Erdős-Rényi random graphs + forest fires = self organized criticality

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Erdős-Rényi forest fire model

The $\lambda(n) \equiv 0$ case: the Erdős-Rényi model
- Phase transition of the E-R graph process
- The Smoluchowski coagulation equations

S.O.C. in the forest fire model
- The controlled Smoluchowski equation

Erdős-Rényi forest fire model

- $G(n, t)$ is a continuous time graph-valued Markov chain on n vertices
- At time t an edge of the graph can be „occupied” or „vacant”
- We investigate the asymptotic behaviour of the component-size distribution of the random graph consisting of occupied edges as $n \to \infty$

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- Vacant edges become occupied with rate $\frac{1}{n}$ (Erdős-Rényi random graph evolution)
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- Fire spreads along the occupied edges and burns them (the number of vertices remains unchanged)

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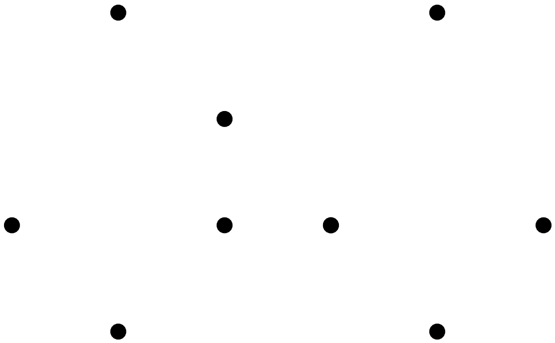
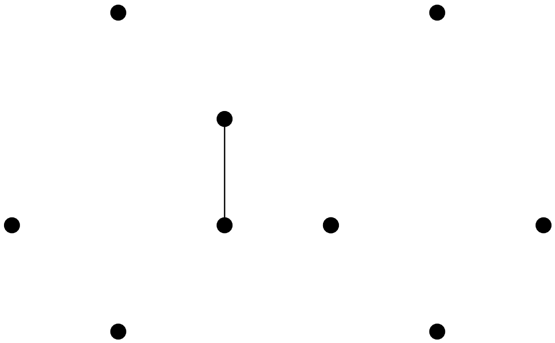
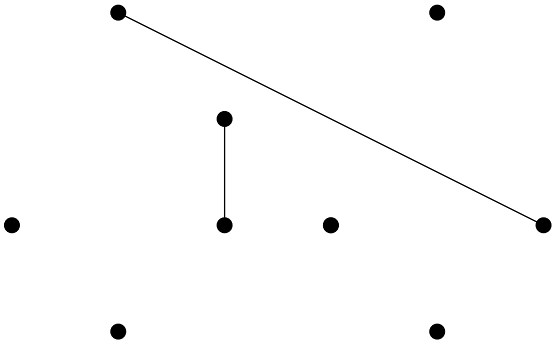


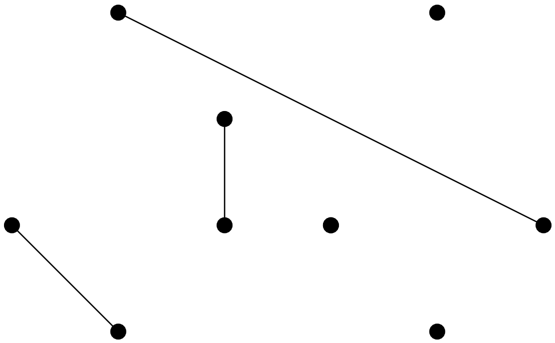
$\mathcal{V}_1 = 9 \quad \mathcal{V}_2 = 0 \quad \mathcal{V}_3 = 0 \quad \mathcal{V}_4 = 0$



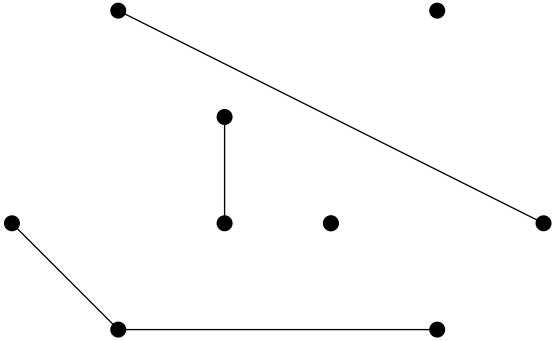
$\mathcal{V}_1 = 7 \quad \mathcal{V}_2 = 2 \quad \mathcal{V}_3 = 0 \quad \mathcal{V}_4 = 0$



