Stochastic dualities and Lie algebras.

Cristian Giardina'

Duality of Markov processes and applications
to spatial population models, *Berlin 6-7 Nov. 2014*

The results presented here have been obtained in a series of works in collaboration with:

Gioia Carinci (Modena)

Claudio Giberti (Modena)

Jorge Kurchan (Paris)

Frank Redig (Delft)

Tomohiro Sasamoto (Tokyo)

Kiamars Vafayi (Eindhoven)

Thanks to

Ellen Baake (Bielefeld)

Frank den Hollander (Leiden)

Outline

- Lie algebraic approach to duality theory.
- Heisenberg algebra: simple dualities in basic models.
- Construction of Markov processes with algebraic structure and symmetries.
- Classical $\mathfrak{su}(1,1)$ algebra: duality for Wright-Fisher and Moran models with symmetric mutations.
- Deformed $\mathfrak{su}(1,1)$ algebra: new processes and new dualities with selection [work in progress!].

1. Lie algebraic approach

to duality

Duality

Definition

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator L,

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator L_{dual}

ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is self-dual if $L_{dual} = L$.

Duality

Assume that

$$D(\cdot, y) \in \mathcal{D}(L) \quad \text{and} \quad S_t D(\cdot, y) \in \mathcal{D}(L) \quad \forall\, y \in \Omega_{dual}$$

$$D(x, \cdot) \in \mathcal{D}(L_{dual}) \quad \text{and} \quad S_t^{dual} D(x, \cdot) \in \mathcal{D}(L_{dual}) \quad \forall\, x \in \Omega\ ,$$

with $\mathcal{D}(L)$ and $\mathcal{D}(L_{dual})$ the generators domain,
S_t and S_t^{dual} the processes semigroup.

Then duality is equivalent to

$$LD(\cdot, \xi)(\eta) = L_{dual} D(\eta, \cdot)(\xi)$$

Lie algebra

A Lie algebra is a vector space $\mathfrak{g}$ over a field F together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (Lie bracket):

- $\forall\ a, b$ in F and $\forall\ u, v, w$ in $\mathfrak{g}$

  $$[au + bv, w] = a[u, w] + b[v, w], \quad [w, au + bv] = a[w, u] + b[w, v]$$

- $\forall\ u$ in $\mathfrak{g}$: $[u, u] = 0$
- [Jacobi identity]: $\forall\ u, v, w$ in $\mathfrak{g}$

  $$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$

Elements of a Lie algebra $\mathfrak{g}$ are said to be generators of the Lie algebra if the smallest subalgebra of $\mathfrak{g}$ containing them is $\mathfrak{g}$ itself.

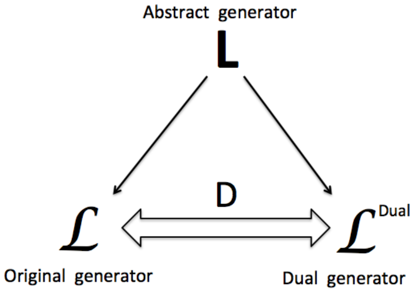
Algebraic approach

1. Write the Markov generator in abstract form, i.e. as an element of a Lie algebra, using the algebra generators (typically creation and annihilation operators).

2. Duality is related to a change of representation, i.e. new operators that satisfy the same algebra. Duality functions are the intertwiners.

3. Self-duality is associated to symmetries, i.e. conserved quantities.

Duality


Abstract generator
L
D
Dual
Original generator
Dual generator

Self-duality

For Markov chain with countable state space

$$LD(\cdot,\xi)(\eta) = LD(\eta,\cdot)(\xi)$$

amounts to

$$\mathbf{LD} = \mathbf{DL}^T$$

Indeed

$$\sum_{\eta'} \mathbf{L}(\eta,\eta')\mathbf{D}(\eta',\xi) = LD(\cdot,\xi)(\eta) = LD(\eta,\cdot)(\xi) = \sum_{\xi'} \mathbf{L}(\xi,\xi')\mathbf{D}(\eta,\xi')$$

Trivial self-duality functions from reversible measures

From a reversible measure μ, i.e.

$$\mathbf{L}(\eta,\xi)\mu(\eta) = \mathbf{L}(\xi,\eta)\mu(\xi)$$

a trivial (i.e. diagonal) self-duality function is

$$\mathbf{d}(\eta,\xi) = \frac{1}{\mu(\eta)}\delta_{\eta,\xi}$$

Indeed

$$\frac{\mathbf{L}(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'} \mathbf{L}(\eta,\eta')\mathbf{d}(\eta',\xi) = \sum_{\xi'} \mathbf{L}(\xi,\xi')\mathbf{d}(\eta,\xi') = \frac{\mathbf{L}(\xi,\eta)}{\mu(\eta)}$$

