STRUCTURED PSEUDOSPECTRA, μ-VALUES AND EIGENVALUE CONDITION NUMBERS (SUMMARY OF CUMULATIVE HABILITATION)

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1. Introduction. The contributions submitted for cumulative habilitation are

[A1] Eugene Gutkin, Edmond Jonckheere, Michael Karow.
Convexity of the joint numerical range:
Topological and differential geometric viewpoints.
Linear Algebra Appl., 376: 143-171, 2004.

[A2] Michael Karow, Diederich Hinrichsen, Anthony J. Pritchard.
Interconnected systems with uncertain couplings:
explicit formulae for μ-values, spectral value sets and stability radii.
SIAM J. Control Opt. 45(3):856-884, 2006.

[A3] Michael Karow.
μ-values and spectral value sets for linear perturbation classes
defined by a scalar product.
MATHEON-Preprint 406. (2007)

[A4] Michael Karow.
Structured Pseudospectra for small perturbations.
MATHEON-Preprint 530. (2008)

[A5] Michael Karow.
Pseudospectra and the condition of a nonderogatory eigenvalue.
MATHEON-Preprint 407. (2007)

[A6] Michael Karow.
Eigenvalue Conditon Numbers and a Formula of Burke, Lewis and Overton.
Electronic Journal of Linear Algebra. ELA Vol. 15, 2006.

[A7] Michael Karow, Daniel Kressner, Francoise Tisseur.
Structured Eigenvalue Condition Numbers.
SIAM J. Matrix Anal. Appl. 28(4):1052-1068, 2006.

The content of these papers is summarized on the following pages. Let me first explain the concepts which are mentioned in the title of this habilitation.

Following Hinrichsen and Pritchard [18, 19] we study the variation of the spectrum, $\sigma(A)$, of a matrix $A \in \mathbb{C}^{n\times n}$ under perturbations of the form

$$A \rightsquigarrow A_\Delta := A + B\Delta C, \tag{1.1}$$

where $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{q\times n}$ are fixed matrices and Δ is an element of a subset $\mathbf{\Delta}$ of $\mathbb{C}^{l\times q}$. It is assumed that $\mathbf{\Delta}$ is closed, connected and contains the zero matrix. The size of a perturbation $\Delta \in \mathbf{\Delta}$ is measured by a norm $\|\cdot\|$ on $\mathbb{C}^{l\times q}$.

(a) The structued pseudospectrum (also called spectral value set) of the triple (A, B, C) with respect to the perturbation class $\mathbf{\Delta}$, the underlying norm $\|\cdot\|$ and the perturbation level $\delta > 0$ is defined as

$$\sigma_{\mathbf{\Delta}}(A, B, C; \delta) := \{s \in \mathbb{C};\ \ s \in \sigma(A_\Delta) \text{ for some } \Delta \in \mathbf{\Delta} \text{ with } \|\Delta\| < \delta\}.$$

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Hence, $\sigma_{\mathbf{\Delta}}(A,B,C;\delta)$ is the set of eigenvalues of all matrices A_Δ with $\Delta\in\mathbf{\Delta}$ and $\|\Delta\|<\delta$.

(b) Let $\lambda\in\mathbb{C}$ and let the eigenvalues $\nu_1,\dots,\nu_n$ of $\widetilde{A}\in\mathbb{C}^{n\times n}$ be ordered such that $|\nu_1-\lambda|\le|\nu_2-\lambda|\le\ldots\le|\nu_n-\lambda|$. Then we set $d_m(\widetilde{A},\lambda):=|\nu_m-\lambda|$. Now suppose that λ is an eigenvalue of $A\in\mathbb{C}^{n\times n}$ of algebraic multiplicity m. Then we define the structured Hölder condition number of λ of order $\gamma>0$ by

$$\operatorname{cond}^{\gamma}_{\mathbf{\Delta}}(A,B,C,\lambda):=\lim_{\delta\searrow 0}\sup_{\substack{\|\Delta\|\le\delta\\ \Delta\in\mathbf{\Delta}}}\frac{d_m(A_\Delta,\lambda)}{\|\Delta\|^{\gamma}}. \tag{1.2}$$

(c) Let $\mathbb{C}_g$ be an open and non-empty subset of $\mathbb{C}$. Then $\widetilde{A}\in\mathbb{C}^{n\times n}$ is called $\mathbb{C}_g$-stable if $\sigma(\widetilde{A})\in\mathbb{C}$. The structured stability radius of a $\mathbb{C}_g$-stable matrix $A\in\mathbb{C}^{n\times n}$ is defined as

$$r_{\mathbf{\Delta}}(A,B,C,\mathbb{C}_g):=\inf\{\|\Delta\|;\ \ \Delta\in\mathbf{\Delta},\ A_\Delta \text{ is not } \mathbb{C}_g\text{-stable}\}.$$

Though stability radii are not mentioned in the title of this habilitation the paper [A2] contains some results on these quantities.

(d) The μ-value of $M\in\mathbb{C}^{q\times l}$ is defined as

$$\mu_{\mathbf{\Delta}}(M):=[\,\inf\{\|\Delta\|;\ \ \Delta\in\mathbf{\Delta},\ 1\in\sigma(\Delta\,M)\}\,]^{-1}. \tag{1.3}$$

We also define a μ-value of second kind by

$$\widetilde{\mu}_{\mathbf{\Delta}}(M):=\inf\{\|\Delta\|;\ \ \Delta\in\mathbf{\Delta},\ \operatorname{rank}(M+\Delta)<\min\{l,q\}\ \}.$$

If $l=q$ then $\widetilde{\mu}_{\mathbf{\Delta}}(M)$ is the structured distance of M to the set of singular matrices, and we have $\widetilde{\mu}_{\mathbf{\Delta}}(M)=\mu_{\mathbf{\Delta}}(M^{-1})^{-1}$.

Stability radii have been introduced by Hinrichen and Pritchard [18], structured pseudospectra as defined above have been introduced by Hinrichsen and Kelb (under the name 'spectral value sets') [17]. In that papers the usage of the structure matrices B and C was motivated by applications in linear system theory: consider a linear system

$$\dot{x}(t)=Ax(t)+Bu(t),\quad y(t)=Cx(t). \tag{1.4}$$

Then by introducing the feedback law $u(t)=\Delta y(t)$ one obtains a closed loop system $\dot{x}(t)=A_\Delta\,x(t)$, with A_Δ given by (1.1). This point of view has been extended in [A2], see Section 3. The matrices B,C can be used to model a lot of structured perturbations. For instance, perturbations in the class of companion matrices of matrix polynomials can be written as [1]

$$\begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ A_0+\Delta_0 & A_1+\Delta_1 & \ldots & A_{n-1}+\Delta_{n-1}\end{bmatrix}=\underbrace{\begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ A_0 & A_1 & \ldots & A_{n-1}\end{bmatrix}}_{A}+\underbrace{\begin{bmatrix}0\\ \vdots\\ 0\\ I\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}\Delta_0 & \ldots & \Delta_{n-1}\end{bmatrix}}_{\Delta}$$

