On extensions of Gray's and Siegmund's dualities and applications

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Outline

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The classical duality concept

Let X and Y be two stochastic processes on some state spaces S and S'.

X and Y are **dual to each other with duality function** ψ if for $x \in S$ and $y \in S'$

$$\mathbb{E}^x[\psi(X_t, y)] = \mathbb{E}^y[\psi(x, Y_t)], \qquad t \geq 0.$$

(Roughly) equivalent:

- $G\psi = H\psi$ for G and H the generators of X and Y
- $s \mapsto \mathbb{E}[\psi(X_s, Y_{t-s})]$ is constant on $[0, t]$ with $t \geq 0$ when X and Y are independent.

Generalization of the concept: subduality

Y is a **subdual** to X with duality function ψ if for $x \in S$ and $y \in S'$

$$\mathbb{E}^x[\psi(X_t, y)] \geq \mathbb{E}^y[\psi(x, Y_t)], \qquad t \geq 0.$$

(Roughly) equivalent:

- $G\psi \geq H\psi$ for G and H the generators of X and Y
- $s \mapsto \mathbb{E}[\psi(X_s, Y_{t-s})]$ is nondecreasing on $[0, t]$ with $t \geq 0$ when X and Y are independent.

Generalization of the concept: pathwise duality/subduality

Y is a **(strong) pathwise dual/subdual** to X with duality function ψ if X and Y can be coupled auch that

$$s \mapsto \psi(X_s, Y_{t-s})$$

is almost surely constant/nondecreasing on $[0, t]$ with $t \geq 0$, and X_{s-} is independent of $Y_{t-s}, s \in [0, t]$.

Terminology: Jansen and Kurt '14

Outline

Random mapping representations of Markov processes

Let X be a continuous-time Markov chain with (finite) state space S and generator G. Then G can be written in the form of a **random mapping representation:**

Let $\mathcal{G} \subset \mathcal{F}(S,S) := \{m : S \to S\}$ and let $(r_m)_{m \in \mathcal{G}}$ be nonnegative constants.

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x)\big) \quad , x \in S.$$

Note: This kind of representation is not unique.

The random mapping representation can be used for a Poissonian construction of the Markov process.

Poissonian construction of Markov processes

Let Δ be a Poisson point subset of $\mathcal{G} \times \mathbb{R}$ with local intensity $r_m \mathrm{d}t$.
For $s \leq u$, set $\Delta_{s,u} := \Delta \cap (\mathcal{G} \times (s, u])$.
Define random maps $\mathbf{X}_{s,t} : S \to S$ $(s \leq t)$ by

$$\mathbf{X}_{s,t}(x) := m_n \circ \cdots \circ m_1(x) \text{ when}$$

$$\Delta_{s,t} := \{(m_1, t_1), \ldots, (m_n, t_n)\}, \quad t_1 < \cdots < t_n.$$

Note that $\mathbf{X}_{t,u} \circ \mathbf{X}_{s,t} = \mathbf{X}_{s,u}$ for all $s \leq t \leq u$.

Poisson construction of Markov processes
Let X_0 be an S-valued r.v., independent of Δ. Setting for $s \in \mathbb{R}$,

$$X_t := \mathbf{X}_{s,s+t}(X_0), \qquad t \geq 0$$

defines a Markov process $X = (X_t)_{t \geq 0}$ with generator G.

Pathwise subduality from the Poissonian construction

Let X and Y be continuous-time Markov chains with (finite) state spaces S and S' and generators

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big),$$

$$Hf(y) = \sum_{m \in \mathcal{G}} r_m \big(f(\hat{m}(y)) - f(y) \big).$$

Let $\psi : S \times S' \to \mathbb{R}$ be a function such that

$$\psi\big(m(x), y\big) \geq \psi\big(x, \hat{m}(y)\big) \qquad x \in S,\ y \in S',\ m \in \mathcal{G}.$$

Pathwise subduality from the Poissonian construction

Let $\Delta, \hat{\Delta}$ be graphical representations for X and Y with

$$\hat{\Delta} := \{(\hat{m}, -t) : (m, t) \in \Delta\}.$$

Let $\mathbf{X}_{s,t-}$ and $\mathbf{Y}_{s,t}$ be the respective Poissonian constructions. Then, for each $s, u \in \mathbb{R}$ with $s \leq u$ and $x \in S$, $y \in S'$, the function

$$[s, u] \mapsto \psi\big(\mathbf{X}_{s,t-}(x), \mathbf{Y}_{-u,-t}(y)\big)$$

is a.s. nondecreasing. Consequently, for each $u \geq 0$ we can couple X and Y in such a way that X_{t-} is independent of Y_{u-t} for all $t \in [0, u]$ and

$$[0, u] \ni t \mapsto \psi(X_{t-}, Y_{u-t})$$

is a.s. nondecreasing.

