Local degree distribution in scale free random graphs

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Scale free graphs

- definition of Barabási and Albert for evolving random graphs ([1], 1999)
- step by step, one new vertex is added to the graph and it is connected to some of the old vertices randomly
- the proportion of the vertices of degree d tends to some c_d almost surely for all $d \geq 0$
  (c_d) is the *asymptotic degree distribution*
- *scale free property*: $c_d \sim K \cdot d^{-\gamma} \quad (d \to \infty)$
- $\gamma > 0$ is the *characteristic exponent*

Barabási tree

Example (Barabási tree)

We start from one edge, one of its endpoints is the root. At each step, the new vertex is connected to one old vertex. The probability that an old vertex of degree d is chosen is proportional to d.
It is well known that the proportion of vertices of degree d almost surely tends to

$$c_d = \frac{4}{(d+2)(d+1)d} \quad (d \geq 1)$$

and

$$\gamma = 3.$$

See e. g. Barabási, Albert, 1999, [1].

Local degree distribution

- in many scale free random graph models the characteristic exponent can be estimated empirically from the whole graph
- is this an appropriate method if only a part of the graph is known?
- for example, we sit in a fixed vertex and we only know the degrees of our neighbours
- we can determine the proportion of vertices of degree d among our neighbours
- when the number of vertices tends to infinity, we get the *asymptotic local degree distribution*

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- when the number of vertices tends to infinity, we get the *asymptotic local degree distribution*
- *in several models, this differs from the original degree distribution, moreover, the characteristic exponent decreases*

Local degree distribution on the Barabási tree

Example

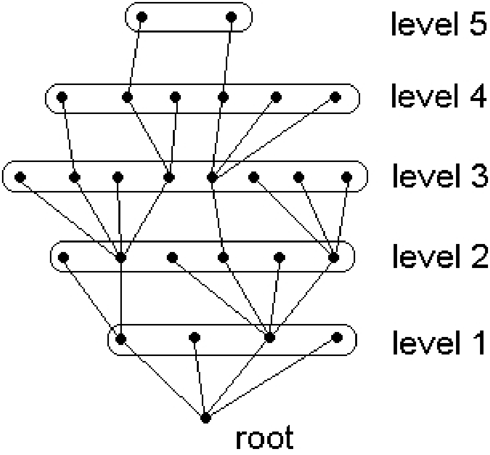
Constrained on the neighbours of the root, the proportion of vertices of degree d tends to

$$x_d = \frac{1}{d\,(d+1)} \quad (d \geq 1)$$

almost surely. The characteristic exponent of the asymptotic local degree distribution is 2.

The jth level of the tree consists of the vertices that are at distance j from the root.
Examining one level of the tree, the characteristic exponent is also 2 (Móri, 2006, [2]).

Levels of the Barabási tree



The problem

The phenomenon described above appears in several scale free random graph models.
Our aim was to give sufficient conditions for the existence of local degree distribution, based on the common properties of the known models:

- scale free property with characteristic exponent $\gamma > 1$;
- the set of the selected vertices is regularly growing with exponent $0 < \alpha \leq 1$.

Another question was to determine the new characteristic exponent from γ and α.

Notations

- $G_n = (V_n, E_n)$ is a sequence of random graphs
- $V_n = \{v_1, \ldots, v_n\}$
- every edge from $E_n \setminus E_{n-1}$ starts from v_n
- $S_n \subseteq V_n$ is the set of selected vertices
- $X_s[n, d]$ is the number of vertices of degree d in S_n
  the degree is counted in G_n
- m is the minimal initial degree of the new vertex

Conditions on the graph model

- scale free asymptotic degree distribution (c_d), such that $c_d > 0$ for $d \geq m$
- at each step, old vertices of the same degree get new edges with the same probability
- the initial degree of the new vertex has an a.s. asymptotic degree distribution (q_d) in the sense that the proportion of vertices with initial degree d converges to q_d a.s.
- uniformly in n, the distribution of the initial degree of the new vertex is bounded by a distribution with finite moment generating function

Conditions on the set of selected vertices

- $S_n \subseteq S_{n+1}$
- after choosing its neighbours, decide immediately if the new vertex is selected
- $|S_n| \sim n^\alpha \zeta_n$ a.s., where $0 < \alpha \leq 1$, and ζ_n almost surely is a slowly varying sequence of positive random variables
- the initial degree of the selected vertices has an a.s. asymptotic distribution (p_d), with exponentially decreasing tail

Main results

$X_s[n,d]$ is the number of vertices of degree d in S_n. If all the conditions are satisfied, then

$$\lim_{n\to\infty} \frac{X_s[n,d]}{|S_n|} = x_d$$

exists with probability 1, and the following holds for $d \geq m+1$:

$$x_d = \frac{x_{d-1} \cdot \dfrac{k_{d-1}}{c_{d-1}} + \alpha\ q_d}{\alpha + \dfrac{k_d}{c_d}}, \quad \text{where} \quad k_d = -\sum_{j=0}^{d}(c_j - p_j).$$

The local degree distribution (x_d) is polynomially decreasing with characteristic exponent $\gamma_s = \alpha(\gamma - 1) + 1$.

Methods of the proof

The proof is based on the methods of martingale theory, for example the following.

