On ballisticity criteria for RWRE

Alexander Drewitz

TU Berlin, IRTG SMCP

Chorin

September 01, 2009

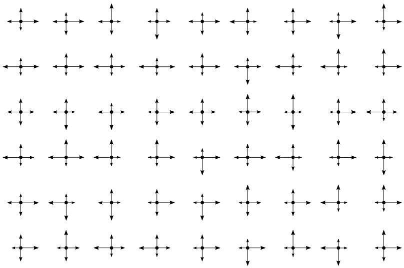
(joint work with Alejandro Ramírez)

Contents

Model

Heuristics:

- *RWRE:* $\mathbb{Z}^d$-valued nearest-neighbour "random walk" with transition probabilities randomly chosen for each site;
- *environment:* family of transition probabilities (indexed by $\mathbb{Z}^d$);



Model

Formally:

- *environment* $\omega \in \Omega := (\mathcal{M})^{\mathbb{Z}^d}$, $\mathcal{M}$ the set of probability measures on $\{e \in \mathbb{Z}^d : |e| = 1\}$;

  given environment $(\omega(z, \cdot))_{z \in \mathbb{Z}^d}$, *quenched RWRE* is the Markov chain (X_n) with state space $\mathbb{Z}^d$ starting in x with law $P_{x,\omega}$ s.t.

$$P_{x,\omega}(X_{n+1} = z + e | X_n = z) := \omega(z, e),$$

  $z \in \mathbb{Z}^d$, $|e| = 1$.

- *environment measure* $\mathbb{P} = \mu^{\mathbb{Z}^d}$, μ measure on $\mathcal{M}$ with $\mu(\inf_e \omega(0, e) \geq \kappa) = 1$ some $\kappa > 0$.

  Annealed (or *averaged*) *measures*

$$P_x = \int_\Omega P_{x,\omega} \, \mathbb{P}(d\omega).$$

Model

Formally:

- *environment* $\omega \in \Omega := (\mathcal{M})^{\mathbb{Z}^d}$, $\mathcal{M}$ the set of probability measures on $\{e \in \mathbb{Z}^d : |e| = 1\}$;

  given environment $(\omega(z, \cdot))_{z \in \mathbb{Z}^d}$, *quenched RWRE* is the Markov chain (X_n) with state space $\mathbb{Z}^d$ starting in x with law $P_{x,\omega}$ s.t.

  $$P_{x,\omega}(X_{n+1} = z + e | X_n = z) := \omega(z, e),$$

  $z \in \mathbb{Z}^d$, $|e| = 1$.
- *environment measure* $\mathbb{P} = \mu^{\mathbb{Z}^d}$, μ measure on $\mathcal{M}$ with $\mu(\inf_e \omega(0, e) \geq \kappa) = 1$ some $\kappa > 0$.
  Annealed (or *averaged*) *measures*

  $$P_x = \int_\Omega P_{x,\omega} \, \mathbb{P}(d\omega).$$

LLN

Theorem (Sznitman, Zerner, [SZ99], [Zer02])

Assume

$$P_0(\lim_n X_n \cdot e = \infty) \in \{0, 1\} \quad \forall e \in \mathbb{Z}^d \textit{ with } |e| = 1.$$

Then there exists $v \in \mathbb{R}^d$ deterministic s.t. P_0-a.s.

$$\lim_n \frac{X_n}{n} = v.$$

Question: For $d \geq 2$, can we find characterisations of *ballisticity*, i.e. of the case $v \neq 0$?

LLN

Theorem (Sznitman, Zerner, [SZ99], [Zer02])

Assume

$$P_0(\lim_n X_n \cdot e = \infty) \in \{0, 1\} \quad \forall e \in \mathbb{Z}^d \text{ with } |e| = 1.$$

Then there exists $v \in \mathbb{R}^d$ deterministic s.t. P_0-a.s.

$$\lim_n \frac{X_n}{n} = v.$$

Question: For $d \geq 2$, can we find characterisations of *ballisticity*, i.e. of the case $v \neq 0$?

Ballisticity criteria

Definition (Sznitman)

For $\gamma \in (0, 1)$, $l \in \mathbb{S}^{d-1}$ *condition* $(T)_\gamma | l$ *is satisfied* if for each l' in a neighbourhood of l and for all $b > 0$ one has

$$\limsup_{L \to \infty} L^{-\gamma} \log P_0(X_{T_{U_{l',b,L}}} \cdot l' < 0) < 0,$$

where $U_{l',b,L} := \{x \in \mathbb{Z}^d : -bL < x \cdot l' < L\}$.
Furthermore: $(T)' | l :\Longleftrightarrow (T)_\gamma | l$ holds for all $\gamma \in (0, 1)$.

