Filtering and the existence of intertwining relations

- Martingale problems
- Conditional distributions for martingale problems
- Intertwining
- Partially observed processes
- Markov mapping theorem
- Burke's theorem
- Lookdown constructions with real-valued levels

Material taken from

- Kurtz and Nappo (2011)
- Kurtz and Rodrigues (2011)
- Etheridge and Kurtz (2014)

with roots in

- Kurtz and Ocone (1988)
- Donnelly and Kurtz (1996, 1999)
- Kliemann, Koch, and Marchetti (1990)

Transition semigroups, generators, and martingale problems

$P(t, x, \Gamma)$ a transition function for a Markov process in $\mathbb{S}$.

Chapman-Kolmogorov

$$P(t+s, x, \Gamma) = \int_{\mathbb{S}} P(s, y, \Gamma) P(t, x, dy)$$

implies the semigroup property for

$$T(t)f(x) \equiv \int_{\mathbb{S}} f(y) P(t, x, dy) = E[f(X(s+t))|X(s) = x]$$

Note that $T(t) : B(\mathbb{S}) \to B(\mathbb{S})$.

Generator

$$A = \{(f, g) \in B(\mathbb{S}) \times B(\mathbb{S}) : T(t)f = f + \int_0^t T(r) g dr\}$$

(In general, A is multivalued, but we will still write Af rather than g.)

Martingale properties

If X is a Markov process with transition function $P(t,x,\Gamma)$, then

$$E[f(X(s+t))-f(X(s))-\int_s^{s+t}Af(X(r))dr|\mathcal{F}_s^X]$$
$$=T(t)f(X(s))-f(X(s))-\int_s^{s+t}T(r-s)Af(X(s))ds=0$$

which implies

$$f(X(t))-f(X(0))-\int_0^t Af(X(s))ds$$

is a $\{\mathcal{F}_t^X\}$-martingale.

Martingale problems

$\mathbb{S}$ the state space (a complete, separable metric space)

A the generator (a linear operator with domain and range in $B(\mathbb{S})$ (can be relaxed).

$\mu \in \mathcal{P}(\mathbb{S})$

X is a solution of the martingale problem for (A, μ) if and only if $\mu = PX(0)^{-1}$ and there exists a filtration $\{\mathcal{F}_t\}$ such that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds \qquad (1)$$

is an $\{\mathcal{F}_t\}$-martingale for each $f \in \mathcal{D}(A)$.

Conditions on generators

Condition 1 *1.* $A : \mathcal{D}(A) \subset C_b(\mathbb{S}) \to M(\mathbb{S})$ *with* $1 \in \mathcal{D}(A)$ *and* $A1 = 0$.

2. $\mathcal{D}(A)$ *is closed under multiplication and separates points.*

3. Either $\mathcal{R}(A) \subset C(\mathbb{S})$ *or there exists a complete separable metric space* U*, a transition function* η *from* $\mathbb{S}$ *to* U*, and an operator* $A_1 : \mathcal{D}(A) \subset C_b(\mathbb{S}) \to C(\mathbb{S} \times U)$ *such that*

$$Af(x) = \int_U A_1 f(x,u)\eta(x,du), \quad f \in \mathcal{D}(A). \tag{2}$$

4. There exist $\psi \in C(\mathbb{S})$, $\psi \geq 1$, *and constants* a_f *such that* $f \in \mathcal{D}(A)$ *implies* $|Af(x)| \leq a_f \psi(x)$, *or if* A *is of the form (2), there exist* $\psi_1 \in C(\mathbb{S} \times U)$, $\psi_1 \geq 1$, *and constants* a_f *such that, for all* $(x,u) \in \mathbb{S} \times U$ $|A_1 f(x,u)| \leq a_f \psi_1(x,u)$. *(If* A *is of the form (2), then define* $\psi(x) \equiv \int_U \psi_1(x,u)\eta(x,du)$.*)*

5. *A is* separable *in the sense that there exists a countable collection* $\{(f_k, g_k)\} \subset A$ *such that the martingale problem for* $\{(f_k, g_k)\}$ *is equivalent to the martingale problem for* A.