Furthermore, perturbations of the form

$$A\rightsquigarrow A(\Delta_1,\dots,\Delta_m,\delta_1,\dots,\delta_r):=A+\sum_{j=1}^{m}B_j\Delta_jC_j+\sum_{j=1}^{r}\delta_jE_j,\qquad \delta_j\in\mathbb{C},$$

[1] Here and in the sequel I denotes the identity matrix of appropriate dimension.

can be written as $A(\Delta_1, \ldots, \Delta_m, \delta_1, \ldots, \delta_r) = A + B\Delta C$ with

$$\begin{aligned} \Delta &= \operatorname{diag}(\Delta_1, \ldots, \Delta_m, \delta_1 I, \ldots, \delta_r I), \\ B &= [B_1, \ldots, B_m, E_1, \ldots, E_r], \\ C &= [C_1^T, \ldots, C_m^T, \underbrace{I, \ldots, I}_{r \text{ times}}]^\top. \end{aligned} \tag{1.5}$$

The μ-values (of first kind) for perturbations of the form (1.5) have been introduced by Doyle, see e.g. [32, 44] and the references therein. The connection of pseudospectra and stability radii with μ-values is as follows. Let $G(s) = B(sI - A)^{-1}C$ be the transfer function of the linear system (1.4). Then

$$\sigma_{\mathbf{\Delta}}(A, B, C, \delta) = \sigma(A) \cup \{s \in \mathbb{C} \setminus \sigma(A);\ \ \mu_{\mathbf{\Delta}}(G(s)) > \delta\}, \tag{1.6}$$

$$r_{\mathbf{\Delta}}(A, B, C, \mathbb{C}_g) = [\sup_{s \in \partial\mathbb{C}_g} \mu_{\mathbf{\Delta}}(G(s))]^{-1} \qquad \text{if } A \text{ is } \mathbb{C}_g\text{-stable.} \tag{1.7}$$

This follows from the definition of μ and the equivalence

$$s \in \sigma(A_\Delta) \quad \Leftrightarrow \quad 1 \in \sigma(\Delta\, G(s)),$$

which holds for all $s \in \mathbb{C} \setminus \sigma(A)$. In the case that $B = C = I$ we have

$$\sigma_{\mathbf{\Delta}}(A, I, I, \delta) = \{s \in \mathbb{C};\ \ \widetilde{\mu}_{\mathbf{\Delta}}(sI - A) < \delta\}, \tag{1.8}$$

$$r_{\mathbf{\Delta}}(A, I, I, \mathbb{C}_g) = \inf_{s \in \partial\mathbb{C}_g} \widetilde{\mu}_{\mathbf{\Delta}}(sI - A) \qquad \text{if } A \text{ is } \mathbb{C}_g\text{-stable.} \tag{1.9}$$

The relationship of spectral value sets and μ-values with eigenvalue condition numbers is explained in the paper [A4]. In order to make the formulas (1.6)-(1.9) useful one needs computable formulae for μ-values. Here are some basic results.

(i) Let $\|\cdot\|_{\alpha,\beta}$ denote the operator norm on $\mathbb{C}^{l \times q}$ subordinate to the vector norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$, i.e.

$$\|\Delta\|_{\alpha,\beta} = \max_{x \in \mathbb{C}^q \setminus \{0\}} \frac{\|\Delta x\|_\beta}{\|x\|_\alpha}. \tag{1.10}$$

Then the μ-value of $M \in \mathbb{C}^{q \times l}$ with respect to unstructured perturbations and the underlying norm $\|\cdot\| = \|\cdot\|_{\alpha,\beta}$ satisfies

$$\mu_{\mathbb{C}^{l \times q}}(M) = \|M\|_{\beta,\alpha}.$$

(ii) Let $\sigma_1(X) \geq \sigma_2(X) \geq \ldots$ denote the singular values of the matrix X in decreasing order. Furthermore, let $\Re X$ and $\Im X$ denote the real and imaginary part of X. If the underlying norm is the spectral norm then for any $M \in \mathbb{C}^{q \times l}$, [1, 34]

$$\mu_{\mathbb{C}^{l \times q}}(M) = \sigma_1(M), \qquad \mu_{\mathbb{R}^{l \times q}}(M) = \inf_{\gamma \in (0,1]} \sigma_2\left(\begin{bmatrix} \Re M & -\gamma\, \Im M \\ \gamma^{-1}\, \Im M & \Re M \end{bmatrix}\right).$$

If additionally $l = q =: n$ then

$$\widetilde{\mu}_{\mathbb{C}^{n \times n}}(M) = \sigma_n(M), \qquad \widetilde{\mu}_{\mathbb{R}^{n \times n}}(M) = \sup_{\gamma \in (0,1]} \sigma_{2n-1}\left(\begin{bmatrix} \Re M & -\gamma\, \Im M \\ \gamma^{-1}\, \Im M & \Re M \end{bmatrix}\right). \tag{1.11}$$

The problem of computing μ-values with respect to the classes (1.5) has been addressed by Doyle and coworkers [13, 32, 44]. In [A2] and [A3] we derive computable formulae for μ with respect to other important perturbation classes.

2. Content of [A1]. The paper [A1] deals with the convexity of the joint numerical range $\mathcal{F}(A)$ of an m-tuple of Hermitian matrices $A = (A_1, \ldots, A_m)$, $A_k \in \mathbb{C}^{n\times n}$, $A_k^* = A_k$. By definition,

$$\mathcal{F}(A) = \{\, (x^* A_1 x, \ldots, x^* A_m x)^\top;\ \ x \in \mathbb{C}^n,\ \|x\|_2 = 1\} \subset \mathbb{R}^m.$$

Before discussing the content of [A1] in detail let me briefly explain the relationship between numerical ranges and μ-values with respect to block diagonal perturbations and the spectral norm, $\|\cdot\|_2$. Let

$$\mathbf{\Delta} = \{\, \mathrm{diag}(\Delta_1, \ldots, \Delta_m);\ \ \Delta_j \in \mathbb{C}^{l_j \times q_j}\},$$

where l_j, q_j are fixed numbers with $\sum l_j = l$ and $\sum q_j = q$. Furthermore, let $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}_+^m$, $M = [M_{jk}]_{j,k\le m} \in \mathbb{C}^{q\times l}$ with $M_{jk} \in \mathbb{C}^{l_j \times q_k}$, $M_\gamma = [(\gamma_j/\gamma_k)\, M_{jk}]_{j,k\le m}$ and $\mathcal{D}_\gamma = \mathrm{diag}(\gamma_1 I, \ldots, \gamma_m I)$. Then for all $\gamma \in \mathbb{R}_+^m$ and all $\Delta \in \mathbf{\Delta}$, $\Delta M_\gamma = \mathcal{D}(\Delta M)\mathcal{D}^{-1}$. Hence the matrices ΔM and ΔM_γ have the same eigenvalues. This yields $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(M_\gamma) \le \|M_\gamma\|_2$. Hence, the quantity

