DUALITY

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Duality in population models: from configurations to genealogies

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Goals

We want to study the *longtime behaviour* of

population models (Fleming-Viot type)

via *duality*, both on the level of

- configurations,
- genealogies,

based on the same **moment dual** with dual based on **coalescents.**

Model

A population consisting of multitype individuals divided into *colonies* (demes) that are located at sites labelled by a

$$\text{countable group } \Omega \tag{1.1}$$

(modelling geographic space) and whose *types* (genotypes) belong to a set

$$\text{the type space} \quad \mathbb{K}. \tag{1.2}$$

The state space of a single component (describing frequencies of types) will be

$$\mathcal{P}(\mathbb{K}) = \text{set of probability measures on } \mathbb{K}. \tag{1.3}$$

The set of colonies (sites or components) will be indexed by a set Ω, which is countable.

The *state space* $\mathcal{X}$ of the system is therefore

$$\mathcal{X} = (\mathcal{P}(\mathbb{K}))^{\Omega}, \tag{1.4}$$

$$X = (x_\xi)_{\xi \in \Omega} \text{ with } x_\xi \in \mathcal{P}(\mathbb{K}). \tag{1.5}$$

We consider a *spatial Fleming-Viot model* describing a population of individuals distributed in

- geographic space (locations)
- carrying a type

which is described as a process

$$(X_t)_{t \geq 0},$$

with values in

$$((\mathcal{P}(\mathbb{K}))^{\Omega}.$$

Dynamics

The dynamic entails

- resampling at rate d
- mutation
- selection
- migration.

The process is defined by a *martingale problem*.

Parameters

$$ca(\cdot,\cdot) \quad , \quad c \in \mathbb{R}^+ \text{ and } a \text{ probability transition kernel on } \Omega \times \Omega. \tag{1.6}$$

Then we can say that c is the migration rate and $a(\xi, \xi')$ the probability that a jump from ξ to ξ' occurs.

In addition to describe *mutation* and *selection* we need two further objects. Let

$$M(\cdot,\cdot) \text{ be a probability transition kernel on } \mathbb{K} \times \mathbb{K}, \tag{1.7}$$

modelling mutation probabilities from one type to another.

Furthermore the fitness function on types

$$\chi(\cdot) \text{ a bounded function on } \mathbb{I}, 0 \leq \chi(\cdot) \leq 1, \quad 0 = \min \chi, 1 = \sup \chi. \tag{1.8}$$

Test functions

Polynomials of the form:

$$F(x) = \int f(u_1, \cdots, u_n) x(du_1) \cdots x(du_n) \quad , \quad x \in \mathcal{M}_{\mathrm{fin}}(\mathbb{K}), \qquad (1.9)$$

where $\mathcal{M}_{\mathrm{fin}}(\mathbb{K})$ denotes finite signed measures and

$$f \in C_b(\mathbb{K}^n, \mathbb{R}), \quad n \in \mathbb{N}. \qquad (1.10)$$

Evaluation of a function of *n sampled individuals*.

Operator

Define the linear operator G acting on functions $F \in \mathcal{A} \subseteq C_b^2(\mathcal{P}(\mathbb{K}), \mathbb{R})$, with values in $C_b(\mathcal{P}(\mathbb{K}), \mathbb{R})$, as follows :

$$\begin{aligned}(GF)(x) = \quad & m \int_{\mathbb{K}} \left(\int_{\mathbb{K}} \frac{\partial F(x)}{\partial x}[v](M(u,dv) - \delta_u(dv)) \right) x(du) \\ & + s \int_{\mathbb{K}} \frac{\partial F(x)}{\partial x}[v](\chi(v) - \int_{\mathbb{K}} \chi(u)x(du))x(dv) \\ & + d \int_{\mathbb{K}} \int_{\mathbb{K}} \frac{\partial^2 F(x)}{\partial x^2}[u,v] Q_x(du,dv),\end{aligned} \tag{1.11}$$

$$Q_x(du,dv) = x(du)\delta_u(dv) - x(du)x(dv). \tag{1.12}$$

$$
\begin{aligned}
(LF)(x) = \sum_{\xi \in \Omega_N} \Bigg[& c \sum_{\xi' \in \Omega} a(\xi, \xi') \int_{\mathbb{K}} \frac{\partial F(x)}{\partial x_\xi}(u)(x_{\xi'} - x_\xi)(du) \\
& + s \int_{\mathbb{K}} \left\{ \frac{\partial F(x)}{\partial x_\xi}(u) \left(\chi(u) - \int_{\mathbb{K}} \chi(w) x_\xi(dw) \right) \right\} x_\xi(du) \\
& + m \int_{\mathbb{K}} \left\{ \int_{\mathbb{K}} \frac{\partial F(x)}{\partial x_\xi}(v) M(u, dv) - \frac{\partial F(x)}{\partial x_\xi}(u) \right\} x_\xi(du) \\
& + d \int_{\mathbb{K}} \int_{\mathbb{K}} \frac{\partial^2 F(x)}{\partial x_\xi \partial x_\xi}(u, v) Q_{x_\xi}(du, dv) \Bigg], \qquad x \in (\mathcal{P}(\mathbb{K}))^\Omega.
\end{aligned}
$$

