# Maximum-Likelihood-Estimation of Lévy driven Ornstein-Uhlenbeck Processes

Hilmar Mai
Humboldt-Universität zu Berlin

IRTG Summer School Chorin

3rd Sept 2009

# Ornstein-Uhlenbeck (OU) Process

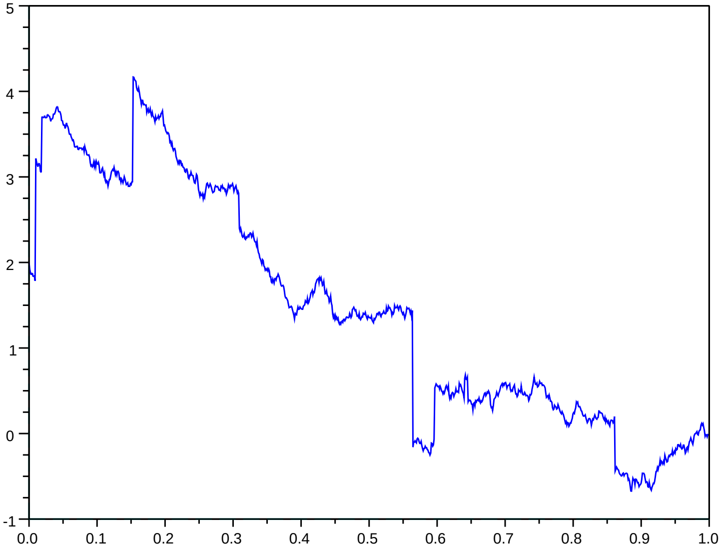
Let $(L_t, t \geq 0)$ be a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ . For every $a \in \mathbb{R}$

$$dX_t = -aX_t\,dt + dL_t, \quad t \in \mathbb{R}_+, \quad X_0 = x, \tag{1}$$

defines an Ornstein Uhlenbeck process driven by the Lévy process $L$ with initial distribution $\pi = \mathcal{L}(X_0)$.
Equivalently,

$$X_t = e^{-at} X_0 + \int_0^t e^{-a(t-s)} dL_s. \tag{2}$$

# Sample path from compound Poisson plus Wiener process driver



# Setting

Problem: Estimation of $a$ from continuous observations $X_t$, $0 \le t \le T$ and known Lévy-Khintchine triplet of $L$.

We work throughout in the canonical setting:

- $\Omega = D(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{R};\ f \text{ càdlàg}\}$
- $X(\omega, t) = \omega(t)$ for all $\omega \in \Omega$ coordinate process
- Filtration generated by $X$

$$\mathcal{F}_t = \bigcap_{s>t} \sigma(X_u : u \le s) \text{ and } \mathcal{F} = \bigvee_t \mathcal{F}_t$$

For every $a \in \mathbb{R}$ we obtain a solution measure $P^a$ of the OU equation on $D(\mathbb{R}_+)$.

# Absolute Continuity/Singularity (ACS) Problem

$P_t^a := P^a_{|\mathcal{F}_t}$ denotes the restriction of $P^a$ to $\mathcal{F}_t$.
Local absolute continuity:

$$P^{a'} \overset{loc}{\ll} P^a \Longleftrightarrow P_t^{a'} \ll P_t^a \quad \forall t \in \mathbb{R}_+$$

1. Does
$$P^{a'} \overset{loc}{\ll} P^a \text{ hold for all } a, a' \in \mathbb{R}?$$
2. Can we derive $Z_t = \frac{dP_t^{a'}}{dP_t^a}$ explicitly?

## First Problem

### Theorem

*Let $P^a, P^{a'}$ be two solution measures of the OU equation for the driving Lévy process L with characteristic triplet $(b, \sigma^2, \rho)$ and initial distributions $\pi$ and $\pi'$. Suppose that $\sigma^2 > 0$ and $\pi' \ll \pi$, then we have*

$$P^{a'} \overset{loc}{\ll} P^a.$$

# Semimartingale characteristics of $X$

Let $(b, \sigma^2, \mu)$ denote the Lévy-Khintchine triplet of $L$.

Then the semimartingale characteristics $(B, C, \nu)$ of $X$ are given by

$$
\begin{aligned}
B(\omega, t) &= bt - a \int_0^t X_{s-}(\omega) ds, \\
C(\omega, t) &= \sigma^2 t, \\
\nu(\omega, dt, dx) &= \mu(dx)\, \lambda(dt),
\end{aligned}
$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$.

# The Hellinger Process of X

## Proposition

*A version of the Hellinger process of two solution measures $P^a, P^{a'}$ is*

$$h_t(\alpha; a, a') = \frac{\alpha(1-\alpha)}{2\sigma^2} \int_0^t \left[ \int_0^u \left( a' e^{-a'(u-s)} - a e^{-a(u-s)} \right) L(ds) \right]^2 du.$$

*for $a, a' \in \mathbb{R}$.*

# Hellinger Process and Absolute Continuity Problems

Let $P, P'$ be two probability measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

**Theorem (Jacod and Mémin (1979))**

*Let $h(\alpha)$, $\alpha \in (0,1)$ be a version of the Hellinger process $h(\alpha; P, P')$. Then for every stopping time $T$ there is equivalence between*

1. $P'_T \ll P_T$;
2. $\exists \alpha \in (0,1)$ *such that* $P'(h(\alpha)_T < \infty) = 1$ *and* $P'_0 \ll P_0$ *and* $P'(h(0)_T = 0) = 1$.