$$\bar{\mu}_{\mathbf{\Delta}}(M) = \inf_{\gamma\in\mathbb{R}_+^m} \|M_\gamma\|_2$$

is an upper bound for $\mu_{\mathbf{\Delta}}(M)$. The computation of $\bar{\mu}_{\mathbf{\Delta}}(M)$ is a convex optimization problem which can be solved by standard methods. In general, $\bar{\mu}_{\mathbf{\Delta}}(M)$ can be a very conservative bound for $\mu_{\mathbf{\Delta}}(M)$ [42]. However, it turns out that $\bar{\mu}_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(M)$ for $m \le 3$. The proof of this fact uses the characterization of both quantities by numerical ranges [13, 14, 15]. For $c > 0$ let $A^c = (A_1^c, \ldots, A_m^c)$, where $A_j^c \in \mathbb{C}^{l\times l}$ are Hermitan matrices such that

$$x^* A_j^c x = \|\sum_{k=1}^m M_{jk} x_k\|_2^2 - c^2\, \|x_j\|_2^2,$$

for all $x = [x_1^\top, \ldots, x_m^\top]^\top$, $x_j \in \mathbb{C}^{l_j}$. Then one can show that

$$\bar{\mu}_{\mathbf{\Delta}}(M) = \inf\{c > 0;\ \ \text{The numerical range } \mathcal{F}(A^c) \text{ does not meet the nonnegative orthant } [0,\infty)^m\} \tag{2.1}$$

and

$$\mu_{\mathbf{\Delta}}(M) = \inf\{c > 0;\ \ \text{There exists a nonnegative vector } p \in \mathbb{R}^m \text{ with } p^\top y < 0 \text{ for all } y \in \mathcal{F}(A^c)\,\} \tag{2.2}$$

If for all $c > 0$, $\mathcal{F}(A^c)$ is convex or the boundary of a convex set then the conditions on the right hand side of (2.1) and (2.2) are equivalent. However, for any Hermitian $A_1, A_2, A_3 \in \mathbb{C}^{n\times n}$ we have that 1. $\mathcal{F}(A_1)$ is an interval, 2. $\mathcal{F}(A_1, A_2)$ is convex (Hausdorff-Töplitz Theorem), 3. $\mathcal{F}(A_1, A_2, A_3)$ is a (possibly degenerated) ellipsoid for $n = 2$ and $\mathcal{F}(A_1, A_2, A_3)$ is convex for $n > 2$, see e.g. [15]. This yields $\bar{\mu}_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(M)$ for $m \le 3$. On the other hand if $m \ge 4$ then it is easy to find Hermitian matrices $A_j \in \mathbb{C}^{n\times n}$, $j = 1, \ldots, m$, such that $\mathcal{F}(A_1, \ldots, A_m)$ in non-convex. This explains why we can have $\mu_\Delta(M) < \bar{\mu}_\Delta(M)$ if $\mathbf{\Delta}$ is a set of matrices with more than 3 blocks.

Motivated by the facts displayed above we invesigate in [A1] the convexity of joint numerical ranges for $m \ge 4$ and provide a new proof of convexity for $m \le 3$. For $A = (A_1, \ldots, A_m)$ and $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$ let

$$A_\eta = \sum_{k=1}^m \eta_k A_k.$$

Furthermore, let $\bar{\lambda}_\eta$ denote the maximum eigenvalue of A_η and let $E_\eta = \{v \in \mathbb{C}^n;\ \ A_\eta v = \bar{\lambda}_\eta\, v\}$ be the associated eigenspace. The main results of the paper [A1] are the statements below.

(i) The map $\mathbb{R}^m \ni \eta \mapsto \bar{\lambda}_\eta$ is the support function of the convex hull of $\mathcal{F}(A)$. The intersection of $\mathcal{F}(A)$ with the supporting hyperplane $H_\eta = \{y \in \mathbb{R}^m;\ \ y^\top \eta = \bar{\lambda}_\eta\}$, $\|\eta\|_2 = 1$, is the numerical range $\mathcal{F}(V_\eta^* A_1 V_\eta, \dots, V_\eta^* A_m V_\eta)$, where V_η is a matrix whose columns form an orthonormal basis of E_η.

(ii) Suppose that $\mathcal{F}(A)$ is not a singleton and that the eigenvalue $\bar{\lambda}_\eta$ has constant multiplicity for $\eta \in \mathbb{R}^m \setminus \{0\}$ (i.e. the dimension of E_η is independent of η). Then the following holds.
   (a) The spheres $S_\eta = \{v \in E_\eta;\ \|v\|_2 = 1\}$, $\|\eta\|_2 = 1$, are pairwise disjoint. The union of these spheres, $\mathcal{M} := \bigcup_{\|\eta\|_2=1} S_\eta$, is a real analytic submanifold of $\mathbb{C}^n$.
   (b) The boundary $\partial\mathcal{F}(A)$ of $\mathcal{F}(A)$ is both a real analytic submanifold of $\mathbb{R}^m$ and a convex surface. Let $f_A(x) = (x^* A_1 x, \dots, x^* A_m x)^\top$, $\|x\|_2 = 1$, denote the numerical range map. Then $\partial\mathcal{F}(A) = f_A(\mathcal{M})$, and to any $y \in \partial\mathcal{F}(A)$ there exists a unique η such that $S_\eta = f_A^{-1}(y)$. Hence $\mathcal{M} \overset{f_A}{\mapsto} \partial\mathcal{F}(A)$ is a sphere bundle over $\partial\mathcal{F}(A)$.
   (c) If $\mathbb{R}\mathcal{M} = \mathbb{C}^n$ then $\mathcal{F}(A) = \partial\mathcal{F}(A)$. If $\mathbb{R}\mathcal{M} \neq \mathbb{C}^n$ then $\mathcal{F}(A)$ is convex. (The proof of the latter statement uses homotopy theoretic methods.)

(iii) Suppose $m \geq 4$ and there exists an $\eta \neq 0$ such that $\bar{\lambda}_\eta$ is not a simple eigenvalue of A_η. Then in each neighborhood of A there exists an m-tuple of Hermitian matrices $\widetilde{A}$ such that $\mathcal{F}(\widetilde{A})$ is not convex.

(iv) If $m = 2, 3$ then generically $\bar{\lambda}_\eta$ is a simple eigenvalue of A_η for all $\eta \neq 0$. This combined with (ii) yields that $\mathcal{F}(A)$ is convex for all $A = (A_1, A_2)$ and $A = (A_1, A_2, A_3)$.