Ballisticity criteria

[Szn02]:

1. $(T')|l$ implies ballisticity.
2. For $\gamma \in (\frac{1}{2}, 1)$: $\quad (T)_\gamma|l \iff (T')|l.$

Theorem (D, Ramírez, [DR09])

Let $d \geq 2$ and

$$\gamma_d := \frac{\sqrt{3d^2 - d} - d}{2d - 1}.$$

Then, for each $\gamma \in (\gamma_d, 1)$ and $l \in \mathbb{S}^{d-1}$, $(T)_\gamma|l$ is equivalent to $(T')|l$.

Remark

Depending on $d \in \mathbb{N}$, γ_d takes values in the interval $(0.366, 0.388)$.

Ballisticity criteria

[Szn02]:

1. $(T')|l$ implies ballisticity.
2. For $\gamma \in (\frac{1}{2}, 1)$: $\quad (T)_\gamma|l \iff (T')|l.$

Theorem (D, Ramírez, [DR09])

Let $d \geq 2$ and

$$\gamma_d := \frac{\sqrt{3d^2 - d} - d}{2d - 1}.$$

Then, for each $\gamma \in (\gamma_d, 1)$ and $l \in \mathbb{S}^{d-1}$, $(T)_\gamma|l$ is equivalent to $(T')|l$.

Remark

Depending on $d \in \mathbb{N}$, γ_d takes values in the interval $(0.366, 0.388)$.

Main ideas of the proof

- *Recall:* In $d = 1$, Solomon [Sol75] showed that (X_n) is ballistic (to the right) iff
$$\mathbb{E}\,\frac{\omega(0,-1)}{\omega(0,1)} < 1.$$
- *Higher-dimensional "analogue":*

Theorem (Sznitman, [Szn02])

Effective criterion $\iff (T')|\hat{v}$.

Sufficient for establishing effective criterion: show that for some $\alpha > 0$,
$$\mathbb{E}\,\rho^{L^{-\alpha}} \to 0$$
fast enough as $L \to \infty$, where
$$\rho(\omega) := \frac{P_{0,\omega}(X_{T_B} \notin \partial_+ B)}{P_{0,\omega}(X_{T_B} \in \partial_+ B)}.$$

Main ideas of the proof

- *Recall:* In $d = 1$, Solomon [Sol75] showed that (X_n) is ballistic (to the right) iff

$$\mathbb{E}\,\frac{\omega(0,-1)}{\omega(0,1)} < 1.$$

- *Higher-dimensional "analogue":*

Theorem (Sznitman, [Szn02])

Effective criterion $\iff (T')|\hat{v}$.

Sufficient for establishing effective criterion: show that for some $\alpha > 0$,

$$\mathbb{E}\,\rho^{L^{-\alpha}} \to 0$$

fast enough as $L \to \infty$, where

$$\rho(\omega) := \frac{P_{0,\omega}(X_{T_B} \notin \partial_+ B)}{P_{0,\omega}(X_{T_B} \in \partial_+ B)}.$$

Main ideas of the proof

Decomposition

$$\mathbb{E}\,\rho^{L^{-\alpha}} = (I) + (II) + (III),$$

where

$$(I) := \mathbb{E}\left(\rho^{L^{-\alpha}}, P_{0,\omega}(X_{T_B} \in \partial_+ B) > e^{-k_0 L^{\gamma}}\right)$$

as well as

$$(II) := \mathbb{E}\left(\rho^{L^{-\alpha}}, e^{-k_1 L^{\beta_1}} < P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_0 L^{\gamma}}\right)$$

and

$$(III) := \sum_{j=1}^{n} \mathbb{E}\left(\rho^{L^{-\alpha}}, e^{-k_{j+1} L^{\beta_{j+1}}} < P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}\right),$$

with constants β_j s.t.

$$1 = \beta_{n+1} > \beta_n > \cdots > \beta_1 > (1+\gamma)^{-1}.$$

Estimate first summand

$$(I) := \mathbb{E}\left(\rho^{L^{-\alpha}}, P_{0,\omega}(X_{T_B} \in \partial_+ B) > e^{-k_0 L^{\gamma}}\right)$$

Lemma

For all $L > 0$,

$$(I) \leq e^{k_0 L^{\gamma-\alpha} - \delta_1 L^{\gamma-\alpha} + o(L^{\gamma-\alpha})},$$

where

$$\delta_1 := -\limsup_{L\to\infty} L^{-\gamma} \log P_0(X_{T_B} \notin \partial_+ B) > 0.$$

Proof.

