KASH

Reference Manual

KANT–Group
Technische Universität Berlin
Fachbereich 3 Mathematik
Straße des 17. Juni 136
10623 Berlin, Germany

February 22, 2004

Copyright © Prof. Dr. Michael E. Pohst, 1987–2003

TU Berlin
Fachbereich 3 Mathematik
Straße des 17. Juni 136
10623 Berlin, Germany

KASH can be copied and distributed freely for any non-commercial purpose.
If you copy KASH for somebody else, you may ask this person to refund your expenses. This should cover cost of media, copying and shipping. You are not allowed to ask for more than this. In any case you must give a copy of this copyright notice along with the program.
If you obtain KASH please send us a short notice to that effect, e.g., an e-mail message to the address *kant@math.tu-berlin.de* containing your full name and address. This allows us to keep track of the number of KASH users.
If you publish a mathematical result that was partly obtained using KASH, please cite

M. Daberkow, C. Fieker, J. Klüners, M. Pohst, K. Roegner
and K. Wildanger, *KANT V4*, in J. Symbolic Comp. **24** (1997), 267-283.

Also we would appreciate it if you could inform us about such a paper.
You are permitted to modify and redistribute KASH, but you are not allowed to restrict further redistribution. That is to say proprietary modifications will not be allowed. We want all versions of KASH to remain free.
If you modify any part of KASH and redistribute it, you must supply a 'README' document. This should specify what modifications you made in which files. We do not want to take credit or be blamed for your modifications.
Of course we are interested in all of your modifications. In particular we would like to see bug-fixes, improvements and new functions. So again we would appreciate it if you would inform us about all modifications you make.
KASH is distributed by us without any warranty, to the extent permitted by applicable state law. We distribute KASH *as is* without warranty of any kind, either expressed or implied, including, but not limited to, the implied warranties of merchantability and fitness for a particular purpose.
The entire risk as to the quality and performance of the program is with you. Should KASH prove defective, you assume the cost of all necessary servicing, repair or correction. In no case unless required by applicable law will we, and/or any other party who may modify and redistribute KASH as permitted above, be liable to you for damages, including lost profits, lost monies or other special, incidental or consequential damages arising out of the use or inability to use KASH.

Acknowledgements

This program could not have been written without the support of Prof. J. Cannon, W. Bosma, S. Collins and A. Steele at the University of Sydney. Additionally, we would like to thank Prof. Dr. J. Neubüser at the RWTH Aachen, Germany for his permission to use and modify a large portion of the GAP source code. Special thanks also go to M. Schönert, a main contributor to the creation of GAP, for his kind support and aid. Finally, we would like to thank A. Weber at Cornell University, who developed the database for algebraic number fields and M. Klebel who did the (Ray) class fields for imaginary quadratic number fields.

KANT V4	Copyright ©	Prof. Dr. M. Pohst, 1987-2003 TU Berlin, Fachbereich 3 Mathematik Straße des 17. Juni 136, 10623 Berlin, Germany EMail: kant@math.tu-berlin.de
KASH	Copyright ©	Prof. Dr. M. Pohst, 1994-2003 TU Berlin, Fachbereich 3 Mathematik Straße des 17. Juni 136, 10623 Berlin, Germany EMail: kant@math.tu-berlin.de
MAGMA	Copyright ©	Prof. J. Cannon, 1995 -2003 Sydney, Australia Email: john@maths.usyd.edu.au
GAP	Copyright ©	Lehrstuhl D für Mathematik, 1992 RWTH Aachen Templergraben 64, 52062 Aachen, Germany

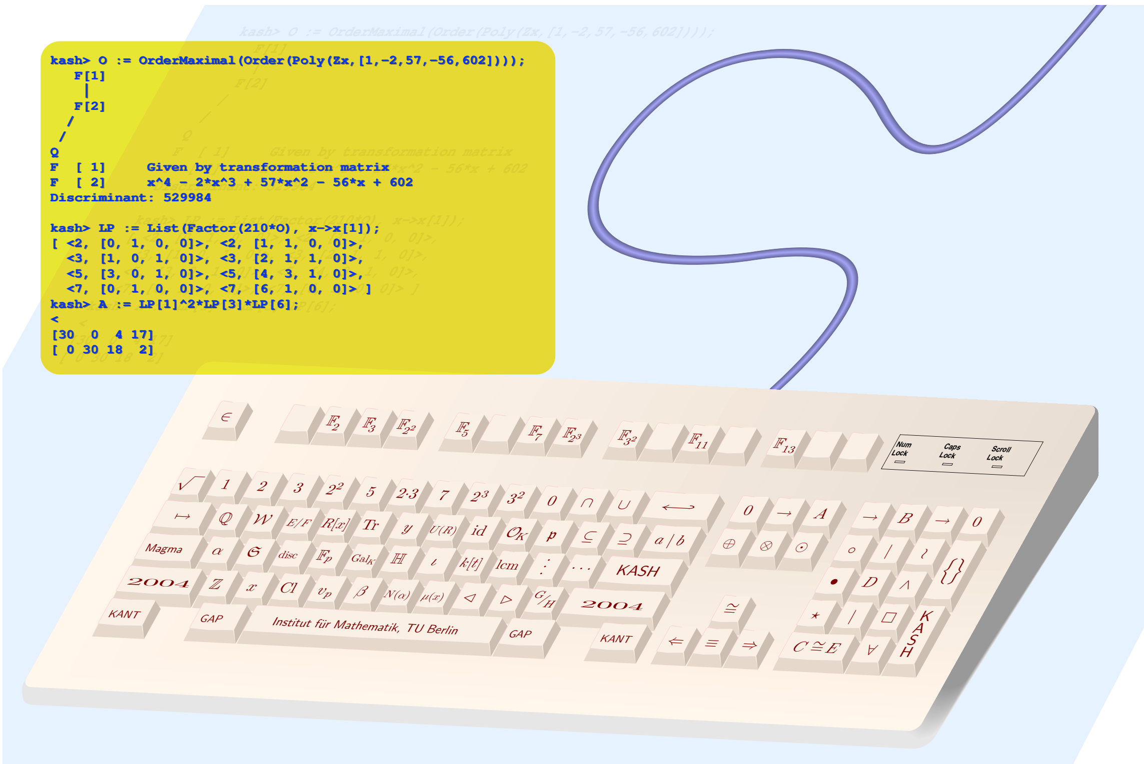
Short Contents

Contents

Chapter 1

KASH functions

This chapter contains an alphabetically sorted line-up of relevant functions available in KASH.



NAME `AbelianDualGroup`

PURPOSE Returns the dual group of a given finite abelian group.

SYNTAX `d := AbelianDualGroup(g);`

`groups g, d`

DESCRIPTION Returns the dual group d of a group g, i.e. the group consisting of the isomorphisms from the group g to $\mathbb{C}$. The isomorphisms are presented by rational numbers q corresponding to $e^{(i*2*\pi*q)}$.

EXAMPLE Compute the dual group and apply one homomorphism:

```
kash> g := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> d := AbelianDualGroup(g);
Group with relations:
[15]
kash> eltd := AbelianGroupEltCreate(d, [1]);
[1]
kash> hom := AbelianGroupDiscreteExp(eltd);
HomMatrix =
[ -6   6]
[ 10 -10]
[ -5   5]
from Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
to Group with relations:
[3 0]
[0 5]

kash> eltg := AbelianGroupEltCreate(g, [1,0,0]);
[1 0 0]
kash> AbelianGroupDiscreteExp(hom*eltg);
1/5
```

It is allowed to apply an elelment of the group to an element of the corresponding dual group:

```
kash> eltd*eltg;
1/5
```

NAME `AbelianDualHom`

PURPOSE Creates the dual homomorphism of a given homomorphism between two finite Abelian groups.

SYNTAX `dualhom := AbelianDualHom(hom);`

`homomorphism`	`hom`	homomorphism from $g1$ to $g2$

DESCRIPTION Creates the dual homomorphism from the group $g2^*$ to group $g1^*$ of a homomorphism hom: $g1 \to g2$. The dual homomorphism is defined by: $\mathrm{hom}^* : g2^* \to g1^*$ and $\mathrm{hom}^* : \chi \to \chi * hom$.