Construction of a pathwise (sub)dual

Goal: Construct in a general setting $\hat{m}$ and find ψ such that

$$(*) \quad \psi\big(m(x), y\big) \geq \psi\big(x, \hat{m}(y)\big) \qquad x \in S,\ y \in S',\ m \in \mathcal{G}.$$

General possibility Let $S' = \mathcal{P}(S)$ be the set of all subsets of S. Set

$$\hat{m}(A) = m^{-1}(A) := \{x \in S : m(x) \in A\}, \quad A \in \mathcal{P}(S).$$

Then equality holds in (*) above with respect to the duality function

$$\psi(x, A) := 1_{\{x \in A\}}, \quad x \in S, A \in \mathcal{P}(S).$$

This dual with state space $\mathcal{P}(S)$ may be too unwieldy.
$\Rightarrow$ Restrict the setting further!

Outline

Partially ordered sets and monotone functions

Focus:

- S is a (finite) partially ordered set $(S, \leq)$.
- The functions m are monotone:

$$x \leq y \text{ implies } m(x) \leq m(y), \quad x, y \in S.$$

Some basics on partially ordered sets and monotone functions

- For $A \subset S$ define $A^{\uparrow} := \{x \in S : x \geq y \text{ for some } y \in A\} \subset A$.
- A is **increasing** if $A^{\uparrow} \subset A$. $(\to \mathcal{P}_{\mathrm{inc}}(S))$
- A is a **principal filter** if $A = \{z\}^{\uparrow}$ for some $z \in S$. $(\to \mathcal{P}_{!\mathrm{inc}}(S))$

The analogous definitions are made for $A^{\downarrow}$.
$\mathcal{P}_{\mathrm{dec}}(S)$ and $\mathcal{P}_{!\mathrm{dec}}(S)$ are the **decreasing** sets and **principal ideals**.

We have the equivalences:

- m is monotone.
- $m^{-1}(A) \in \mathcal{P}_{\mathrm{inc}}(S)$ for all $A \in \mathcal{P}_{\mathrm{inc}}(S)$.
- $m^{-1}(\{x\}^{\uparrow}) \in \mathcal{P}_{\mathrm{inc}}(S)$ for all $x \in S$.

(Analogously, for decreasing sets.)

A special class of monotone functions: Additive functions

We have the equivalences:

- $m^{-1}(A) \in \mathcal{P}_{\text{!dec}}(S)$ for all $A \in \mathcal{P}_{\text{!dec}}(S)$.
- There exists a map $m' : S \to S$ such that

$$m(x) \leq x' \Leftrightarrow x \leq m'(x'), \quad x, x' \in S.$$

At least for finite lattices S ($\mathcal{P}_{\text{!dec}}(S)$ is closed under finite intersections) these are also equivalent to:

- For $x \vee y$ defined via $\{x \vee y\}^{\uparrow} := \{x\}^{\uparrow} \cap \{y\}^{\uparrow}$ we have

$$m(x \vee y) = m(x) \vee m(y) \text{ for all } x, y \in S$$

  as well as $m(0) = 0$ where 0 is the unique minimal element.

In the interacting particle systems context such functions are called **additive**.

(Realizably) monotone processes

Let X be a continuous-time Markov chain with values in the partially ordered (finite) set S and generator G as well as transition probability kernel $(P_t)_{t\geq 0}$.

We call X a **monotone process** if we have the equivalent conditions:

- P_t is for each $t \geq 0$ a monotone probability kernel: $P_t f$ is monotone for all monotone f.
- For each monotone $f : S \to \mathbb{R}$ and $x, y \in S$ such that $x \leq y$ and $f(x) = f(y)$, one has $Gf(x) \leq Gf(y)$.

These are implied by the *stronger* condition that X is a **realizably monotone process**:

- G has a random mapping representation with all maps $m \in \mathcal{G}$ monotone.

Pathwise duality for realizably monotone processes

Let X be realizably monotone and let Y be the pathwise dual of X to the duality function $\psi(x, A) = 1_{\{x \in A\}}$ as before. Let G and H be the respective generators.

Recall:

- m^{-1} occurs in the random mapping representation of H
- m^{-1} maps $\mathcal{P}_{\text{inc}}(S)$ into itself since m are monotone

$\Rightarrow$ $Y_0 = y \in \mathcal{P}_{\text{inc}}(S)$ implies $Y_t \in \mathcal{P}_{\text{inc}}(S), t \geq 0$
$\Rightarrow$ Define a process Z such that $Y_t = Z_t^{\uparrow}, t \geq 0$.