$\mathcal{V}_1 = 5 \quad \mathcal{V}_2 = 4 \quad \mathcal{V}_3 = 0 \quad \mathcal{V}_4 = 0$



$\mathcal{V}_1 = 3 \quad \mathcal{V}_2 = 6 \quad \mathcal{V}_3 = 0 \quad \mathcal{V}_4 = 0$



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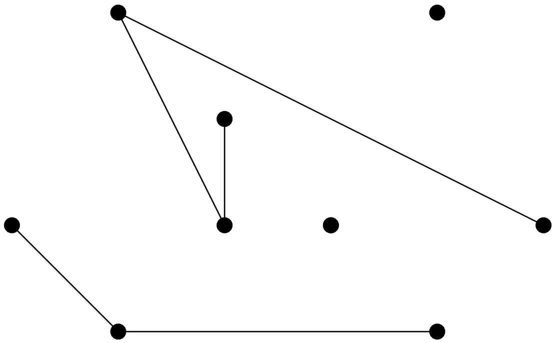

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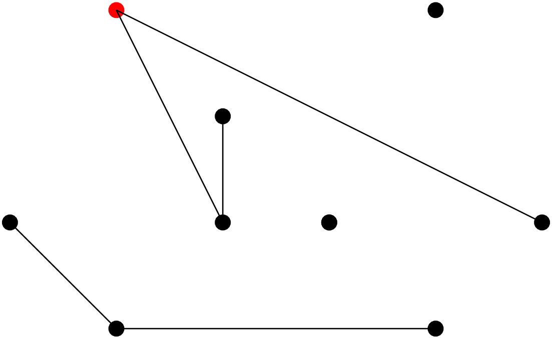

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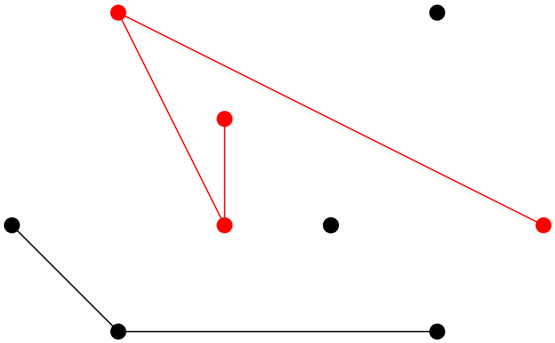

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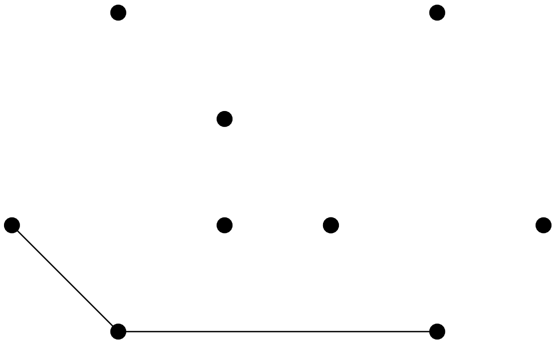

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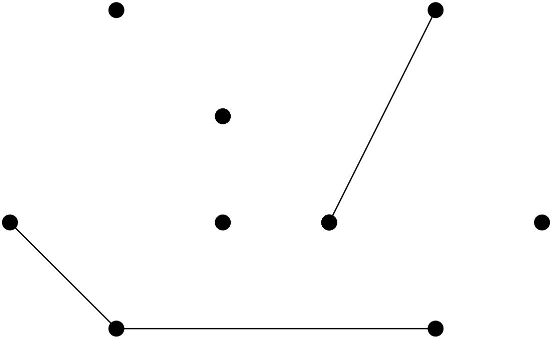

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The phase transition of the Erdős-Rényi model can be treated in a different way:

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$\sum_{k=1}^{n} v_k^n(t) \equiv 1$

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More generally: $v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1}$

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If $k \neq l$ then

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$$\mathcal{C}^n_k(t)\mathcal{C}^n_l(t)\frac{1}{n}kl = \mathcal{V}^n_k(t)\mathcal{V}^n_l(t)\frac{1}{n} = (nv^n_k(t))(nv^n_l(t))\frac{1}{n} = nv^n_k(t)v^n_l(t)$$

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$$\frac{\mathcal{C}^n_k(t)(\mathcal{C}^n_k(t)-1)}{2}\frac{1}{n}k^2 \approx \frac{1}{2}nv^n_k(t)^2$$

$\mathcal{W}^n_{k,l}(t)$ is counting the number of times when a component of size k coagulated with a component of size l from before t

$\mathcal{V}^n_k$ is the total weight of the components of size k

$$d\mathcal{V}_k = k \cdot \sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} d\mathcal{W}_{l,k-l} - k \cdot \sum_{l=1}^{n} d\mathcal{W}_{l,k}$$

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How to solve these differential equations?