Symmetries and self-duality

S: symmetry of the transposed of the generator, i.e. $[\mathbf{L^T}, \mathbf{S}] = 0$,
$\mathbf{d}$: trivial self-duality function,

$$\longrightarrow \quad \mathbf{D} = \mathbf{dS} \text{ self-duality function.}$$

Indeed

$$\mathbf{LD} = \mathbf{LdS} = \mathbf{dL}^T\mathbf{S} = \mathbf{dSL}^T = \mathbf{DL}^T$$

Self-duality is related to the action of a symmetry

2. Basic examples

Moran model with two types

Population of N individuals, each of which can be of types 1 or type 2. A pair of individuals are sampled uniformly at random, one dies with probability $1/2$, the other reproduces.

Define

$$K^{(N)}(t) = \text{number of individuals of type 1 at time } \ t \geq 0$$

$(K^{(N)}(t))_{t\geq 0}$ is a continuos time Markov chain with state space $\Omega_N = \{0, 1, \ldots N\}$ and generator

$$L_N^{Moran} f(k) = \frac{1}{2} k(N-k)(f(k+1) + f(k-1) - 2f(k))$$

Wright-Fisher diffusion with two types

Diffusive scaling limit: the process $\left(X^{(N)}(t) = \frac{K^{(N)}(N^2 t)}{N}\right)_{t\geq 0}$ with state space $\Omega'_N = \{0, 1/N, \ldots, 1\}$ has generator

$$L'_N f(\frac{k}{N}) = N^2 \frac{1}{2}\frac{k}{N}(1 - \frac{k}{N})\left(f(\frac{k}{N} + \frac{1}{N}) + f(\frac{k}{N} - \frac{1}{N}) - 2f(\frac{k}{N})\right)$$

In the limit $N \to \infty$ the process $(X^{(N)}(t))_{t\geq 0}$ converges to the Wright-Fisher diffusion $(X(t))_{t\geq 0}$ with state space $[0, 1]$ and generator

$$L^{WF} f(x) = \frac{1}{2}x(1 - x)\frac{\partial^2 f}{\partial x^2}(x)$$

Counting blocks of Kingman coalescence

For each $k \in \mathbb{N}$, the k-coalescence is a continuous time Markov chain on the space of equivalence relations on $\{1, 2, \ldots, k\}$ with transition rates

$$c(x, y) = \begin{cases} 1 & \text{if } y \text{ is obtained by coalescing} \\ & \text{two equivalence classes of } x, \\ 0 & \text{otherwise.} \end{cases}$$

By extension the Kingman coalescent on $\mathbb{N}$ is defined by requiring that for each k its restriction to $\{1, \ldots, k\}$ is a k-coalescence.

Define

$$N(t) = \text{ number of blocks in the } k\text{-coalescence at time } t \geq 0.$$

It is a death process on $\{1, \ldots, k\}$ defined by the Markov generator

$$(L^{King} f)(n) = \frac{n(n-1)}{2}(f(n-1) - f(n))$$

Heisenberg algebra

The Lie bracket is given by the commutator, i.e. for u, v in the algebra

$$[u, v] = uv - vu$$

The algebra is generated by the elements (a^+, a^-) with commutator

$$[a^-, a^+] = \mathbf{1}$$

Two representations are:

$$\begin{cases} \mathrm{a}^+ = x \\ \\ \mathrm{a}^- = \frac{\partial}{\partial x} \end{cases} \qquad \begin{cases} a^+|n\rangle = |n+1\rangle \\ \\ a^-|n\rangle = n|n-1\rangle \end{cases}$$

where, for $n \in \{0, 1, 2, \ldots\}$, $|n\rangle = e_n$ denote the column vectors

$$(e_n)_i = \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{if } i \neq n \end{cases} \qquad e_n^T \cdot e_m = \langle n|m\rangle = \delta_{n,m}$$

Duality Wright-Fisher / Kingman

Proposition

The process $\{X(t)\}_{t\geq 0}$ with generator L^{WF} and the process $\{N(t)\}_{t\geq 0}$ with generator L^{King} are dual on $D(x,n) = x^n$, i.e.

$$\mathbb{E}_x(X(t)^n) = \mathbb{E}_n(x^{N(t)})$$

Indeed:

$$\begin{aligned}
L^{WF} D(\cdot, n)(x) &= \frac{1}{2} x(1-x) \frac{\partial^2}{\partial x^2} x^n \\
&= \frac{n(n-1)}{2} (x^{n-1} - x^n) \\
&= \frac{n(n-1)}{2} (D(x, n-1) - D(x,n)) \\
&= L^{King} D(x, \cdot)(n)
\end{aligned}$$