3. Content of [A2]. Ths subject of [A2] are pseudospectra, stability radii and μ-values for coupled linear systems. The setting is as follows. Suppose we are given m linear time invariant systems of the form

$$\Sigma_j: \qquad \dot{x}_j(t) = A_j x_j(t) + B_j u(t), \qquad y_j(t) = C_j x_j(t),$$

where $A_j \in \mathbb{C}^{n_j \times n_j}, B_j \in \mathbb{C}^{l_j \times n_j}, C_j \in \mathbb{C}^{n_j \times q_j},\ \ j = 1, \dots, m$. By introducing the couplings $u_j(t) = \sum_{k=1}^m \Delta_{jk} y_k(t),\ \Delta_{jk} \in \mathbb{C}^{q_j \times l_j}$, one obtains the closed loop system

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} = \left(\underbrace{\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{bmatrix}}_{=:A} + \underbrace{\begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_m \end{bmatrix}}_{=:B} \underbrace{\begin{bmatrix} \Delta_{11} & \dots & \Delta_{1m} \\ \vdots & & \vdots \\ \Delta_{m1} & \dots & \Delta_{mm} \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_m \end{bmatrix}}_{=:C} \right) \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}. \tag{3.1}$$

For some index pairs (j, k) the coupling Δ_{jk} may not be present. Thus in (3.1) we consider only block matrices $\Delta = [\Delta_{jk}]$ which are elements of the class

$$\mathbf{\Delta}_\mathcal{I} = \{[\Delta_{jk}]_{j,k \leq m};\ \ \Delta_{jk} = 0 \text{ if } (j,k) \notin \mathcal{I}\},$$

where $\mathcal{I} \subset \{1, \dots, m\} \times \{1, \dots, m\}$ is a prescribed index set. The pairs $(j, k) \in \mathcal{I}$ can be regarded as the oriented edges of a directed graph whose vertices are the numbers $1, \dots, m$. This is illustrated in Figure 3.1 for the case where $m = 3, \mathcal{I} = \{\,(1,2),\,(1,3),\,(2,1),\,(3,2),(3,3)\}$. Observe that in the directed graph the endpoint of the edge (j, k) is the first component, j. This orientation reflects the interconnection structure. In order to calculate pseudospectra and stability radii for systems of the form (3.1) by means of the formulas (1.6) and (1.7) we need to determine the quantity $\mu_{\mathbf{\Delta}_\mathcal{I}}(G(s))$. However, since A, B and C are block diagonal, $G(s)$ is also block diagonal:

$$G(s) = C(sI - A)^{-1}B = \operatorname{diag}(C_1(sI - A_1)^{-1}B_1, \dots, C_m(sI - A_m)^{-1}B_m).$$

Thus, it is enough to calculate $\mu_{\mathbf{\Delta}_\mathcal{I}}(M)$ for block diagonal M. In [A2] we provide formulas for $\mu_{\mathbf{\Delta}_\mathcal{I}}(M)$ with respect to two families of norms on $\mathbf{\Delta}_\mathcal{I}$. The definition of these norms presumes that the sizes of the blocks Δ_{jk} are measured by operator norms $\|\cdot\|_{\alpha_j,\beta_k}$, which are defined as in (1.10).

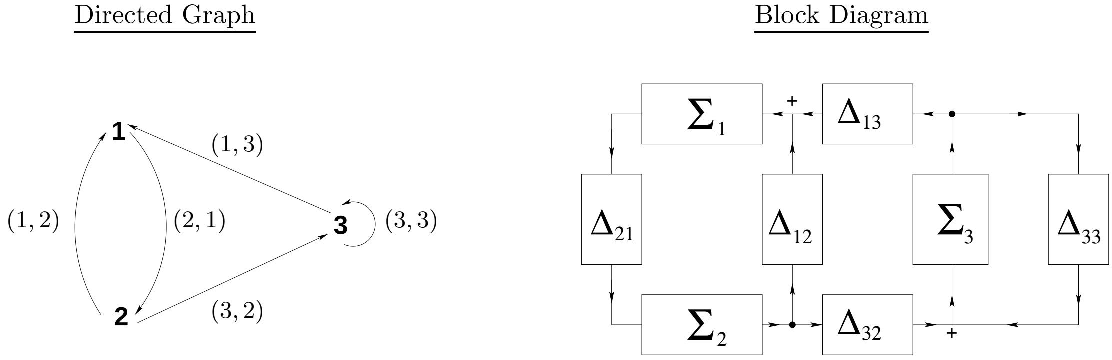

FIG. 3.1. *Composite System*

(i) Let $R = [r_{jk}] \in \mathbb{R}^{m\times m}$ be a nonnegative matrix, and let $\mathcal{I} := \{\, (j,k);\;\; r_{jk} > 0\,\}$. We define a weighted maximum norm on $\mathbf{\Delta}_{\mathcal{I}}$ by the formula

$$\|\Delta\| := \max_{(j,k)\in\mathcal{I}} r_{jk}^{-1}\|\Delta_{jk}\|_{\alpha_j,\beta_k}, \tag{3.2}$$

where $\Delta = [\Delta_{jk}]$ is a matrix as in (3.1). Then, with respect to the norm (3.2) we have

$$\mu_{\mathbf{\Delta}_{\mathcal{I}}}(\mathrm{diag}(M_1,\ldots,M_m)) = \varrho(\, R\ \mathrm{diag}(\|M_1\|_{\beta_1,\alpha_1},\ldots,\|M_m\|_{\beta_m,\alpha_m})\ ), \tag{3.3}$$

where $\varrho(\cdot)$ denotes the spectral radius.

(ii) The second norm type is given by

$$\|\Delta\| := \mathcal{N}_\alpha\left(\begin{bmatrix} \|\Delta_{11}\|_{\alpha_1,\beta_1} & \ldots & \|\Delta_{1m}\|_{\alpha_m,\beta_1} \\ \vdots & & \vdots \\ \|\Delta_{m1}\|_{\alpha_1,\beta_m} & \ldots & \|\Delta_{mm}\|_{\alpha_m,\beta_m} \end{bmatrix}\right), \tag{3.4}$$

where $\mathcal{N}_\alpha = \|\cdot\|_{\alpha,\alpha}$ is an operator norm induced by a norm $\|\cdot\|_\alpha$ on $\mathbb{R}^{m\times m}$. Let $\mathcal{Z}(\mathcal{I})$ denote the set of cycles of the oriented graph whose edges are the pairs $(j,k)\in\mathcal{I}$. If $\mathcal{Z}(\mathcal{I}) = \emptyset$ then $\mu_{\mathbf{\Delta}_{\mathcal{I}}}(\mathrm{diag}(M_1,\ldots,M_m)) = 0$. If $\mathcal{Z}(\mathcal{I}) \neq \emptyset$ then with respect to the norm (3.4),

$$\mu_{\mathbf{\Delta}_{\mathcal{I}}}(\mathrm{diag}(M_1,\ldots,M_m)) = \max_{\gamma\in\mathcal{Z}(\mathcal{I})}\left(\prod_{j\in\gamma}\|M_j\|_{\beta_j,\alpha_j}\right)^{\frac{1}{|\gamma|}}, \tag{3.5}$$

where $|\gamma|$ denotes the length of the cycle γ. See [A2] for details.