SEE ALSO `AbelianGroupHomCreate`, `AbelianHomGroup`,

EXAMPLE

```
kash> g1 := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);;
kash> g2 := AbelianGroupCreate([[0,2],[3,0]]);;
kash> mat := Mat(Z, [[-24,24],[20,-20],[-10,10]]);;
kash> hom := AbelianGroupHomCreate(g1, g2, mat, true);
HomMatrix =
[-24  24]
[ 20 -20]
[-10  10]
from Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
to Group with relations:
[0 2]
[3 0]

kash> dualhom:=AbelianDualHom(hom);
HomMatrix =
[5]
from Group with relations:
[6]
to Group with relations:
[15]
```

NAME `AbelianFieldToRCF`

PURPOSE Computes the data neccessary to get an abelian field as a Ray Class Field.

SYNTAX `rc := AbelianFieldToRCF(o [, I]);`

`list`	`rc`	[ideal, inf, relations]
`order`	`o`	of an abelian field with known automorphisms
`ideal`	`I`	of the coef. ring of `o`. Must be a multiple of the conductor.

DESCRIPTION

EXAMPLE Suppose we know that the field generated by a root of $x^4 - 4x^2 + 1$ is abelian, and we'd like to work with this field:

```
kash> o := OrderMaximal(x^4-4*x^2+1);
Generating polynomial: x^4 - 4*x^2 + 1
Discriminant: 2304

kash> OrderAutomorphisms(o);
[ [0, 1, 0, 0], [0, -4, 0, 1], [0, 4, 0, -1], [0, -1, 0, 0] ]
kash> rc1 := AbelianFieldToRCF(o, Disc(o));
Quotient of RayClassGroupToAbelianGroup(<2304>, [  ])
Group with relations:
[  2   0]
[  0   2]
[  2   0]
[  0 192]
kash> f := RayConductor(rc1);
[ <24>, [  ] ]
kash> rc2 := AbelianFieldToRCF(o, f[1]);
Quotient of RayClassGroupToAbelianGroup(<24>, [  ])
Group with relations:
[2 0]
[0 2]
[2 0]
[0 2]
kash> rcf := RayClassField(rc2);
[ x^2 - 2, x^2 - 3 ]
```

NAME `AbelianFixPointGroup`

PURPOSE Computes the fix point group of endomorphisms.

SYNTAX
```
f := AbelianFixPointGroup(hom);
f := AbelianFixPointGroup(L);
```

group	f	
homomorphism	hom	
list	L	list of homomorphisms

DESCRIPTION Computes the group consisting of elements that are (pointwise) fix under each given endomorphism.

EXAMPLE Compute the fix point group and apply an endomorphism:

```
kash> g := AbelianGroupCreate([[0,0],[0,60]]);;
kash> mat := Mat(Z, [[1,1], [0,1]]);;
kash> hom := AbelianGroupHomCreate(g, g, mat, true);
HomMatrix =
[1 1]
[0 1]
from Group with relations:
[ 0  0]
[ 0 60]
to Group with relations:
[ 0  0]
[ 0 60]

kash> f := AbelianFixPointGroup(hom);
Group with relations:
[60  0]
[ 0  0]
kash> elt := AbelianGroupEltCreate(f, [1,2]);
[1 2]
kash> elt := AbelianGroupEltMove(elt, g);
[120   1]
kash> hom*elt;
[120   1]
```

NAME `AbelianGroupBasis`

PURPOSE Returns a basis of a finite abelian group.

SYNTAX `L := AbelianGroupBasis(g [, exp]);`

`group`	`g`	
`list`	`L`	list of basis elements of g
`boolean`	`exp`	

DESCRIPTION Returns a list of basis elements of the group g. If the second (optional) parameter *exp* is `true` then the discrete exp is applied.

SEE ALSO `AbelianGroupMinNumberGenerators`,

EXAMPLE Compute the basis of a ray class group:

```
kash> O := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> g := RayClassGroupToAbelianGroup(3*O, [1, 2]);
RayClassGroupToAbelianGroup(<3>, [ 1, 2 ])
Group with relations:
[4 0]
[0 2]
kash> l := AbelianGroupBasis(g);
[ [0 1], [1 0] ]
kash> l := AbelianGroupBasis(g, true);
[ <2, [0, 1]>, <[1, 3]> ]
```

NAME	`AbelianGroupCanonicalQuotient`
PURPOSE	Returns the canonical quotient group of a homomorphism.
SYNTAX	`q := AbelianGroupCanonicalQuotient(hom);`

`group`	`q`
`homomorphism`	`hom`

DESCRIPTION Let `hom` be a homomorphism from a group g to a group h. Then `AbelianGroupCanonicalQuotient` returns the quotient group q of g and the kernel of `hom`. All homomorphisms and isomorphisms are installed. This is useful if you frequently consider elements of g, h and the kernel and the image of `hom`.

SEE ALSO `AbelianGroupHomImage`, `AbelianGroupHomKernel`, `AbelianQuotientGroup`,

EXAMPLE Compute the canonical quotient of a homomorphism:

```
kash> g := AbelianGroupCreate([[1,2,3],[2,4,0], [2,0,0]]);;
kash> h := AbelianGroupCreate([[1,0,0],[0,4,0], [1,1,1]]);;
kash> mat := Mat(Z, [[0,0,0],[6,0,-6],[4,0,4]]);;
kash> hom := AbelianGroupHomCreate(g, h, mat, true);
HomMatrix =
[ 0  0  0]
[ 6  0 -6]
[ 4  0  4]
from Group with relations:
[1 2 3]
[2 4 0]
[2 0 0]
to Group with relations:
[1 0 0]
[0 4 0]
[1 1 1]

kash> q := AbelianGroupCanonicalQuotient(hom);
Group with relations:
[1 0 0]
[0 2 0]
[0 0 1]
[1 2 3]
```

```
[2 4 0]
[2 0 0]
kash> AbelianGroupSmithCreate(q);
Group with relations:
[2]
```

Name	`AbelianGroupCreate`
Purpose	Creates a finite abelian group via a given relation matrix.

Syntax	`g := AbelianGroupCreate(mat);`
	`g := AbelianGroupCreate(mat, "*");`
	`g := AbelianGroupCreate(mat, "+");`

group	g
matrix or list of lists	mat

Description	Creates the group `g` with the relation matrix `mat`. Use "*" as second parameter to create a multiplicative group and "+" to create an additive group. This is needed by the discrete exp. The default is "*".
See Also	AbelianSubGroup, AbelianGroupSmithCreate,
Example	

```
kash> g := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
```

Name AbelianGroupCyclicFactors

Purpose Returns cyclic factors and their orders from the Smith normal form.

Syntax `L := AbelianGroupCyclicFactors(g);`

group	g	
list	L	list of cyclic factors of g

Description Returns a list of lists of cyclic factors of the group and the orders of the cyclic factors obtained from the Smith normal form of the relations for g.

See Also AbelianGroupSmithCreate,

Example Compute the cyclic factors of a ray class group:

```
kash> O := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> g := RayClassGroupToAbelianGroup(3*O, [1, 2]);
RayClassGroupToAbelianGroup(<3>, [ 1, 2 ])
Group with relations:
[4 0]
[0 2]
kash> L := AbelianGroupCyclicFactors(g);
[ [ <2, [0, 1]>, 2 ], [ <[1, 3]>, 4 ] ]
```

NAME `AbelianGroupDirectProduct`

PURPOSE Returns the direct product of two finite abelian groups.

SYNTAX
```
d := AbelianGroupDirectProduct(g1, g2);
d := AbelianGroupDirectProduct(L);
```

`groups`	`g1, g2, d`	
`list`	`L`	list of groups

DESCRIPTION In the first case the direct product d of the group $g1$ and the group $g2$ is computed. In the second case the direct product d of all groups in the list is computed.
Instead of using `AbelianGroupDirectProduct` you may use the * operator (see example).

EXAMPLE Compute the direct product using AbelianGroupDirectProduct:

```
kash> g1 := AbelianGroupCreate([[5]]);
Group with relations:
[5]
kash> g2 := AbelianGroupCreate([[7,7],[6,8]]);
Group with relations:
[7 7]
[6 8]
kash> d := AbelianGroupDirectProduct([g1, g2]);
Group with relations:
[5 0 0]
[0 7 7]
[0 6 8]
```

Compute the direct product using *:

```
kash> d := g1*g2;
Group with relations:
[5 0 0]
[0 7 7]
[0 6 8]
```

Compute the power $\bigoplus_{i=1}^{4} g_1$:

```
kash> p := g1^4;
Group with relations:
[5 0 0 0]
[0 5 0 0]
[0 0 5 0]
[0 0 0 5]
```

NAME `AbelianGroupDiscreteExp`

PURPOSE Returns the object corresponding to a representation in a group.