PDE

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$$V(t,x) := \sum_{k=1}^{\infty} v_k(t) e^{-kx}$$

Laplace-transform/generating function

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The transformed equation:

$$\partial_t V(t,x) = \partial_x V(t,x) - \frac{1}{2}\partial_x V^2(t,x)$$

Burgers-equation

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This equation can be solved explicitly, eg. with the method of characteristics, or with the following method:

Solution of the Burgers equation

$$\partial_t V(t,x) = \partial_x V(t,x) - V(t,x)\partial V(t,x)$$

Denote by $X(t,v)$ the inverse function of $V(t,x)$ in the second variable $V(t,X(t,v)) = v$.

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Denote by $X(t,v)$ the inverse function of $V(t,x)$ in the second variable $V(t,X(t,v)) = v$.

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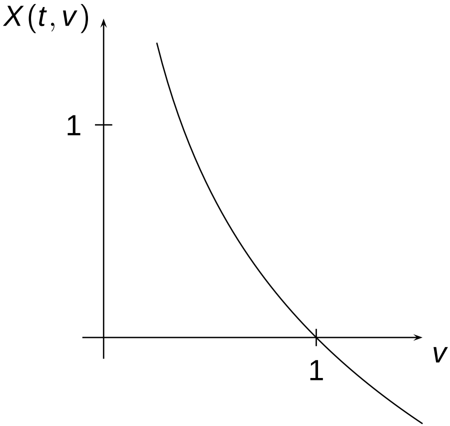
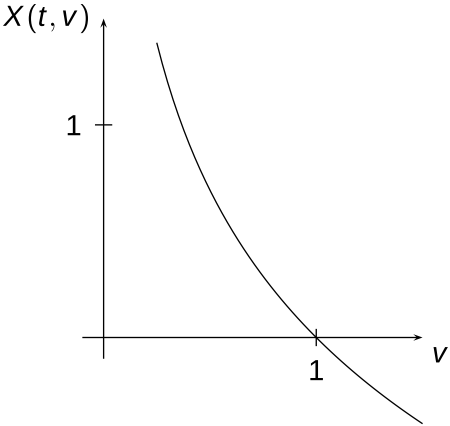
Solution:

$$X(t,v) = X(0,v) + t \cdot (v-1)$$



$$V(0,x) = e^{-x} \quad X(0,v) = -\log(v)$$



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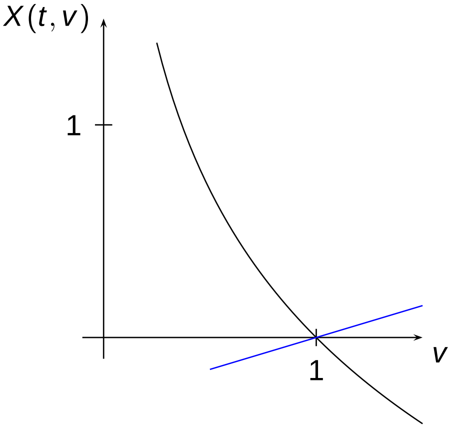

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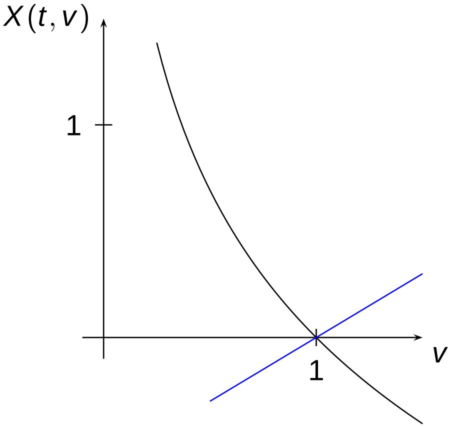

