UNITS IN SOME PARAMETRIC FAMILIES OF QUARTIC FIELDS

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ABSTRACT. In this article we compute fundamental units for three parametric families of number fields of degree 4 with unit rank 2 and 3 generated by polynomials with Galois group D_4 and S_4.

1. INTRODUCTION

Let F be a number field generated by a zero ρ of a monic irreducible polynomial $f \in \mathbb{Z}[x]$. Let n_F be the degree of F and r_F the unit rank of F. The computation of the unit group of an order of F can be done by several methods like the Voronoi algorithm ($r_F \leq 2$), successive minima and other geometric methods using parallelotopes and ellipsoids. If f defines a parametric family of polynomials it is a problem to give the fundamental units of F in a parametric form, in particular for increasing degree n_F and rank r_F.

In this article we only consider parametric families of quartic fields. In this case $n_F = 4$ Stender ([16], [17]) has obtained families with unit rank 2. Some families with unit rank 3 are described in the biquadratic case ([15], [1], [3], [18]). In the non-biquadratic case families are published in several articles for example by Washington ([19]), by Lecacheux ([5, 4]), by Lettl and Pethö ([7]), by Nakamula ([10]) and by Niklasch and Smart ([11]). These families are different from the three presented here: In [19, 7] cyclic number fields are studied, and the families in [5] are also abelian with Galois group C_4 or V_4. The polynomials in [11] have Galois group S_4, and the generated number fields have unit rank 2. While in [4] the generating polynomials have Galois group D_4, the generated number fiels are totally real with unit rank 3. And in [10] there are parametric polynomials with Galois group D_4 considered: while the first family of number fields has unit rank 1 and the last has unit rank 3, the second has unit rank 2. This family generates for almost all choices of the parameter number fields with signature $(2, 1)$, but the polynomials with Galois group D_4 of our first family have for different

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choices of the parameter infinitely often signature $(2,1)$ and $(4,0)$. The other two families presented here have Galois group S_4 and signature $(4,0)$.

In section 2 of this article we compute parametric units for a family of number fields presented in [6]. There we have constructed polynomials $F_n(x)$ of degree n by using elliptic curves with rational points of order n. The polynomials have Galois group either the dihedral group D_n of order $2n$, or the cyclic group C_n of order n. Here we consider the case $n_F = 4$, and we compute parametric units which form a system of fundamental units under some conditions. In [14] the case $n_F = 5$ is examined.

In the last two sections, we present two new families of totally real quartic number fields and compute parametric systems of fundamental units. The first family arieses from the same idea as the families in [10] but is not included there.

2. Family with Galois group D_4 or C_4

For $n \in \mathbb{Z}$ we consider polynomials

$$F_b(x) \;:=\; x^4 - nx^3 + b(n-1)x^2 + 2b^2x - b^3.$$

These polynomials were already considered in [6] for other purposes. They have discriminants

$$d_b = d(F_b) = (4(n-4b)+1)(n^2+4b)^2.$$

To compute parametric units of the number fields F generated by F_b we consider only $b = \pm 1$. Furthermore we assume from now on that $(b,n) \in \{(-1,\pm 2),(1,0),(1,4)\}$, hence the polynomials F_b are irreducible.

Theorem 2.1. *The polynomial F_1 has the signature $(2,1)$ for $n \le 3$ and the signature $(4,0)$ for $n \ge 4$. The polynomial F_{-1} has the signatures $(2,1)$ for $n \le -5$, $(0,2)$ for $n \in \{-4,-3,-1,0,1\}$ and $(4,0)$ for $n \ge 3$.*

For $n \le 3$ the discriminant d_1 is negative, for $n \ge 4$ it is positive. Because of $F_1(0) = -1$ the polynomial F_1 has at least one real zero, hence all zeros are real.
The discriminant d_{-1} is negative for $n \le -5$, and positive for $n \ge -4$. For $n \ge 3$ we have $F_{-1}(1) = 1 - n + (1-n) + 2 + 1 = 5 - 2n < 0$ so that F_{-1} again has one and therefore 4 real zeros. In the remaining cases $n \in \{-4,-3,-1,-0,1\}$ one easily checks that the signature is $(0,2)$.