SYNTAX `b := AbelianGroupDiscreteExp(a);`

`group`	`g`	
`group element`	`a`	representation of b in the abstract group g
`object or boolean`	`b`	

DESCRIPTION Returns the concrete object b corresponding to the abstact representation of a in g. For example, if the abstract group g represents a concrete class group of a number field, then `AbelianGroupDiscreteExp(a)` calculates an Ideal `b`, that is representing a class of ideals in the class group. If g is the dual group of a given group, $a \in g$, then b :=`AbelianGroupDiscreteExp(a)` is the homomorphism represented by a. In this form it can directly be used to evaluate element of the original group, to witch g is dual.
If no element is found, $false$ is returned.

SEE ALSO `AbelianGroupDiscreteLog`,

EXAMPLE

```
kash> O := OrderMaximal(x^2-2*x-5);
Generating polynomial: x^2 - 2*x - 5
Discriminant: 24

kash> g := RayResidueRingToAbelianGroup(27*O, [2]);
Group with relations:
[2 0 0 0 0]
[0 3 0 0 0]
[0 0 9 0 0]
[0 0 0 9 0]
[0 0 0 0 2]
kash> a := AbelianGroupEltCreate(g, [1, 2, 3, 0, 0]);
[1 2 3 0 0]
kash> b := AbelianGroupDiscreteExp(a);
[-337479530000, -232975502000]
kash> a := AbelianGroupDiscreteLog(g, b);
[1 2 3 0 0]
```

NAME `AbelianGroupDiscreteLog`

PURPOSE Returns the representation of an object in a group.

SYNTAX `a := AbelianGroupDiscreteLog(g, b);`

`group`	`g`	
`group element`	`a`	representation of b in the abstract group g
`object`	`b`	

DESCRIPTION Returns the abstract representation a (exponent vector corresponding to the generators of g) of an element b in the group g. For example if g represents a class group of a number field, then a given ideal is represented by `AbelianGroupDiscreteLog(g,a)` as an abstract element of this group.

SEE ALSO `AbelianGroupDiscreteExp`,

EXAMPLE

```
kash> O := OrderMaximal(x^2-10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> g := RayResidueRingToAbelianGroup(27*O, [2]);
Group with relations:
[2 0 0 0 0]
[0 9 0 0 0]
[0 0 2 0 0]
[0 0 0 9 0]
[0 0 0 0 2]
kash> a := AbelianGroupEltCreate(g, [1, 2, 3, 0, 0]);
[1 2 3 0 0]
kash> b := AbelianGroupDiscreteExp(a);
[22271000, -10959000]
kash> a := AbelianGroupDiscreteLog(g, b);
[1 2 1 0 0]
```

```
kash> g := RayClassGroupToAbelianGroup(1*O);
RayClassGroupToAbelianGroup(<1>, [  ])
Group with relations:
[2]
kash> I := Ideal(2,Elt(O,[0,1]));
<2, [0, 1]>
kash> b := AbelianGroupDiscreteLog(g,I);
[1]
```

Name `AbelianGroupEltCreate`

Purpose Creates an element of a group.

Syntax `elt := AbelianGroupEltCreate(g, vector);`

`group element`	`elt`
`group`	`g`
`list or matrix with one row of integers`	`vector`

Description Assume that g is generated by $g_1, \ldots, g_r$. Then `vector` contains $v_1, ..., v_r \in \mathbb{Z}$ and the element `elt`$\in g$ is generated as $\prod |i = 1^r g_i^{v_i}$. Only the exponent vector is stored.

See Also `AbelianGroupEltReduce`,

Example Create an element in an abstract group:

```
kash> g := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> elt := AbelianGroupEltCreate(g, [3,2,1]);
[3 2 1]
```

NAME `AbelianGroupEltMove`

PURPOSE Moves an element from a subgroup into a supergroup.

SYNTAX `elt2 := AbelianGroupEltMove(elt1, g);`

`group`	`g`	
`group element`	`elt1`	
`group element`	`elt2`	element of the group g

DESCRIPTION Returns the representation `elt2` of `elt1` in the group g.

EXAMPLE Move an element from a subgroup s of g into g:

```
kash> g := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> s := AbelianSubGroup(g, [3,3,3]);
Group with relations:
[5]
kash> elt1 := AbelianGroupEltCreate(s, [2]);
[2]
kash> elt2 := AbelianGroupEltMove(elt1, g);
[6 6 6]
```

NAME AbelianGroupEltOrder

PURPOSE Returns the order of an element.

SYNTAX `a := AbelianGroupEltOrder(elt);`

group element	elt
integer	a

DESCRIPTION Returns the order of the element `elt`.

SEE ALSO AbelianGroupOrder, AbelianGroupEltCreate,

EXAMPLE

```
kash> g := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> elt := AbelianGroupEltCreate(g, [1,0,0]);
[1 0 0]
kash> a := AbelianGroupEltOrder(elt);
5
```

NAME `AbelianGroupEltRandom`

PURPOSE Generates a random element of an abelian group.

SYNTAX `e := AbelianGroupEltRandom(G);`

AbelianGroupElt	e
AbelianGroup	G

DESCRIPTION

EXAMPLE We'll produce some representatives for random rays modulo 9 in $\mathbb{Q}(\sqrt{10})$:

```
kash> o := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> G := RayClassGroupToAbelianGroup(9*o);
RayClassGroupToAbelianGroup(<9>, [  ])
Group with relations:
[2 0]
[0 3]
kash> for i in [1..10] do
> a := AbelianGroupEltRandom(G);
> Print(a, "\t", AbelianGroupDiscreteExp(a), "\n");
> od;
[0 0] <1>
[1 2] <[-1096, -612], [-3060, -548]>
[0 1] <[-2, -3]>
[1 0] <[-4, -6], [-30, -2]>
[1 1] <[188, 24], [120, 94]>
[1 0] <[-4, -6], [-30, -2]>
[0 0] <1>
[0 0] <1>
[0 1] <[-2, -3]>
[1 1] <[188, 24], [120, 94]>
```

Name `AbelianGroupEltReduce`

Purpose Returns a reduced representation of a group element.

Syntax `elt2 := AbelianGroupEltReduce(elt1 [, positive]);`

`group element`	`elt1`	
`group element`	`elt2`	reduced representation
`boolean`	`positive`	

Description Returns the reduced representation `elt2` of a group element (exponent vector) `elt1`. If the second (optional) parameter is `true` then all entries of `elt2` are non negative. Only in this case it is sure that a good reduction is performed.

Example

```
kash> g := AbelianGroupCreate(Mat(Z, [[2,1], [1,3]]));
Group with relations:
[2 1]
[1 3]
kash> elt1 := AbelianGroupEltCreate(g, [-100, -217]);
[-100 -217]
kash> elt2 := AbelianGroupEltReduce(elt1);
[-100 -217]
kash> AbelianGroupSmithCreate(g);
Group with relations:
[5]
kash> elt2 := AbelianGroupEltReduce(elt1);
[-1  0]
kash> elt2 := AbelianGroupEltReduce(elt1, true);
[4 0]
```

NAME	`AbelianGroupEnumInit`
PURPOSE	Generates a environment for the enumeration of all elements of a finite abelian group G.
SYNTAX	`s := AbelianGroupEnumInit(G [, "gen"]);`

`record`	`s`	
`AbelianGroup`	`G`	
`arbitray`	`gen`	if present, only a set of generators will be enumerated.

DESCRIPTION

SEE ALSO `AbelianGroupEnumNext`,

EXAMPLE See `AbelianGroupEnumNext` for an example.

Name `AbelianGroupEnumNext`

Purpose Computes "the next" element of an abelian group.

Syntax `flag := AbelianGroupEnumNext(s);`

`boolean`	`flag`	`true` iff there is a valid element in `s`
`record`	`s`	generated by `AbelianGroupEnumInit`
`AbelianGroupElt`	`s.elt`	

Description

See Also AbelianGroupEnumInit,

Example We'll produce an ideal in every ray modulo 9 in $\mathbb{Q}(\sqrt{10})$:

```
kash> o := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> G := RayClassGroupToAbelianGroup(9*o);
RayClassGroupToAbelianGroup(<9>, [  ])
Group with relations:
[2 0]
[0 3]
kash> s := AbelianGroupEnumInit(G);
Record of type AbelianGroupEnum

kash> while AbelianGroupEnumNext(s) do
> Print(s.elt, "\t", AbelianGroupDiscreteExp(s.elt), "\n");
> od;
[0 0] <1>
[1 1] <[188, 24], [120, 94]>
[0 2] <[94, 12]>
[1 0] <[-4, -6], [-30, -2]>
[0 1] <[-2, -3]>
[1 2] <[-1096, -612], [-3060, -548]>
```

NAME	`AbelianGroupEqual`
PURPOSE	Tests if two groups are equal.
SYNTAX	`test := AbelianGroupEqual(g1, g2);`

groups	g1, g2
boolean	test

DESCRIPTION	Returns `true` iff the groups `g1` and `g2` are contained in each other. Otherwise `false` is returned.
EXAMPLE	

```
kash> g1 := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> g2 := AbelianSubGroup(g1, [1,1,1]);
Group with relations:
[15]
kash> test := AbelianGroupEqual(g1, g2);
true
```

Note:

```
kash> g1 = g2;
false
```

NAME `AbelianGroupExponent`

PURPOSE Returns the exponent of a group.