6. $A_0 = \psi^{-1}A$ *is a pre-generator (for each fixed* u*, if* A *is of the form (2)), that is,* A_0 *is dissipative and there are sequences of functions* $\mu_n : \mathbb{S} \to \mathcal{P}(\mathbb{S})$ *and* $\lambda_n : \mathbb{S} \to [0, \infty)$ *such that for each* $(f, g) \in A$

$$g(x) = \lim_{n\to\infty} \lambda_n(x) \int_{\mathbb{S}} (f(y) - f(x))\mu_n(x, dy) \tag{3}$$

for each $x \in \mathbb{S}$.

A martingale lemma

Let $\{\mathcal{F}_t\}$ and $\{\mathcal{G}_t\}$ be filtrations with $\mathcal{G}_t \subset \mathcal{F}_t$.

Lemma 2 *Suppose U and V are $\{\mathcal{F}_t\}$-adapted and*

$$U(t) - \int_0^t V(s)ds$$

is an $\{\mathcal{F}_t\}$-martingale. Then

$$E[U(t)|\mathcal{G}_t] - \int_0^t E[V(s)|\mathcal{G}_s]ds$$

is a $\{\mathcal{G}_t\}$-martingale.

Proof. The lemma follows by the definition and properties of conditional expectations. □

Martingale properties of conditional distributions

Corollary 3 *If X is a solution of the martingale problem for A with respect to the filtration $\{\mathcal{F}_t\}$ and π_t is the conditional distribution of $X(t)$ given $\mathcal{G}_t \subset \mathcal{F}_t$, then*

$$\pi_t f - \pi_0 f - \int_0^t \pi_s A f ds$$

is a $\{\mathcal{G}_t\}$-martingale for each $f \in \mathcal{D}(A)$.

Martingale characterization of conditional distributions

Theorem 4 *(Kurtz and Nappo (2011)) Suppose that* $\{\widetilde{\pi}_t, t \geq 0\}$ *is a cadlag,* $\mathcal{P}(\mathbb{S})$*-valued process adapted to* $\{\widetilde{\mathcal{G}}_t\}$ *satisfying*

$$E[\int_0^t \widetilde{\pi}_s \psi ds] < \infty, \quad t > 0$$

and that

$$\widetilde{\pi}_t f - \widetilde{\pi}_0 f - \int_0^t \widetilde{\pi}_s A f ds$$

is a $\{\widetilde{\mathcal{G}}_t\}$*-martingale for each* $f \in \mathcal{D}(A)$*. Then there exists a solution* X *of the martingale problem for* A*, a* $\mathcal{P}(\mathbb{S})$*-valued process* $\{\pi_t, t \geq 0\}$ *with the same distribution as* $\{\widetilde{\pi}_t, t \geq 0\}$*, and a filtration* $\{\mathcal{G}_t\}$ *such that* π_t *is the conditional distribution of* $X(t)$ *given* $\mathcal{G}_t$*.*

Conditioning on a process

Corollary 5 *If $\{\widetilde{\mathcal{G}}_t\}$ in Theorem 4 is generated by a cadlag process $\widetilde{Y}$ with no fixed points of discontinuity and $\widetilde{\pi}(0)$, that is,*

$$\widetilde{\mathcal{G}}_t = \mathcal{F}_t^{\widetilde{Y}} \vee \sigma(\widetilde{\pi}(0)),$$

then there exists a solution X of the martingale problem for A, a $\mathcal{P}(\mathbb{S})$-valued process $\{\pi_t, t \geq 0\}$, and a process Y such that $\{\pi_t, t \geq 0\}$ and Y have the same joint distribution as $\{\widetilde{\pi}_t, t \geq 0\}$ and $\widetilde{Y}$ and π_t is the conditional distribution of $X(t)$ given $\mathcal{F}_t^Y \vee \sigma(\pi(0))$.

