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SFB TRR 109 Discretization in Geometry and Dynamics

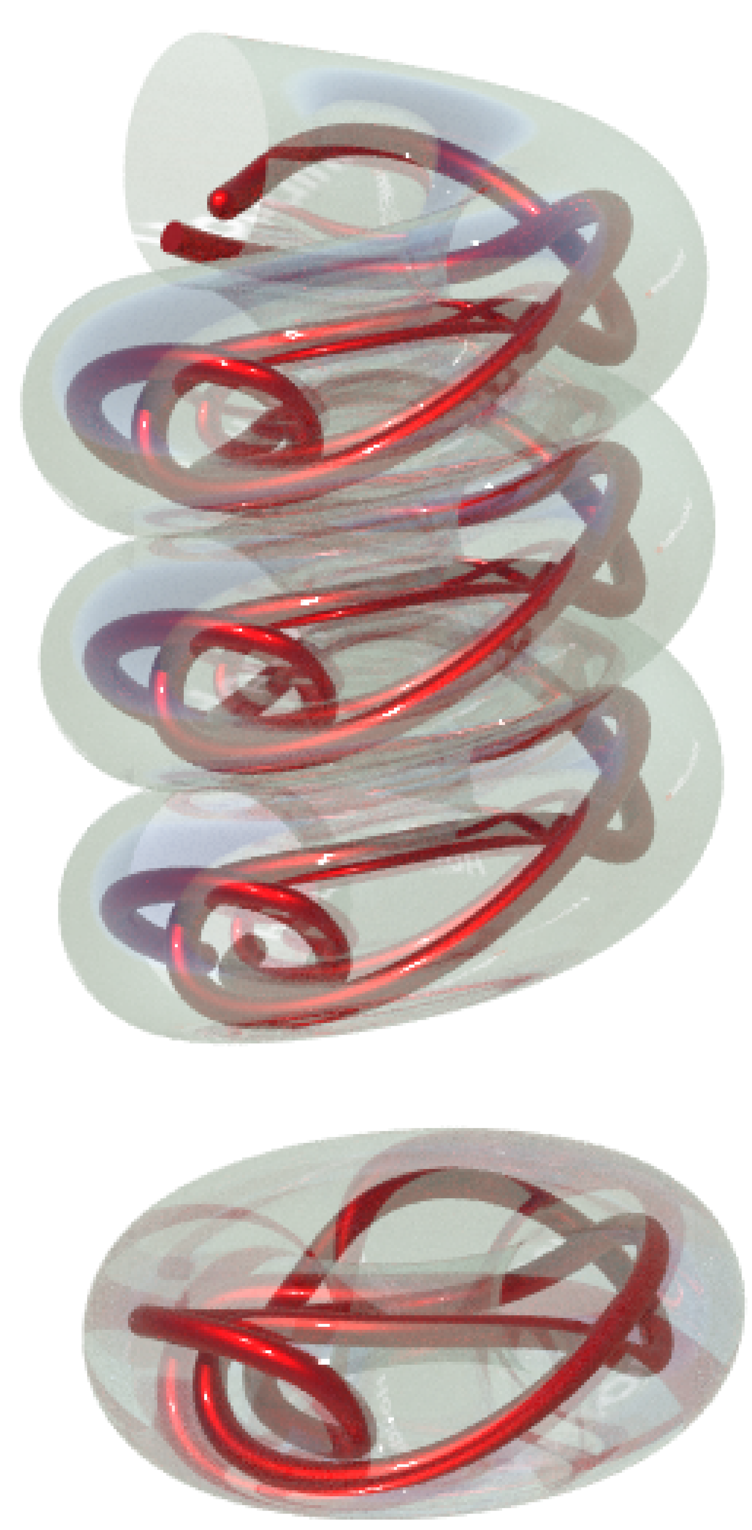
Classification of Singly Periodic Links

Periodic links

A **periodic link** is a link L in the three-torus T^3, that is, a finite collection of embeddings $K_i : S^1 \to T^3$ whose images are pairwise disjoint. A periodic link admits a periodic structure in $\mathbb{R}^3$,via its preimage under the universal covering map $p : \mathbb{R}^3 \to T^3$ - hence the name.
A special subclass are periodic links that only have one periodic direction in $\mathbb{R}^3$ - or, equivalently, links that are contained in an embedded solid torus $T_f \subseteq T^3$. We call those **singly periodic**. Our goal is to produce a table of such links.

Open question

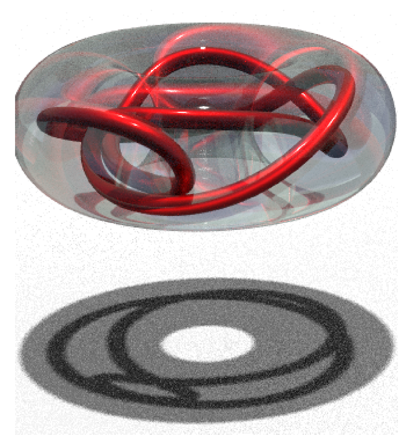
How can good expressive diagrams for links in T^3 be obtained?