Duality Wright-Fisher / Kingman : algebraic approach

The duality is a consequence of the change of representation:

$$\begin{cases} \mathrm{a}^+ = x \\ \mathrm{a}^- = \frac{d}{dx} \end{cases} \qquad \begin{cases} a^+|n\rangle = |n+1\rangle \\ a^-|n\rangle = n|n-1\rangle \end{cases}$$

The abstract element $L = \frac{1}{2} a^+(1-a^+)(a^-)^2$

$L = L^{WF}$ in the first representation

$L^T = L^{King}$ in the second representation

Duality fct. $D(x,n) = x^n$ is the intertwiner:

$$xD(x,n) = D(x,n+1) \qquad \frac{d}{dx}D(x,n) = nD(x,n-1)$$

Duality Moran / Kingman

Proposition

The process $\{K_N(t)\}_{t\geq 0}$ with generator L_N^{Moran} and the process $\{N(t)\}_{t\geq 0}$ with generator L^{King} are dual on

$$D_N(k,n) = \frac{\binom{k}{n}}{\binom{N}{n}} = \frac{k(k-1)\cdots(k-(n-1))}{N(N-1)\cdots(N-(n-1))}\,.$$

with the convention $D_N(k,0)=1$, $D_N(k,N+1)=0$.

Indeed:

$$L_N^{Moran} D_N(\cdot,n)(k) = L^{King} D_N(k,\cdot)(n)$$

Duality Moran / Kingman : algebraic approach

The duality is a consequence of a change of representation.
For functions $f : \{0, \dots, N\} \to \mathbb{R}$

$$\begin{cases} \mathrm{a}_N^+ f(k) = \sum_{r=0}^{k-1} (-1)^{k-1-r} \frac{\binom{N}{r}}{\binom{N}{k}} f(r) \\ \mathrm{a}_N^- f(k) = (N-k)\, f(k+1) + (2k-N)\, f(k) - kf(k-1) \end{cases}$$

with the convention $f(-1) = f(N+1) = 0$, and

$$\begin{cases} a^+ |n\rangle = |n+1\rangle \\ a^- |n\rangle = n|n-1\rangle \end{cases}$$

The abstract element $\quad L = \frac{1}{2} a^+ (1 - a^+)(a^-)^2$

$L = L_N^{Moran}$ in the first representation

$L^T = L^{King}$ in the second representation

Duality Moran / Kingman : algebraic approach

The intertwiner

$$D_N(k,n) = \frac{\binom{k}{n}}{\binom{N}{n}} = \frac{k(k-1)\cdots(k-(n-1))}{N(N-1)\cdots(N-(n-1))}\ .$$

with the convention $D_N(k,0) = 1$, $D_N(k, N+1) = 0$ is such that on the vector space generated by the functions $k \mapsto D_N(k,n), 0 \le n \le N$

$$\begin{aligned} \mathrm{a}_N^- D_N(\cdot, n)(k) &= nD_N(k, n-1),\ \forall\, 1 \le n, \forall\, k \ge n-1 \\ \mathrm{a}_N^- D_N(\cdot, 0)(k) &= 0,\ \forall\, 0 \le k \le N \\ \mathrm{a}_N^+ D_N(\cdot, n)(k) &= D_N(k, n+1),\ \forall\ 0 \le n \le N, k \ge n \end{aligned}$$

Mutation

Consider the Moran model where each individual of type 2 mutates to an individual of type 1 at rate θ/N. Then in the diffusive limit one has

$$\begin{aligned} L^{WF,mut} &= x(1-x)\frac{d^2}{dx^2} + \theta(1-x)\frac{d}{dx} \\ &= a^+(1-a^+)(a^-)^2 + \theta(1-a^+)a^- \end{aligned}$$

By changing to a discrete representation of the Heisenberg algebra this gives the dual

$$L^{King,mut}f(n) \quad = \quad n(n-1)(f(n-1)-f(n)) + \theta n(f(n-1)-f(n))$$

which corresponds to Kingman's coalescent with extra rate θn to go down from n to $n-1$, due to mutation.

3. Constructive approach

Algebraic approach

1. Write the Markov generator in abstract form, i.e. as an element of a Lie algebra, using the algebra generators (typically creation and annihilation operators).

2. Duality is related to a change of representation, i.e. new operators that satisfy the same algebra. Duality functions are the intertwiners.