We want that the polynomials F_b generate quartic fields containing exactly one quadratic subfield. A candidate for the discriminant of (an order) of such a quadratic field is clearly $n^2 \pm 4$. Therefore we make a

First Assumption: $n^2 + 4b$ **is not a square. Clearly, this is tantamount to** $(n, b) \neq (0, 1)$**.**

Theorem 2.2. *$\Omega_b := \mathbb{Q}(\sqrt{n^2 + 4b})$ defines a quadratic number field. The polynomial F_b splits over this field as follows. We have*

$$F_b(x) = (x^2 + \varepsilon x - \varepsilon b)(x^2 + \bar{\varepsilon} x - \bar{\varepsilon} b)$$

with a unit $\varepsilon = \frac{1}{2}(-n + \sqrt{n^2 + 4b}) \in \Omega_b$ of norm $-b$. (By $^-$ we denote the non-trivial automorphism of a quadratic field.)

The proof is by a straightforward calculation.

Remark It is well known [9] that $n^2 \pm 4$ is square-free for infinitely many $n \in \mathbb{Z}$, hence ε is the fundamental unit of Ω_b in those cases, except for $n = 3, b = -1$, where ε is the cube of the fundamental unit.

Remark If F_b is irreducible with Galois group V_4 then $4(n - 4b) + 1$ is a square.

Theorem 2.3. *If $4(n - 4b) + 1$ is not a square in $\mathbb{Z}$ the polynomial F_b has Galois group D_4 or C_4.*

The polynomial F_b is irreducible over $\mathbb{Q}$ if and only if the polynomial

$$x^2 + \varepsilon x - \varepsilon b$$

is irreducible in $\Omega_b[x]$. That polynomial is reducible if and only if $\alpha := \varepsilon^2 + 4\varepsilon b$ is a square in Ω_b. But in that case $N(\alpha) = N(\varepsilon(\varepsilon + 4b)) = 4n + 1 - 16b$ is a square in $\mathbb{Q}$ which is in contradiction with our premises. Together with the preceding remark we obtain the theorem.

We note that $4n + 1 - 16b$ is a square if and only if $n = u^2 + u + 4b$ for some $u \in \mathbb{Z}$.

Because of Theorem 2.3. and because we want to have Galois group D_4 or C_4 we make a

Second Assumption: $4(n - 4b) + 1$ **is not a square in** $\mathbb{Z}$**.**

Theorem 2.4. *The polynomial F_b generates a Galois extension over $\mathbb{Q}$ (with Galois group C_4) if and only if for $\alpha := \varepsilon^2 + 4\varepsilon b$ the quotient $\alpha/\bar{\alpha}$ is a square in Ω_b. The latter is tantamount to $4(n - 4b) + 1$ being a square in Ω_b.*

At this stage we know that a root ρ of F_b generates a quartic extension of $\mathbb{Q}$. Hence, the square-roots of $\alpha = \varepsilon^2 + 4\varepsilon b$ and of $\bar{\alpha}$ generate quadratic extensions of Ω_b. If and only if these extensions coincide, either of them will be a cyclic extension of $\mathbb{Q}$. In that case, we have $\sqrt{\alpha} = \mu + \nu\sqrt{\bar{\alpha}}$ with elements μ, $\nu \in \Omega_b$. Squaring this equation leads to $\mu\nu = 0$, hence $\mu = 0$. Therefore $\alpha/\bar{\alpha}$ must be a square in Ω_b. Because of

$$\frac{\alpha}{\bar{\alpha}} = \frac{N(\alpha)}{(\bar{\alpha})^2}$$

and $N(\alpha) = 4n + 1 - 16b$ the theorem follows.

As mentioned in Theorem 1.4 the polynomial F_b has Galois group C_4 if and only if $4n + 1 - 16b$ is a square in Ω_b. The latter is tantamount to $v^2(1 + 4n - 16b) = n^2 + 4b$ with $n, v \in \mathbb{Q}$.