SYNTAX `a := AbelianGroupExponent(g);`

group	g
integer	a

DESCRIPTION Returns the exponent a of the group g.

SEE ALSO AbelianGroupOrder, AbelianGroupEltOrder,

EXAMPLE

```
kash> g := AbelianGroupCreate(Mat(Z, [[2,0], [0,4]]));
Group with relations:
[2 0]
[0 4]
kash> a := AbelianGroupExponent(g);
4
```

Name	`AbelianGroupGenerators`
Purpose	Returns generators of a group.
Syntax	`L := AbelianGroupGenerators(g [, exp]);`

`group`	`g`	
`list`	`L`	list of generators of g
`boolean`	`exp`	

Description	Returns a list of generators of the group g. If the second (optional) parameter *exp* is `true` then the discrete exp is applied.
See Also	`AbelianGroupNumberGenerators`, `AbelianGroupBasis`,
Example	

```
kash> O := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> g := RayClassGroupToAbelianGroup(3*O, [1, 2]);
RayClassGroupToAbelianGroup(<3>, [ 1, 2 ])
Group with relations:
[4 0]
[0 2]
kash> l := AbelianGroupGenerators(g);
[ [1 0], [0 1] ]
kash> l := AbelianGroupGenerators(g, true);
[ <2, [0, 1]>, <[1, 3]> ]
```

NAME `AbelianGroupHomCreate`

PURPOSE Creates the homomorphism corresponding to a matrix.

SYNTAX
```
hom := AbelianGroupHomCreate(g1, g2, mat [, check]);
hom := AbelianGroupHomCreate(g1, g2, mat, [matinv]);
```

groups	g1, g2	
homomorphism	hom	homomorphism from $g1$ to $g2$
boolean	check	
matrix	matinv	

DESCRIPTION Creates the homomorphism from group `g1` to group `g2` corresponding to the matrix `mat`. The columns of `mat` are the images of the generators of `g1`. If the fourth (optional) parameter `check` is true then it is checked whether the object `hom` is a homomorphism, i.e. `mat` has the correct number of rows and columns. If the fourth parameter is a matrix then it has to be the inverse of `mat`.

SEE ALSO `AbelianHomGroup`,

EXAMPLE

```
kash> g1 := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);;
kash> g2 := AbelianGroupCreate([[0,2],[3,0]]);;
kash> mat := Mat(Z, [[-24,24],[20,-20],[-10,10]]);;
kash> hom := AbelianGroupHomCreate(g1, g2, mat);
HomMatrix =
[-24  24]
[ 20 -20]
[-10  10]
from Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
to Group with relations:
[0 2]
[3 0]
```

If you feel insecure, you can use the fourth parameter *check* to test, whether the matrix corresponds to a homomorphism:

```
kash> mat := Mat(Z, [[3,0],[3,-2],[-1,2]]);;
kash> hom := AbelianGroupHomCreate(g1, g2, mat, true);
false
```

NAME `AbelianGroupHomImage`

PURPOSE Returns the image of a group homomorphism.

SYNTAX `g := AbelianGroupHomImage(hom [, generators]);`

group	g
homomorphism	hom
boolean	generators

DESCRIPTION Returns the image g of the homomorphism `hom`. g is a group. If the second (optional) parameter `generators` is `true` then the generators of the image g are generated, too.

SEE ALSO `AbelianGroupKernel`, `AbelianQuotientGroup`, `AbelianGroupCanonicalQuotient`,

EXAMPLE

```
kash> g1 := AbelianGroupCreate([[1,2,3],[2,4,0], [2,0,0]]);;
kash> g2 := AbelianGroupCreate([[1,0,0],[0,4,0], [1,1,1]]);;
kash> mat := Mat(Z, [[0,0,0],[6,0,-6],[4,0,4]]);;
kash> hom := AbelianGroupHomCreate(g1, g2, mat, true);
HomMatrix =
[ 0  0  0]
[ 6  0 -6]
[ 4  0  4]
from Group with relations:
[1 2 3]
[2 4 0]
[2 0 0]
to Group with relations:
[1 0 0]
[0 4 0]
[1 1 1]

kash> g := AbelianGroupHomImage(hom);
Group with relations:
[1 0 0]
[0 2 0]
[0 0 1]
kash> AbelianGroupSmithCreate(g);
Group with relations:
[2]
```

NAME `AbelianGroupHomKernel`

PURPOSE Returns the kernel of a group homomorphism.

SYNTAX `g := AbelianGroupHomKernel(hom [,generators]);`

`group`	`g`
`homomorphism`	`hom`
`boolean`	`generators`

DESCRIPTION Returns the kernel g of the homomorphism `hom`. g is a group. If the second (optional) parameter `generators` is `true` then the generators of the kernel g are generated, too.

SEE ALSO `AbelianGroupHomImage`, `AbelianQuotientGroup`, `AbelianGroupCanonicalQuotient`,

EXAMPLE

```
kash> g1 := AbelianGroupCreate([[1,2,3],[2,4,0], [2,0,0]]);;
kash> g2 := AbelianGroupCreate([[1,0,0],[0,4,0], [1,1,1]]);;
kash> mat := Mat(Z, [[0,0,0],[6,0,-6],[4,0,4]]);;
kash> hom := AbelianGroupHomCreate(g1, g2, mat, true);
HomMatrix =
[ 0  0  0]
[ 6  0 -6]
[ 4  0  4]
from Group with relations:
[1 2 3]
[2 4 0]
[2 0 0]
to Group with relations:
[1 0 0]
[0 4 0]
[1 1 1]

kash> g := AbelianGroupHomKernel(hom);
Group with relations:
[1 1 3]
[0 2 0]
[0 0 6]
kash> AbelianGroupSmithCreate(g);
```

```
Group with relations:
[2 0]
[0 6]
```

NAME AbelianGroupIndex

PURPOSE Returns the index of a subgroup in a group.

SYNTAX `a := AbelianGroupIndex(g, s);`

group	g	
group	s	subgroup of g
integer	a	

DESCRIPTION Returns the index a of s in g. s has to be a subgroup of g.

SEE ALSO AbelianSubGroup, AbelianGroupOrder,

EXAMPLE

```
kash> g := AbelianGroupCreate([[0]]);
Group with relations:
[0]
kash> bas := AbelianGroupBasis(g);
[ [1] ]
kash> s := AbelianSubGroup(g, 2*bas[1]);
Group with relations:
[0]
kash> a := AbelianGroupIndex(g, s);
2
```

NAME `AbelianGroupIntersect`

PURPOSE Returns the intersection of two groups.

SYNTAX `g := AbelianGroupIntersect(g1, g2);`

`groups g, g1, g2`

DESCRIPTION Returns the intersection g of the groups $g1, g2$. g is a group.

SEE ALSO `AbelianGroupUnite`,

EXAMPLE

```
kash> g0 := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> g1 := AbelianSubGroup(g0, [1,0,0]);
Group with relations:
[5]
kash> g2 := AbelianSubGroup(g0, [[0,1,0], [0,0,5]]);
Group with relations:
[1 1]
[0 3]
kash> g := AbelianGroupIntersect(g1, g2);
Group with relations:
[1]
```

NAME `AbelianGroupIsAut`

PURPOSE Tests whether a function is a group automorphism.

SYNTAX `L := AbelianGroupIsAut(hom [, check]);`

homomorphism	hom
boolean	check
list	L

DESCRIPTION Tests whether the function `hom` is an automorphism. `AbelianGroupIsAut` returns the list `[true, hom`$^{-1}$`]` iff `hom` is an automorphism, and the list `[false]` otherwise. If the second (optional) parameter `check` is `false` then there is no check, whether `hom` is an endomorphism.