Intertwining

Corollary 6 *(Kurtz (1998)) Let α be a transition function from $\mathbb{S}_0$ into $\mathbb{S}$, and for $\mu_0 \in \mathcal{P}(\mathbb{S}_0)$, let $\nu_0 = \int \alpha(y,\cdot)\mu_0(dy)$. Define*

$$C = \{(\int_{\mathbb{S}} f(z)\alpha(\cdot,dz), \int_{\mathbb{S}} Af(z)\alpha(\cdot,dz)) : f \in \mathcal{D}(A)\}\ .$$

If $\widetilde{Y}$ is a cadlag solution of the MGP for (C,μ_0), then there exists a solution X of the MGP for (A,ν_0) and a process Y such that Y and $\widetilde{Y}$ have the same distribution and

$$E[f(X(t))|\mathcal{F}_t^Y] = \int_{\mathbb{S}} f(x)\alpha(Y(t),dx)$$

Proof. Take $\widetilde{\pi}_t = \alpha(\widetilde{Y}(t),\cdot)$. □

Partially observed processes

Let $\gamma : \mathbb{S} \to \mathbb{S}_0$ be Borel measurable.

Corollary 7 *If in Corollary 5, $\widetilde{Y}$ and $\widetilde{\pi}$ satisfy*

$$\int_{\mathbb{S}} h \circ \gamma(x) \widetilde{\pi}_t(dx) = h(\widetilde{Y}(t)) \quad a.s.$$

for all $h \in B(\mathbb{S}_0)$ and $t \geq 0$, then $Y(t) = \gamma(X(t))$.

(cf. Kurtz and Ocone (1988) and Kurtz (1998))

Markov mapping theorem

Theorem 8 *If in Corollary 7,*

$$\widetilde{\pi}_t = \alpha(\widetilde{Y}(t), \cdot)$$

for α a transition function from $\mathbb{S}_0$ to $\mathbb{S}$ satisfying $\alpha(y, \gamma^{-1}(y)) = 1$ and uniqueness holds for the martingale problem for A, then $Y(t) = \gamma(X(t))$ is a Markov process.

Proof. Let $\{\mathcal{G}_t\} = \{\mathcal{F}_t^Y \vee \sigma(\pi(0))\}$. For $h \in B(\mathbb{S}_0)$ and $s < t$,

$$\begin{aligned} E[h(Y(t))|\mathcal{G}_s] &= E[h(\gamma(X(t)))|\mathcal{G}_s] = E[E[h(\gamma(X(t)))|X(s)]|\mathcal{G}_s] \\ &= E[E[h(\gamma(X(t)))|X(s)]|Y(s)] \end{aligned}$$

□

Burke's output theorem

Kliemann, Koch, and Marchetti (1990)

$X = (Q, D)$, an $M/M/1$ queue and its departure process

$$Af(k,l) = \lambda(f(k+1,l) - f(k,l)) + \mu \mathbf{1}_{\{k>0\}}(f(k-1,l+1) - f(k,l))$$

$\gamma(k,l) = l$

Assume $\lambda < \mu$ and define

$$\alpha(l,\{(k,l)\}) = (1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^{k-1}, \quad k = 0,1,2,\ldots \quad \alpha(l,\{(k,m)\}) = 0,\ m \neq l$$

Then

$$\begin{aligned} \alpha Af(l) &= \mu \sum_{k=1}^{\infty}(1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^{k-1}(f(k-1,l+1) - f(k-1,l)) \\ &= \lambda(\alpha f(l+1) - \alpha f(l)) \end{aligned}$$

Poisson output

Therefore, there exists a solution (Q, D) of the martingale problem for A such that D is a Poisson process with parameter λ and

$$P\{Q(t) = k | \mathcal{F}_t^D\} = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\mu})^{k-1},$$

that is, $Q(t)$ is independent of $\mathcal{F}_t^D$ and is geometrically distributed.

A population model

Let $U_1(0), \ldots, U_{N_\lambda(0)}$ be iid uniform $[0, \lambda]$ and independent of $N_\lambda(0)$. and for $d_0 > 0$, let

$$\dot{U}_i(t) = d_0 U_i(t).$$

Define $N_\lambda(t) = \#\{i : U_i(t) < \lambda\}$.