3. Self-duality is associated to symmetries, i.e. conserved quantities.

Conversely, Step 1. can be turned into a constructive step.

Construction of Markov generators with algebraic structure and symmetries

i) (*Lie Algebra*): Start from a (representation of a) Lie algebra $\mathfrak{g}$.

ii) (*Casimir*): Pick an element in the center of $\mathfrak{g}$, e.g. the Casimir C.

iii) (*Co-product*): Consider a co-product $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ making the algebra a bialgebra and conserving the commutation relations.

iv) (*Quantum Hamiltonian*): Compute the co-product $H = \Delta(C)$.

v) (*Markov generator*): Apply a ground state transform (often a similarity transformation) to turn H into a Markov generator L.

vi) (*Symmetries*): $S = \Delta(X)$ with $X \in \mathfrak{g}$ is a symmetry of H:

$$[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.$$

4. Classical $\mathfrak{su}(1,1)$ algebra.

Classical $\mathfrak{su}(1,1)$ Lie algebra

It turns out that Wright-Fisher diffusion and Moran models have more structure than only the Heisenberg algebra.

In the (multi-type) setting with parent independent mutations their Markov generator can be written using classical $\mathfrak{su}(1,1)$ Lie algebra.

The generators K^+, K^-, K^o of $\mathfrak{su}(1,1)$ algebra satisfy the commutation relations

$$[K^o, K^\pm] = \pm K^\pm \qquad [K^+, K^-] = -2K^o$$

$\mathfrak{su}(1,1)$ Heisenberg ferromagnet as a population model

The abstract operator

$$\mathcal{L}_m = \frac{1}{2} \sum_{1 \le i < j \le d} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2 K_i^o K_j^o + \frac{m^2}{8} \right)$$

1. written in terms of a continuous representation, is the generator of the d-type Wright-Fisher diffusion with mutation rate $\frac{m}{4}(d-1)$
2. written in terms of a discrete representation, is the generator of the d-type Moran model with mutation rate $\frac{m}{4}(d-1)$
3. therefore the two processes are dual; in addition - from the symmetries - we will find self-duality.

Let us apply the construction ....

step i): representation in terms of matrices

A discrete representation of $\mathfrak{su}(1,1)$ algebra is

$$\begin{cases} K^{+}|n\rangle = \left(n+\frac{m}{2}\right)|n+1\rangle \\ K^{-}|n\rangle = n|n-1\rangle \\ K^{o}|n\rangle = \left(n+\frac{m}{4}\right)|n\rangle \end{cases}$$

In a canonical base

$$K^{+} = \begin{pmatrix} 0 & & & \\ \frac{m}{2} & \ddots & & \\ & \frac{m}{2}+1 & \ddots & \\ & & \ddots & \ddots \end{pmatrix} \quad K^{-} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \quad K^{o} = \begin{pmatrix} \frac{m}{4} & 0 & & \\ & \frac{m}{4}+1 & \ddots & \\ & & \frac{m}{4}+2 & \ddots \\ & & & \ddots \end{pmatrix}$$

step ii): Casimir element

For the $\mathfrak{su}(1,1)$ algebra the Casimir is

$$C = \frac{1}{2}(K^- K^+ + K^+ K^-) - (K^0)^2$$

C is in the center of the algebra:

$$[C, K^+] = [C, K^-] = [C, K^o] = 0$$

$$\begin{aligned} C|n\rangle &= \frac{1}{2}\left((n+1)(\frac{m}{2}+n) + (\frac{m}{2}+n-1)n\right)|n\rangle - (n+\frac{m}{4})^2|n\rangle \\ &= \frac{m}{4}(1-\frac{m}{4})|n\rangle \end{aligned}$$

step iii): Co-product

The co-product is a morphism that turns the algebra into a bialgebra:

$$\Delta : \mathfrak{su}(1,1) \to \mathfrak{su}(1,1) \otimes \mathfrak{su}(1,1)$$

and conserves the commutations relations

$$[\Delta(K^o), \Delta(K^{\pm})] = \pm\Delta(K^{\pm})$$
$$[\Delta(K^-), \Delta(K^+)] = 2\Delta(K^o)$$