Definition

(Martingale problem)
(a) The law P on a space of E-valued path for a Polish space E, either $D([0,\infty),E)$ or $C([0,\infty),E)$, is a solution to the martingale problem for (L,ν) w.r.t. $\mathcal{A}$ if and only if

$$\left(F(X(t)) - \int_0^t (LF)(X(s))ds\right)_{t\geq 0} \quad \text{is a martingale under } P \text{ for all } F \in \mathcal{A} \tag{1.13}$$

and

$$\mathcal{L}(X(0)) = \nu. \tag{1.14}$$

The martingale problem is called wellposed, if the finite dimensional distributions of P are uniquely determined by the property (1.13) and (1.14). □

Theorem

(Existence and Uniqueness)
Let ν be a probability measure on $(\mathcal{P}(\mathbb{K}))^{\Omega}$ specifying the initial state which is independent of the evolution.
(a) Then the $(L;\nu)$-martingale problem w.r.t. $\mathcal{A}$, on the space $C([0,\infty)\ (\mathcal{P}(\mathbb{K}))^{\Omega})$, is well-posed. For fixed value of the parameter N the resulting canonical stochastic process is denoted

$$(X_t)_{t\geq 0}. \tag{1.15}$$

(b) The solution defines a strong Markov process with the Feller property.
□

Definition

(Duality)
A process $(X_t)_{t\geq 0}$ with state space E and $(X'_t)_{t\geq 0}$ with state space E' with E and E' both Polish spaces satisfies a *duality relation* if there exists a function

$$H : E \times E' \to \mathbb{R}, \quad H \in C_b(E \times E', \mathbb{R}), \tag{2.1}$$

such that

$$E[H(X_t, X'_0)] = E[H(X_0, X'_t)] \text{ for all } (X_0, X'_0) \in E \times E'. \tag{2.2}$$

Usually with dualities we assume that the family of functions

$$\{H(\cdot, X'_0), X'_0 \in E'\}, \tag{2.3}$$

is distribution and convergence-determining on the state space E. □

Generator criterion for duality

The duality relation between two Markov processes and associated semigroups with generators G_1 respectively G_2 w.r.t. to the function H is implied by the properties :

$$H(\cdot, X_0') \in \mathcal{D}(G_1), \quad H(X_0, \cdot) \in \mathcal{D}(G_2) \quad \forall\, (X_0, X_0') \in E \times E', \tag{2.4}$$

$$(G_1(H(\cdot, X_0')))(X_0) = (G_2(H(X_0, \cdot)))(X_0'), \quad \forall\, X_0 \in E, \quad X_0' \in E'. \tag{2.5}$$

Representation of f.d.d. (Fleischmann & Greven, 1996) [FG96]

We can represent for Markov processes with a dual based on *frozen states* the f.d.d. Consider initial states for dual at times

$$0 = t_0 < t_1 < \cdots < t_n = t$$

denoted

$$X'^{,1}, X'^{,2}, \cdots, X'^{,n}$$

which are frozen till time

$$t - t_1, t - t_2, \cdots, t - t_{n-1}, 0.$$

Then space-time processes

$$(s, X_s)_{s \in [0,t]}, (t - s, X'_{t-s})_{s \in [0,t]}$$

are dual w.r.t. $\widetilde{H}$, in case of moment duals

$$\widetilde{H}(\widetilde{X}, \widetilde{X}') = \Pi_{i=1}^{n} H((t_i, X_i), (t - t_i, X'^{,i})).$$

Definition

(Feynman-Kac duality)
The processes are in Feynman-Kac duality with respect to H, satisfying (2.1) and (2.3), if there exists a function V on E' which is bounded and continuous such that:

$$E[H(X_t, X_0')] = E[H(X_0, X_t') exp(\int_0^t V(X_s') ds)]. \qquad \square \qquad (2.6)$$

Checking for Feynman-Kac duality entails to show:

$$(G_1 H(\cdot, X_0'))(X_0) = (G_2 H(X_0, \cdot))(X_0') + V(X_0')H(X_0, X_0'), \quad \forall\, (X_0, X_0') \in E \times E'. \tag{2.7}$$

This is a criterion one can typically check by explicit calculation.

