

Quantum spin systems: a dynamical approach

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1 Introduction

- Quantum spin systems
- Quantum models with a transverse field

2 Spectral gap estimates

- The Bochner-Bakry-Emery (or Γ_2) method

3 Decay of correlations

- Basic definitions
- Exponential decay of correlations

Classical spin systems

A classical spin system consists of

- $\Lambda \subset \mathbb{Z}^d$, $d \geq 1$;
- a spin configuration $\sigma \in \Omega_\Lambda := \{-1, +1\}^\Lambda$;
- an Hamiltonian $H : \Omega_\Lambda \rightarrow \mathbb{R} \rightsquigarrow$ the Gibbs measure

$$\mu(\sigma) := \frac{1}{Z} e^{-\beta H(\sigma)},$$

where $\beta := 1/T$.

How can we view it in quantum terms?

Quantum reformulation of the classical model

- ± 1 are dressed as quantum eigenfunctions of the Pauli matrix

$$\hat{\sigma}^z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and are indicated as}$$

$$|1\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad | -1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

- $\Omega_\Lambda \ni \sigma \rightsquigarrow |\sigma\rangle := \otimes_{i \in \Lambda} |\sigma_i\rangle \in \mathbb{X}_\Lambda := \otimes_{i \in \Lambda} \mathbb{R}^2$;
- \mathbb{X}_Λ becomes a Hilbert space with a scalar product: $\langle \sigma | \sigma' \rangle := \prod_{i \in \Lambda} \langle \sigma_i | \sigma'_i \rangle_2$. $\mathcal{B} := \{|\sigma\rangle\}_{\sigma \in \Omega_\Lambda}$ is an orthonormal basis of the space.
- For $i \in \Lambda$, $\hat{\sigma}_i^z |\sigma\rangle := |\sigma_1\rangle \otimes \dots \otimes \hat{\sigma}^z |\sigma_i\rangle \otimes \dots \otimes |\sigma_{|\Lambda} \rangle$. So

$$\hat{\sigma}_i^z \hat{\sigma}_j^z |\sigma\rangle = \sigma_i \sigma_j |\sigma\rangle$$

Rephrasing of functions

Definition

Given $f : \Omega_\Lambda \rightarrow \mathbb{R}$,

$$\mathcal{F} : \mathbb{X}_\Lambda \rightarrow \mathbb{X}_\Lambda \quad \text{s. t.} \quad \langle \sigma | \mathcal{F} | \sigma \rangle := f(\sigma), \quad \forall |\sigma\rangle \in \mathcal{B} \quad (1)$$

Notation When $f := H$, the associated quantum operator is \mathcal{H}_1 .

- The partition function $Z := \sum_{\sigma \in \Omega_\Lambda} e^{-\beta H(\sigma)} = \text{Tr}(e^{-\beta \mathcal{H}_1})$.
- $\langle \mathcal{F} \rangle = \frac{\text{Tr}(\mathcal{F} e^{-\beta \mathcal{H}_1})}{\text{Tr}(e^{-\beta \mathcal{H}_1})}$.
- $\langle \mathcal{F}; \mathcal{G} \rangle = \langle \mathcal{F} \mathcal{G} \rangle - \langle \mathcal{F} \rangle \langle \mathcal{G} \rangle$.

Definition of a quantum model with transverse field

We now change the Hamiltonian in this way: setting

$\hat{\sigma}^x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\hat{\sigma}_i^x$ as before, we introduce a *transverse field* of intensity $\lambda \geq 0$:

$$-\mathcal{H} := -\mathcal{H}_1 + \lambda \sum_{i \in \Lambda} \hat{\sigma}_i^x$$

\mathcal{H}_1 and $\sum_{i \in \Lambda} \hat{\sigma}_i^x$ do not commute.

Goal: represent the new model in classical terms.

Path-integral representation

We do this in two steps:

- 1 for every $i \in \Lambda$ throw points $\sim PPP(\lambda)$ on the loop \mathbb{S}_β . The arrival configuration is denoted with ξ_i .



- 2 Color the connected components of each loop with two colors, independently and with probability $1/2$. The coloring is coded by a function $\tau_i(t)$, $t \in [0, \beta)$.



Let $\mathcal{D} := \{\sigma : \mathbb{S}_\beta \rightarrow \{-1, +1\} : \text{piece - wise constant, right continuous}\}$ and \mathcal{S} the set of finite subsets of \mathbb{S}_β .

Definition (*Spin configuration*)

$$\{(\xi_i, \tau_i(\cdot))\}_{i \in \Lambda} \in (\mathcal{S} \times \mathcal{D})^\Lambda.$$

$P(d\xi_i) :=$ Poisson measure of intensity λ on one site.

$$\pi(\tau, d\xi) := \frac{1}{Z} \otimes P(d\xi) \mathbf{1}_{\{\tau \sim \xi\}} \exp \left[\int_0^\beta -H(\tau(t)) dt \right]$$

Theorem (Quantum mean of an observable)

For \mathcal{F} as in (1)

$$\langle \mathcal{F} \rangle = \int f(\tau(0)) \pi(\tau, d\xi)$$

that is the quantum mean of \mathcal{F} is an average w. r. t. a classical probability measure.

Definition of the dynamics

Definition

If $\sigma \in \mathcal{S}^\Lambda$, $a \in \mathcal{S}$, we denote with $\sigma^i a$ the element of \mathcal{S}^Λ obtained from σ by replacing σ_i by a .