A knot in the solid torus and its periodic cover

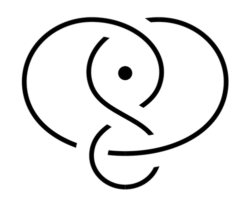
Diagrams and basic invariants

There are two ways to create a knot diagram of a singly periodic link: Projecting to solid torus to its shadow gives an **annulus diagram**, projecting a fundamental domain of the universal cover $\mathbb{R}^3$ to a plane gives a **periodic diagram**.
Some basic invariants can be read from such diagrams analogously to classical knot diagrams, e.g. crossing numbers. But there are also numerical invariants inherent to the ambient space:



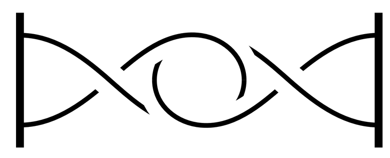
The annulus diagram is obtained from the projection of the link

Definition *For a link L and a periodic diagram D of it, the **wrapping number** of D number of intersections with the boundary. The wrapping number of L is the minimum across all diagrams.*
*The **winding number** of L is the multiplicity of the homology class of the link in $H_1(T_f)$.*



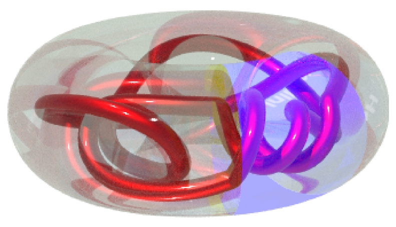
Corresponding annulus diagram, with hole marked

There is also a version of the Kauffman polynomial that works in the setting of annular diagrams **[1]**.



Corresonding periodic diagram

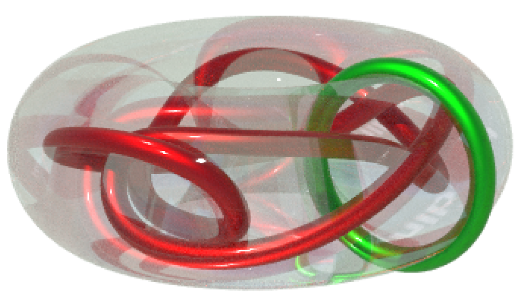
Notions of equivalence



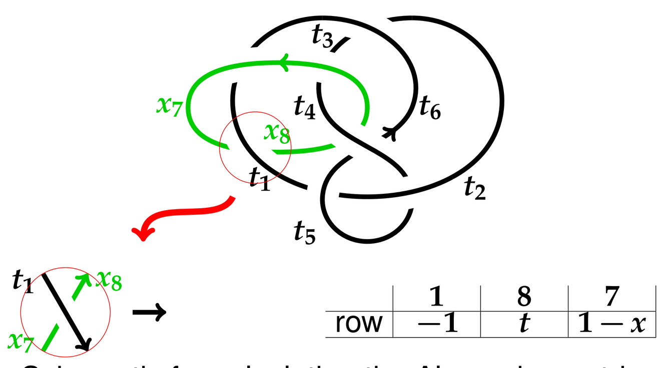
The Dehn twist of a knot obtained by twisting the colored region

Unlike in the classical case, links related by a self-homeomorphism of T_f are not necessarily ambiently isotopic. For a general 3-manifold M, the homeomorphic equivalence class of a link is given by the mapping class group of M. For T_f, these are just the Dehn twists.

Multivariable Alexander polynomial



Augmentation component is added (green)



Schematic for calculating the Alexander matrix

Every singly periodic link has an associated **augmentation** in $\mathbb{R}^3$. This colored link is an invariant of the periodic link. The multivariable Alexander polynomial $\Delta_L(t,x)$ is a colored link invariant calculated from the link diagram by extracting an $(n \times n)$-matrix using the process indicated in the diagram, where n is the crossing number.
More formally, $\Delta_L(t,x)$ is an invariant of the homology of the universal Abelian cover of the link complement X, considered as a $\mathbb{Z}H_1(X)$-module **[2]**.

Theorem *If L_1, L_2 are singly periodic links that are homeomorphically equivalent, then $\Delta_{L_1}(t,x) = \Delta_{L_2}(t,xt^k)$ for some $k \in \mathbb{Z}$.*

To prove this theorem, we first note that given a suitable set of generators of $H_1(X)$, a Dehn twist keeps all but one of them the same. We can trace the way it changes through the related module structure on the homology of the universal abelian cover via a commuting diagram of their free resolutions.

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{Z}H_1(X))^n \arrow[r, "A"] \arrow[d, "g"'] & (\mathbb{Z}H_1(X))^m \arrow[r, "f"] \arrow[d, "g"'] & H_{\star_2} \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & (\mathbb{Z}H_1(X))^n \arrow[r, "B"] & (\mathbb{Z}H_1(X))^m \arrow[r, "\tilde{f}"] & H_{\star_1} \arrow[r] & 0 \end{tikzcd}$$

The matrices A, B appearing give rise to $\Delta_L(t,x)$, and we can see that $A(t,x) = B(t,xt^k)$, where k is the product of the link's winding number and the multiplicity of the Dehn twist applied.