We say $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are in *strong duality* if for fixed t

$$\mathcal{L}[X_t] = \mathcal{L}[Y_t], \tag{2.8}$$

i.e. Y represents X at a *fixed* time.

This happens for moment duals in population models for entrance laws of the dual with countably many individuals.

Dual process

We introduce now the dual process for the neutral population model. The Kingman coalescent is a process with values in the partitions of $\mathbb{N}$ with the rule,

$$\text{two partition elements coalesce (i.e. } \pi_1, \pi_2 \to \pi_1 \cup \pi_2\text{) at rate } d. \tag{3.1}$$

Consider also the entrance law at time $t = 0$ from infinite dual populations starting in the partition $(\{1\}, \{2\}, \cdots)$.

Choose f a bounded continuous function on $\mathbb{K}^{\mathbb{N}}$ which depends only on the L first entries for some $L \in \mathbb{N}$ and this object does not change under the dual dynamic (for the neutral model).

$$E = \mathcal{P}(\mathbb{K}), \quad E' = \{ \text{ partitions of } \mathbb{N}\} \cup C_b(\mathbb{K}^{\mathbb{N}}, \mathbb{R}), \tag{3.2}$$

$$H(X,Z) = \int\limits_{\mathbb{K}^{\mathbb{N}}} f(u_{\pi(1)}, \cdots) X^{\otimes \mathbb{N}}(du_1, du_2, \cdots), \qquad Z = (\pi, f), \quad X \in \mathcal{P}(\mathbb{K}), \tag{3.3}$$

where the partition Π of $\mathbb{N}$ in Z is represented by a map

$$\pi : \mathbb{N} \to \mathbb{N}. \tag{3.4}$$

Lemma

(Duality Fleming-Viot)
For $(Y_t)_{t\geq 0}$ *and* $(\eta_t)_{t\geq 0}$ *w.r.t. H in (3.3), relations (2.2) and (2.3) hold.*
$\square$

Remark on Moran model

For the Moran model we modify $H(,)$. The idea is to sample k-individuals at random,

$$(\mu^N)^{\otimes k,\downarrow N}(du_1,\cdots,du_k) = (Z_N^k)^{-1}\mu^N \otimes (\mu^N - \tfrac{1}{N}\delta_{u_1}) \otimes \cdots (\mu^N - \tfrac{1}{N}(\delta_{u_1} + \cdots + \delta_{u_{k-1}}))(du_1,\cdots,du_k), \tag{3.5}$$

$$Z_N^k = (1 - \frac{1}{N}) \cdots (1 - \frac{N-k+1}{N}), \tag{3.6}$$

$$H_N(X, Z) = \int_{\mathbb{K}^N} f(u_{\pi(1)}, u_{\pi(2)}, \cdots, u_{\pi(n)}) X^{\odot n}(du_1, du_2, \cdots, du_n), \tag{3.7}$$

$\odot n$ is shortened for $\otimes n, \downarrow N$.

Lemma

(Duality Moran model)

For Y^N and $(\eta_t)_{t \geq 0}$ and $H_N(\cdot, \cdot)$ the relations (2.2) and (2.3) hold.
$\square$

Function-valued dual

The dual process for the *Fleming-Viot diffusion with selection and mutation* is a *function-valued process* driven by a *particle system* denoted $(\eta_t, \mathcal{H}_t)_{t \geq 0}$,

$$\begin{array}{l}\eta \text{ is a (partition-valued)-Kingman coalescent (at rate } d) \\ \text{with birth of new individual at rate } s \text{ per individual,}\end{array} \tag{4.1}$$

$$(\mathcal{H}_t)_{t \geq 0} \text{ is a \textit{function-valued process} driven by } \eta. \tag{4.2}$$

$$E' = \bigcup_{N \in \mathbb{N}} \mathbb{P}(\lfloor N \rfloor) \times \left(\bigcup_{n=1}^{\infty} C_b((\mathbb{K})^n, \mathbb{R}) \right) = (\bigcup_{N \in \mathbb{N}} \mathbb{P} \lfloor N \rfloor) \times \mathbb{S} \subseteq \mathbb{P}(\mathbb{N}) \times C_b(\mathbb{K}^{\mathbb{N}}, \mathbb{R}). \tag{4.3}$$

Start with n individuals creating the partition $\{\{1\}, \cdots, \{n\}\}$ and with every individual (partition element) we associate a distinct variable and then $\mathcal{H}_0 = f$ is a function of n-variables defined in $(\mathbb{K})^n$ which is positive and bounded.