For classical Lie-algebras the co-product is just the symmetric tensor product with the identity

$$\Delta(X) = X \otimes \mathbf{1} + \mathbf{1} \otimes X := X_1 + X_2$$

step iv): Quantum Hamiltonian

$$\Delta(C) = \frac{1}{2}\Big(\Delta(K^-)\Delta(K^+) + \Delta(K^+)\Delta(K^-)\Big) - \Big(\Delta(K^0)\Big)^2$$

$$= \frac{1}{2}\Big((K_1^- + K_2^-)(K_1^+ + K_2^+) + (K_1^+ + K_2^+)(K_1^- + K_2^-)\Big) - \Big(K_1^o + K_2^o\Big)^2$$

$$= K_1^- K_2^+ + K_1^+ K_2^- - 2K_1^o K_2^o + C_1 + C_2$$

$$= \mathfrak{su}(1,1) \text{ Heisenberg ferromagnet } + \text{ diagonal}$$

step v): Markov generator

There is no need of a “ground state transformation”. In the discrete representation we find

$$\Delta(C) = (L_{1,2}^{SIP(m)})^* + (\frac{m}{2}(1 - \frac{m}{2}))\mathbf{1} \otimes \mathbf{1}$$

where

$$L_{1,2}^{SIP(m)} f(\eta_1, \eta_2) = \eta_1 \left(\eta_2 + \frac{m}{2}\right) [f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2)] + \eta_2 \left(\eta_1 + \frac{m}{2}\right) [f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2)]$$

is the generator of the Symmetric Inclusion Process SIP(m).
If $\eta_1 + \eta_2 = N$ it is the Moran model with N individuals, two types and symmetric mutation rate $m/2$.

step vi): symmetries

As a consequence of the construction,

$\Delta(K^\alpha)$ with $\alpha \in \{+, -, o\}$ are symmetries of the process:

$$[(L_{1,2}^{SIP(m)})^*, K_1^o + K_2^o] = 0$$

$$[(L_{1,2}^{SIP(m)})^*, K_1^+ + K_2^+] = 0$$

$$[(L_{1,2}^{SIP(m)})^*, K_1^- + K_2^-] = 0$$

Self-duality of the multi-type Moran model

Theorem [Carinci, G., Giberti, Redig (2013), to appear on SPA]

On the simplex $\sum_{i=1}^{d} \eta_i = N$, the d-types Moran model with N individuals and parent-independent mutation at rate $\frac{m}{4}(d-1)$ coincides with the $SIP(m)$ on the complete graph with d sites

$$\mathscr{L}^{Moran}_{N,d,\frac{m}{4}(d-1)} = \mathscr{L}^{SIP(m)}_{d}$$

$$\begin{aligned}\mathcal{L}^{SIP(m)}_{d} f(\eta) &= \frac{1}{2} \sum_{1\le i<j\le d} \eta_i \left(\eta_j + \frac{m}{2}\right) \left(f(\eta + e_i - e_j) - f(\eta)\right) \\ & \frac{1}{2} \sum_{1\le i<j\le d} \eta_j \left(\eta_i + \frac{m}{2}\right) \left(f(\eta - e_i + e_j) - f(\eta)\right)\end{aligned}$$

The process is self-dual with self-duality function

$$D(\eta_1,\ldots,\eta_d;\xi_1,\ldots,\xi_d) = \prod_{i=1}^{d} \frac{\eta_i!}{(\eta_i-\xi_i)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+\xi_i\right)}$$

Trivial self-duality function d

Reversible product measure are product of Negative Binomial $(p, \frac{m}{2})$

$$\mu_{rev}(\eta) = \prod_{i=1}^{d} \frac{p^{\xi_i}(1-p)^{\frac{m}{2}}}{\eta_i!} \frac{\Gamma(\frac{m}{2}+\eta_i)}{\Gamma(\frac{m}{2})}$$

and a trivial (i.e. diagonal) self-duality function was obtained as

$$\mathbf{d}(\eta, \xi) = \frac{1}{\mu_{rev}(\eta)} \delta_{\eta,\xi}$$

Since the total number of particles is constant, one can take

$$\mathbf{d}(\eta, \xi) = \prod_{i=1}^{d} \frac{\eta_i! \, \Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\eta_i)} \delta_{\eta_i,\xi_i}$$

The symmetry $S = \exp(\sum_{i=1}^{d} K_i^+)$

$$
\begin{aligned}
S(\eta,\xi) &= \prod_{i=1}^{d} \langle \eta_i | \exp(K_i^+) | \xi_i \rangle \\
&= \prod_{i=1}^{d} \langle \eta_i | \sum_{s_i \geq 0} \frac{(K_i^+)^{s_i}}{s_i!} | \xi_i \rangle \\
&= \prod_{i=1}^{d} \langle \eta_i | \sum_{s_i \geq 0} \frac{(\frac{m}{2} + \xi_i + s_i - 1)!}{(\frac{m}{2} + \xi_i - 1)! s_i!} | \xi_i + s_i \rangle \\
&= \prod_{i=1}^{d} \frac{(\frac{m}{2} + \eta_i - 1)!}{(\frac{m}{2} + \xi_i - 1)!(\eta_i - \xi_i)!} \\
&= \prod_{i=1}^{d} \frac{\Gamma(\frac{m}{2} + \eta_i)}{\Gamma(\frac{m}{2} + \xi_i)} \frac{1}{(\eta_i - \xi_i)!}
\end{aligned}
$$

