Studying periodic entanglements via links in the 3-torus

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Definition

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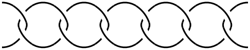
- Any periodic entanglement projects to a link $L \subset \mathbb{R}^3/\mathcal{L}$.
- K is called i-periodic if $\mathcal{L}$ has rank i.

1-periodic **2-periodic** **3-periodic**



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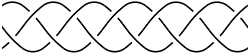

braid

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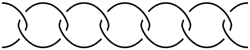

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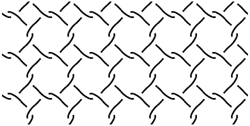

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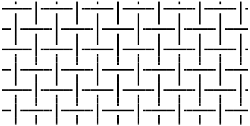
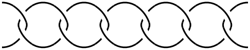


weave

1-periodic



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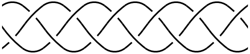

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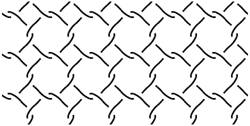
2-periodic



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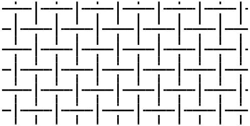
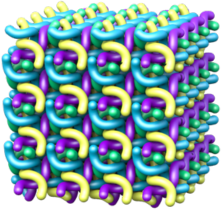


weave

3-periodic



space weave

What is our notion of equivalence for periodic entanglements?

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Two periodic entanglements (with same lattice $\mathcal{L}$) are *a-equivalent* if there is a periodic isotopy $a_t : \mathbb{R}^3 \to \mathbb{R}^3$ taking one to the other, i.e. $a_t \circ \tau = \tau \circ a_t$ for any $\tau \in \mathcal{L}$.

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They are *h-equivalent* if there is an ambient homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ taking one to the other, such that $h \circ \tau = \tau \circ h$ for any $\tau \in \mathcal{L}$.

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They are *commensurable* if there is an ambient homeomorphism $g : \mathbb{R}^3 \to \mathbb{R}^3$ taking one to the other, such that for any $\tau \in \mathcal{L}$, there are $k, l \in \mathbb{N}$ such that $g \circ \tau^k = \tau^l \circ g$.

Existing results for classification of periodic entanglements:

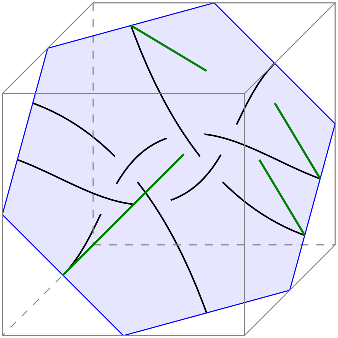
- Knots in the solid torus (Gabrovšek, Mroczkowski, 2012): 1-periodic entanglements up to a-equivalence
- Virtual knots of genus 1 (Akimova, Matveev, 2014): 2-periodic entanglements up to h-equivalence

Definition

The *Alexander polynomial* of a 2-periodic entanglement is the multivariate Alexander polynomial of the link obtained by embedding $\mathbb{R}^3/\mathcal{L}$ into S^3 and adding components along the boundary.

- The Alexander polynomial is well-behaved under h-equivalence and for commensurable links.

Projection of the link in a cube



- We obtain a diagram within a hexagon.
- Important: keep track of front/back faces!

What moves are allowed?

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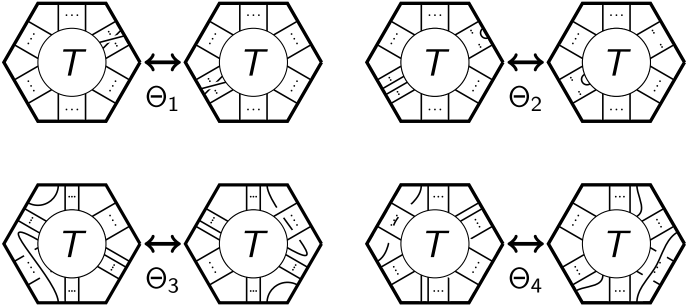
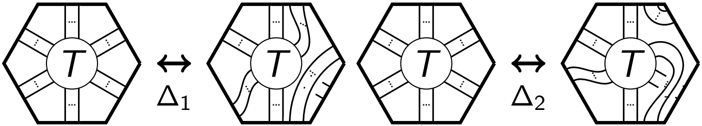
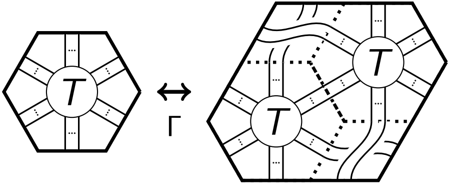
- Reidemeister moves
- New moves from pulling across the cube boundary:



- Shearing (h-equivalence) moves:



- Cyclic cover (commensurability) moves:



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Theorem

T^3 is homeomorphic to the manifold obtained by performing 0-surgery along the Borromean links $B \subset S^3$.

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Theorem

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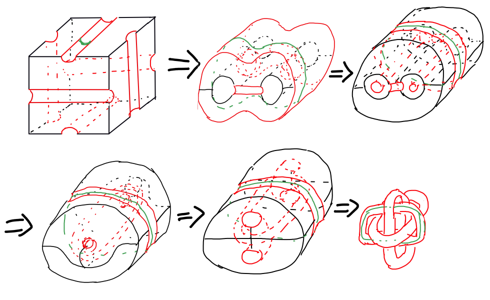
Proof.



Our recipe for a hyperbolic structure:

- Remove link $\hat{B}$ from T^3
- Obtain homeomorphism $T^3 \setminus (\hat{B} \cup L) \cong S^3 \setminus (B \cup \tilde{L})$.
- Find incomplete structure on $S^3 \setminus (B \cup \tilde{L})$.
- Ensure that the completion corresponds to the 0-Dehn filling of B.

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Properties of hyperbolic volume:

- $\mathrm{vol}(L)$ is an h-equivalence invariant.
- If L_1, L_2 are commensurable, then we have

$$\frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)} = \frac{p}{q}$$

for some $p, q \in \mathbb{N}$.

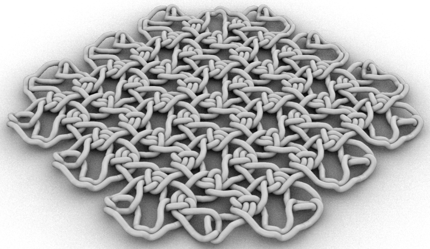
Further research

- For a given n, find a bound C_n such that for any two commensurable links L_1, L_2 with at most n crossings and

$$\frac{p}{q} := \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)},$$

we have that $p, q \leq C_n$.

- Find invariants that work for non-hyperbolic links.



Thank you for your attention!