

# Pólya's Urn and its Perspective on Point Processes

Mathias Rafler<sup>1</sup>

joint work with Sylvie Roelly<sup>1</sup> and Hans Zessin<sup>2</sup>

<sup>1</sup>Universität Potsdam

<sup>2</sup>Universität Bielefeld

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# The Poisson Process

## Construction of the Poisson Process

Model events in time ( $\mathbb{R}_+$ ) or objects space ( $\mathbb{R}^d$ ).

### Definition (Poisson Process)

There exists exactly one point process with

- 1 for each bounded, measurable  $B$ ,  $N_B \sim Poi(\rho(B))$
- 2  $B_1, \dots, B_n$  disjoint, then  $N_{B_1}, \dots, N_{B_n}$  independent

This point process is the *Poisson process* with *intensity measure*  $\rho$ ,  $\mathbf{P}_\rho$ .

Example:  $\rho = a\lambda$ ,  $a > 0$ .

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## Some Properties of the Poisson Process

- 1 simple iff intensity measure  $\rho$  non-atomic
- 2 conditioned on  $\{N_B = n\}$ , location of points in  $B$  is *iid*.

$$\begin{aligned} \mathbf{P}_\rho(\varphi | N_B = n) \\ = \rho(B)^{-n} \int_B \cdots \int_B \varphi(\delta_{x_1} + \cdots + \delta_{x_n}) \rho(dx_n) \cdots \rho(dx_1) \end{aligned}$$

- 3 local representation

$$\begin{aligned} \mathbf{P}_{\rho, B}(\varphi) = e^{-\rho(B)} \sum_{n \geq 1} \frac{1}{n!} \\ \times \int_B \cdots \int_B \varphi(\delta_{x_1} + \cdots + \delta_{x_n}) \rho(dx_n) \cdots \rho(dx_1) \end{aligned}$$

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## Further Properties of the Poisson Process

5 infinitely divisible:

$$\text{for each } k \geq 1, \mathbf{P}_\rho = \underbrace{\mathbf{P}_{\rho/k} * \dots * \mathbf{P}_{\rho/k}}_{k \text{ times}}$$

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## Motivation and Questions

Motivation and main object

- relax *iid*: replace by Pólya-like dynamic

Questions concern:

- local representation
- independent increments
- infinite divisibility
- fidi distribution
- typical point

# The Pólya Sum Process

## Generalisations of Pólya's Urn Scheme

### Pólya's Urn

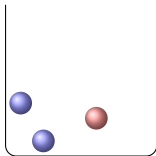
Marbles of two colours, after each draw additional marble.

### Hoppe (1984): Pólya-like Urn

Introduction of new colours by a special marble.

### Blackwell/MacQueen (1973)

Extension to a continuum of colours.



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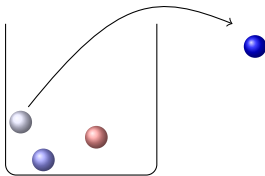
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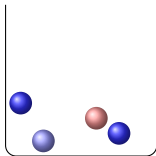
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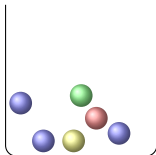
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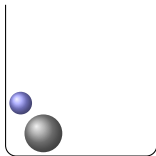
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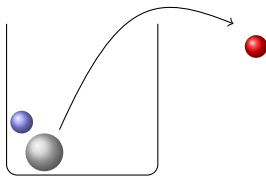
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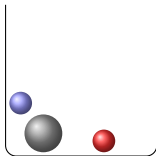
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# The Pólya Sum Process

A detailed view

Inductive construction:

- 1 realise  $x_1$  according to  $\rho$ ,  $\mu_0 := 0$
- 2 set  $\mu_k := \mu_{k-1} + \delta_{x_k}$
- 3 realise  $x_{k+1}$  according to  $\rho + \mu_k$

$$\begin{aligned} S_{z,\rho}(\varphi | N_B = n) &= \rho(B)^{-[n]} \\ &\times \int_B \cdots \int_B \varphi(\mu_n) (\rho + \mu_{n-1})(dx_n) \cdots (\rho + \mu_1)(dx_2) \rho(dx_1) \end{aligned}$$

where

$$a^{[n]} := (a + n - 1) \cdots (a + 1)a$$

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## Local representation of the Pólya Sum Process

Let  $z \in (0, 1)$ ,  $\rho$  a locally bounded measure

$$\begin{aligned} S_{z,\rho,B}(\varphi) &= (1-z)^{\rho(B)} \sum_{n \geq 0} \frac{z^n}{n!} \\ &\times \int_B \cdots \int_B \varphi(\mu_n) (\rho + \mu_{n-1})(dx_n) \cdots (\rho + \mu_1)(dx_2) \rho(dx_1) \end{aligned}$$

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Finite-dimensional distribution

(Zessin 09)

Let  $0 < z < 1$ . Then

$$S_{z,\rho}(N_B = k) = (1 - z)^{\rho(B)} \frac{z^k}{k!} \rho(B)^{[k]}.$$

If  $B_1, \dots, B_m$  disjoint, the  $N_{B_1}, \dots, N_{B_m}$  are independent.

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Further properties

- 1  $S_{z,\rho}$  is *not* simple

$$S_{z,\rho}(\cdot | N_{\{x\}} > 0) = \frac{1-z}{z} \sum_{j \geq 1} z^j S_{z,\rho} * \underbrace{\Delta_x * \dots * \Delta_x}_{j \text{ times}}$$

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Is this the end?

## Pólya Sum Process

- further properties
- models and applications

## Papangelou Processes

- replace  $\rho + \mu$  by more general  $\eta(\mu, \cdot)$   
interacting gases, Cox processes