Finite coverings

The preimage of L in some k-fold cover of the solid torus gives a new link. Both induce the same structure in $\mathbb{R}^3$. We can detect such pairs by the hyperbolic volume of their complement, which will be a k-multiple of the original volume.

Open questions

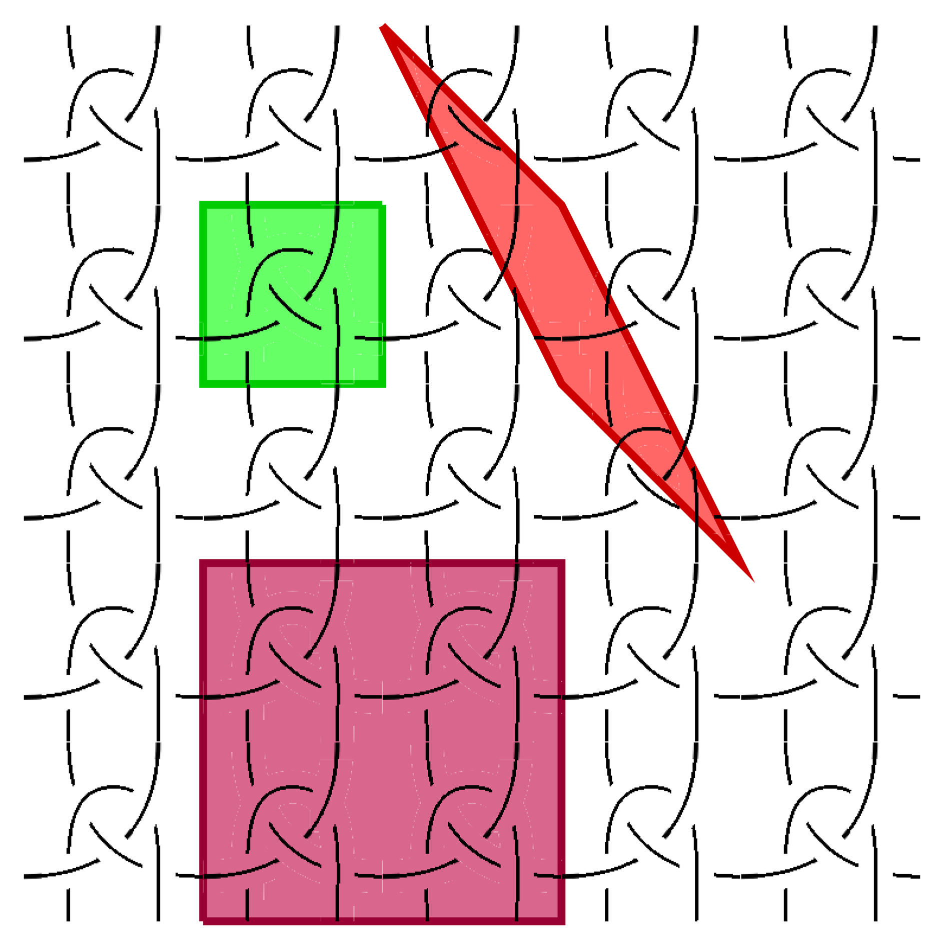
Is there an invariant that can guarantee that two links will not become the same after taking a large-fibered finite coverings of both?
Can we see how the number of components changes (and whether it stabilizes) under taking finite covers repeatedly?

Classification algorithm

All ingredients come together in an algorithm to automatically generate a link table for singly periodic links. A version of this was already done **[3]**, but without regard for homeomorphic equivalence or covering equivalence. The output is a table of links and their invariants, with automatic diagram drawing using circle packings.

Doubly periodic analogues

It is reasonable to expect that our results generalize to a doubly periodic setting. These links live in $T^2 \times [0,1]$. We can think of their associated periodic structures as **weaving patterns**. Augmentation happens by adding a colored Hopf link in the holes of the torus.
In this case, we can understand our 2 operations on the level of the periodic structure:
Homeomorpic equivalence corresponds to choosing a different fundamental domain of the same volume and passing to the quotient, and finite coverings correspond to passing to a sublattice.



Periodic pattern with fundamental domains corresponding to the base link (green), a homeomorphic one (red) and a finite cover (purple).

Open questions

Does the theorem on the multivariable Alexander polynomial generalize to ambient homeomorphisms of doubly periodic links?
Is there a 3-variable analogue of the 2-variable Kauffman polynomial?

References & acknowledgments

[1] J. Hoste, J. Przytycki, *An invariant of dichromatic links*, Proc. Amer. Math. Soc. 105 (4), 1989

[2] A. Kawauchi, *A survey of knot theory*, Birkhäuser, 1996

[3] B. Gabrovšek, M. Mroczkowski, *Knots in the solid torus up to 6 crossings*, J. Knot Theory Ramifications 21 (11), 2012

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