Theorem

$(X_i, \mathcal{G}_i)$ is a square integrable submartingale ($i = 1, 2, \ldots$),

$$A_n = \sum_{i=2}^{n} \left(X_i - E\left(X_i \middle| \mathcal{G}_{i-1}\right)\right),$$

$$B_n = \sum_{i=2}^{n} Var\left(X_i \middle| \mathcal{G}_{i-1}\right).$$

If $B_n^{1/2} \log B_n = O(A_n)$, then $X_n \sim A_n$ on the event $A_n \to \infty$ with probability 1.

See e. g. Neveu, 1975, [4].

Generalized PORT

A tree is built randomly, the initial degree of the new vertex is always 1. At the nth step, the probability that an old vertex of degree d is connected to the new vertex is $\frac{d+\beta}{T_{n-1}}$, where $\beta > -1$, $T_{n-1} = (2+\beta)\,n + \beta$.
S_n is a fixed level of the tree. Using the results of Móri, 2006, [2], one can see that the conditions are satisfied with

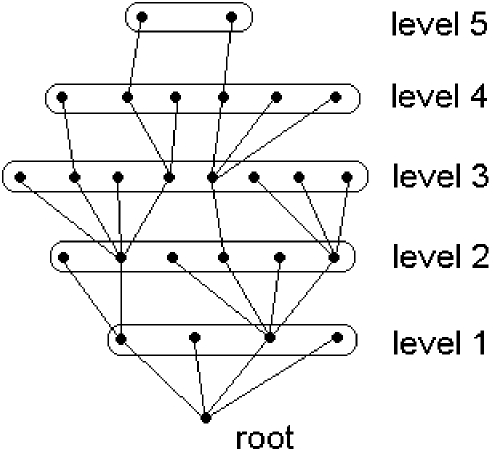
$$\gamma = 3 + \beta, \qquad \alpha = \frac{1}{2+\beta},$$

Thus the characteristic exponent on the selected set is

$$\gamma_s = \alpha\,(\gamma - 1) + 1 = \frac{1}{2+\beta}\,(3 + \beta - 1) + 1 = 2.$$

We get Barabási tree with $\beta = 0$.

Generalized PORT



Independent edges

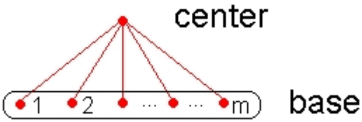
At the nth step, every old vertex is connected to the new vertex with probability $\lambda d / T_{n-1}$, independently of each other. $\lambda > 0$ is fixed. T_{n-1} is the sum of the degrees in G_{n-1}.
S_n is the neighbourhood of a fixed vertex. The initial degree of the new vertex is not fixed, but its distribution is asymptotically a Poisson distribution with expected value λ.
Based on Móri, 2007, [3], we see that the conditions are satisfied with

$$\gamma = 3, \qquad \alpha = \frac{1}{2},$$

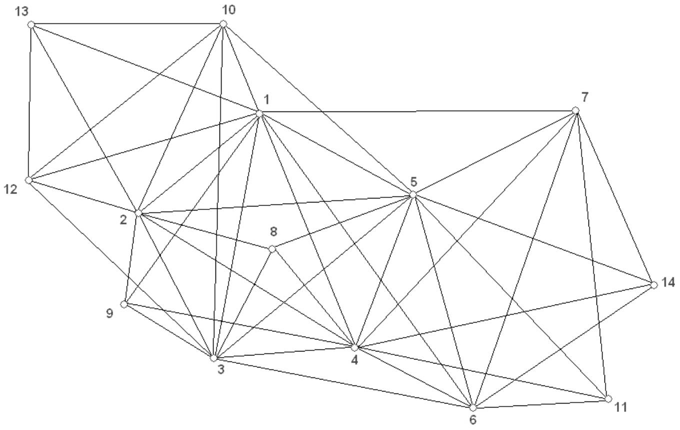
$$\gamma_s = \alpha\,(\gamma - 1) + 1 = 2.$$

Random multitree

- m-multicherry: a hypergraph that constist of $m+1$ vertices, m edges and a hyperedge of m vertices ($m \geq 2$);
- center: the vertex outside the hyperedge, connected to all the other vertices;
- base: the hyperedge that consists of the vertices excluded the center;
- at each step, the new vertex is the center of an m-multicherry; the base is chosen uniformly randomly from the existing bases; m new bases are added to the new m-multicherry



Random multitree



Random multitree

S_n consists of the vertices that are at distance j from the initial configuration.

$$\gamma = 2 + \frac{1}{m-1}, \quad \alpha = \frac{m-1}{m},$$

$$\gamma_s = \alpha(\gamma - 1) + 1 = \frac{m-1}{m}\left(2 + \frac{1}{m-1} - 1\right) + 1 = 2.$$

A variant: S_n consists of the common neighbours of k fixed vertices, $1 \leq k < m$.

$$\gamma = 2 + \frac{1}{m-1}, \quad \alpha = 1 - \frac{k}{m},$$

$$\gamma_s = 2 - \frac{k-1}{m-1} > 1.$$

Summary

We have sufficient conditions for the existence of local degree distribution on scale free random graphs.
The characteristic exponent of the local degree distribution is

$$\alpha(\gamma - 1) + 1,$$

where the characteristic exponent is γ, and the number of selected vertices is regularly growing with exponent α.
We have several graph models and methods of selection that satisfy the conditions.

Remarks

The characteristic exponent of the local degree distribution is usually less than γ. This can happen because the set of selected vertices can contain “older“ vertices than the ”typical” vertex, and in these models “older vertices have larger degree”.

For the existence of the local degree distribution, the existence of asymptotic degree distribution is sufficient, and the scale free property ensures that local degree distribution is also polynomially decreasing.

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