quadratic in z

For $(x, m) \in \mathbb{R}^n \times \mathbb{R}^d$ and $t \in [0, T]$ we consider

$$X_t^{x,m} = x + \int_0^t \sigma(s, X_s^{x,m}, M_s^m) dM_s + \int_0^t b(s, X_s^{x,m}, M_s^m) dC_s$$

$$Y_t^{x,m} = F(X_T^{x,m}) - \int_t^T Z_r^{x,m} dM_r - \int_t^T dL_r^{x,m} + \frac{\eta}{2} \int_t^T d\langle L^{x,m}, L^{x,m} \rangle_r + \int_t^T f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m} q_r^*) dC_r$$

with solution processes $(X^{x,m}, Y^{x,m}, Z^{x,m}, L^{x,m})$.

Theorem (IRR '09)

Let $M^{t,m}$ be a strong Markov process.
Then there exist deterministic functions u and v such that for $s \in [t, T]$

$$Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m}), \quad Z_s^{t,x,m} = v(s, X_s^{t,x,m}, M_s^{t,m}).$$

Differentiability of Y

Difficulty: Presence of $\langle L^{x,m}, L^{x,m}\rangle$ in

$$Y_t^{x,m} = F(X_T^{x,m}) - \int_t^T Z_r^{x,m} dM_r - \int_t^T dL_r^{x,m} + \frac{\eta}{2}\int_t^T d\langle L^{x,m}, L^{x,m}\rangle_r + \int_t^T f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m} q_r^*) dC_r$$

Introduce assumption (MRP):
There exists a square-integrable martingale N with $\langle M, N\rangle = 0$ and such that (M, N) satisfies the martingale representation property.

$\Longrightarrow \exists$ process $U^{x,m}$ such that $L^{x,m} = U^{x,m} \cdot N$

Differentiability of Y

Theorem (IRR '09)

Let (MRP) be satisfied, $\partial_i f$ of linear growth in z,...

Then there exists a modification $(Y^{x,m}, Z^{x,m}, U^{x,m})$ such that

- $Y^{x,m}$ *is continuously differentiable in x and m,*
- *and there exist processes $\partial_x Z^{x,m}$, $\partial_m Z^{x,m}$ and $\partial_x U^{x,m}$, $\partial_m U^{x,m}$ such that the derivatives*

$$(\partial_x Y^{x,m}, \partial_x Z^{x,m}, \partial_x U^{x,m}) \text{ and } (\partial_m Y^{x,m}, \partial_m Z^{x,m}, \partial_m U^{x,m})$$

solve BSDEs.

The representation formula

Theorem (IRR '09)

If

- $Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m})$ *and*
- $Y^{t,x,m}$ *continuously differentiable in* (x, m).

Then

$$Z_s^{t,x,m} = \partial_2 u(s, X_s^{t,x,m}, M_s^{t,m})\sigma(s, X_s^{t,x,m}, M_s^{t,m}) + \partial_3 u(s, X_s^{t,x,m}, M_s^{t,m}).$$

Hedging with stochastic correlation [Ankirchner,Heyne '09]

- $F(R_T) =$ derivative of risk R with payoff function F
- $S =$ correlated traded asset

$$dS_t = S_t(\mu_S dt + \sigma_S dW_t^1)$$
$$dR_t = R_t(\mu_R dt + \sigma_R(\rho_t dW_t^1 + \sqrt{1-{\rho_t}^2} dW_t^2)),$$

where ρ is the stochastic correlation with dynamics

$$d\rho_t = a(\rho_t)dt + g(\rho_t)(\gamma dW_t^1 + \delta dW_t^2 + \sqrt{1-\gamma^2-\delta^2} dW_t^3).$$

Aim: Find a local risk minimizing hedge for $F(R_T)$.

FS decomposition and BSDEs

- Standard method for deriving the local risk minimizing strategy π is based on FS decomposition

$$F(R_T) = C + \int_0^T \pi_u dS_u + L_T.$$

- Let (Y, Z) be the solution of the linear BSDE

$$Y_t = F(R_T) - \int_t^T Z_u dW_u - \int_t^T Z_u^1 \frac{\mu_S}{\sigma_S} du.$$

Then FS decomposition of $F(R_T)$ is given by

$$F(R_T) = Y_0 + \int_0^T \frac{Z_u^1}{\sigma_S S_u} dS_u + \int_0^T Z_u^2 dW_u^2 + \int_0^T Z_u^3 dW_u^3.$$

Optimal hedge

Hedging strategy $\pi = \frac{Z^1}{\sigma_S S}$ $\rightsquigarrow$ Explicit description of π?

The solution of

$$Y_t = F(R_T) - \int_t^T Z_u dW_u - \int_t^T Z_u^1 \frac{\mu_S}{\sigma_S} du.$$

can be described in terms of

$$u(t,x,v) = E^Q\left[h(R_T^{t,x,v})\right], \text{ i.e. } Y_t = u(t, R_t, \rho_t).$$

Difficulty: $x \mapsto u(t,x,v)$ is only locally Lipschitz continuous
$\rightsquigarrow$ no (direct) access to the chain rule of Malliavin Calculus

With the representation formula

$$Z_t = \sigma(t, R_t, \rho_t)^* \begin{pmatrix} \partial_2 u(t, R_t, \rho_t) \\ \partial_3 u(t, R_t, \rho_t) \end{pmatrix}.$$

Literature

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Thank you for your attention!