The formulas (3.3) and (3.5) are the main results of [A2]. They yield a collection of corollaries on the eigenvalues of perturbed block diagonal matrices. By specializing (3.5) to the diagonal case we obtain a new proof of the eigenvalue inclusion theorem of Brualdi.

4. Content of [A3]. In this paper we consider μ-values for perturbations which are self- or skew-adjoint with respect to an inner product on $\mathbb{C}^n$. The inner product is defined by a unitary matrix. The underlying norm is the spectral norm. First, we treat the Hermitian, complex symmetric and complex skew-symmetric perturbations. For $M \in \mathbb{C}^{n\times n}$ let $M_h :=$

$\frac{M-M^*}{2i}$, $M_s := \frac{M+M^\top}{2}$. In [A3] we show that

$$\mu_{\mathrm{Herm}}(M) = \sup\{\|Mv\|_2;\ v \in \mathbb{C}^n, \|v\|_2 = 1,\ v^* M_h v = 0\}$$
$$= \begin{cases} 0 & \text{if } M_h \text{ is definite,} \\ \sqrt{\inf_{t\in\mathbb{R}} \lambda_1(M^*M + t\,M_h)}, & \text{otherwise,} \end{cases} \tag{4.1}$$

$$\widetilde{\mu}_{\mathrm{Herm}}(M) = \inf\{\|Mv\|_2;\ v \in \mathbb{C}^n, \|v\|_2 = 1,\ v^* M_h v = 0\},$$
$$= \begin{cases} \infty & \text{if } M_h \text{ is definite,} \\ \sqrt{\sup_{t\in\mathbb{R}} \lambda_n(M^*M + t\,M_h)}, & \text{otherwise,} \end{cases} \tag{4.2}$$

$$\mu_{\mathrm{Skew}}(M) = \sup\{\|Mv\|_2;\ v \in \mathbb{C}^n, \|v\|_2 = 1,\ v^\top M_s v = 0\},$$
$$= \sqrt{\inf_{t\in\mathbb{R}} \lambda_2\left(\begin{bmatrix} M^*M & t\,M_s \\ t\,\overline{M_s} & \overline{M^*M} \end{bmatrix}\right)}, \tag{4.3}$$

$$\widetilde{\mu}_{\mathrm{Skew}}(M) = \inf\{\|Mv\|_2;\ v \in \mathbb{C}^n, \|v\|_2 = 1,\ v^\top M_s v = 0\},$$
$$= \sqrt{\sup_{t\in\mathbb{R}} \lambda_{2n-1}\left(\begin{bmatrix} M^*M & t\,M_s \\ t\,\overline{M_s} & \overline{M^*M} \end{bmatrix}\right)}, \tag{4.4}$$

$$\mu_{\mathrm{Sym}}(M) = \sigma_{\max}(M),$$
$$\widetilde{\mu}_{\mathrm{Sym}}(M) = \sigma_{\min}(M),$$

where $\lambda_1(\cdot) \geq \lambda_2(\cdot) \geq \ldots$ denote the eigenvalues of a Hermitian matrix in decreasing order, and $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ denote the maximum and the minimum singular value, respectively. The functions to be extremized in (4.1)-(4.4) have the property that each local extremum is global. By means of the formulas (4.1)-(4.4) one can compute $\mu_{\mathbf{\Delta}}$ and $\widetilde{\mu}_{\mathbf{\Delta}}$ if $\mathbf{\Delta}$ is the set of self-adjoint or the set of skew-adjoint matrices with respect to an inner product of the form

$$\langle x, y\rangle_\Pi := x^\bullet \Pi y, \qquad x, y \in \mathbb{C}^n,$$

where $\bullet \in \{*, \top\}$, Π is unitary and $\Pi^\bullet = \pm\Pi$. By definition,

$$\begin{aligned} \Delta \in \mathbb{C}^{n\times n} \text{ is self-adjoint with respect to } \langle\cdot,\cdot\rangle_\Pi &\Leftrightarrow \forall x, y \in \mathbb{C}^n :\ \langle x, \Delta y\rangle_\Pi = \langle \Delta x, y\rangle_\Pi \\ &\Leftrightarrow \Pi\Delta = \Delta^\bullet\Pi = \pm(\Pi\Delta)^\bullet, \\ \Delta \in \mathbb{C}^{n\times n} \text{ is skew-adjoint with respect to } \langle\cdot,\cdot\rangle_\Pi &\Leftrightarrow \forall x, y \in \mathbb{C}^n :\ \langle x, \Delta y\rangle_\Pi = -\langle \Delta x, y\rangle_\Pi \\ &\Leftrightarrow \Pi\Delta = -\Delta^\bullet\Pi = \mp(\Pi\Delta)^\bullet. \end{aligned}$$

As an important example let us discuss the case of complex Hamiltonian matrices. These matrices are skew-adjoint with respect to the inner product.

$$\langle x, y\rangle_J = x^* J y, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

For the μ-value of second kind with respect to Hamiltonian perturbations we obtain

$$\begin{aligned} \widetilde{\mu}_{\mathrm{Ham}}(M) &= \inf\{\|\Delta\|;\ \det(M - \Delta) = 0,\ \Delta^* J = -J\Delta,\ \Delta \in \mathbb{C}^{n\times n}\} \\ &= \inf\{\|J\Delta\|;\ \det(JM - J\Delta) = 0,\ (J\Delta)^* = J\Delta,\ \Delta \in \mathbb{C}^{n\times n}\} \qquad (4.5) \\ &= \widetilde{\mu}_{\mathrm{Herm}}(JM) \\ &= \sqrt{\sup_{t\in\mathbb{R}} \lambda_n(M^*M + (t/2i)\,((JM) - (JM)^*))} \qquad (4.6) \end{aligned}$$

The pseudospectra of a Hamiltonian matrix $H \in \mathbb{C}^{2n\times 2n}$ with respect to Hamiltonian perturbations satisfy

$$\sigma_{\mathrm{Ham}}(H,\delta) = \{s \in \mathbb{C};\ \widetilde{\mu}_{\mathrm{Ham}}(sI - H) < \delta\},$$

and from (4.6) it follows that

$$\widetilde{\mu}_{\mathrm{Ham}}(sI - H) = \begin{cases} \sigma_{\min}(sI - H) & \text{if } s \in i\mathbb{R}, \\ \sqrt{\sup_{t\in\mathbb{R}} \lambda_n((sI - H)^*(sI - H) + itJ)} & \text{otherwise.} \end{cases} \tag{4.7}$$

It is possible to specify a finite interval on which the supremum in (4.7) is attained. This enables us to compute $\sigma_{\mathrm{Ham}}(H,\delta)$ by evaluating the function $s \mapsto \widetilde{\mu}_{\mathrm{Ham}}(sI - H)$ on a grid.