Theorem 2.5. *The polynomial F_b generates a Galois extension over $\mathbb{Q}$ with Galois group C_4 only for $(b, n) \in \{(1, 8), (-1, -3), (-1, 7)\}$.*

To prove this we first consider $b = 1$. That means we want to solve $v^2(4n - 15) = n^2 + 4$ which implicates $n_{1/2} = 2v^2 \pm \sqrt{4v^4 - 15v^2 - 4}$. We have $n \in \mathbb{Q}$ if $4v^4 - 15v^2 - 4$ is a square in $\mathbb{Q}$, in other words if the elliptic curve E_1 of equation $y^2 = 4v^4 - 15v^2 - 4$ has at least one rational point $(v, y) \in \mathbb{Q}^2$.

The Weierstraß form of E_1 is

$$z^2 = t^3 - 11t - 890.$$

Computations with the computer algebra system Magma [8] show that $E_1(\mathbb{Q}) \simeq \mathbb{Z}/4\mathbb{Z} = \{\mathcal{O}, P_1, P_2, P_3\}$, with

	z	t	y	v
P_1	136	27	∞	∞
P_2	0	10	0	-2
P_3	-136	27	∞	∞

That means in the case $b = 1$ we get the Galois group C_4 only for $n = 8$ corresponding to the polynomial $x^4 - 8x^3 + 7x^2 + 2x - 1$.

For the second case, $b = -1$ the same considerations yield: $n_{1/2} = 2v^2 \pm \sqrt{4v^4 + 17v^2 + 4}$ has to be a rational number which implies the existence of rational points on the elliptic curve E_{-1} of Weierstraß equation

$$z^2 = t^3 - 12987t - 263466.$$

Computations show that $E_{-1}(\mathbb{Q}) \simeq \mathbb{Z}/8\mathbb{Z} = \{\mathcal{O}, P_1, P_2, \dots, P_7\}$, where

	t	z	v	y
P_1	-21	0	∞	∞
P_2	-102	0	0	2
P_3	-57	-540	-1	5
P_4	-57	540	1	5
P_5	303	4860	1	-5
P_6	123	0	∞	∞
P_7	303	-4860	-1	-5

Hence in the case $b = -1$ we get the Galois group C_4 only for $n = -3, 7$ which corresponds to the polynomials $x^4 + 3x^3 + 4x^2 + 2x + 1$ and $x^4 - 7x^3 - 6x^2 + 2x + 1$.

From now on we assume that E_b is a quartic number field generated by a root ρ of F_b over $\mathbb{Q}$, and F_b has Galois group D_4. Our construction immediately leads to two independent units of E_b, namely ρ itself and the unit ε of Ω_b. We will further restrict our considerations to fields E_b of signature $(2, 1)$. In that case those two units form a maximal independent set of units of E_b. For the signature $(4, 0)$ our efforts to find a third independent unit in parametric form were unsuccessful.

In the remainder of this section we show that ρ and ε form a set of fundamental units for the order $\mathbb{Z}[\rho]$. This also means that they form a system of fundamental units for the field $\mathbb{Q}(\rho)$ whenever $n^2 + 4b$ and $4(n - 4b) + 1$ are square-free and coprime.

Remark From $16(n^2 + 4b) = (4n + (16b - 1))(4n - (16b - 1)) + (16b + 1)^2$ we conclude that a common factor of $n^2 + 4b$ and $4(n - 4b) + 1$ necessarily divides $(16b + 1)^2$.