The dynamics of the system is as follows.

- Two transitions occur in the *particle system* (i.e. η),
  (1) at rate d two partition elements *coalesce* into one new partition element and
  (2) at rate s independently for each partition element a *birth* occurs, i.e. every partition element creates independent of everything else a new individual in the basic set which forms its own partition element.
- The *function-valued* part (i.e. f) makes the transitions corresponding to resampling, selection and mutation:

If a coalescence event occurs, the variables corresponding to the partition elements are set equal,

(4.4)

if a birth occurs for individual i and $\mathcal{H}_t = f$, with $f : (\mathbb{K})^n \to \mathbb{R}$, then $f \to \chi_i f \otimes 1 + f \otimes (1 - \chi_{n+1})$, an element of $C_b(\mathbb{K}^{n+1}, \mathbb{R})$,

(4.5)

at rate m for every variable u_i in $\mathcal{H}_t = f$, $f : (\mathbb{K})^n \to \mathbb{R}$, the following transition occurs:
$f \longrightarrow \int_{\mathbb{K}} f(u_1, \cdots, u_n)(M(v, du_i))$.

(4.6)

Spatial case

To incorporate space we have to modify the duality function and the coalescent as follows:

- The spatial coalescent now has partition elements which have locations in Ω, which carry out $c \cdot a(\cdot,\cdot)$ independent random walks up to coalescence.
- The duality function is now:
  $H : E \times E' \to \mathbb{R}$ by

$$H(X,(\eta,\mathcal{F})) = \int_{\mathbb{K}} \cdots \int_{\mathbb{K}} \mathcal{F}(u_1,\cdots,u_{|\pi|}) x_{\xi(1)}(du_1)\cdots x_{\xi(|\pi_t|)}(du_m). \tag{4.7}$$

Then the collections of bounded measureable functions

$$\{H(\cdot,(\eta,\mathcal{F})),(\eta,\mathcal{F}) \in E'\}, \quad (\{H(X,\cdot), \quad X \in (\mathcal{P}(\mathbb{K}))^{\Omega}\}) \tag{4.8}$$

are measure-determining on $(E,\mathcal{B}(E))$ respectively $(E',\mathcal{B}(E'))$.

Theorem

The IFV-process with mutation and selection $(Y_t)_{t\geq 0}$ *and* $(\eta_t, \mathcal{H}_t)_{t\geq 0}$ *are in duality w.r.t.* $H(\cdot,\cdot)$. $\square$

More dualities

Depending on the purpose one can deduce for finitely many fitness levels (Dawson & Greven, 2011):

- set-valued duals
- modified duals for selection transition.

See Dawson and Greven [DGsel14]: LNM2092.

Remark on Moran model

For the Moran model the duality is now mir subtle and leads to a Feynman-Kac duality. Here we have to modify our objects as follows.

(a) There exists a sequence of linear operators $G^{*,N}$ which are of the form of a sum of generators of jump processes on the state space (4.3) plus a Feynman-Kac term, with domain the monomials of degree $n \leq N$ of the form (3.5) such that on this domain for some constant Const:

$$G^{\mathrm{dual},N} = (1 - \frac{n}{N}) G^{\mathrm{dual}} + \frac{n}{N} G^{*,N} \text{ on configurations with } |n| < N, \quad (4.9)$$

$$|G^{*,N}(\mathcal{H})| \leq c(n) \|\mathcal{H}\|_{\infty}, \quad , \quad \text{if } \mathcal{H} \text{ is of degree } n < N. \quad (4.10)$$

(b) If $|\eta| = N$, in particular the birth rate of new dual particles becomes 0.

(c) The generator $G^{*,N}$ is a bounded operator which corresponds to a generator plus a Feynman-Kac tern and corresponds to a coalescence or coalescence with rebirth in the particle system and instantaneous multiplication operations acting on f, the function which is characterizing the monomial, by the bounded functions $\chi, (1-\chi)$ and -1.

Namely at rate $3sN^{-1}$ partition elements coalesce and potentially induce the instantaneous birth of a new singleton partition element and the function makes a transition corresponding to the generator given as follows. If $\mathcal{H}(\mu, (f, \{1, \cdots, n\}) = \int f(\underline{u})\mu^{\odot n}(d\underline{u})$, where $\underline{u} = (u_1, \cdots, u_n)$, we have

$$(G^{*,N}H(y,\cdot))((\{\{1\},\cdots,\{n\}\},f)) = \langle y^{\odot n}, \widehat{G}^{*,N} f\rangle \qquad (4.11)$$