EXAMPLE

```
kash> g := AbelianGroupCreate(Mat(Z, [[2,2], [1,3]]));;
kash> mat := Mat(Z, [[0,2],[0,2]]);;
kash> hom := AbelianGroupHomCreate(g, g, mat, true);;
kash> AbelianGroupIsAut(hom);
[ false ]
```

Further information:

```
kash> AbelianGroupPrintLevel := 1;;
kash> AbelianGroupIsAut(hom);
homomorphism is not surjective
[ false ]
kash> AbelianGroupPrintLevel := 0;;
```

Compute the inverse homomorphism:

```
kash> mat := Mat(Z, [[-1,2],[3,2]]);;
kash> hom := AbelianGroupHomCreate(g, g, mat, true);;
kash> hominv := hom^-1;
HomMatrix =
[  0 1/5]
```

```
[  0 1/5]
from Group with relations:
[2 2]
[1 3]
to Group with relations:
[2 2]
[1 3]
```

Name `AbelianGroupIsSub`

Purpose Tests whether s is a subgroup of g.

Syntax `test := AbelianGroupIsSub(s, g);`

`groups`	`g, s`
`boolean`	`test`

Description Returns `true` iff the group s is a subgroup of g and `false` otherwise.

See Also AbelianSubGroup,

Example

```
kash> g := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> s1 := AbelianSubGroup(g, [1,0,0]);
Group with relations:
[5]
kash> s2 := AbelianSubGroup(g, [[0,1,0], [0,0,5]]);
Group with relations:
[1 1]
[0 3]
kash> AbelianGroupIsSub(s1, g);
true
kash> AbelianGroupIsSub(s1, s2);
false
```

NAME `AbelianGroupMinNumberGenerators`

PURPOSE Returns the minimal number of generators of a group.

SYNTAX `a := AbelianGroupMinNumberGenerators(g);`

`group`	`g`
`integer`	`a`

DESCRIPTION Returns the minimal number a of generators of the group g.

SEE ALSO `AbelianGroupBasis`,

EXAMPLE

```
kash> g := AbelianGroupCreate(Mat(Z, [[2,1], [1,3]]));
Group with relations:
[2 1]
[1 3]
kash> a := AbelianGroupMinNumberGenerators(g);
1
```

NAME `AbelianGroupMultiHomCreate`

PURPOSE Creates a multilinear mapping.

SYNTAX `f := AbelianGroupMultiHomCreate(d, g, t, mat);`

`group`	`d, g, t`
`matrix or list of lists`	`mat`

DESCRIPTION Creates a multilinear mapping from a direct product of groups d to a group g. This multilinear mapping corresponds to the homomorphism from the tensor product t of the groups to the group g with the matrix `mat`. It is recommended to use the second and third entry returned by `AbelianMultiHomGroup` (see example). It is checked if `mat` corresponds to a homomorphism. For a second possibility to create multilinear mappings see `AbelianMultiHomGroup`.

SEE ALSO `AbelianMultiHomGroup`,

EXAMPLE

```
kash> g1 := AbelianGroupCreate(Mat(Z,[[2]]));;
kash> g2 := AbelianGroupCreate(Mat(Z,[[4]]));;
kash> h := AbelianGroupCreate(Mat(Z,[[8,0],[0,2]]));;
kash> l := AbelianMultiHomGroup([g1,g2], h);
[ Group with relations:
    [2 0]
    [0 2], Group with relations:
    [2 0]
    [0 4], Group with relations:
    [2] ]
```

direct product $d = g1 \times g2$:

```
kash> d := l[2];
Group with relations:
[2 0]
[0 4]
```

tensor product $t = g1 \otimes g2$:

```
kash> t := l[3];
Group with relations:
[2]
```

create the multilinear mapping:

```
kash> f := AbelianGroupMultiHomCreate(d, h, t, [[0,1]]);
MultiHom [ 1, 1 ] from Group with relations:
[2 0]
[0 4] to Group with relations:
[8 0]
[0 2]

kash> elt := AbelianGroupEltCreate(d, [1,1]);
[1 1]
kash> AbelianGroupEltReduce(f*elt, true);
[0 1]
kash> f := AbelianGroupMultiHomCreate(d, h, t, [[1,0]]);
false
kash> AbelianGroupPrintLevel := 1;
1
kash> f := AbelianGroupMultiHomCreate(d, h, t, [[1,0]]);
false
kash> AbelianGroupPrintLevel := 0;
0
```

Name `AbelianGroupName`

Purpose Gives a name to a group.

Syntax `AbelianGroupName(g, s);`

`group`	`g`	
`string`	`s`	name of g

Description Gives the name of a string s to a group g. The name of g is shown everytime when g is printed.

See Also `AbelianGroupPrintLevel`,

Example

```
kash> g1 := AbelianGroupCreate(Mat(Z, [[2,1], [1,3]]));
Group with relations:
[2 1]
[1 3]
kash> g2 := AbelianGroupCreate(Mat(Z, [[2,1], [1,3]]));
Group with relations:
[2 1]
[1 3]
kash> AbelianGroupName(g1, "first group");
kash> AbelianGroupName(g2, "second group");
kash> g1;
first group
Group with relations:
[2 1]
[1 3]
kash> g2;
second group
Group with relations:
[2 1]
[1 3]
```

NAME `AbelianGroupNumberGenerators`

PURPOSE Returns the number of generators.

SYNTAX `a := AbelianGroupNumberGenerators(g);`

group	g
integer	a

DESCRIPTION Returns the number of generators of a group g in the current representation.

SEE ALSO `AbelianGroupGenerators`,

EXAMPLE

```
kash> g := AbelianGroupCreate(Mat(Z, [[2,1], [1,3]]));
Group with relations:
[2 1]
[1 3]
kash> a := AbelianGroupNumberGenerators(g);
2
```

NAME `AbelianGroupOrder`

PURPOSE Returns the order of a group.

SYNTAX `a := AbelianGroupOrder(g);`

```
group    g
integer  a
```

DESCRIPTION Returns the order a of the group g.

SEE ALSO `AbelianGroupExponent`,

EXAMPLE

```
kash> g := AbelianGroupCreate(Mat(Z, [[2,1], [1,3]]));
Group with relations:
[2 1]
[1 3]
kash> a := AbelianGroupOrder(g);
5
```

NAME `AbelianGroupPrintLevel`

PURPOSE Sets printlevel.

SYNTAX `AbelianGroupPrintLevel := a;`

`integer a`

DESCRIPTION Sets printlevel for printing groups:

- AbelianGroupPrintLevel := 0; : few information (standard)
- AbelianGroupPrintLevel := 1; : more information
- AbelianGroupPrintLevel := 2; : all information

EXAMPLE Few information (standard):

```
kash> g := AbelianGroupCreate(Mat(Z, [[4,2], [8,2]]));
Group with relations:
[4 2]
[8 2]
kash> AbelianGroupSmithCreate(g);;
kash> h := AbelianGroupHomCreate(g, g, Mat(Z, [[0,1],[1,0]]), true);
false
```

More information:

```
kash> AbelianGroupPrintLevel := 1;;
kash> g;
Group with relations:
[4 2]
[8 2]
exponent   : 4
order      : 8
has SNF    : Group with relations:
[2 0]
[0 4]
kash> h := AbelianGroupHomCreate(g, g, Mat(Z, [[0,1],[1,0]]), true);
false
```

NAME `AbelianGroupSmithCreate`

PURPOSE Returns the smith normal form of a group.

SYNTAX `s := AbelianGroupSmithCreate(g [, generators]);`

`group`	`g, s`
`boolean`	`generators`

DESCRIPTION Returns the smith normal form s of g: the relation matrix of s is in smith normal form. The groups s and g are isomorphic. If the second (optional) parameter `generators` is `true` then the new generators are computed, too. This is useful if you often use the discrete exp.

EXAMPLE

```
kash> g := AbelianGroupCreate([[15,5,-5,10],[0,0,0,50],[6,6,6,6]]);
Group with relations:
[15  5 -5 10]
[ 0  0  0 50]
[ 6  6  6  6]
kash> s := AbelianGroupSmithCreate(g);
Group with relations:
[ 10   0   0]
[  0 300   0]
[  0   0   0]
```

NAME `AbelianGroupTensorProduct`

PURPOSE Returns the tensor product of groups.

SYNTAX `t := AbelianGroupTensorProduct(L);`

`group`	`t`	
`list`	`L`	list of groups

DESCRIPTION Returns the tensor product t of a list L of groups.