5. Content of [A4]. In this paper we study the shape and growth of structured pseudospectra for small perturbations. Thoughout the paper it is assumed that the perturbation class $\mathbf{\Delta}$ is a cone, i.e. $\Delta \in \mathbf{\Delta}$ implies $t\Delta \in \mathbf{\Delta}$ for all $t \geq 0$. Our main result uses the the partial fraction expansion of the rational function $G(s) = C(sI_n - A)^{-1}B$, i.e.

$$G(s) = \sum_{\lambda\in\sigma(A)} \left(\frac{CP_\lambda B}{s-\lambda} + \sum_{\ell=2}^{\ell_\lambda} \frac{CN_\lambda^{\ell-1}B}{(s-\lambda)^\ell} \right), \tag{5.1}$$

where P_λ is the spectral projector onto the generalized eigenspace of A associated with $\lambda \in \sigma(A)$, $N_\lambda = (A - \lambda I)P_\lambda$ is the eigen-nilpotent, and

$$\ell_\lambda := \begin{cases} 1 & \text{if } CN_\lambda^{\ell-1}B = 0 \text{ for all } \ell \geq 2, \\ \max\{\ \ell \geq 2;\ CN_\lambda^{\ell-1}B \neq 0\ \} & \text{otherwise.} \end{cases} \tag{5.2}$$

We denote the leading coeffients in (5.1) by

$$\Gamma_\lambda := \begin{cases} CP_\lambda B & \text{if } \ell_\lambda = 1, \\ CN_\lambda^{\ell_\lambda - 1}B & \text{otherwise.} \end{cases} \tag{5.3}$$

Note that $\Gamma_\lambda = 0$ if and only if $\ell_\lambda = 1$ and $CP_\lambda B = 0$. Next, we introduce the sets

$$\mathcal{L}_\lambda := \{\, z \in \mathbb{C};\ \ z^{\ell_\lambda} \in \sigma(\Delta\,\Gamma_\lambda) \text{ for some } \Delta \in \mathbf{\Delta} \text{ with } \|\Delta\| \leq 1\,\}. \tag{5.4}$$

In words, $\mathcal{L}_\lambda$ is the set of roots of order ℓ_λ of all eigenvalues of the matrix products $\Delta\,\Gamma_\lambda$, where $\Delta \in \mathbf{\Delta}$ with $\|\Delta\| \leq 1$. Let $\mathcal{C}_\lambda(\delta)$ denote the connected component of the closure of the pseudospectrum $\sigma_\Delta(A,B,C,\delta)$ which contains the eigenvalue λ of A. The main result of [A4] is the identity

$$\lim_{\delta\searrow 0} \frac{\mathcal{C}_\lambda(\delta) - \lambda}{\delta^{1/\ell_\lambda}} = \mathcal{L}_\lambda, \tag{5.5}$$

where the limit is taken with respect to the Hausdorff distance between non-empty compact subsets of $\mathbb{C}$. More explicitely, (5.5) states that to each $\epsilon > 0$ there exists a $\delta_0 > 0$ such that for all positive $\delta < \delta_0$,

$$(1)\quad \mathcal{C}_\lambda(\delta) \subset \lambda + \delta^{1/\ell_\lambda}\,\mathcal{U}_\epsilon(\mathcal{L}_\lambda), \qquad\qquad (2)\quad \lambda + \delta^{1/\ell_\lambda}\mathcal{L}_\lambda \subset \mathcal{U}_{(\epsilon\,\delta^{1/\ell_\lambda})}(\mathcal{C}_\lambda(\delta)),$$

where $\mathcal{U}_\epsilon(\mathcal{M}) = \{\, z \in \mathbb{C};\ |z - s| < \epsilon \text{ for some } s \in \mathcal{M}\,\}$ is an ϵ-neighborhood of $\mathcal{M} \subset \mathbb{C}$. The limit sets $\mathcal{L}_\lambda$ can be computed using μ-values. Specifically we have

$$\mathcal{L}_\lambda = \{re^{i\phi};\ \phi \in [0,2\pi],\ 0 \leq r \leq r(\phi)\}, \qquad \text{where } r(\phi) := \mu_{\mathbf{\Delta}}(e^{-i\ell_\lambda\phi}\Gamma_\lambda)^{1/\ell_\lambda}.$$

If Γ_λ has a factorization $\Gamma_\lambda = XY^*$ with $X \in \mathbb{C}^{q\times r}, Y \in \mathbb{C}^{l\times r}$ then each nonzero eigenvalue of $\Delta\Gamma_\lambda$ is also an eigenvalue of $Y^*\Delta X$, and vice versa. Hence,

$$\mathcal{L}_\lambda = \{\, z \in \mathbb{C};\ \ z^{\ell_\lambda} \in \sigma(Y^*\Delta X) \text{ for some } \Delta \in \mathbf{\Delta} \text{ with } \|\Delta\| \leq 1\,\}. \tag{5.6}$$

As a corollary to (5.5) we obtain the following characterizations for the structured condition number of order $1/\ell_\lambda$ defined in (1.2).

$$\begin{aligned} \mathrm{cond}_{\boldsymbol{\Delta}}^{1/\ell_\lambda}(A,B,C,\lambda) &= \max\{|w|;\ w\in\mathcal{L}_\lambda\} \\ &= \max\nolimits_{\phi\in[0,2\pi]}\ \mu_{\boldsymbol{\Delta}}(e^{-i\ell_\lambda\phi}\,\Gamma_\lambda)^{1/\ell_\lambda} \\ &= [\,\max\{\varrho(\Delta\Gamma_\lambda);\ \Delta\in\boldsymbol{\Delta},\ \|\Delta\|=1\}\,]^{1/\ell_\lambda} \\ &= [\,\max\{\varrho(Y^*\Delta X);\ \Delta\in\boldsymbol{\Delta},\ \|\Delta\|=1\}\,]^{1/\ell_\lambda}, \end{aligned} \tag{5.7}$$

where $\varrho(\cdot)$ denotes the spectral radius.

Example: Figure 5 shows the limit sets $\mathcal{L}_\lambda$ for perturbations of the form

$$A \rightsquigarrow A_\Delta = \begin{bmatrix} \Re(\lambda I+M)+\Delta & -\Im(\lambda I+M) \\ \Im(\lambda I+M) & \Re(\lambda I+M)\end{bmatrix} = A + B\Delta C,$$

where $\lambda\in\mathbb{C}\setminus\mathbb{R}$, $\Delta\in\mathbb{R}^{n\times n}$ and

$$A = \begin{bmatrix} \Re(\lambda I+M)+\Delta & -\Im(\lambda I+M) \\ \Im(\lambda I+M) & \Re(\lambda I+M)\end{bmatrix},\quad B=\begin{bmatrix} I\\ 0\end{bmatrix}\in\mathbb{R}^{2n\times n},\quad C=\begin{bmatrix} I & 0\end{bmatrix}\in\mathbb{R}^{n\times 2n}$$

and $M=M_{jk}$, $j=1,2$, $k=1,2,3$, where

$$M_{1k}=\begin{bmatrix} 0 & Z_k\\ 0 & 0\end{bmatrix},\qquad M_{2k}=\begin{bmatrix} 0 & Z_k & 0\\ 0 & 0 & Z_k\\ 0 & 0 & 0\end{bmatrix}$$

with

$$Z_1=\begin{bmatrix} 1-2i & 2-3i\\ -i & 4-3i\end{bmatrix},\qquad Z_2=\begin{bmatrix} 4-5i & 1-i\\ 3-3i & -i\end{bmatrix},\qquad Z_3=\begin{bmatrix} 3-2i & 2\\ 1-2i & 5-i\end{bmatrix}.$$