Consequently: Z is dual to X with duality function

$$\phi^{\uparrow}(x, B) := 1_{\{x \in B^{\uparrow}\}} = 1_{\{x \geq z \text{ for some } z \in B\}}, \quad x \in S, \; B \in \mathcal{P}(S).$$

Remark: Analogously, one may define $\phi^{\downarrow}$ as a duality function.

Pathwise duality for realizably monotone processes

Proposition:
$\hat{m}^{\uparrow}$ is dual to m with respect to the duality function $\phi^{\uparrow}$ iff

$$\hat{m}^{\uparrow}(B)^{\uparrow} = m^{-1}(B^{\uparrow}), \quad B \in \mathcal{P}(S).$$

Indeed $\hat{m}^{\uparrow}$ is dual to m with respect to the duality function $\phi^{\uparrow}$ if for $x \in S, \; B \in \mathcal{P}(S)$

$$1_{\{m(x) \in B^{\uparrow}\}} = 1_{\{x \in \hat{m}(B)^{\uparrow}\}}.$$

$\Leftrightarrow$

$$1_{\{x \in m^{-1}(B^{\uparrow})\}} = 1_{\{x \in \hat{m}(B)^{\uparrow}\}}$$

Pathwise duality for realizably monotone processes

Two natural functions with the property $\hat{m}^{\uparrow}(B)^{\uparrow} = m^{-1}(B^{\uparrow})$:

- $m^{\uparrow\dagger}(B) := (m^{-1}(B^{\uparrow}))_{\min}$ satisfying

$$m^{\uparrow\dagger}(B)_{\min} = m^{\uparrow\dagger}(B)$$

- $m^{\uparrow *}(B) := \bigcup_{x \in B}(m^{-1}(\{x\}^{\uparrow}))_{\min}$ satisfying

$$m^{\uparrow *}(B \cup C) = m^{\uparrow *}(B) \cup m^{\uparrow *}(C)$$

Remark: For $\phi^{\downarrow}$ analogously $\hat{m}^{\downarrow}, m^{\downarrow\dagger}, m^{\downarrow *}$.

Gray's duality (Gray '86)

The function $m^{\uparrow *}$ is used in Gray's duality for interacting particle systems with state space $(S, \leq) = (\mathcal{P}(\mathbb{Z}^d), \subset)$ and generator:

$$\begin{aligned} Gf(x) &= \sum_{i \in \Lambda} \beta_i(x) \big(f(x \cup \{i\}) - f(x) \big) \\ &\quad + \sum_{i \in \Lambda} \delta_i(x) \big(f(x \backslash \{i\}) - f(x) \big). \end{aligned}$$

Here, $\beta_i(x)$ and $-\delta_i(x)$ are assumed to be monotone.

Gray's duality - construction of the dual

Use the random mapping representation and the Poissonian construction $\mathbf{X}_{s,t}$ from before:
For some $\mathcal{A}_i \subset \mathcal{P}_{\mathrm{inc}}(S)$

$$\beta_i(x) = \sum_{A \in \mathcal{A}_i} r_{+,i,A} 1_{\{x \in A\}}.$$

Also set

$$m_{+,i,A}(x) := \begin{cases} x \cup \{i\} & \text{if } x \in A, \\ x & \text{otherwise.} \end{cases}$$

Similarly, for δ_i. This leads to

$$\begin{aligned} Gf(x) &= \sum_{i \in \Lambda} \sum_{A \in \mathcal{A}_i} r_{+,i,A}\big(f(m_{+,i,A}(x)) - f(x)\big) \\ &\quad + \sum_{i \in \Lambda} \sum_{B \in \mathcal{B}_i} r_{-,i,B}\big(f(m_{-,i,B}(x)) - f(x)\big) \end{aligned}$$

Gray's duality - construction of the dual

Define cadlag $[s,u]$-**path** $\pi : [s,u] \to S$
with $\pi(s) = x$ and $\pi(u) = y$ such that

(i) $\mathbf{X}_{t,t'}(\pi(t)) \supset \pi(t')$ for all $s \leq t \leq t' \leq u$.

(ii) π is *minimal*: if a cadlag function $\tilde{\pi} : [s,u] \to S$ with $\tilde{\pi}(u) = y$ satisfies (i) and $\tilde{\pi}(t) \subset \pi(t)$ for all $t \in [s,u]$, then $\pi = \tilde{\pi}$.

For each $s \leq t$ set

$$\zeta_{s,t}(y) := \{x \in S : \text{ there exists an } [s,t]\text{-path from } x \text{ to } y\}, \quad y \in S.$$

Proposition: For each $s \leq u$

$$\zeta_{s,u}(y) = \mathbf{Z}_{-u,-s}(\{y\}), \quad y \in S$$

where $\mathbf{Z}$ is constructed as above using $m^{\uparrow *}$.

