Knot theory in the thickened torus

Structures on Surfaces - CIRM 2022

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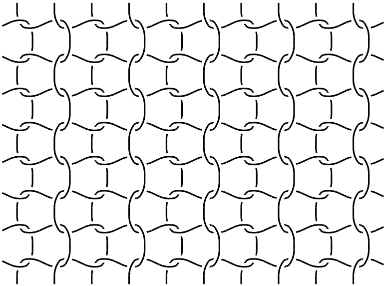
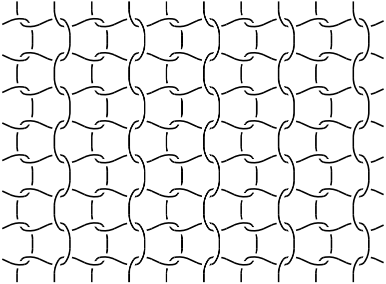
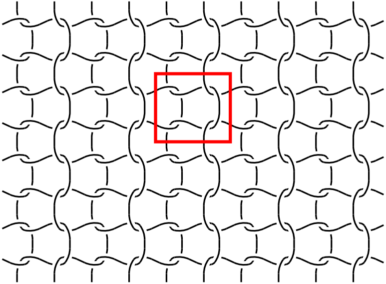
Technische Universität Berlin

May 2, 2022





- Goal: distinguish periodic knottings (pictured here: garter stitch)



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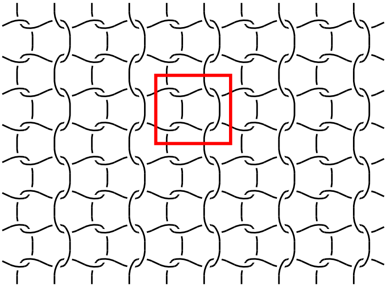
- project to the quotient space!



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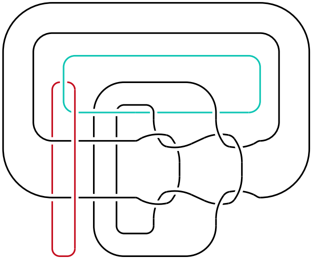

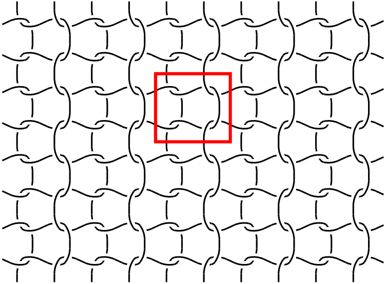

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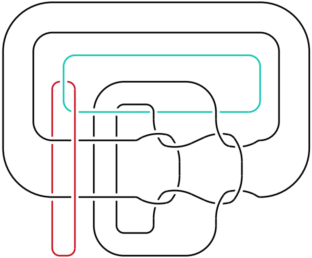

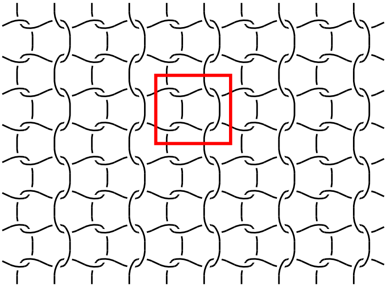

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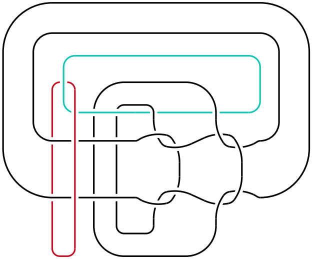
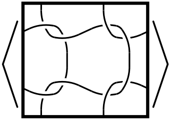


- get a link in $T^2 \times I$
- add extra components in the holes
- result: colored link in $\mathbb{R}^3$ (or S^3)

The Kauffman polynomial



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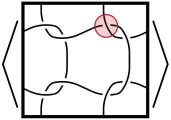

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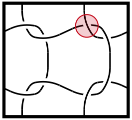
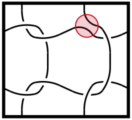
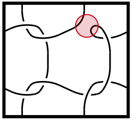
$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

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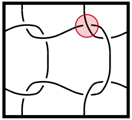
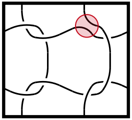
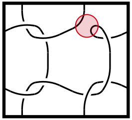
$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