We use a lower regulator bound of Nakamula [10]. Proposition 3 of his article states that the quotient of the regulators of E_b and Ω_b is bounded from below by

$$L := \frac{1}{2} \log \left(\sqrt[3]{|4(n - 4b) + 1|(n^2 + 4b)^2/4 + \left(\frac{317}{27}\right)^3} - \frac{290}{27} \right) .$$

We need to give a lower estimate for L. We start with the radicand of the cubic root. For $n \leq -10$ it is of the form

$$|n|^5(1 + \lambda)$$

with:

$$\lambda > \begin{cases} \frac{15}{4|n|} + \frac{8}{n^2} + 0.048 & \text{for} \quad b = 1 \\ -\frac{17}{4|n|} - \frac{8}{n^2} + 0.051 & \text{for} \quad b = -1 \end{cases}$$

From this we conclude

$$L > \frac{1}{2}\log\left(|n|^{(5/3)}(1+\lambda/3-\lambda^2/6)-\frac{290}{27}\right)$$

resulting in

$$L > \frac{2}{3}\log|n|.$$

Next we compute an upper estimate for the regulator R_{E_b} of the independent units ρ and ε. We choose the first two conjugates $\rho^{(1)}$ and $\rho^{(2)}$ of ρ for this purpose, and get

$$\begin{aligned} R_{E_b} &= |\det\begin{pmatrix} \log|\rho^{(1)}| & \log|\rho^{(2)}| \\ \log|\varepsilon| & \log|\varepsilon| \end{pmatrix}| \\ &= |\log|\varepsilon||\,|\log\frac{|\rho^{(1)}|}{|\rho^{(2)}|}| \ . \end{aligned}$$

We begin by estimating the quotient $|\rho^{(1)}/\rho^{(2)}|$. We have

$$\frac{\rho^{(1)}}{\rho^{(2)}} = \frac{-\varepsilon+\sqrt{\varepsilon^2+4b\varepsilon}}{-\varepsilon-\sqrt{\varepsilon^2+4b\varepsilon}} \ .$$

We easily compute

$$\mu := \varepsilon^2+4b\varepsilon = (n^2-4bn+2b-(n-4b)\sqrt{n^2+4b})/2.$$

One obtains the estimates

$$L_\varepsilon < \varepsilon < U_\varepsilon,$$

where

$$L_\varepsilon := |n|+b/|n|-2/n^3\ ,\ \ U_\varepsilon := |n|+b/|n|,$$

and

$$L_\mu < \sqrt{\mu} < U_\mu$$

where

$$L_\mu := |n|+2b+(b-4)/|n|-2/n^2-(8b+2)/|n|^3-4b/n^4$$

and

$$U_\mu := |n|+2b+b/|n|+2/n^2.$$

By considering the cases $b=\pm1$ separately, one obtains

$$\left|\frac{\rho^{(1)}}{\rho^{(2)}}\right| < C = \frac{|n|+1.11}{0.779}$$

for $|n|\geq 10$.

If the unit group $U := \langle -1,\rho,\varepsilon\rangle$ is a proper subgroup of the full unit group $U_{\mathbb{Z}[\rho]}$ of $\mathbb{Z}[\rho]$, then the regulator of E_b divided by the regulator of Ω_b is

$\leq \log(C)/2$. Showing $\log(C)/2 < L$ therefore proves that ρ, ε are a set of fundamental units for $\mathbb{Z}[\rho]$. Again, it is easy to see that

$$\frac{1}{2}\log\frac{|n|+1.11}{0.779} \;<\; \frac{2}{3}\log|n|$$

is tantamount to

$$\frac{|n|+1.11}{0.779|n|^{4/3}} \;<\; 1$$

and the latter is satisfied for all $n < -5$.

Theorem 2.6. *In case the field $E_b = \mathbb{Q}(\rho)$ is generated by F_b with dihedral Galois group, then ρ and ε are fundamental units of the order $\mathbb{Z}[\rho]$. They are even fundamental units of E_b, when $4(n-4b)+1$ and n^2+4b are both square-free and coprime.*

The estimates above prove the theorem for $n \leq -10$. For larger values of n the prove is by directly calculating the unit group of E_b with KANT [2].