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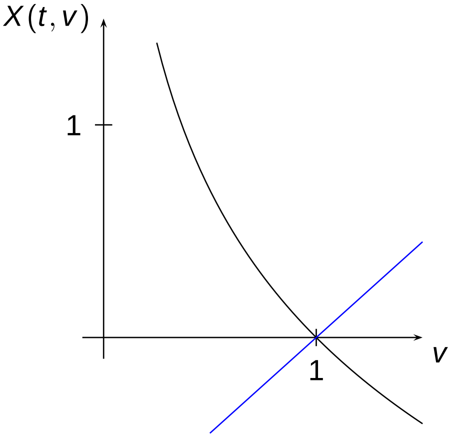

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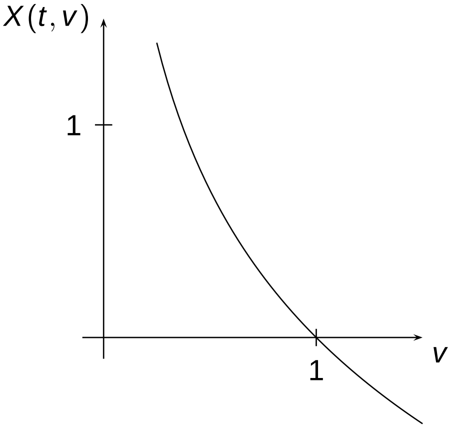

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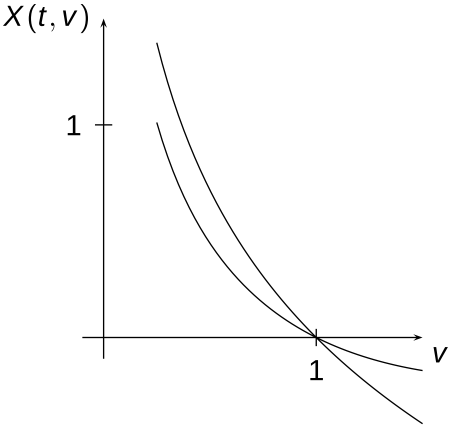

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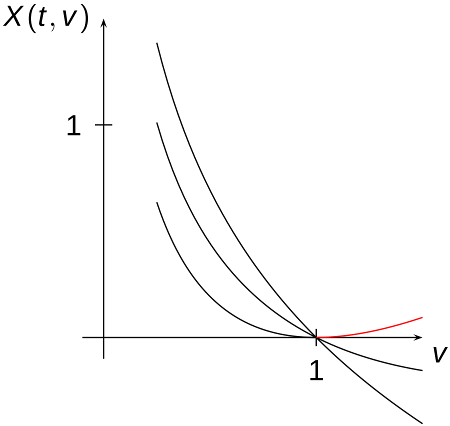

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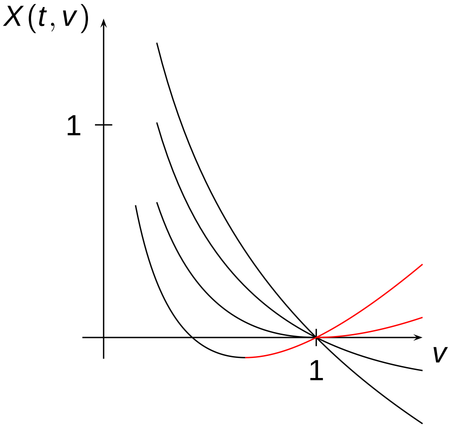

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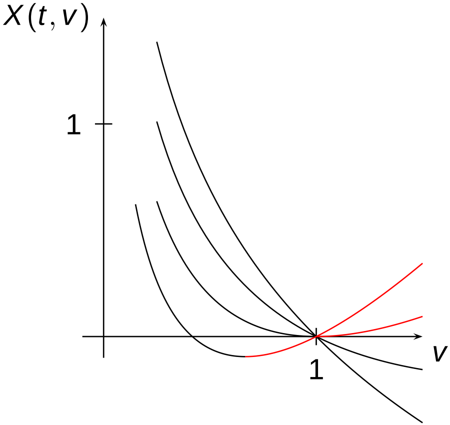
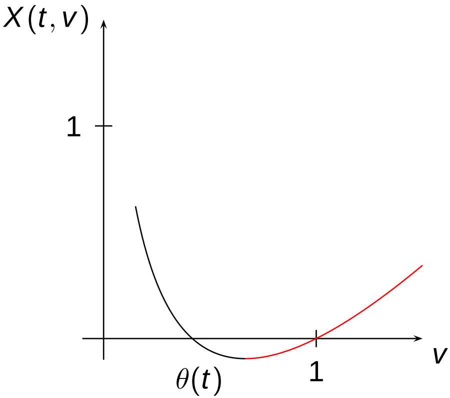
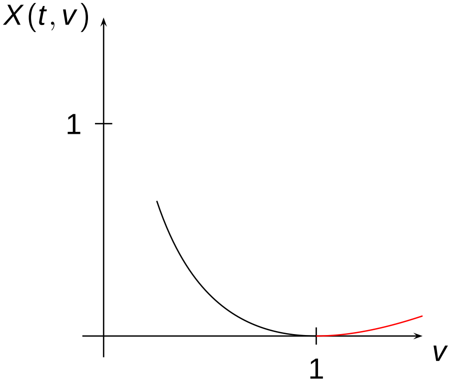