EXAMPLE

```
kash> g1 := AbelianGroupCreate(MatDiag(Z, [1,2,3]));
Group with relations:
[1 0 0]
[0 2 0]
[0 0 3]
kash> g2 := AbelianGroupCreate(MatDiag(Z, [4,5]));
Group with relations:
[4 0]
[0 5]
kash> g3 := AbelianGroupCreate(MatDiag(Z, [4, 5, 0]));
Group with relations:
[4 0 0]
[0 5 0]
[0 0 0]
```

Compute the tensor prosuct $t = g1 \otimes g2 \otimes g3$:

```
kash> t := AbelianGroupTensorProduct([g1, g2, g3]);
Group with relations:
[2 0]
[0 2]
```

NAME AbelianGroupUnite

PURPOSE Returns the smallest group containing two given groups.

SYNTAX g := AbelianGroupUnite(g1, g2);

groups g, g1, g2

DESCRIPTION Returns the group g generated by the elements of the groups $g1, g2$.

SEE ALSO AbelianGroupIntersect,

EXAMPLE

```
kash> g0 := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> g1 := AbelianSubGroup(g0, [1,0,0]);
Group with relations:
[5]
kash> g2 := AbelianSubGroup(g0, [[0,1,0], [0,0,5]]);
Group with relations:
[1 1]
[0 3]
kash> g := AbelianGroupUnite(g1, g2);
Group with relations:
[15]
```

NAME `AbelianHomGroup`

PURPOSE Returns the group consisting of homomorphisms between two groups.

SYNTAX `h := AbelianHomGroup(g1, g2);`

`groups  g1, g2, h`

DESCRIPTION Returns the group h consisting of homomorphisms from g_1 to g_2.

SEE ALSO AbelianGroupHomCreate,

EXAMPLE

```
kash> g1 := AbelianGroupCreate([[1,2,3],[2,4,0], [2,0,0]]);;
kash> g2 := AbelianGroupCreate([[1,0,0],[0,4,0], [1,1,1]]);;
kash> h := AbelianHomGroup(g1, g2);
Group with relations:
[2 0]
[0 4]
```

`elt` is an element of h:

```
kash> elt := AbelianGroupEltCreate(h, [0,1]);
[0 1]
```

`hom` is a homomorphism from `g1` to `g2`:

```
kash> hom := AbelianGroupDiscreteExp(elt);
HomMatrix =
[ 0  0  0]
[ 0  3  0]
[ 0 -2  0]
from Group with relations:
[1 2 3]
[2 4 0]
[2 0 0]
to Group with relations:
```

```
[1 0 0]
[0 4 0]
[1 1 1]

kash> elt1 := AbelianGroupEltCreate(g1, [1,1,0]);
[1 1 0]
kash> elt2 := hom*elt1;
[0 3 0]
```

NAME `AbelianMultiHomGroup`

PURPOSE Computes the group of multilinear mappings.

SYNTAX `L := AbelianMultiHomGroup(`$[g_1, \ldots, g_n]$`, h);`

`groups`	$g_1, \ldots, g_n, h$
`list`	`L`

DESCRIPTION Computes the group of multilinear mappings from $d = g_1 \times \cdots \times g_n$ to h. `AbelianMultiHomGroup` returns a list with three entries:

1. group of multilinear mappings from $g_1 \times \cdots \times g_n$ to h.
2. direct product $d = g_1 \times \cdots \times g_n$
3. tensor product $t = g_1 \otimes \cdots \otimes g_n$.

You need d and t only if you want to create multilinear mappings from $g_1 \times \cdots \times g_n$ to h corresponding to a given matrix using the function `AbelianGroupMultiHomCreate`.

SEE ALSO `AbelianGroupMultiHomCreate`,

EXAMPLE

```
kash> z := AbelianGroupCreate(Mat(Z,[[0]]));;
kash> z2 := AbelianGroupCreate(Mat(Z,[[2]]));;
kash> z4 := AbelianGroupCreate(Mat(Z,[[4]]));;
kash> z8 := AbelianGroupCreate(Mat(Z,[[8]]));;
kash> l := AbelianMultiHomGroup([z,z4,z8], z2*z8);
[ Group with relations:
    [2 0]
    [0 4], Group with relations:
    [0 0 0]
    [0 4 0]
    [0 0 8], Group with relations:
    [4] ]
```

mg is the group of multilinear mappings from $d = z \times z4 \times z8$ to $h = z2 \times z8$:

```
kash> mg := l[1];
Group with relations:
[2 0]
[0 4]
```

d is the direct product $z \times z4 \times z8$:

```
kash> d := l[2];
Group with relations:
[0 0 0]
[0 4 0]
[0 0 8]
```

t is the tensor product $z \otimes z4 \otimes z8$:

```
kash> t := l[3];
Group with relations:
[4]
```

elt is element of group of multilinear mappings from $d = z \times z4 \times z8$ to $h = z2 \times z8$:

```
kash> elt := AbelianGroupEltCreate(mg, [1,0]);
[1 0]
```

f is a multilinear mapping from $d = z \times z4 \times z8$ to $h = z2 \times z8$:

```
kash> f := AbelianGroupDiscreteExp(elt);
MultiHom [ 1, 1, 1 ] from Group with relations:
[0 0 0]
[0 4 0]
[0 0 8] to Group with relations:
[2 0]
[0 8]

kash> a := AbelianGroupEltCreate(d, [1,1,1]);
[1 1 1]
kash> f * a;
[1 0]
```

Name `AbelianQuotientGroup`

Purpose Returns the quotient of a group and a subgroup.

Syntax `q := AbelianQuotientGroup(g, s);`

`groups`	`g, q`	
`group`	`s`	subgroup of g

Description Returns the quotient group q of the group g and its subgroup s.

See Also `AbelianGroupCanonicalQuotient`,

Example

```
kash> g := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);
Group with relations:
[0 1 2]
[5 6 0]
[0 4 5]
kash> s := AbelianSubGroup(g, [3,3,3]);
Group with relations:
[5]
kash> q := AbelianQuotientGroup(g, s);
Group with relations:
[3 3 3]
[0 1 2]
[5 6 0]
[0 4 5]
kash> AbelianGroupSmithCreate(q);
Group with relations:
[3]
```

NAME `AbelianRayClassGroupAutoCreate`

PURPOSE Given a field automorphism, this function generates the corresponding automorphism of abelian groups.

SYNTAX `aut := AbelianRayClassGroupAutoCreate(G, sigma);`

`AbelianGroupHom`	`aut`	
`AbelianGroup`	`G`	must be RayClassGroupToAbelianGroup
`KASH function`	`sigma`	acting on ideals

DESCRIPTION

EXAMPLE We demonstrate this in $\mathbb{Q}(\sqrt{10})$:

```
kash> o := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> G := RayClassGroupToAbelianGroup(27*o);
RayClassGroupToAbelianGroup(<27>, [  ])
Group with relations:
[18]
kash> OrderAutomorphisms(o);
[ [0, 1], [0, -1] ]
kash> aut := AbelianRayClassGroupAutoCreate(G, x->IdealAutomorphism(x, 2));
HomMatrix =
[1]
from RayClassGroupToAbelianGroup(<27>, [  ])
Group with relations:
[18]
to RayClassGroupToAbelianGroup(<27>, [  ])
Group with relations:
[18]
```

This should be equivalent to (but is faster for larger groups):

```
kash> l := AbelianGroupGenerators(G);
[ [1] ]
kash> Apply(l, AbelianGroupDiscreteExp);
```

```
kash> Apply(l, x->IdealAutomorphism(x, 2));
kash> Apply(l, x->AbelianGroupDiscreteLog(G, x));
kash> Apply(l, x->x.expvec);
kash> IsMat(l);
true
kash> aut2 := AbelianGroupHomCreate(G, G, l);
HomMatrix =
[1]
from RayClassGroupToAbelianGroup(<27>, [  ])
Group with relations:
[18]
to RayClassGroupToAbelianGroup(<27>, [  ])
Group with relations:
[18]
```

NAME `AbelianRayGroupImbed`

PURPOSE Imbeds a ClassGroup into a RayClassGroup.