3. A parametric family of number fields of degree 4

In this part we consider the parametric family of polynomials of degree 4 defined by $f(x) = x^4 + ax^3 - 2x^2 + (1-a)x + 1$. This family arises by the same idea of construction as the families in [10], but there only the cases with Galois group D_4 are presented. The constructive idea is the assumption that $\rho, \rho+1$ and $\rho-1$ are units of the number fields generated by $x^4+ax^3+bx^2+cx+1$ (with ρ a zero). In this way one gets three families, two of them are studied in [10] ($x^4+ax^3-bx^2-ax+1$ with $b \in \{1,3\}$), the third family $f(x)$ is presented here. By straightforward calculation it is easily seen that these polynomials are irreducible and have (for $a \geq 3$) four real roots. They generate for $a \in \mathbb{N}$, $a \geq 3$ number fields $F = \mathbb{Q}[\rho]$ of signature $[4,0]$ with rank $r_F = 3$. For $a \in \{\pm 1, 0, 2\}$ the number fields have signature $[2,1]$. And for $k \in \mathbb{Z}$ the polynomial f generates the same number field F for $a = k$ and $a = 1-k$, hence there is no need to consider $a < -1$.
In the following we therefore only consider the case $a \geq 3$.

Remark Computation of examples suggests that for infinitely many a the discriminant $d_f = 4a^6 - 12a^5 + 28a^4 - 36a^3 - 56a^2 + 72a - 283$ of f has no quadratic factors which implies that the order $\mathbb{Z}[\rho]$ is maximal and $D_F = d_f$, and the polynomials f generate infinitely many number fields.

Theorem 3.1. *The index of $\mathbb{Z}[\rho]$ in the maximal order of the number field F generated by f is not divisible by 5 or 13 for all $a \geq 3$.*

For $a \equiv 3 \ (mod\ 5)$ (and only for these a) we have $d_f \equiv 0 \ (mod\ 25)$ but $d_f \not\equiv 0 \ (mod\ 5^3)$. The Dedekind test shows that the order $\mathbb{Z}[\rho]$ is in this case (and therefore in all cases) already 5-maximal. Similarly for $a \equiv 7 \ (mod\ 13)$ (and only for these a) we have $d_f \equiv 0 \ (mod\ 13^2)$ but $d_f \not\equiv 0 \ (mod\ 13^3)$. Again $\mathbb{Z}[\rho]$ is already 13-maximal. Thus this order is maximal if the discriminant is divided by only the quadratic factors 25 and/or 169.

Remark Computations show that for $3 \leq a \leq 2000$ there are only 26 number fields with non-maximal order $\mathbb{Z}[\rho]$: $a \in \{80, 143, 326, 380, 406, 425, 450, 537, 609, 620, 699, 979, 984, 1044, 1049, 1106, 1138, 1235, 1386, 1498, 1508, 1540, 1667, 1695, 1825, 1906\}$. These fields are partly described with $k \in \mathbb{N}$ by $a = (3 + k \cdot 23) \cdot 23 + 11$ (that are $a = 80, 609, 1138, 1667$) where d_f is divisible by 23^2, and by $a = (19 + k \cdot 23) \cdot 23 + 13$, (that are $a = 450, 979, 1508$) where d_f is again divisible by 23^2. The discriminant d_f is divided by 29^2 for $a = (4 + k \cdot 29) \cdot 29 + 27$ (that are $a = 143, 984, 1825$) or $a = (24 + k \cdot 29) \cdot 29 + 3$ (that are $a = 699, 1540$). Or d_f is divisible by 31^2 and we have $a = (13 + k \cdot 31) \cdot 31 + 22$ (that are $a = 425, 1386$) or $a = (17 + k \cdot 31) \cdot 31 + 10$ (that are $a = 537, 1498$). On the other hand, with the choice of a in one of these sets of parametric natural numbers we always have that d_f is divisible by the corresponding square.

Theorem 3.2. *The four zeros of f lie in the following four intervals:*

$$\rho_1 \in [-a - \frac{1}{a} - \frac{1}{a^2}, -a]$$
$$\rho_2 \in [-1 + \frac{1}{a^2}, -1 + \frac{1}{a}]$$
$$\rho_3 \in [\frac{1}{a}, \frac{1}{a} + \frac{1}{a^2}]$$
$$\rho_4 \in [1 - \frac{2}{3a}, 1 - \frac{1}{2a}]$$

For $a \geq 4$, one shows that $f(x_{min})f(x_{max}) < 0$, where $(x_{min}, x_{max}) \in \{(-a-\frac{1}{a} - \frac{1}{a^2}, -a), (-1 + \frac{1}{a^2}, -1 + \frac{1}{a}), (\frac{1}{a}, \frac{1}{a} + \frac{1}{a^2}), (1 - \frac{2}{3a}, 1 - \frac{1}{2a})\}$. This proves the theorem.