# Second Problem

## Proposition

*There exists a local martingale* $N : \Delta \to \mathbb{R}$ *on a random interval* $\Delta \subset \Omega \times \mathbb{R}_+$ *such that the density process is given by*

$$Z_t = \frac{dP_t^{a'}}{dP_t^a} = Z_0 \exp\left(N_t - \frac{(a'-a)^2}{2\sigma^2}\int_0^t X_{s-}^2 ds\right).$$

*Furthermore, for every stopping time* $S$ *such that* $[0, S] \subset \Delta$ *the stopped process* $N^S$ *is of the form*

$$N^S = \left(\frac{(a'-a)}{\sigma^2} X_{t-} 1_{[0,S]}\right) \cdot X^c,$$

*where* $X^c$ *denotes the continuous martingale part of* $X$ *under* $P^a$.

# Maximum-Likelihood-Estimator (MLE)

For continuous observations of the Ornstein-Uhlenbeck process $X$ the likelihood function $\mathcal{L}$ for the statistical experiment $(\Omega, \mathcal{F}, (\mathcal{F}_t), (P^a)_{a\in\mathbb{R}})$ takes the form

$$\mathcal{L}(a, X^T) = \frac{dP_t^a}{dP_t^0} = \exp\left(-\frac{a}{\sigma^2}\int_0^T X_{s-}\, dX_s^c - \frac{a^2}{2\sigma^2}\int_0^T X_{s-}^2\, ds\right).$$

Hence, the MLE for $a$ is explicitly given by

$$\hat{a}_T = -\frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_{s-}^2 ds}.$$

# The continuous martingale part $X^c$

By the Lévy-Itô decompostition of $L$ we can write $X$ as

$$X_t = X_0 - a \int_0^t X_{s-}\, ds + W_t + J_t \quad , t \geq 0,$$

where $W$ is a Wiener Process and $J$ a quadratic pure jump process given by

$$J_t = \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) + bt + \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}.$$

$N$ is the jump measure of $L$ with compensator $\mu$.

Under $P^0$ it follows that $X^c = W$, but under $P^a$

$$\tilde{W}_t = W_t - a \int_0^t X_{s-}\, ds$$

defines a Wiener process such that $X^c = \tilde{W}$ under $P^a$. Hence, given observations $(X_t(\omega), t \in [0, T])$

$$X_t^c = X_t - \int_{\{|x|<1\}} x(N_t(dx) - t\mu(dx)) - bt - \sum_{0 \le s \le t} \Delta X_s 1_{\{|\Delta X_s| \ge 1\}}.$$

which can be reconstructed from continuous observations. Hence, the MLE can be rewritten as

$$\hat{a}_T = -\frac{\int_0^T X_{s-}(dW_s - aX_{s-} ds)}{\int_0^T X_{s-}^2 ds} = a - \frac{\int_0^T X_{s-} dW_s}{\int_0^T X_{s-}^2 ds}.$$

under $P^a$.

# Curved Exponential Families

Let $\{P^\theta, \theta \in \Theta\}$ be a family of measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$.

## Definition (Küchler and Sørensen (1997))

*We say that a statistical experiment* $\{P^\theta, \theta \in \Theta\}$ *forms a* ***curved exponential family*** *if the likelihood function exists and is of the form*

$$\frac{dP_t^\theta}{dP_t^{\theta_0}} = \exp\left(\theta' A_t - \kappa(\theta) S_t\right)$$

*where* $\kappa : \Theta \to \mathbb{R}$*, for* $\theta_0 \in \Theta$ *arbitrary but fixed and* $A : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *is a càdlàg process. Moreover,* $S : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *is assumed to be a non-decreasing continuous process with* $S_0 = 0$ *and* $S_t \xrightarrow{t\to\infty} \infty$*.*

# Strong Consistency

## Theorem

*Under the condition $\sigma^2 > 0$ the MLE $\hat{a}_T$ for any $a \in \mathbb{R}$ based on continuous observations $X_t$, $t \in [0, T]$ exists and is given by*

$$\hat{a}_T = -\frac{\int_0^T X_{s-} dX_s^c}{\int_0^T X_{s-}^2 ds}.$$

*Furthermore, under $P_a$ the maximum likelihood estimator is unique for $T$ sufficiently large and*

$$\hat{a}_T \to a \quad \textit{almost surely}$$

*as $T \to \infty$.*

# Asymptotic Normality

Theorem

*Let $X$ be a stationary OU process and $a > 0$ such that $c = E[X^2] < \infty$. Then under $P^a$*

$$\sqrt{T}(\hat{a}_T - a) \to N(0, \frac{\sigma^2}{c}) \quad \textit{weakly}$$

*as $T \to \infty$.*

# Discrete Observations

Given discrete observations $X_\Delta, X_{2\Delta}, \ldots, X_{n\Delta}$ with step size $\Delta$ a dicretized version of $\hat{a}$ is

$$\hat{a}_n = -\frac{\sum_{m=0}^{n-1} X_{m\Delta}\ \delta X_m^c}{\Delta \sum_{m=0}^{n-1} X_{m\Delta}}$$

with increments $\delta X_m^c = X_{(m+1)\Delta}^c - X_{m\Delta}^c$.