SYNTAX `dualhom := AbelianRayGroupImbed(G,g);`

`abstract rayclassgroup`	`G`
`abstract classgroup`	`g`

DESCRIPTION If $I^{(\mathfrak{f})}/P_{\mathfrak{f}}$ is a ray class group with conductor $\mathfrak{f}$ and $I^{\mathfrak{f}'}/U_{\mathfrak{f}'}$ is a classgroup with conductor $\mathfrak{f}'$, then we may imbed $U_{\mathfrak{f}'}$ canonically into $I^{(\mathfrak{f})}/P_{\mathfrak{f}}$: $U_{\mathfrak{f}'}^{(\mathfrak{f})}/P_{\mathfrak{f}}$ is a subgroup of $I^{(\mathfrak{f})}/P_{\mathfrak{f}}$.

SEE ALSO RayClassGroupToAbelianGroup,

EXAMPLE

```
kash> O:=OrderMaximal(x^3-x^2-9*x+8);;
kash> G :=RayClassGroupToAbelianGroup(25*O);;
kash> g:=RayClassGroupToAbelianGroup(1*O);;
kash> u:=AbelianRayGroupImbed(G,g);
Group with relations:
[2 0]
[0 5]
```

NAME AbelianSubGroup

PURPOSE Creates a subgroup.

SYNTAX

```
s := AbelianSubGroup([g,] L [, generators]);
s := AbelianSubGroup(g, mat [, generators]);
```

groups	g, s	
list	L	list of elements of g or list of exponent vectors
matrix	mat	
boolean	generators	

DESCRIPTION Creates a subgroup s of the group g. The subgroup s is generated by the elements of the list `L`. If you use `s := AbelianSubGroup(g, mat);` then the subgroup s is generated by the elements represented by the rows of `mat`. If the last (optional) parameter `generators` is `true` then the new generators are computed, too.

EXAMPLE

```
kash> g := AbelianGroupCreate([[0,1,2],[5,6,0],[0,4,5]]);;
```

Create subgroup with list of exponent vectors:

```
kash> s := AbelianSubGroup(g, [[0,1,0], [0,0,5]]);
Group with relations:
[1 1]
[0 3]
```

Create subgroup with list of group elements:

```
kash> elt1 := AbelianGroupEltCreate(g, [0,1,0]);;
kash> elt2 := AbelianGroupEltCreate(g, [0,0,5]);;
kash> s := AbelianSubGroup([elt1, elt2]);
Group with relations:
[1 1]
[0 3]
```

Create subgroup with matrix:

```
kash> mat := Mat(Z, [[0,1,0], [0,0,5]]);;
kash> s := AbelianSubGroup(g, mat);
Group with relations:
[1 1]
[0 3]
```

NAME `Abs`

PURPOSE Returns the absolute value of a number.

SYNTAX `a := Abs(x);`

```
complex  a
complex  x
```

EXAMPLE

```
kash> Abs(-3);
3
kash> Abs(Comp(1,1));
1.4142135623730950488016887242096980785696718753777
```

NAME `Alff`

PURPOSE Creates an algebraic function field $F/k(T)$.

SYNTAX `F := Alff(f);`

`algebraic function field`	`F`
`polynomial`	`f`

DESCRIPTION Let $f \in k[T, y]$ be an irreducible polynomial which is separable in y, where k is a finite field or a number field (represented by $\mathbb{Q}$ or by number field orders). This function defines the algebraic function field

$$F := k(T, \rho) \quad \text{where} \quad f(T, \rho) = 0.$$

In KASH, F is often also considered as a finite extension of function fields $F/k(T)$ such as in minimal polynomial computations or in determining the places of F over places of $k(T)$. For further information, see the KASH manual.

SEE ALSO `AlffInit`, `AlffOrders`, `AlffGenus`, `AlffPlaceSplit`,

EXAMPLE The definition of an algebraic function field:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^4+1);
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
```

Name `AlffCanonicalDivisor`

Purpose Computes a canonical divisor of an algebraic function field F/k.

Syntax `W := AlffCanonicalDivisor(F);`

`alff divisor`	`W`
`algebraic function field`	`F`

Description Let F/k be an algebraic function field and ω be a differential of F/k. The divisor $W = (\omega)$ of the differential ω is called a canonical divisor of F/k. For W holds deg $W = 2$ g -2 and dim $W = g$, where g is the genus of F/k.

See Also `AlffDifferentDeg`, `AlffDifferent`, `AlffDiffDivisor`, `AlffDivisorLDim`, `AlffGenus`, `AlffInit`, `AlffOrders`,

Example

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> AlffCanonicalDivisor(F);
Alff divisor
[ [ Alff place < [ 1/T, 0, 0, 0 ], [ 0, 1, 0, 0 ] >, -5 ],
[ Alff place < [ T + 1, 0, 0, 0 ], [ 0, 1, 0, 0 ] >, 3 ],
[ Alff place < [ T^6 + 4*T^5 + T^4 + 4*T^3 + T^2 + 4*T + 1, 0, 0, 0 ], [ 0, 1,\
 0, 0 ] >, 3 ] ]
```

NAME `AlffClassGroup`

PURPOSE Computes the structure of the divisor class group of a global function field.

SYNTAX `Cl := AlffClassGroup( F [, b ] [, A ] [, "fast" ] );`

`list`	`Cl`	of integer and list (class group structure)
`alff`	`F`	global function field
`integer`	`b`	bound for degree of places to be used
`alff divisor`	`A`	for reduction

DESCRIPTION Let F/k be a global function field. The divisor class group is the factor group of all divisors factored by the principle divisors. This function returns a list `Cl`: `Cl[1]` is the class number and `Cl[2]` is a list of integers $1 < c_1 \mid \ldots \mid c_m$ such that the class group of F/k is isomorphic to $\mathbb{Z}/c_1\mathbb{Z} \times \cdots \times \mathbb{Z}/c_m\mathbb{Z}$. The function requires that the constant field of definition is the exact constant field.

The following additional parameters can be given:

- b is a bound for the places to be considered in the factor base. For large q or g it can be reasonable to take smaller values for b than returned by `AlffClassGroupGenBoundStrong(F)`.
- A is a reduction divisor. It must factor over the factor base (which is tested automatically) and should have small, positive degree. If the degree $[F : k(T)]$ is large than it can be reasonable to take e.g. `A1 := AlffDivisorDegOne(F)`.
- If b is chosen very small and if the class group is a product of many cyclic factors then the option `"fast"` gives a speedup. The result of the computation however is not proven be correct anymore.

The algorithm is described in [Heß99].

SEE ALSO `AlffClassGroupGenBound`, `AlffClassNumberApprox`, `AlffDivisorDegOne`, `AlffLPoly`, `Alff`, `AlffInit`, `AlffOrders`,

EXAMPLE

```
kash> AlffInit(FF(7));;
kash> AlffOrders(y^2 + T^3 + T + 1);;
kash> AlffClassGroup(F);
[ 11, [ 11 ] ]
```

Name `AlffClassGroupGenBound`

Purpose Returns a bound for the degree of prime divisors which generate the divisor class group.

Syntax

```
b := AlffClassGroupGenBound(q, g);
b := AlffClassGroupGenBound(F);
```

`integer`	`b`	generation bound
`integers`	`q, g`	exact constant field size and genus of F
`alff`	`F`	global function field

Description Let F/k be a global function field of genus g over the exact constant field of q elements, let A_1 be a divisor of degree one and let S be the set of prime divisors of degree $\leq b$. This function computes $b \leq \lceil 2 \log_q(4g-2) \rceil$ without using further information on F such that the classes of the elements of $S \cup \mathrm{supp}(A_1)$ generate the divisor class group of F/k. See [Heß99] for details.

See Also `AlffClassGroupGenBoundStrong`, `AlffClassGroupProdBound`, `AlffClassGroup`, `AlffPlacesNum`, `AlffInit`, `AlffOrders`, `AlffGenus`,

Example

```
kash> AlffInit(FF(7, 2));;
kash> AlffOrders(y^4 + T^7 + T);;
kash> b := AlffClassGroupGenBound(F);
2
kash> Sum([1..b], i->AlffPlacesNum(F, i));
848
```

NAME `AlffClassGroupGenBoundStrong`

PURPOSE Return a bound for the degree of prime divisors which generate the divisor class group.

SYNTAX `b := AlffClassGroupGenBoundStrong(F);`

`integer`	`b`	generation bound
`alff`	`F`	global function field

DESCRIPTION Let F/k be a global function field of genus g over the exact constant field of q elements, let A_1 be a divisor of degree one and let S be the set of prime divisors of degree $\leq b$. This function computes $b \leq \lceil 2\log_q(4g-2)\rceil$ using information on the number of places of F of bounded degree such that the classes of the elements of $S \cup \operatorname{supp}(A_1)$ generate the divisor class group of F/k. See [Heß99] for details.