The self-duality function D

Combining trivial self-duality and symmetry leads to

$$
\begin{aligned}
D(\eta,\xi) &= dS(\eta,\xi) \\
&= \prod_{i=1}^{d} \frac{\eta_i!\,\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\eta_i)} \cdot \frac{\Gamma(\frac{m}{2}+\eta_i)}{\Gamma(\frac{m}{2}+\xi_i)} \frac{1}{(\eta_i-\xi_i)!} \\
&= \prod_{i=1}^{d} \frac{\eta_i!}{(\eta_i-\xi_i)!} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+\xi_i)}
\end{aligned}
$$

Duality between multi-type

Wright-Fisher diffusion and Moran model

Theorem [Carinci, G., Giberti, Redig (2013), to appear on SPA]

On the simplex $\sum_{i=1}^d x_i = 1$, the d-types Wrigh-Fisher diffusion with parent-independent mutation at rate $\frac{m}{4}(d-1)$ coincides with the $BEP(m)$ on the complete graph with d sites

$$\mathscr{L}^{WF}_{d,\frac{m}{4}(d-1)} = \mathscr{L}^{BEP(m)}_d$$

$$\mathscr{L}^{WF}_{d,\theta} = \sum_{i=1}^{d-1} \frac{1}{2} x_i(1-x_i)\frac{\partial^2}{\partial x_i^2} - \sum_{1\le i<j\le d-1} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\theta}{d-1}\sum_{i=1}^{d-1}(1-dx_i)\frac{\partial}{\partial x_i}$$

$$\mathscr{L}^{BEP(m)}_d = \frac{1}{2}\sum_{1\le i<j\le d} x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2 - \frac{m}{4}\sum_{1\le i<j\le d}(x_i - x_j)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)$$

The process is dual to Moran model with $N = \sum_{i=1}^d \xi_i$ individuals on duality function

$$D_N(x_1,\ldots,x_d;\xi_1,\ldots,\xi_d) = \prod_{i=1}^d x_i^{\xi_i}\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+\xi_i\right)}$$

Duality explained

The abstract operator (quantum Heisenberg ferromagnet)

$$\mathscr{L} = \sum_{(i,j)\in E} \left(K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^o K_j^o + \frac{m^2}{8} \right)$$

with $\{K_i^+, K_i^-, K_i^o\}_{i\in V}$ satisfying $\mathfrak{su}(1,1)$ commutation relations:

$$[K_i^o, K_j^\pm] = \pm\delta_{i,j} K_i^\pm \qquad [K_i^+, K_j^-] = -2\delta_{i,j} K_i^o$$

can be looked at in different representations.

Duality between L^{WF} e L^{Moran} corresponds to two different representations of the operator $\mathscr{L}$.
Duality fct is the intertwiner.

Representation of $\mathfrak{su}(1,1)$ algebra in terms of differential operators

Continuous representation

$$\mathcal{K}_i^+ = x_i \qquad \mathcal{K}_i^- = x_i \frac{\partial^2}{\partial x_i^2} + \frac{m}{2}\frac{\partial}{\partial x_i} \qquad \mathcal{K}_i^o = x_i \frac{\partial}{\partial x_i} + \frac{m}{4}$$

satisfy commutation relations

$$[\mathcal{K}_i^o, \mathcal{K}_j^\pm] = \pm\delta_{i,j}\mathcal{K}_i^\pm \qquad [\mathcal{K}_i^-, \mathcal{K}_j^+] = 2\delta_{i,j}\mathcal{K}_i^o$$

In this representation

$$\mathscr{L} = L^{BEP(m)}$$

Duality function as intertwiner

Intertwiner

$$\mathcal{K}_i^+ D_i(\cdot, \xi_i)(x_i) = K_i^+ D_i(x_i, \cdot)(\xi_i)$$
$$\mathcal{K}_i^- D_i(\cdot, \xi_i)(x_i) = K_i^- D_i(x_i, \cdot)(\xi_i)$$
$$\mathcal{K}_i^o D_i(\cdot, \xi_i)(x_i) = K_i^o D_i(x_i, \cdot)(\xi_i)$$

Example

$$\begin{aligned}
\mathcal{K}_i^+ D_i(\cdot, \xi_i)(x_i) &= x_i D_i(x_i, \xi_i) \\
&= x_i^{\xi_i+1} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + \xi_i)} \\
&= (\frac{m}{2} + \xi_i) x_i^{\xi_i+1} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + \xi_i + 1)} \\
&= (\frac{m}{2} + \xi_i) D_i(x_i, \xi_i + 1) = K_i^+ D_i(x_i, \cdot)(\xi_i)
\end{aligned}$$

5. Deformed $\mathfrak{su}(1,1)$ algebra

q-numbers

For $q \in (0,1)$ and $n \in \mathbb{N}_0$ introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q\to 1}[n]_q = n$.