$$\begin{aligned}
\widehat{G}^{*,N}(f) = s \sum_{i=1}^{n} \Big[\sum_{k=1}^{n} \big(\chi(u_i) f(u_1,\cdots,u_{k-1},u_i,u_{k+1},\cdots,u_n) - f(u_1,\cdots,u_n)\big) \\
+ \big((1(u_k) - \chi(u_k)\big) f(u_1,\cdots,u_{k-1},u_i,u_{k+1},\cdots,u_n) - f(u_1,\cdots,u_n))\Big] \\
+ \sum_{i=1}^{n} \Big[\sum_{k=1}^{n} \big(- 1(u_i) f(u_1,\cdots,u_{k-1},u_i,u_{k+i},\cdots,u_n) - f(u_1,\cdots,u_n)\big) \\
+ 3\binom{n}{2} f(u_1,\cdots,u_n). \qquad \square
\end{aligned} \qquad (4.12)$$

In a Moran, Fleming-Viot model we can define *genealogies*.

Consider a *Moran model with N individuals*. Between individuals we have an

ancestor - descendent

relationship, giving in the Moran model a "genealogical tree", marked with types and locations.

We consider the genealogical information about a population given by a set U with ancestor - descendence relation by specifying

$$\textit{ultrametric } r(\cdot,\cdot) \tag{5.1}$$

$$r(\iota,\iota') = 2 \text{ time back to MRCA } (\iota,\iota'), \tag{5.2}$$

a *mark function* $\kappa : U \to \Omega \times \mathbb{K}$

$$\kappa(\iota) = \text{ (type } (\iota), \text{ location } (\iota)), \tag{5.3}$$

sampling measure μ on $(U, \mathcal{B})$

$$\mu = \frac{1}{|U|} \sum_{\iota \in U} \delta_\iota. \tag{5.4}$$

Then

$$(U, r, \kappa, \mu) \tag{5.5}$$

is a $(\Omega \times \mathbb{K})$-marked ultrametric probability measure space.

Next we "forget" individuals' names and consider

$$\overline{(U, r, \kappa, \mu)} \tag{5.6}$$

the equivalence classes of mumm-spaces under

$$\text{mark-measure-preserving isometries.} \tag{5.7}$$

Consider measure ν on $U \times (\Omega \times \mathbb{K})$ via the $U \times (\Omega \times \mathbb{K})$-kernel $K(\iota, dv) = \delta_{\kappa(\iota)}(dv)$ by

$$\nu = \mu \otimes K, \tag{5.8}$$

leading to

$$\overline{(U, r, \nu)}. \tag{5.9}$$

This object makes still sense in the infinitely many particles limit.

Recall we can introduce a topology by defining convergence of sequences $(\mathcal{U}_n)_{n\in\mathbb{N}}$ in Gromov-weak topology by requiring convergence of all *finite sampled subspaces*.

Definition (Ω-mmm-spaces)

Let (Ω, r_Ω) be a complete separable metric space with metric r_Ω, and let o be a distinguished point in Ω.

1. We call (X, r, μ) a Ω-marked metric measure space (Ω-mmm space for short) if:
   (i) (X, r) is a complete separable metric space,
   (ii) μ is a σ-finite measure on the Borel σ-algebra of $X \times \Omega$, with $\mu(X \times B_o(R)) < \infty$ for each ball $B_o(R) \subset \Omega$ of finite radius R centered at o.

Definition

2. We say two Ω-mmm spaces (X, r_X, μ_X) and (Y, r_Y, μ_Y) are equivalent if there exists a measureable map $\varphi : X \to Y$, such that

$$r_X(x_1, x_2) = r_Y(\varphi(x_1), \varphi(x_2)) \qquad \text{for all} \quad x_1, x_2 \in \operatorname{supp}(\mu_X(\cdot \times \Omega)), \tag{5.10}$$

and if $\widetilde{\varphi} : X \times \Omega \to Y \times \Omega$ is defined by $\widetilde{\varphi}(x, v) := (\varphi(x), v)$, then

$$\mu_Y = \mu_X \circ \widetilde{\varphi}^{-1}. \tag{5.11}$$

In other words, φ is an isometry between *supp*$(\mu_X(\cdot \times \Omega))$ and *supp*$(\mu_Y(\cdot \times \Omega))$, and the induced map $\widetilde{\varphi}$ is mark and measure preserving. We denote the equivalence class of (X, r, μ) by

$$\overline{(X, r, \mu)}. \tag{5.12}$$

Definition

3. The space of (equivalence classes of) Ω-mmm spaces is denoted by

$$\mathbb{M}^{\Omega} := \{\overline{(X, r, \mu)} : (X, r, \mu) \text{ is a } \Omega-\textit{mmm space}\}. \qquad (5.13)$$