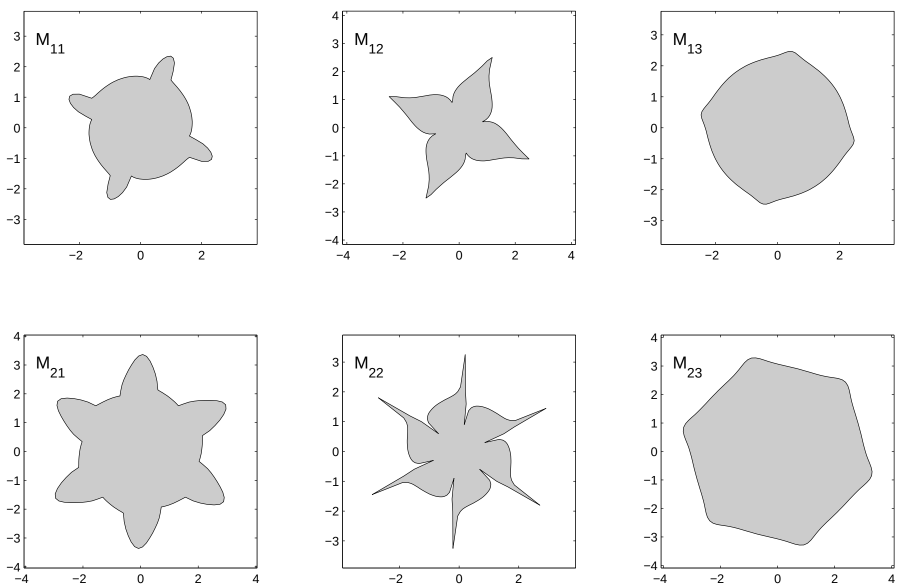

FIG. 5.1. *Some limit sets $\mathcal{L}_\lambda$.*

6. Content of [A5]. Suppose the matrix Γ_λ defined in (5.3) has rank 1, i.e. $\Gamma_\lambda = xy^*$, $x \in \mathbb{C}^q$, $y \in \mathbb{C}^l$. Then (5.6) and (5.7) yield,

$$\mathcal{L}_\lambda = \{ z \in C;\ z^{\ell_\lambda} \in K_{\mathbf{\Delta}}(x,y)\}, \tag{6.1}$$

$$\mathrm{cond}_{\mathbf{\Delta}}^{1/\ell_\lambda}(A,B,C,\lambda) = \max\{|w|;\ w \in K_{\mathbf{\Delta}}(x,y)\}^{1/\ell_\lambda}, \tag{6.2}$$

where

$$K_{\mathbf{\Delta}}(x,y) := \{ y^*\Delta x;\ \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1\}.$$

In [A5] we investigate the sets $K_{\mathbf{\Delta}}(x,y)$ for the case that $\mathbf{\Delta}$ is a vector space over $\mathbb{R}$. Then $K_{\mathbf{\Delta}}(x,y)$ is a convex subset of $\mathbb{C}$. As is well known a convex set is uniquely determined by its support function. The support function of $K_{\mathbf{\Delta}}(x,y)$ is given by

$$\begin{aligned} s_{\mathbf{\Delta}}(z) &= \max\{\Re(\bar z \xi);\ \xi \in K_{\mathbf{\Delta}}(x,y)\,\} \\ &= \max\{\langle \Delta, zyx^*\rangle;\ \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1\,\}, \end{aligned} \tag{6.3}$$

where

$$\langle F, G\rangle := \Re\mathrm{tr}(F^*G) \tag{6.4}$$

denotes the inner product between the matrices F and G. If $|z| = 1$ and $\Delta_0 \in \mathbf{\Delta}$ is a maximizer for (6.3) then $x^*\Delta_0 y$ is a boundary point of $K_{\mathbf{\Delta}}(x,y)$. Let $\mathcal{P}_{\mathbf{\Delta}} : \mathbb{C}^{l\times q} \to \mathbb{C}^{l\times q}$ denote the orthogonal projection onto $\mathbf{\Delta}$ with respect to the inner product (6.4). Furthermore, let $\|\cdot\|'$ denote the dual to the norm $\|\cdot\|$, i.e. $\|M\|' = \max\{\langle M, \Delta\rangle;\ \Delta \in \mathbb{C}^{q\times l},\ \|\Delta\| = 1\}$. For instance, the dual to the Frobenius norm is the Frobenius norm, and the dual to the spectral norm is the Schatten-1-norm. The basic result of [A5] is the identity

$$s_{\Delta}(z) = \|\mathcal{P}_{\mathbf{\Delta}}(z\,yx^*)\|', \qquad z \in \mathbb{C}, \tag{6.5}$$

which holds whenever the map $\mathcal{P}_{\mathbf{\Delta}}(\cdot)$ is a contraction with respect to the norm $\|\cdot\|$. The latter condition trivially holds if $\|\cdot\|$ is the Frobenius norm.

If $\mathbf{\Delta}$ is a vector space over $\mathbb{C}$ then it follows from (6.5) that

$$s_{\Delta}(z) = \|\mathcal{P}_{\mathbf{\Delta}}(yx^*)\|'\,|z|, \qquad z \in \mathbb{C}.$$

Hence in this case $K_{\mathbf{\Delta}}(x,y)$ is a disk of radius $\|\mathcal{P}_{\mathbf{\Delta}}(yx^*)\|'$ about 0. However, many important perturbation classes $\mathbf{\Delta}$ are only vector spaces over $\mathbb{R}$. In [A5] we discuss the cases that $\mathbf{\Delta}$ is the set of real matrices, the set of Hermitian matrices, the set of real symmetric matrices or the set of real skew-symmetric matrices. The underlying norm is either the Frobenius norm or the spectral norm. For these cases (with one exception[2]) we obtain that the support function is of the form

$$s_{\Delta}(z) = \sqrt{a\,|z|^2 + \Re(b\,z^2)}, \quad a \ge 0, b \in \mathbb{C}, |b| \le a.$$

This is the support function of the ellipse

$$E_{a,b} = \left\{ e^{i\phi/2}\left(\sqrt{a+|b|}\,\xi_1 + \sqrt{a-|b|}\,\xi_2\, i\right);\ \xi_1,\xi_2 \in \mathbb{R},\ \xi_1^2+\xi_2^2 \le 1\right\}, \quad \phi = \arg(b).$$

It follows that $K_{\mathbf{\Delta}}(x,y) = E_{a,b}$ and $\mathrm{cond}_{\mathbf{\Delta}}^{1/\ell_\lambda}(A,B,C,\lambda) = (\sqrt{a+|b|})^{1/\ell_\lambda}$. We derive formulae for the constants a and b in terms of x and y. For instance, if $\mathbf{\Delta} = \mathbb{R}^{l\times q}$ and the underying norm is the spectral norm then

$$a = \frac{1}{2}\left[\|x\|_2^2\,\|y\|_2^2 + \sqrt{(\|x\|_2^4 - |x^\top x|^2)(\|y\|_2^4 - |y^\top y|^2)}\right], \qquad b = \frac{1}{2}\,(x^\top x)(\overline{y^\top y}).$$

[2]The exceptional case is $\mathbf{\Delta}$=set of real symmetric matrices, $\|\cdot\|$ =spectral norm.