Ignoring order, $f(u, n) = \prod_{i=1}^n g(u_i)$, where $0 \le g \le 1$, g is continuously differentiable, and $g(u_i) = 1$ for $u_i \ge \lambda$.

The "generator" for $U(t) = (U_1(t), \ldots, U_{N_\lambda(t)})$ is

$$Af(u, n) = f(u, n) \sum_{i=1}^n d_0 u_i g'(u_i)/g(u_i)$$

$$f(U(t), N_\lambda(t)) - f(U(0), N_\lambda(0)) - \int_0^t Af(U(s), N_\lambda(s))ds = 0$$

The conditional distribution of $U(t)$ given $\mathcal{F}_t^{N_\lambda} = \sigma(N_\lambda(s) : s \le t)$, should be iid uniform $[0, \lambda]$.

A calculation

Let $\alpha(n,du)$ be the joint distribution of n iid uniform $[0,\lambda]$ random variables. Setting $\overline{g}=\lambda^{-1}\int_0^\lambda g(z)dz$, $\alpha f(n)=\int f(u,n)\alpha(n,du)=\overline{g}^n$ and

$$\begin{aligned}\int Af(u,n)\alpha(n,du) &= n\overline{g}^{n-1}d_0\lambda^{-1}\int_0^\lambda zg'(z)dz \\ &= d_0n\overline{g}^{n-1}\lambda^{-1}\int_0^\lambda (g(\lambda)-g(z))dz \\ &= d_0n\overline{g}^{n-1}(1-\overline{g}) = d_0n(\overline{g}^{n-1}-\overline{g}^n) \\ &= C\alpha f(n)\end{aligned}$$

where $C\widehat{f}(n)=d_0n(\widehat{f}(n-1)-\widehat{f}(n)$, and it follows that

$$\alpha f(N_\lambda(t))-\alpha f(N_\lambda(0))-\int_0^t C\alpha f(N_\lambda(s))ds$$

is a martingale.

Type dependent death rates

We could let d_0 depend on a "type" or "location" taking the state to be $\eta = \sum \delta_{(x,u)}$,

$$f(x,u) = \prod_{(x,u)\in\eta} g(x,u) = \exp\{\int \log g(x,u)d\eta\}$$

$$Af(u,\eta) = f(u,\eta) \sum_{(x,u)\in\eta} d_0(x)u\frac{\partial_u g(x,u)}{g(x,u)}$$

$$\alpha f(\overline{\eta}) = \prod_{x\in\overline{\eta}} \overline{g}(x), \quad \overline{\eta} = \sum_{(x,u)} \delta_x, \quad \overline{g}(x) = \lambda^{-1}\int_0^\lambda g(x,u)du$$

$$\alpha A f(\overline{\eta}) = \alpha f(\overline{\eta}) d_0(x) \sum_{x\in\overline{\eta}} (\frac{1}{\overline{g}(x)} - 1)$$

Thinning

$$f(x,u) = \prod_{(x,u)\in\eta} g(x,u) = \exp\{\int \log g(x,u)d\eta\}$$

$$A_{th}f(u,\eta) = \mu(\overline{\eta})(\prod_{(x,u)\in\eta} g(x,\rho(x)u) - f(u,\eta))$$

Take $\rho(x) = \frac{1}{1-p(x)}$ so for U uniformly distributed on $[0,\lambda]$,

$$P\{\rho(x)U > \lambda\} = p(x).$$

$$\alpha f(\eta) = \prod_{x\in\overline{\eta}} \overline{g}(x), \quad \overline{\eta} = \sum_{(x,u)} \delta_x, \quad \overline{g}(x) = \lambda^{-1}\int_0^\lambda g(x,u)du$$

$$\alpha A_{th}f(\overline{\eta}) = \mu(\overline{\eta})(\prod_{x\in\overline{\eta}}((1-p(x))\overline{g}(x) + p(x)) - \alpha f(\overline{\eta})).$$