Localisation

Definition (Ω-marked Gromov-weak$^{\#}$ Topology)

Fix a sequence of continuous functions $\psi_k : \Omega \to [0,1]$, $k \in \mathbb{N}$, such that $\psi_k = 1$ on $B_o(k)$, the ball of radius k centered at $o \in \Omega$, and $\psi_k = 0$ on $B_o^c(k+1)$. Let $\chi := \overline{(X, r, \mu)}$ and $\chi_n := \overline{(X_n, r_n, \mu_n)}$, $n \in \mathbb{N}$, be elements of $\mathbb{M}^\Omega$. Let $\psi_k \cdot \mu$ be the measure on $X \times \Omega$ defined by $(\psi_k \cdot \mu)(\mathrm{d}(x,v)) := \psi_k(v)\mu(\mathrm{d}(x,v))$, and let $\psi_k \cdot \mu_n$ be defined similarly. We say that $\chi_n \underset{n\to\infty}{\longrightarrow} \chi$ in the Ω-marked Gromov-weak$^{\#}$ topology if and only if $k \in \mathbb{N}$:

$$\overline{(X_n, r_n, \psi_k \cdot \mu_n)} \underset{n\to\infty}{\Longrightarrow} \overline{(X, r, \psi_k \cdot \mu)} \quad \text{in the Gromov-weak topology .}$$

Theorem (Polish space)

The space $\mathbb{M}^{\Omega}$, equipped with the Ω-marked Gromov-weak$^{\#}$ topology, is a Polish space.

Definition (Polynomials)

Let $n \in \mathbb{N}$. Let $g \in C_{bb}(\Omega^n, \mathbb{R})$, the space of real-valued bounded continuous function on Ω^n with bounded support. For $k \in \mathbb{N} \cup \{0, \infty\}$, let $\phi \in C_b^k(\mathbb{R}_+^{\binom{n}{2}}, \mathbb{R})$, the space of k-times continuously differentiable real-valued functions on $\mathbb{R}_+^{\binom{n}{2}}$ with bounded derivatives up to order k. We call the function $\Phi^{n,\phi,g} : \mathbb{M}^\Omega \to \mathbb{R}$ defined by

$$\Phi^{n,\phi,g}(\overline{(X, r, \mu)}) := \int \cdots \int \phi(\underline{\underline{r}}) g(\underline{v}) \mu^{\otimes n}(\mathrm{d}(\underline{x}, \underline{v})), \tag{5.14}$$

a monomial of order n, where $\underline{v} := (v_1, \dots, v_n)$, $\underline{x} := (x_1, \dots, x_n)$, $\underline{\underline{r}} = \underline{\underline{r}}(\underline{x}) := (r(x_i, x_j))_{1 \le i < j \le n}$, and $\mu^{\otimes n}(\mathrm{d}(\underline{x}, \underline{v}))$ denotes the n-fold product measure of μ on $(X \times \Omega)^n$.

Definition

(a) Let $\Pi_n^k := \{\Phi^{n,\phi,g} : \phi \in C_b^k(\mathbb{R}_+^{\binom{n}{2}}, \mathbb{R}), g \in C_{bb}(\Omega^n, \mathbb{R})\}$, which we call the space of monomials of order n (with differentiability of order k). Let Π_0^k be the set of constant functions. We then denote by

$$\Pi^k := \cup_{n \in \mathbb{N}_0} \Pi_n^k$$

the set of all monomials (with differentiability of order k).

(b) For $\Omega = \mathbb{R}^d$ and $k, l \in \mathbb{N} \cup \{0, \infty\}$, we define

$$\Pi_n^{k,l} := \{\Phi^{n,\phi,g} : \phi \in C_b^k(\mathbb{R}_+^{\binom{n}{2}}, \mathbb{R}), g \in C_{bb}^l(\Omega^n, \mathbb{R})\},$$

and $\Pi^{k,l} := \cup_{n \in \mathbb{N}_0} \Pi_n^{k,l}$.

Definition

(c) We call the linear spaces

$$\widetilde{\Pi}^{k,\ell} \text{ generated by } \Pi^{k,\ell}, \tag{5.15}$$

the polynomials (with differentiability of ϕ, resp. g, of order k, resp. ℓ).