Remark Because $\rho_3 < \frac{1}{a} + \frac{1}{a^2} < \frac{1}{2} < 1 - \frac{2}{3a} < \rho_4$ we get the following inequalities for the zeros of f:

$$-a - 1 < \rho_1 < -a < -1 < \rho_2 < 0 < \rho_3 < \frac{1}{2} < \rho_4 < 1$$

Theorem 3.3. *The polynomial f has Galois group S_4.*

To show this we first look at the cubic resolvent r_f of f. As in [13], we get $r_f(x) = x^3 + 4x^2 + a(1-a)x + 1$ with discriminant $d(r_f) = d(f) =$

$-4\alpha^3+16\alpha^2+72\alpha-283$ with $\alpha=a(1-a)$. The resolvent r_f is irreducible and we observe that $d(r_f)>0$ for $a\geq 3$. Moreover the discriminant is not a square in $\mathbb{Q}$ because $y^2=d(r_f)$ defines an elliptic curve which has no rational point except ∞. This implies that r_f has Galois group S_4 and the theorem follows.

Let ρ be a zero of f. In the number field $\mathbb{Q}(\rho)$ the element ρ is obviously a unit. Moreover, by definition of f the elements $\rho+1,\ \rho-1\ \in\mathbb{Z}[\rho]$ are units as well, and $(\rho+1)^{-1}=\rho^3+(a-1)\rho^2-(a+1)\rho+2$, and $(\rho-1)^{-1}=-\rho(\rho^2+(a+1)\rho+(a-1))$.
With $\rho-1$ and ρ being units, their quotient $\vartheta:=\frac{\rho-1}{\rho}$ is a unit too.

Theorem 3.4. *The three units $\{\rho,\rho+1,1-\frac{1}{\rho}\}$ form a system of independent units of the order $\mathbb{Z}[\rho]$. Moreover this set is a system of fundamental units for $a\geq 3$.*

To show this, we first assume $(\rho+1)^k=\pm\rho^l$ with $k\in\mathbb{N},l\in\mathbb{Z}$. This implies that $|\rho+1|^k=|\rho|^l$. Let $k>0$. Because of $1<\rho_4+1<2$ and $0<\rho_4<1$ we get $l<0$; with $a-1<|\rho_1+1|<a<|\rho_1|<a+1$ we get $l>0$ which yields a contradiction.
The pairwise independency for the other two cases is shown in a similar way with the help of the sequence of inequalities for ϑ (for $a>3$):

$$2-a<\vartheta_3<1-a<-1<-\tfrac{1}{a}<\vartheta_4<-\tfrac{1}{2a}<0<1<\vartheta_1<\tfrac{3}{2}<\vartheta_2<3.$$

Now we assume that $\vartheta^k=\pm\rho^l(\rho+1)^m$ where $k,l,m\in\mathbb{Z}$. Without loss of generality let $k>0$. If $l,m>0$ then the image of the canonical embedding φ_2 with $\rho\longmapsto\rho_2$ yields $|\vartheta_2|^k=|\rho_2|^l|\rho_2+1|^m$ which is impossible because the left hand side is >1 and the right is <1. The consideration of the other canonical embeddings φ_1,φ_3 and φ_4 leads also to contradictions in the remaining cases.
Thus we have shown that the three units $\rho,\rho+1$ and ϑ are a maximally independent set of units of $\mathbb{Q}(\rho)$.