Set of transitions G :

$$\gamma_{i,+}^{x,a,b}(\sigma, \xi) := \begin{cases} (\sigma^i b, \xi \overset{i}{\cup} \{x\}) & \text{if } \sigma_i = a, b \sim \xi_i \cup \{x\} \\ (\sigma, \xi) & \text{otherwise} \end{cases}$$

$$\gamma_{i,-}^{x,a,b}(\sigma, \xi) := \begin{cases} (\sigma^i b, \xi \overset{i}{\setminus} \{x\}) & \text{if } x \in \sigma_i, \sigma_i = a, b \sim \xi_i \setminus \{x\} \\ (\sigma, \xi) & \text{otherwise} \end{cases}$$

$$\nabla_\gamma f(\sigma, \xi) := f(\gamma(\sigma, \xi)) - f(\sigma, \xi).$$

We construct a dynamics reversible w. r. t. π with generator

$$\begin{aligned} \mathcal{L}f(\sigma, \xi) := & \\ & \lambda \sum_{i \in \Lambda} \int_0^\beta dx \frac{1}{2^{\#\xi_i + 1}} \sum_{a \sim \xi_i \cup \{x\}} \exp \left[\int_0^\beta \left(\nabla_{\gamma_{i,+}^{x, \sigma_i, a}} H(\sigma(t)) \right) dt \right] \times \\ & \times \left[\nabla_{\gamma_{i,+}^{x, \sigma_i, a}} f((\sigma, \xi)) \right] + \\ & + \sum_{i \in \Lambda} \frac{1}{2^{\#\xi_i}} \sum_{y \in \xi_i} \sum_{\eta \sim \xi_i \setminus \{y\}} \left[\nabla_{\gamma_{i,-}^{y, \sigma_i, \eta}} f((\sigma, \xi)) \right]. \end{aligned}$$

Properties of the Hamiltonian

We require that $\forall \gamma \in G$ the following hold:

- Bounded gradient

$$\|\nabla_\gamma H\|_{L^\infty(\pi)} \leq C \quad (\text{P1})$$

- Local function of range R : if γ acts on the i -th site,

$$\nabla_\gamma H(\sigma) = \nabla_\gamma H(\sigma') \quad (\text{P2})$$

if $\sigma_k = \sigma'_k, \forall k : d(i, k) \leq R$.

Spectral gap

Using the Bochner-Bakry-Emery method we obtain the following

Theorem

If (P1)-(P2) are satisfied, for every β sufficiently small there exists $k > 0$ such that

$$\text{gap}(\mathcal{L}) > k$$

uniformly in the volume $|\Lambda|$.

Local functions and decay of correlations

Let $\Lambda_f \subset \Lambda$, $\Lambda_g \subset \Lambda$, and $\Lambda_f \cap \Lambda_g = \emptyset$; let f (resp. g) be a local function on Λ_f (resp. Λ_g); \mathcal{F} , \mathcal{G} as in (1).

Definition

A quantum spin model has *decay of covariances* if

$$\langle \mathcal{F}; \mathcal{G} \rangle = o(d(\Lambda_f, \Lambda_g))$$

Definition

A quantum model has an *exponential decay of correlations* if there exist C , C' constants s. t.

$$|\langle \mathcal{F}; \mathcal{G} \rangle| \leq C e^{-C' d(\Lambda_f, \Lambda_g)}$$

Positivity of the spectral gap \Rightarrow decay of correlations

Theorem (Exponential decay of correlations)

Let H be an Hamiltonian with (P1)-(P2). If $\text{gap}(\mathcal{L}) > 0$ then the following holds: for every $\Lambda_f \subset\subset \Lambda$, $\Lambda_g \subset\subset \Lambda$ and $\Lambda_f \cap \Lambda_g = \emptyset$, f (resp. g) local on Λ_f (resp. Λ_g), there exist $C < +\infty$, $C' < +\infty$ constants not depending on $|\Lambda|$ s. t.

$$|\langle \mathcal{F}; \mathcal{G} \rangle| = |\pi[f; g]| \leq C e^{-C' d(\Lambda_f, \Lambda_g)}$$

Sketch of the proof.

Lemma There are constants $A < +\infty$, $B < +\infty$, $\delta > 0$ that depend on $|\Lambda_f|$, $|\Lambda_g|$, $\|f\|_{L^2(\pi)}$, $\|g\|_{L^2(\pi)}$, $\|f\|$ and $\|g\|$ s. t. for all t

$$|\pi[S_t(fg) - S_t f S_t g]| \leq A e^{Bt - \delta d(\Lambda_f, \Lambda_g)}$$

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$$\begin{aligned} |\pi[f; g]| &= |\pi[S_t(fg)]| = \\ &= |\pi[S_t f S_t g] + \pi[S_t(fg) - S_t f S_t g]| \leq \\ &\leq \|S_t f\|_{L^2(\pi)} \|S_t g\|_{L^2(\pi)} + |\pi[S_t(fg) - S_t f S_t g]| \end{aligned}$$

by Cauchy-Schwarz inequality.

- By Poincaré inequality ($G := 1/k$)

$$|\pi[f; g]| \leq e^{-\frac{2t}{G}} \|f\|_{L^2(\pi)} \|g\|_{L^2(\pi)} + Ae^{Bt - \delta d(\Lambda_f, \Lambda_g)}$$

- If A and B are chosen as in the Lemma, set $t := \frac{\delta}{2B} d(\Lambda_f, \Lambda_g)$ and the proof is concluded.