Jensen implies $(I) \leq e^{k_0 L^{\gamma-\alpha}} P_0(X_{T_B} \notin \partial_+ B)^{L^{-\alpha}}$.
$(T)_\gamma$ entails $P_0(X_{T_B} \notin \partial_+ B)^{L^{-\alpha}} \leq e^{-\delta_1 L^{\gamma-\alpha} + o(L^{\gamma-\alpha})}$. □

$\Longrightarrow$ Choose $\alpha < \gamma$.

Estimate second summand

$$(II) := \mathbb{E}\left(\rho^{L^{-\alpha}}, e^{-k_1 L^{\beta_1}} < P_{0,\omega}(X_{T_B} \in \partial_+ B) \le e^{-k_0 L^{\gamma}}\right)$$

Lemma

For all $L > 0$,

$$(II) \le e^{k_1 L^{\beta_1 - \alpha} - \delta_1 L^{\gamma} + o(L^{\gamma})}.$$

Proof.

$$\begin{aligned}(II) &\le e^{k_1 L^{\beta_1 - \alpha}} \mathbb{E}(P_{0,\omega}(X_{T_B} \notin \partial_+ B)^{L^{-\alpha}}, P_{0,\omega}(X_{T_B} \notin \partial_+ B) \ge 1 - e^{-k_0 L^{\gamma}}) \\ &\le e^{k_1 L^{\beta_1 - \alpha}} P_0(X_{T_B} \notin \partial_+ B)(1 - e^{-k_0 L^{\gamma}})^{-1}.\end{aligned}$$

□

$\Longrightarrow$ Choose $\beta_1 < 2\gamma,\ \alpha \in (\beta_1 - \gamma, \gamma)$.

Estimate third summand

Lemma

Let the β_j's be chosen as above. Then, for all $j \in \{1, \ldots, n\}$ and $\zeta \in (0, f(\beta_j))$,

$$\mathbb{E}\left(\rho^{L^{-\alpha}}, e^{-k_{j+1}L^{\beta_{j+1}}} \leq P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-k_j L^{\beta_j}}\right) \leq e^{k_{j+1}L^{\beta_{j+1}-\alpha} - \delta_2 L^\zeta + o(L^\zeta)}$$

where f is defined by

$$f : ((1+\gamma)^{-1}, 1) \ni \beta \mapsto d\Big(\beta - \frac{1}{1+\gamma}\Big)\frac{1+\gamma}{\gamma},$$

and

$$\delta_2 := -\limsup_{L\to\infty} L^{-\zeta} \log \mathbb{P}(P_{0,\omega}(X_{T_B} \in \partial_+ B) \leq e^{-cL^{\beta_j}}) > 0.$$

(Proof omitted, via renormalisation estimate from [Szn01])

$$\Longrightarrow \text{Need } \beta_{j+1} - \alpha < f(\beta_j)$$

Fixed-point iteration

1. Conditions from estimating (*I*) to (*III*) :

$$\beta_1 < 2\gamma, \quad \alpha \in (\beta_1 - \gamma, \gamma), \quad \beta_{j+1} - \alpha < f(\beta_j).$$

2. Leads to

$$\beta_{j+1} < f(\beta_j) + \gamma.$$

3. Unstable fixed point of the iteration of $x \mapsto f(x) + \gamma$ is given by

$$x^* := \frac{d - \gamma^2}{(1+\gamma)d - \gamma}.$$

4. Finally, the starting condition $\beta_1 > x^*$ yields

$$\gamma > \gamma_d = \frac{-2d + \sqrt{12d^2 - 4d}}{2(2d-1)}.$$

Open questions

1. Can one lower γ_d to 0 (for all $d \geq 2$)?
2. Does ballisticity imply $(T)'|\hat{v}$?

Alexander Drewitz and Alejandro F. Ramírez.
Ballisticity conditions for random walk in random environment.
arXiv:0903.4465, 2009.

Fred Solomon.
Random walks in a random environment.
Ann. Probab., 3:1–31, 1975.

Alain-Sol Sznitman and Martin Zerner.
A law of large numbers for random walks in random environment.
Ann. Probab., 27(4):1851–1869, 1999.

Alain-Sol Sznitman.
On a class of transient random walks in random environment.
Ann. Probab., 29(2):724–765, 2001.

Alain-Sol Sznitman.
An effective criterion for ballistic behavior of random walks in random environment.
Probab. Theory Related Fields, 122(4):509–544, 2002.

Martin P. W. Zerner.
A non-ballistic law of large numbers for random walks in i.i.d. random environment.
Electron. Comm. Probab., 7:191–197 (electronic), 2002.