A lower bound for the regulator R of the unit group of the maximal order of $\mathbb{Q}(\rho)$ is given in [12]:

$$R\geq\sqrt{\left(\frac{(\log(\frac{|D_F|}{16}))^2}{20}\right)^3\frac{1}{8}}$$

(In general we have $D_F=c^2\cdot d_f$ for some constant $c\in\mathbb{N}$, but in infinitely many cases (see first Remark of this section) the order $\mathbb{Z}[\rho]$ seems to be already maximal, so $c=1$ as assumed. The inequality holds in general for $\mathbb{Z}[\rho]$ replacing D_F by d_f.)

With $D_f > \frac{64}{17}a^6$ for $a \geq 49$ we get the lower bound R_{low} of the regulator:

$$\frac{1}{\sqrt{64000}}\Big(\log(\frac{D_f}{16})\Big)^3 \;\geq\; \frac{1}{253}\Big(6\log(a)+\log(\frac{4}{17})\Big)^3 \;\geq\; \frac{(6\log(a)-1.5)^3}{253} =: R_{low}$$

The regulator R_ρ for a system of independent units $\{\rho, \rho+1, \vartheta\}$ of $\mathbb{Z}[\rho]$ is defined by

$$R_\rho = \left|\det\begin{pmatrix} \log(|\rho_1+1|) & \log(|\rho_1|) & \log(|\vartheta_1|) \\ \log(|\rho_3+1|) & \log(|\rho_3|) & \log(|\vartheta_3|) \\ \log(|\rho_4+1|) & \log(|\rho_4|) & \log(|\vartheta_4|) \end{pmatrix}\right|$$

Computing the determinant and taking into account the size of the arguments of the logarithms, respectively the signs of the values of the logarithms, we can estimate R_ρ from above:

$$\begin{aligned} R_\rho \leq\ & \log(|\rho_1+1|)\log(\tfrac{1}{|\rho_3|})\log(\tfrac{1}{|\vartheta_4|}) + \log(|\rho_1|)\log(|\vartheta_3|)\log(|\rho_4+1|) + \\ & \log(|\rho_4+1|)\log(\tfrac{1}{|\rho_3|})\log(|\vartheta_1|) + \log(\tfrac{1}{|\rho_4|})\log(|\vartheta_3|)\log(|\rho_1+1|) + \\ & \log(\tfrac{1}{|\vartheta_4|})\log(|\rho_3+1|)\log(|\rho_1|). \end{aligned}$$

Now all factors are positive. Using the approximations of ρ and ϑ and the inequalities $\log(2) < 0.7$, $\log(1+\frac{1}{a}) < 0.02$ and $\log(1+\frac{1}{a}+\frac{1}{a^2}) < \log(1+\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}) < 0.021$, one shows that for $a \geq 50$:

$$R_\rho \leq \log(a)^3 + 1.461\cdot\log(a)^2 + 0.05822\cdot\log(a) + 0.00042 =: R_{up}$$

Finally, we obtain

$$1 < \frac{R}{R_{low}} < \frac{R_{up}}{R_{low}} < 2,$$

where the last inequality holds for $a > 44$. This comes from the inequality $\frac{R_{up}}{R_{lo}}(\log(44)) < 2$ and because the quotient is monotonic decreasing for $a > 44$. So the index of the unit system $\{\rho, \rho+1, \vartheta\}$ in a system of fundamental units is lower than 2, which implies that for $a > 50$ the units $\{\rho, \rho+1, 1-\frac{1}{\rho}\}$ are fundamental units of $\mathbb{Z}[\rho]$.

The remaining cases $3 \leq a \leq 44$ are proved by direct calculations with KANT [2].

4. A second family of number fields of degree 4

In an analogous way as in Section 3 we show that for the family of polynomials $f_a(x) = x^4 - (a^2+a+1)x^2 + (a^2+a)x - 1$ the set $\{\rho, \rho-1, \rho-a\}$ forms a system of fundamental units of the number field generated by a root of f_a.

Calculations show that the $f_a(x)$ are irreducible and have four real roots for $a \notin \{0, \pm 1, -2\}$. Computations of examples suggests that for $a \in \mathbb{Z}^{\geq 2}$ the f_a generate infinitely many number fields of signature $[4, 0]$ with unit rank 3. For $a \in \{0, \pm 1, -2\}$ the number fields have signature $[2, 1]$. Moreover f_a and f_{-a-1} generate the same number field, hence there is no need to consider $a < -2$.