Continuous Births

$$f(\eta) = \prod_{(x,u)\in\eta} g(x,u)$$

$$A_{cb}f(\eta) = f(\eta)\sum_{(x,u)\in\eta} r(x)\Big[\frac{2}{\lambda}\int_u^\lambda (g(x,v)-1)dv + G_1^\lambda(u)\frac{\partial_u g(x,u)}{g(x,u)}\Big]$$

$$G_1^\lambda(u) = -\frac{u}{\lambda}(\lambda - u),$$

As before, $\alpha f(\overline{\eta}) = \prod_{x\in\overline{\eta}} \overline{g}(x)$ and

$$\alpha A_{cb} f(\overline{\eta}) = \alpha f(\overline{\eta}) \sum_{x\in\overline{\eta}} r(x)\left[\overline{g}(x) - 1\right].$$

Discrete birth event with $k-1$ offspring

At the time of a birth event, k points are chosen independently and uniformly on $[0,\lambda]$.

Let v^* denote the minimum of the k new levels.

For old points $(x,u)\in\eta$ with $u>v^*$ and $r(x)>0$, let τ_x be defined by

$$e^{-r(x)\tau_x}=\frac{\lambda-u}{\lambda-v^*}.$$

For $(x,u)\in\eta$ satisfying $u<v^*$ and $r(x)>0$, let τ_x be determined by $e^{-r(x)\tau_x}=\frac{u}{v^*}$.

In both cases, τ_x is exponentially distributed with parameter $r(x)$ and the τ_x are independent.

Taking (x^*,u^*) to be the point in η with $\tau_{x^*}=\min_{(x,u)\in\eta}\tau_x$, we have

$$P\{(x^*,u^*)=(x',u')\}=\frac{r(x')}{\int r(x)\overline{\eta}(dx)},\qquad (x',u')\in\eta.$$

The population after the birth event

Assuming that the offspring are all given the type of the parent, after the birth event, the population becomes

$$\eta^* = \{(x, \lambda - (\lambda - u)e^{r(x)\tau_{x^*}}) : (x,u) \in \eta, \tau_x > \tau_{x^*}, u > v^*\}$$
$$\cup\{(x, ue^{r(x)\tau_{x^*}}) : (x,u) \in \eta, \tau_x > \tau_x^*, u < v^*\}$$
$$\cup\{(x^*, v_i), i = 1, \ldots, k\}.$$

Other models with real (as opposed to integer) levels

Donnelly, Evans, Fleischmann, Kurtz, and Zhou (2000) and Buhr (2002) give constructions of Moran-like models with multiple sites in which particles do random walk among the sites. One dimensional Brownian approximations have lookdowns controlled by intersection local times.

Greven, Limic, and Winter (2005) gives a construction for a Moran/Fleming-Viot model with multiple sites.

Veber and Wakolbinger (2013) gives a construction for the spatial Λ-Fleming-Viot process.

Filtering equations

X a Markov process with generator $A \subset C_b(\mathbb{S}) \times C_b(\mathbb{S})$

$$Y(t) = Y(0) + W(t) + \int_0^t h(X(s))ds$$

Assume for $t \geq 0$, $E\left[\int_0^t |h(X(s))|)\, ds\right] < \infty$, and $\int_0^t (\pi_s |h|)^2) ds < \infty$.

Then assuming independence of X and W, the conditional distribution π_t of $X(t)$ given $\mathcal{F}_t^Y$ satisfies

$$\begin{aligned} \pi_t f &= \pi_0 f + \int_0^t \pi_s A f\, ds \\ &\quad + \int_0^t \left[\pi_s (hf - \pi_s h\, \pi_s f\right] \left[dY(s) - \pi_s h\, ds\right], \end{aligned} \tag{4}$$

Note that $Y(t) - \int_0^t \pi_s h ds$ is a Brownian motion.

Abstract

Filtering and the existence of intertwining relations

The general filtering problem for a Markov process is concerned with identifying the conditional distribution of the process state given partial information about the process. Martingale characterizations of these conditional distributions have as a corollary a general result on the existence of intertwining relations. The basic characterization will be given and a variety of examples discussed.

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