The first q-number's are:

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q+q^{-1}, \qquad [3]_q = q^2+1+q^{-2}, \quad \ldots$$

The deformed Lie algebra $\mathfrak{su}_q(1,1)$

For $q \in (0,1)$ consider the algebra with generators K^+, K^-, K^0 with commutation relations

$$[K^+, K^-] = -[2K^0]_q, \qquad [K^0, K^\pm] = \pm K^\pm$$

where

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

The Casimir element is

$$C = [K^o]_q[K^o - 1]_q - K^+K^-$$

A standard discrete representation is given by

$$\begin{cases} K^+|n\rangle &= \sqrt{[n+\frac{m}{2}]_q[n+1]_q}\;|n+1\rangle \\ K^-|n\rangle &= \sqrt{[n]_q[n+\frac{m}{2}-1]_q}\;|n-1\rangle \\ K^o|n\rangle &= (n+\frac{m}{4})\;|n\rangle \end{cases}$$

Co-product

A co-product $\Delta : \mathfrak{su}_q(1,1) \to \mathfrak{su}_q(1,1) \otimes \mathfrak{su}_q(1,1)$ is defined as

$$\begin{aligned} \Delta(K^{\pm}) &= K^{\pm} \otimes q^{-K^o} + q^{K^o} \otimes K^{\pm} \\ \Delta(K^o) &= K^o \otimes 1 + 1 \otimes K^o \end{aligned}$$

The co-product is an isomorphism such that

$$[\Delta(K^+), \Delta(K^-)] = -[2\Delta(K^o)]_q \qquad [\Delta(K^o), \Delta(K^{\pm})] = \pm\Delta(K^{\pm})$$

The co-product applied to the Casimir in the discrete representation gives, after a suitable ground state transformation, the generator of a new asymmetric process which we called Asymmetric Inclusion Process $ASIP(q, m)$.

ASIP(q,m) process

For $0 < q \leq 1$ the generator is given by

$$
\begin{aligned}
&(\mathcal{L}_{1,2}^{ASIP} f)(\eta_1, \eta_2) \\
&= q^{\eta_1 - \eta_2 + (\frac{m}{2} - 1)} [\eta_1]_q [\eta_2 + \frac{m}{2}]_q (f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2)) \\
&+ q^{\eta_1 - \eta_2 - (\frac{m}{2} - 1)} [\eta_2]_q [\eta_1 + \frac{m}{2}]_q (f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2))
\end{aligned}
$$

By construction, the process has natural symmetries:

$$
\begin{aligned}
\Delta(K^{\pm}) &= K^{\pm} \otimes q^{-K^o} + q^{K^o} \otimes K^{\pm} \\
\Delta(K^o) &= K^o \otimes 1 + 1 \otimes K^o
\end{aligned}
$$

Self-duality of ASIP(q,m)

Theorem [Carinci, G., Redig, Sasamoto (2014 + in progress)]

The ASIP(q,m) on $[1,L]\cap\mathbb{Z}$ with generator $\sum_{i=0}^{L-1} L_{i,i+1}^{ASIP(q,m)}$ is self-dual on

$$D_q(\eta,\xi) = \prod_{i=1}^{L} \frac{[\eta_i]_q!}{[\eta_i-\xi_i]_q!} \frac{\Gamma_q(\frac{m}{2})}{\Gamma_q(\frac{m}{2}+\xi_i)} \cdot q^{(\eta_i-\xi_i)[2\sum_{k=1}^{i-1}\xi_k+\xi_i]-mi\xi_i}$$

The proof is an immediate consequence of the constructive method.