In the following we therefore only consider the case $a \geq 2$.

The discriminant of f_a is $d_f = 4a^{10} + 20a^9 + 9a^8 - 84a^7 - 74a^6 + 156a^5 + 169a^4 - 60a^3 - 396a^2 - 320a - 400$. Computations show that $d_f \equiv 0 \mod 2^4$ but $d_f \not\equiv 0 \mod 2^5$ for any $a \in \mathbb{Z}$, and $d_f \equiv 0 \mod 5^2$ for $a \equiv 0, 4 \mod 5$ but $d_f \not\equiv 0 \mod 5^3$ for any $a \equiv 0, 4 \mod 5$. Using the Dedekind test for the maximality of an order we get:

Theorem 4.1. *The index of $\mathbb{Z}[\rho]$ in the maximal order of the number field generated by f_a is not divisible by 2 or 5 for all $a \geq 2$.*

Numerical approximations of the roots of f_a lead to:

Theorem 4.2. *The four roots of f_a lie in the four intervals:*

$$\rho_1 \in [-a-2, -a-1]$$

$$\rho_2 \in [\frac{1}{a^3}, \frac{1}{a^2}]$$

$$\rho_3 \in [1 - \frac{1}{a^2}, 1 - \frac{1}{a^3}]$$

$$\rho_4 \in [a + \frac{1}{a^4}, a + \frac{1}{a^3}]$$

As in Section 3 we compute the Galois group of f_a with the cubic resolvent $r_{f_a} = x^3 + 2(a^2 + a + 1)x^2 + ((a^2 + a + 1)^2 + 4)x + a^2(a + 1)^2$ to S_4. The roots of f_a are units and we have:

Theorem 4.3. *The three units $\{\rho, \rho - 1, \rho - a\}$ are independent units of the order $\mathbb{Z}[\rho]$. They form a system of fundamental units for $a \geq 2$.*

To prove this theorem the following proposition is helpful:

Proposition 4.4. *The three units $\{\rho, \rho - 1, \rho - a\}$ are independent if and only if $\{\rho, \frac{\rho-1}{\rho}, \rho(\rho - a)\}$ are independent.*

The independency of $\{\rho, \frac{\rho-1}{\rho}, \rho(\rho - a)\}$ is proved similarly to Theorem 3.4. The fundamentality of the set of Theorem 4.3 is proved by approximations of the regulator as in 3.4:

$$R_{lo} = \frac{(10\ln a + \ln(\frac{1}{4}) + \ln(1 + \frac{5}{a}))^3}{\sqrt{64000}}$$

and

$$R_{up} = 8.07\ln^3 a + 3\ln^2 a$$

which implies

$$\frac{R_{up}}{R_{lo}} < 3$$

for $a \geq 150$. Finally we have to show that any unit of the form $\theta = \pm\rho^{m_1}(\rho-1)^{m_2}(\rho-a)^{m_3}$ with $m_i \in \{0,1\}$ is not a square in the order $\mathbb{Z}[\rho]$. For $(m_1, m_2, m_3) \in \{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (1,1,1)\}$ there exists for all $a \in \mathbb{Z}$ a negative conjugate of θ which implies that θ cannot be a square. In the remaining case $(m_1, m_2, m_3) = (0,1,1)$ the unit $(\rho-1)(\rho-a) = \rho^2 - (a+1)\rho + a$ cannot be a square too for $a \equiv 0 \mod 2$: consider $\alpha \in Z[\rho]$ with $\alpha^2 = \rho^2 - (a+1)\rho + a$; this implies for every choice of $a \in \mathbb{Z}$ a contradiction concerning the coefficients of α^2 and $\rho^2 - (a+1)\rho + a$ modulo 2. For $a \not\equiv 1, 7 \mod 8$ the considered unit can also not be a square for the same reasons modulo 8. (Even for other choices of the parameter a computations show that $\{\rho, \rho-1, \rho-a\}$ are fundamental.)

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