Theorem (Convergence-determining class)

We have the following properties for Π^k, for each $k \in \mathbb{N} \cup \{0, \infty\}$:

(i) *Π^k is convergence determining in $\mathbb{M}^\Omega$. Namely, $\overline{(X_n, r_n, \mu_n)} \to \overline{(X, r, \mu)}$ in $\mathbb{M}^\Omega$ if and only if $\Phi(\overline{(X_n, r_n, \mu_n)}) \to \Phi(\overline{(X, r, \mu)})$ as $n \to \infty$ for all $\Phi \in \Pi^k$.*

(ii) *Π^k is also convergence determining in $\mathcal{M}_1(\mathbb{M}^\Omega)$, the space of probability measures on $\mathbb{M}^\Omega$. Namely, a sequence of $\mathbb{M}^\Omega$-valued random variables $(\mathcal{X}_n)_{n\in\mathbb{N}}$ converges weakly to a $\mathbb{M}^\Omega$-valued random variable $\mathcal{X}$ if and only if $\mathbb{E}[\Phi(\mathcal{X}_n)] \to \mathbb{E}[\Phi(\mathcal{X})]$ as $n \to \infty$ for all $\Phi \in \Pi^k$.*

Tree-valued Fleming-Viot dynamic

$$L^{\mathrm{FV}}\Phi(\mathcal{X}) = 2\int_{(X\times\Omega)^n} \mu^{\otimes n}(\mathrm{d}(\underline{x},\underline{v}))\, g(\underline{v}) \sum_{1\le k<\ell\le n} \frac{\partial}{\partial r_{k,\ell}}\phi(\underline{\underline{r}}) \tag{6.1}$$

$$+ \int_{(X\times\Omega)^n} \mu^{\otimes n}(\mathrm{d}(\underline{x},\underline{v}))\, \phi(\underline{\underline{r}}) \sum_{j=1}^{n} \sum_{v'\in\Omega} \bar{a}(v_j, v')(M_{v_j,v'}g - g)(uv)$$

$$+ 2\gamma \int_{(X\times\Omega)^n} \mu^{\otimes n}(\mathrm{d}(\underline{x},\underline{v}))\; g(\underline{v}) \sum_{1\le k<\ell\le n} 1_{\{v_k=v_\ell\}}(\theta_{k,\ell}\phi - \phi)(\underline{\underline{r}}),$$

where $\underline{x} = (x_1, \cdots, x_n)$, $\underline{v} = (v_1, \cdots, v_n)$,
$\underline{\underline{r}} := (r_{k,\ell})_{1\le k<\ell\le n} := (r(x_k, x_\ell)_{1\le k<\ell\le n})$, and

$$(M_{v_j,v'}g)(v_1, \cdots, v_n) := g(v_1, \cdots, v_{j-1}, v', v_{j+1}, \cdots, v_n) \tag{6.2}$$

$$(\theta_{k,\ell}\phi)(\underline{\underline{r}}) := \phi(\theta_{k,\ell}\underline{\underline{r}}), \quad \text{with} \quad (\theta_{k,\ell}\underline{\underline{r}})_{i,j} := \begin{cases} r(x_i, x_j) & \text{if } i, j \neq \ell, \\ r(x_i, x_k) & \text{if } j = \ell, \\ r(x_k, x_j) & \text{if } i = \ell. \end{cases} \tag{6.3}$$

Theorem (Martingale problem characterization of IFV Genealogy processes)

For any $\mathcal{X}_0 = \overline{(X_0, r_0, \mu_0)} \in \mathbb{U}_1^{\Omega}$, *we have:*

(i) *The* $(L^{\mathrm{FV}}, \Pi^{1,0}, \delta_{\mathcal{X}_0})$*-martingale problem is well-posed, i.e. there exists a* $\mathbb{U}_1^{\Omega}$*-valued process* $\mathcal{X}^{\mathrm{FV}} := (\mathcal{X}_t^{\mathrm{FV}})_{t \geq 0}$, *unique in its distribution, which has initial condition* $\mathcal{X}_0$ *and cádlág path, such that for all* $\Phi \in \Pi^{1,0}$ *and w.r.t. the natural filtration generated by* $(\mathcal{X}_t^{\mathrm{FV}})_{t \geq 0}$,

$$\left(\Phi(\mathcal{X}_t^{\mathrm{FV}}) - \Phi(\mathcal{X}_0^{\mathrm{FV}}) - \int_0^t (L^{\mathrm{FV}} \Phi)(\mathcal{X}_s^{\mathrm{FV}}) \mathrm{d}s \right)_{t \geq 0} \text{ is a martingale.} \tag{6.4}$$

Theorem

(ii) *The solutions (for varying initial conditions) define a strong Markov, and Feller process with continuous path.*

(iii) *If the initial state admits a mark function, then so does the path for all $t > 0$ almost surely. Furthermore for every $t > 0$ and $\varepsilon > 0$, every localization of the state $\mathcal{X}_t^{\mathrm{FV}}$ to a site can be covered by finitely many balls of radius ε except for a set with μ-measure ε.*

Dual process

The dual process is driven by a system of n coalescing random walks for some $n \in \mathbb{N}$, labeled from 1 to n starting at $\xi_{0,1}, \cdots, \xi_{0,n} \in \Omega$ at time 0. Each walk i is identified with the partition element $\{i\}$ for a partition of the set $\{1, 2, \cdots, n\}$. We label partition elements by their smallest element they contain.