Applications in non-equilibrium statistical physics

Let $\xi^{(i)}$ be the configuration with only one dual particle

$$\xi^{(i)}_m = \begin{cases} 1 & \text{if } m = i \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad N_i(\eta) := \sum_{k \geq i} \eta_k$$

then

$$D_q(\eta, \xi^{(i)}) = \frac{q^{-4mi+1}}{q^{\frac{m}{2}} - q^{-\frac{m}{2}}} \cdot (q^{2N_i(\eta)} - q^{2N_{i+1}(\eta)})$$

$N_i(\eta(t))$ is related to the total current $J_i(t)$ in the time interval $[0, t]$ across bond $(i-1, i)$ in the right direction: $J_i(t) = N_i(\eta(t)) - N_i(\eta(0))$

The duality relation gives

$$\mathbb{E}_\eta D_q(\eta(t), \xi^{(i)}) = \mathbb{E}_i \frac{q^{-4mY(t)+1}}{q^{\frac{m}{2}} - q^{-\frac{m}{2}}} \cdot (q^{2N_{Y(t)}(\eta)} - q^{2N_{Y(t)+1}(\eta)})$$

where $Y(t)$ is a continuos time random walk jumping to the right at rate $q^{\frac{m}{2}}[\frac{m}{2}]_q$ and to the left at rate $q^{-\frac{m}{2}}[\frac{m}{2}]_q$.

Applications in population dynamics

Let $(\eta(t))_{t\geq 0}$ be the $ASIP(1-\frac{\sigma}{N}, m)$ process with N particle and weak asymmetry.

Scaling limit: let $X_i(t) = \lim_{N\to\infty} \frac{\eta_i(t)}{N}$, this process is a diffusion with generator

$$\mathcal{L}_{i,i+1}^{ABEP(\sigma,m)} = \frac{1}{4\sigma^2}(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1)\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i+1}}\right)^2 + \frac{1}{2\sigma}\left\{(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1)+\frac{m}{2}\left(2-e^{-2\sigma x_i}-e^{2\sigma x_{i+1}}\right)\right\}\left(\frac{\partial}{\partial x_{i+1}}-\frac{\partial}{\partial x_i}\right)$$

If $\sigma \to 0$ we recover the BEP(m), i.e. the Wright-Fisher model.

Applications in population dynamics

$$\mathcal{L}_{i,i+1}^{ABEP(\sigma,m)} = \frac{1}{4\sigma^2}(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1)\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i+1}}\right)^2$$
$$+\frac{1}{2\sigma}\left\{(1-e^{-2\sigma x_i})(e^{2\sigma x_{i+1}}-1)+\frac{m}{2}\left(2-e^{-2\sigma x_i}-e^{2\sigma x_{i+1}}\right)\right\}\left(\frac{\partial}{\partial x_{i+1}}-\frac{\partial}{\partial x_i}\right)$$

Expanding to first order in m and σ one has

$$\mathcal{L}_{i,i+1}^{ABEP(\sigma,m)} = x_i x_{i+1}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i+1}}\right)^2+\left\{2\sigma x_i x_{i+1}+\frac{m}{2}(x_i-x_{i+1})\right\}\left(\frac{\partial}{\partial x_{i+1}}-\frac{\partial}{\partial x_i}\right)+\dots$$

To first order, we recover the Wright-Fisher model with mutation and selection

Duality between $ABEP(\sigma, m)$ and $SIP(m)$

Theorem [Carinci, G., Redig, Sasamoto (2014 + in progress)]

The ABEP(σ, m) on $[1, L] \cap \mathbb{Z}$ is dual to the SIP(m) on $[1, L] \cap \mathbb{Z}$ with self-duality function

$$D^\sigma(x, \xi) = \prod_{i=1}^{L} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2} + \xi_i)} \cdot (\sinh(\sigma x_i))^{\xi_i} \cdot e^{-\sigma x_i [2 \sum_{k=1}^{i-1} \xi_k + \xi_i]}$$

The proof follows from the self-duality of *ASIP* and the scaling limit:

$$\tilde{D}_q(\eta, \xi) := (1 - q)^{|\xi|} D_q(\eta, \xi) \qquad \lim_{N \to \infty} \tilde{D}_{1-\sigma/N}(Nx, \xi) = D^\sigma(x, \xi)$$

$$\begin{aligned}
\left[(\mathcal{L}^{ABEP(\sigma,m)} D^\sigma)(\cdot, \xi)\right](x) &= \lim_{N \to \infty} \left[(\mathcal{L}^{ASIP(1-\sigma/N,m)} \tilde{D}_{1-\sigma/N})(\cdot, \xi)\right](Nx) \\
&= \lim_{N \to \infty} \left[(\mathcal{L}^{ASIP(1-\sigma/N,m)} \tilde{D}_{1-\sigma/N})(Nx, \cdot)\right](\xi) \\
&= \left[(\mathcal{L}^{SIP(m)} D^\sigma)(x, \cdot)\right](\xi)
\end{aligned}$$

Thank you for your attention.