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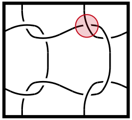
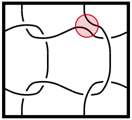
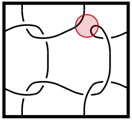
$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

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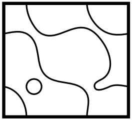

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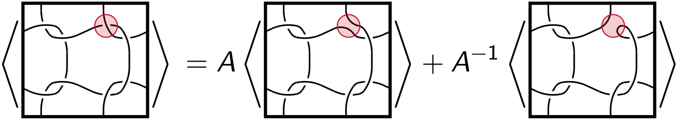

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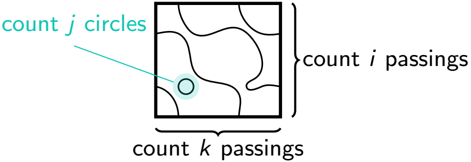

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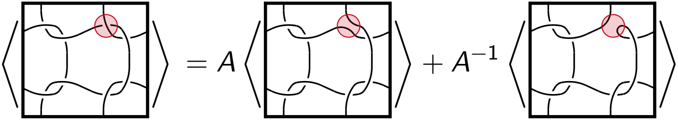

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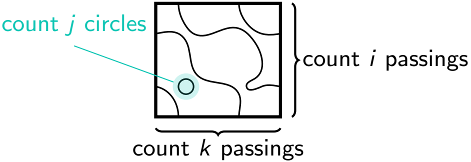
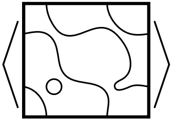
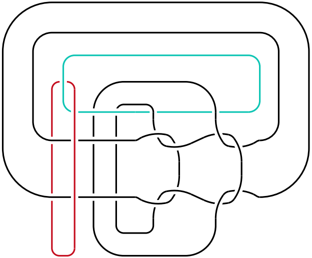




$$= x^k y^i \left(-A^2 - A^{-2}\right)^j \tag{1}$$

The Alexander polynomial



- consider augmented link in S^3

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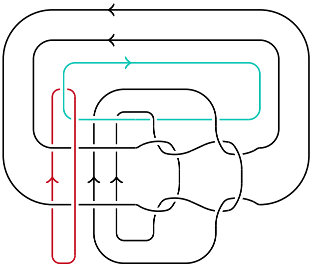

- consider augmented link in S^3
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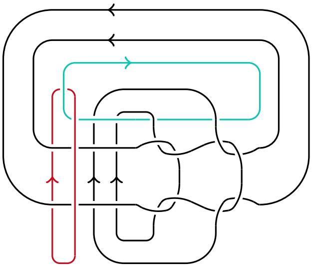

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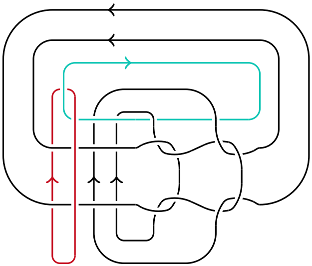
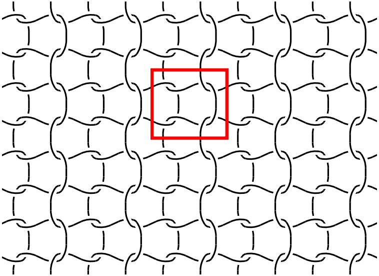


- consider augmented link in S^3
- choose orientation
- Each crossing gives rise to a row in a matrix $A(x, y, t)$
- Determinant of the matrix gives a polynomial

$$(x-1)\Delta_L(x, y, t) = \det A(x, y, t)$$

Shearing the grid



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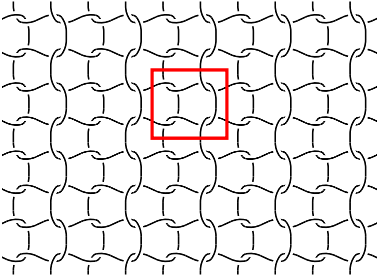

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Theorem

Let L be a periodic knotting, and $C(L)$ its image under the linear map C. Then

$$\langle C(L) \rangle (A, x, y) = \langle L \rangle (A, x^a y^c, x^b y^d)$$

and for some $k, l \in \mathbb{Z}$,

$$\Delta_{C(L)}(x, y, t) = \Delta_L(x^a y^c t^k, x^b y^d t^l, t).$$