if $t \neq 1$ then $\min_v X(t,v) = v_{min} < 0$, so $V(t,x)$ can be extended analytically to the domain $(v_{min}, +\infty)$, thus v_k decays exponentially.



The size of the giant component: $\theta(t) = V(t,0)$, thus
$X(t,\theta(t)) = 0$



At the critical time: $\partial_v X(t=1, v=1) = 0$, but $\partial_{vv} X(t=1, v=1) > 0$, so $|V(t=1,x) - 1| \asymp \sqrt{x}$, so it follows from Tauberian theory that $v_k(t=1) \asymp k^{-\frac{3}{2}}$

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- The functions $v_k(\cdot)$ are indepentent of the exact decay rate of $\lambda(n)$, we do not have to fine-tune the model to see permanent criticality in the limit: S.O.C.

About the proof

Definition

The critical controlled Smoluchowski equations:

- For $k \geq 2$ we have the usual $\dot{v}_k = -kv_k + \frac{k}{2}\sum_{l=1}^{k-1} v_l v_{k-l}$

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Why „controlled”?

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$$\partial_t V(t,x) = \partial_x V(t,x) - \frac{1}{2}\partial_x V^2(t,x) + \varphi(t)e^{-x}$$

$$V(t,0) \equiv 1 \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} v_k(t) \equiv 1 \quad \Longleftrightarrow \quad \theta(t) \equiv 0$$

Critical controlled Burgers equation

Thank you for your attention

Frequently Asked Questions

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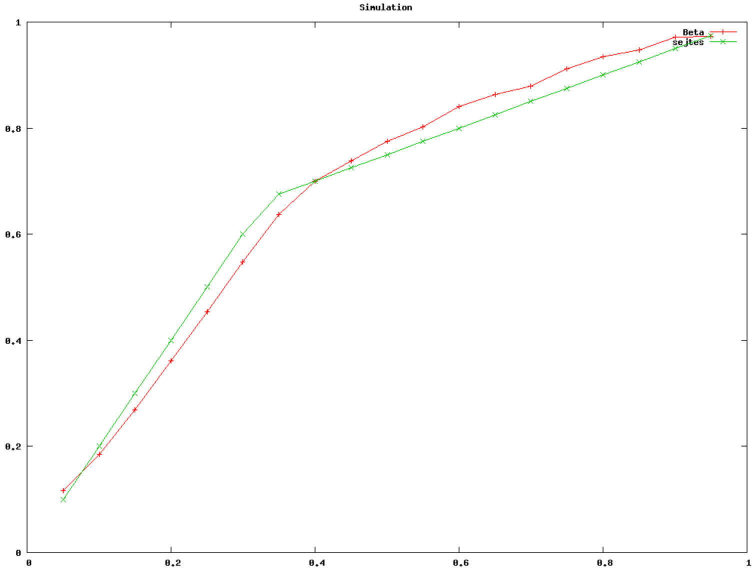
Conjecture:
If $\lambda(n) = n^{-\alpha}$ where $0 < \alpha < 1$, then for $t > 1$ the expected size of the largest component is n^{β}, where

$$\beta = \begin{cases} 2\alpha & \text{if } \alpha \leq \frac{1}{3} \\ \frac{1+\alpha}{2} & \text{if } \alpha \geq \frac{1}{3} \end{cases}$$


Simulation
Beta
sejtes
0
0.2
0.4
0.6
0.8
1
0
0.2
0.4
0.6
0.8
1