## Theorem

*Under the assumption that $X$ is stationary and $K_a(t) = E_a(X_t X_0)$ is continuously differentiable in $t$ the MLE satisfies*

$$\hat{a}_n \stackrel{n\to\infty}{\longrightarrow} a + \Delta a \frac{\dot{K}_a(0)}{E_a(X_0^2)} + O(\Delta^2) \qquad P_a\text{-}a.s.$$

# Proof

We rewrite $\hat{a}_n = \frac{A_n}{I_n} = -\frac{\sum_{m=0}^{n-1} X_{m\Delta}\ \delta X_m}{\Delta \sum_{m=0}^{n-1} X_{m\Delta}}$ as

$$
\begin{aligned}
A_n &= \sum_{m=0}^{n-1} X_{m\Delta}\ \delta W_m - a \sum_{m=0}^{n-1} X_{m\Delta} \int_{m\Delta}^{(m+1)\Delta} X_{s-}\ ds \\
&= \sum_{m=0}^{n-1} X_{m\Delta}\ \delta W_m - a \sum_{m=0}^{n-1} X_{m\Delta} \int_{m\Delta}^{(m+1)\Delta} (X_{s-} - X_{m\Delta})\ ds - a\Delta \sum_{m=0}^{n-1} X \\
&=: Z_n - aR_n - aI_n
\end{aligned}
$$

By ergodicity of $X$ it follows that $P_a$-a.s.

$$\frac{I_n}{n} \stackrel{n\to\infty}{\longrightarrow} \Delta n, \quad \frac{Z_n}{n} \stackrel{n\to\infty}{\longrightarrow} 0,$$

and

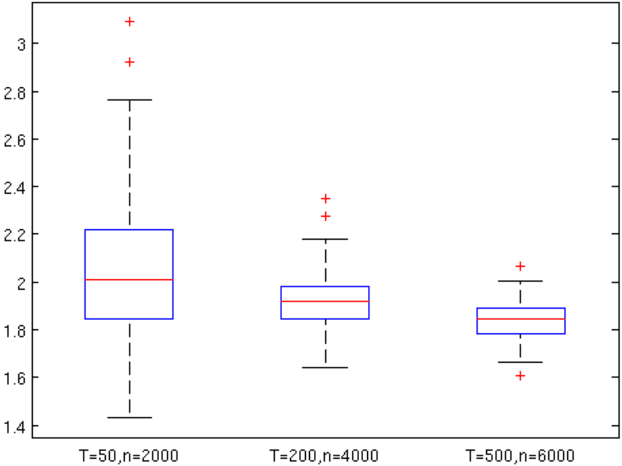
$$\frac{R_n}{n} \stackrel{n\to\infty}{\longrightarrow} \int_0^\Delta K_a(t)\,dt - \Delta E(X_0^2),$$

such that finally

$$\hat{a}_n = \frac{Z_n}{I_n} - a\frac{R_n}{I_n} - a \stackrel{n\to\infty}{\longrightarrow} a\frac{\int_0^\Delta K_a(t)\,dt}{\Delta E_a(X_0^2)} \qquad P_a\text{-a.s.}$$

$\square$

Boxplot for $\hat{a}_n$ from a Wiener process plus compound Poisson ($\lambda = 4$, N(0,1)-jumps) driver and true parameter $a = 2$.



# Summary and Outlook

- Under $\sigma^2 > 0$ the solution measures $\{P^a, a \in \mathbb{R}\}$ are locally equivalent.
- The MLE takes an explicit form and is efficient and asymptotically normal.
- Computation from discrete observations is straight forward.
- Asymptotics for $a < 0$ and without second moments of $X$?
- Delay estimation

# Bibliography


Ole E. Barndorff-Nielsen and Neil Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc.*, 63(2), 2001.

Yaozhong Hu and Hongwei Long. Least squares estimator for ornstein-uhlenbeck processes driven by [alpha]-stable motions. *Stochastic Processes and their Applications*, 2009.

G. Jongbloed, F.H. van der Meulen, and A.W. van der Vaart. Nonparametric inference for Lévy-driven Ornstein-Uhlenbeck processes. *Bernoulli*, 11(5), 2005.

L. Valdivieso, W. Schoutens, and F. Tuerlinckx. Maximum Likelihood Estimation in Processes of Ornstein-Uhlenbeck type. *Stat. Infer. Stoch Process.*, 12:1–19, 2009.