We also fix two functions, $\phi \in C_b(\mathbb{R}_+^{\binom{n}{2}}, \mathbb{R})$, which takes as its arguments the genealogical distances between the n coalescing walks, and $g \in C_{bb}(\Omega^n, \mathbb{R})$, which takes as its arguments the initial positions $\xi_0 := (\xi_{0,1}, \ldots, \xi_{0,n})$ of the walks.

The dynamics of the dual process is as follows:

- Partition elements migrate independently on Ω according to rate 1 continuous time random walks with transition kernel $\bar{a}$.
- Independently, every pair of partition elements at the same location in Ω merge at rate γ.
- At time t, we define the genealogical distance $r_t(i,j)$ of two individuals i and j in $\{1, 2, \cdots, n\}$ as $2\min\{t, T_{i,j}\}$, where $T_{i,j}$ is the first time the two partition elements labelled by i and j coalesce, i.e., i and j are in the same partition element.

The positions of the n walks are denoted by $\xi_t := (\xi_{t,1}, \ldots, \xi_{t,n})$.

This way we obtain a process

$$(\mathcal{K}_t)_{t\geq 0} \text{ with states } (\pi, \xi', \underline{\underline{r}}) \in \mathbb{S}_n, \tag{6.5}$$

where π is a partition of the set $\{1, \cdots, n\}$ with cardinality $|\pi|$, ξ and ξ' record respectively the positions of the walks at the initial and present time, and $\underline{\underline{r}} := (r(i,j))_{1\leq i<j\leq n}$ is the genealogical distance matrix of the individuals at the present time. We set

$$\mathbb{S} = \bigcup_{n\in\mathbb{N}} \mathbb{S}_n. \tag{6.6}$$

Duality function

$$H(\mathcal{X},\mathcal{K}) = \int\limits_{(X\times\Omega)^{|\pi|}} \Big(\bigotimes_{\pi} \mu\Big)(d(\underline{x},\underline{v}))g(\underline{v}^{\pi}) \prod_{j=1}^{n} 1_{\{v_j=\xi_j'\}}\, \phi(\underline{\underline{r}}^{\pi} + \underline{\underline{r}}'), \quad (6.7)$$

where the upper π means that we set x and v equal if the indices belong to one partition element in π.

Theorem (Duality and longtime behaviour)

The following properties hold for the IFV genealogy process $(\mathcal{X}_t^{\mathrm{FV}})_{t\geq 0}$*:*

(a) *For every* $\mathcal{X}_0^{\mathrm{FV}} \in \mathbb{U}_1^{\Omega}$ *and* $\mathcal{K}_0 \in \mathbb{S}$*, we have*

$$\mathbb{E}[H(\mathcal{X}_t^{\mathrm{FV}}, \mathcal{K}_0)] = \mathbb{E}[H(\mathcal{X}_0^{\mathrm{FV}}, \mathcal{K}_t)], \quad t \geq 0. \tag{6.8}$$

(b) *If* $\widehat{a}(\cdot,\cdot) = \frac{1}{2}(a(\cdot,\cdot) + \bar{a}(\cdot,\cdot))$ *is recurrent, then*

$$\mathcal{L}[\mathcal{X}_t^{\mathrm{FV}}] \underset{t\to\infty}{\Longrightarrow} \Gamma \in \mathcal{M}_1(\mathbb{U}_1^{\Omega}), \tag{6.9}$$

where Γ *is the unique invariant measure of the process* $\mathcal{X}^{\mathrm{FV}}$ *on* $\mathbb{U}_1^{\Omega}$*.*

The duality relation and longtime behaviour holds also for **continuum space limit of IFV**, the dual is given as:

Functional of Brownian web - dual Brownian web.

In the *transient case* we have to transform the state

$$r(\cdot,\cdot) \longrightarrow 1 - e^{-r(\cdot,\cdot)} \tag{6.10}$$

to obtain *convergence to the unique invariant measure*.

We have now positive probability that two sampled points from the equilibrium tree have distance 1 which corresponds to not having a common ancestor.

Case with selection and mutation

We can derive a duality relation and use it to obtain the longtime behaviour.

If the measure-valued process of types converges to a certain equilibrium, then the marked metric measure space converges to a corresponding equilibrium.

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