Additive process duality

Consider m additive: $m(0) = 0$ and $m(x \vee y) = m(x) \vee m(y)$
Then it maps $\mathcal{P}_{\mathrm{!dec}}(S)$ into itself. Consider

$$\phi^{\downarrow}(x, B) := 1_{\{x \in B^{\downarrow}\}} = 1_{\{x \leq z \text{ for some } z \in B\}}.$$

and the corresponding $m^{\downarrow *}(B) := \bigcup_{x \in B} m^{-1}(\{x\}^{\downarrow})_{\max}$.
Define $\tilde{m} : S \to S$ by

$$\{\tilde{m}(y)\}^{\downarrow} = m^{-1}(\{y\}^{\downarrow}), \quad y \in S.$$

Proposition:
m is dual to $\tilde{m}$ with respect to the duality function

$$\theta_{\leq}(x, y) := 1_{\{x \leq y\}}, \quad x, y \in S.$$

The map $\tilde{m}$ is additive with respect to the reverse order and

$$m^{*}(B) := \{\tilde{m}(x) : x \in B\}, \quad B \in \mathcal{P}(S).$$

Siegmund's duality (Siegmund '76)

On a (finite) totally ordered space S a map m is additive iff $m(0) = 0$ and m is monotone.

This is in general Siegmund's setting for the duality function

$$\theta_{\leq}(x, y) := 1_{\{x \leq y\}}, \quad x, y \in S.$$

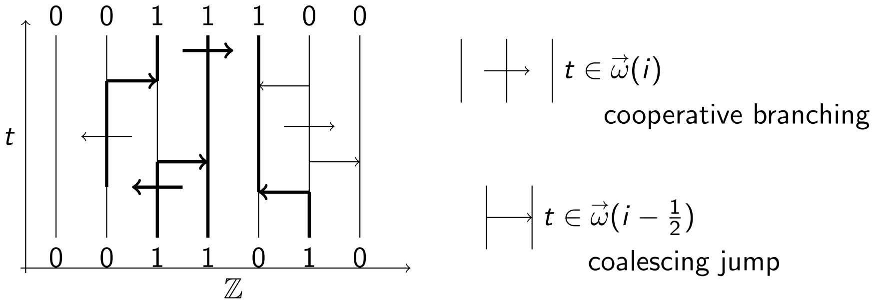
Original motivation: cooperative branching coalescent (S., Swart '14)

The **cooperative branching coalescent** is a continuous-time Markov process $X = (X_t)_{t \geq 0}$ with state space $\{0,1\}^{\mathbb{Z}}$ and transition rates

$$
\begin{array}{llll}
\text{If } x(i) = 1 & (x(i), x(i+1)) & \mapsto (0,1) & \text{at rate } \frac{1}{2} \\
 & (x(i-1), x(i)) & \mapsto (1,0) & \text{at rate } \frac{1}{2} \\
\text{If } (x(i), x(i+1)) = (1,1) & x(i+2) & \mapsto 1 & \text{at rate } \frac{1}{2}\lambda \\
 & x(i-1) & \mapsto 1 & \text{at rate } \frac{1}{2}\lambda
\end{array}
$$

- **Symmetric random walk with coalescence:** particles merge
- **Pairs of particles produce a new particle:** particle is placed on a neighbouring site at cooperative branching rate λ

Cooperative branching coalescent: The graphical representation

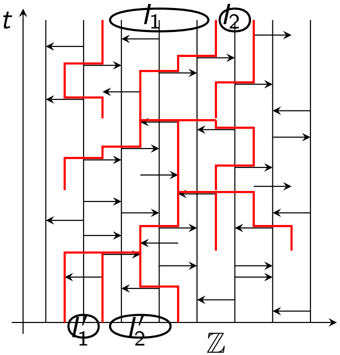


For $i \in \mathbb{Z}$

$$\vec{\omega}(i), \overleftarrow{\omega}(i) \text{ as well as } \vec{\omega}(i-\tfrac{1}{2}), \overleftarrow{\omega}(i-\tfrac{1}{2})$$

are Poisson processes with rate $\frac{1}{2}\lambda$ and $\frac{1}{2}$.

Cooperative branching coalescent: Pathwise superduality



Pathwise superduality: If $\eta_t \cap I_1 \neq \emptyset$ and $\eta_t \cap I_2 \neq \emptyset$, then there must exist a backward 3-path as drawn such that $\eta_0 \cap I'_1 \neq \emptyset$ and $\eta_0 \cap I'_2 \neq \emptyset$.

$\Rightarrow$ Decay of the survival probability and density

Thank you for your attention!