7. Content of [A6]. For a square matrix $M \in \mathbb{C}^{n\times n}$ let $\pi_\Sigma(M)$ denote the product of the nonzero singular values of M, and let $\pi_\Lambda(M)$ denote the product of the nonzero eigenvalues of M. For $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$ we define

$$q(A,\lambda) := \frac{\pi_\Sigma(A-\lambda I_n)}{|\pi_\Lambda(A-\lambda I_n)|}. \tag{7.1}$$

According to (1.8) and (1.11) the unstructured pseudospectrum with respect to the spectral norm, $\|\cdot\|_2$, satisfies

$$\sigma_{\mathbb{C}^{n\times n}}(A,I,I;\delta) = \{s \in \mathbb{C};\ \sigma_{\min}(A-sI) < \delta\}, \tag{7.2}$$

where $\sigma_{\min}(\cdot)$ denotes the minimum singular value.

Suppose that λ is a nonderogatory eigenvalue of A of algebraic multiplicity m. In [6] Burke, Lewis and Overton proved the following expansion of the function $s \mapsto \sigma_{\min}(A-s\,I)$.

$$\sigma_{\min}(A-sI) = \frac{|s-\lambda|^m}{q(A,\lambda)} + \mathcal{O}(|s-\lambda|^{m+1}), \qquad s \in \mathbb{C}. \tag{7.3}$$

This identity combined with (7.2) yields that for small δ the connected component of $\sigma_{\mathbb{C}^{n\times n}}(A,I,I;\delta)$ which contains λ is approximately a disk of radius $(q(A,\lambda)\delta)^{1/m}$. Hence, the condition number defined in (1.2) satisfies

$$\mathrm{cond}^{1/m}_{\mathbb{C}^{n\times n}}(A,I,I,\lambda) = q(A,\lambda)^{1/m}.$$

However, according to the results in [A4] we have

$$\mathrm{cond}^{1/m}_{\mathbb{C}^{n\times n}}(A,I,I,\lambda) = \begin{cases} \|P_\lambda\|_2 & \text{if } m=1, \\ \|N_\lambda^{m-1}\|_2^{1/m} & \text{if } m>1, \end{cases}$$

where P_λ is the eigenprojector associated with λ and $N_\lambda = (A-\lambda)P_\lambda$ is the eigen-nilpotent. Consequently,

$$q(A,\lambda) = \begin{cases} \|P_\lambda\|_2 & \text{if } m=1, \\ \|N_\lambda^{m-1}\|_2 & \text{if } m>1. \end{cases} \tag{7.4}$$

In [A6] we give a direct proof of (7.4) without using (7.3).

8. Content of [A7]. Let $\lambda \in \mathbb{C}$ be a simple eigenvalue of $A \in \mathbb{C}^{n\times n}$ with right eigenvector x and left eigenvector y such that $y^*x = 1$. If $\Delta \in \mathbb{C}^{n\times n}$ is sufficiently small then there is an eigenvalue $\hat\lambda(A+\Delta)$ of $A+\Delta$ satisfying

$$\hat\lambda(A+\Delta) = \lambda + y^*\Delta x + \mathcal{O}(\|\Delta\|^2).$$

Consequently, for any perturbation class $\boldsymbol{\Delta} \subseteq \mathbb{C}^{n\times n}$ with accumulation point 0 the structured condition number of λ satisfies

$$\begin{aligned} \mathrm{cond}_{\boldsymbol{\Delta}}(A,\lambda) &= \lim_{\delta\searrow 0}\sup\left\{\frac{|\hat\lambda(A+\Delta)-\lambda|}{\|\Delta\|};\ \Delta\in\boldsymbol{\Delta},\ 0<\|\Delta\|\le\delta\right\} \\ &= \lim_{\delta\searrow 0}\sup\left\{\frac{|y^*\Delta x|}{\|\Delta\|};\ \Delta\in\boldsymbol{\Delta},\ 0<\|\Delta\|\le\delta\right\}. \end{aligned}$$

Hence, if $\boldsymbol{\Delta}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$ then

$$\mathrm{cond}_{\boldsymbol{\Delta}}(A,\lambda) = \max\{\,|y^*\Delta x|;\ \Delta\in\boldsymbol{\Delta},\ \|\Delta\|=1\,\}. \tag{8.1}$$

This is a special case of the results in [A4] and [A5]. In Section 2 of [A7] we consider the case that A as well as its perturbations $A+\Delta$ are elements of the same smooth (real or complex) submanifold $\mathcal{M}$ od $\mathbb{C}^{n\times n}$. Hence, the perturbation class under consideration is given by

$$\mathbf{\Delta}_{\mathcal{M}} = \{\Delta \in \mathbb{C}^{n\times n};\ A+\Delta \in \mathcal{M}\}.$$

In general $\mathbf{\Delta}_{\mathcal{M}}$ is not a cone. Hence, the methods and results of [A4] and [A5] do not directly apply to this class. However, in [A7] we show that

$$\operatorname{cond}_{\mathbf{\Delta}_{\mathcal{M}}}(A,\lambda) = \max\{\, |y^*\Delta x|;\ \ \Delta \in T_A\mathcal{M},\ \|\Delta\| = 1\,\}, \tag{8.2}$$

where $T_A\mathcal{M}$ denotes the tangent space of $\mathcal{M}$ at $A \in \mathcal{M}$. Based on (8.1) and (8.2) we derive estimates and precise formulae for structured condition numbers of matrices which are skew- or self-adjoint or automorphisms with respect to an inner product $(x,y) \mapsto \langle x,y\rangle_\Pi$ (see Section 4 for the definition of the inner product). The sets of real and complex automorphisms are given by

$$\mathcal{G}_{\mathbb{K}} = \{\, A \in \mathbb{K}^{n\times n};\ \langle Ax, Ay\rangle_\Pi = \langle x,y\rangle_\Pi \text{ for all } x,y \in \mathbb{C}^n\}, \quad \mathbb{K} \in \{\mathbb{C},\mathbb{R}\}.$$

These sets are Lie-groups, i.e. smooth manifolds which are closed under matrix multiplication.

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