SEE ALSO `AlffClassGroupGenBound`, `AlffClassGroupProdBound`, `AlffClassGroup`, `AlffInit`, `AlffOrders`,

EXAMPLE

```
kash> AlffInit(FF(7, 2));;
kash> AlffOrders(y^4 + T^7 + T);;
kash> b := AlffClassGroupGenBoundStrong(F);
1
kash> Sum([1..b], i->AlffPlacesNum(F, i));
176
```

NAME `AlffClassGroupGens`

PURPOSE Computes generators of the divisor class group of a global function field.

SYNTAX `L := AlffClassGroupGens( F, [ b, ] [ A, ] [ "fast" ] );`

`list`	`L`	of generating divisors
`alff`	`F`	global function field
`integer`	`b`	bound for degree of places to be used
`alff divisor`	`A`	for reduction

DESCRIPTION Let F/k be a global function field. The divisor class group is the factor group of all divisors factored by the principle divisors. This function returns a list $[A_1, D_1, \ldots, D_m]$ of divisors of F/k such that $\deg(A_1) = 1$ and $\deg(D_i) = 0$. The classes $[D_i]$ correspond to generators of the cyclic factors of the class group of F/k as returned by `AlffClassGroup(F)`. The calling sequence is identical to that of `AlffClassGroup()`.

SEE ALSO `AlffClassGroup`, `AlffClassGroupGenBound`, `AlffClassGroupGenBoundStrongAlffDivisorDegOne`, `AlffInit`, `AlffOrders`,

EXAMPLE

```
kash> AlffInit(FF(7));;
kash> AlffOrders(y^2 + T^3 + T + 1);;
kash> AlffClassGroupGens(F);
[ Alff divisor
    [ [ Alff place < [ T + 1, 0 ], [ 6, 1 ] >, 1 ] ]
    , Alff divisor
    [ [ Alff place < [ T + 6, 0 ], [ 5, 1 ] >, -1 ],
    [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 1 ] ]
     ]
```

NAME `AlffClassGroupPRank`

PURPOSE Computes the p-rank of the class group of a global function field.

SYNTAX `s := AlffClassGroupPRank(F);`

`integer`	`s`
`global function field`	`F`

DESCRIPTION Let F/k be a global function field of characteristic p. Consider the subgroup $Cl^0(F/k)[p]$ of p-torsion elements of the group of divisor classes of degree zero. This function returns its dimension as an $\mathbb{F}_p$-vector space. Possible values range from 0 to g, where g is the genus of F/k.

SEE ALSO `AlffHasseWittInvariant`,

EXAMPLE

```
kash> AlffInit(FF(2, 1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(T*y^3 + y + T^3);
Algebraic function field defined by
$.1^3*$.2 + $.1 + $.2^3
over
Univariate rational function field over GF(2)
Variables: T

kash> AlffClassGroupPRank(F);
1
```

NAME AlffClassNumberApprox

PURPOSE Compute an approximation of the class number of a global function field.

SYNTAX `hbar := AlffClassNumberApprox(F, b);`

real	hbar	approximated class number
alff	F	global function field
integer	b	bound

DESCRIPTION Let F/k be a global function field of genus g over the exact constant field of q elements. If h is the class number of F/k we have (see [Heß99])

$$\left|\log\bigl(h\,/\,q^g\,\bigr) - \sum_{r=1}^{b} \frac{q^{-r}}{r}\bigl(N_r - (q^r+1)\bigr)\right| \le \frac{2g}{q^{1/2}-1}\cdot\frac{q^{-b/2}}{b+1},$$

where N_r is the number of places of degree one in the constant field extension of degree r of F/k. By this formula an approximation of h is computed using the values of N_r for $1 \le r \le b$. The bound b can be determined from a given multiplicative error bound $a \in \mathbb{R}^{>1}$ via `AlffClassNumberApproxBound()`.

SEE ALSO AlffClassNumberApproxBound, AlffClassGroup, AlffLPoly, AlffInit, AlffOrders,

EXAMPLE

```
kash> AlffInit(FF(7, 2));;
kash> AlffOrders(y^4 + T^7 + T);;
kash> b := AlffClassNumberApprox(F, 1);
21306983003206122.33001105081895164915977901789304
kash> b := AlffClassNumberApprox(F, 2);
17731831072348895.78932172128458952555036536781544
```

NAME `AlffClassNumberApproxBound`

PURPOSE Return a bound for the prime divisors to be considered for the approximation of the class number of a global function field.

SYNTAX

```
b := AlffClassNumberApproxBound(q, g, a);
b := AlffClassNumberApproxBound(F, a);
```

`integer`	`b`	approximation bound
`integers`	`q, g`	exact constant field size and genus
`alff`	`F`	global function field
`real`	`a`	$\in \mathbb{R}^{>1}$

DESCRIPTION Let F/k be a global function field of genus g over the exact constant field of q elements. This function computes $b \in \mathbb{Z}$ such that

$$\frac{2g}{q^{1/2}-1} \cdot \frac{q^{-b/2}}{b+1} \; < \; \log(a)/2$$

holds. See function `AlffClassNumberApprox()`.

SEE ALSO `AlffClassNumberApprox`, `AlffClassGroup`, `AlffLPoly`, `AlffInit`, `AlffOrders`,

EXAMPLE

```
kash> AlffInit(FF(7, 2));;
kash> AlffOrders(y^4 + T^7 + T);;
kash> b := AlffClassNumberApproxBound(F, 2);
1
kash> b := AlffClassNumberApproxBound(F, 1.2);
2
```

NAME AlffConstField

PURPOSE Returns the constant field k of definition of an algebraic function field $F/k(T)$.

SYNTAX `k := AlffConstField(F);`

field	k
algebraic function field	F

SEE ALSO AlffDimExactConstField,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffConstField(F);
Finite field of size 5^2
```

Name `AlffConstFieldSize`

Purpose Returns the size of the constant field of definition of a global function field.

Syntax `q := AlffConstFieldSize(F);`

`integer`	`q`	size of constant field
`algebraic function field`	`F`	

See Also `AlffConstField`, `AlffIsGlobal`, `Characteristic`,

Example

```
kash> AlffInit(FF(2,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^3)
Variables: T

kash> AlffConstFieldSize(F);
8
```

Name AlffDeg

Purpose Returns the degree of the extension $F/k(T)$.

Syntax `n := AlffDeg(F);`

integer	n
algebraic function field	F

See Also AlffOrderPoly,

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> F;
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T

kash> AlffDeg(F);
3
```

NAME AlffDiff

PURPOSE Returns the exact differential dh of an alff element h.

SYNTAX `dh := AlffDiff(h);`

alff differential	dh
alff element	h

SEE ALSO AlffDiffDivisor, Alff,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2 + T^3 + 1);
"Defining global variables: F, o, oi, one"
kash> dT := AlffDiff(AlffEltGenT(F));
Alff Differential
[ 1, 0 ] d[ T, 0 ]
kash> dy := AlffDiff(AlffEltGenY(F));
Alff Differential
[ 0, 4*T^2 ] / (T^3 + 1) d[ T, 0 ]
kash> dy / dT;
[ 0, 4*T^2 ] / (T^3 + 1)
kash> dT / dy;
[ 0, 1 ] / T^2
kash> dT + dy;
Alff Differential
[ T^3 + 1, 4*T^2 ] / (T^3 + 1) d[ T, 0 ]
```

NAME `AlffDiffAlff`

PURPOSE Returns the algebraic function field of an alff differential.

SYNTAX `F := AlffDiffAlff(hdx);`

algebraic function field	F
alff differential	hdx

SEE ALSO AlffDiff,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2 + T^3 + 1);
"Defining global variables: F, o, oi, one"
kash> dT := AlffDiff(AlffEltGenT(F));
Alff Differential
[ 1, 0 ] d[ T, 0 ]
kash> AlffDiffAlff(dT);
Algebraic function field defined by
$.1^2 + $.2^3 + 1
over
Univariate rational function field over GF(5)
Variables: T
```

NAME AlffDiffCartier

PURPOSE Applies the Cartier operator.

SYNTAX `hdx := AlffDiffCartier(gdx, r);`

differentials	hdx, gdx	
integer	r	number of iterated applications

DESCRIPTION Let F/k be a global function field, $x \in F$ be a separating variable and $g\,dx \in \Omega(F/k)$ be a differential. The Cartier operator is defined by

$$C(\omega) := \left(-d^{p-1}g/dx^{p-1}\right)^{1/p}dx.$$

This function computes the the rth iterated application of C to $g\,dx$.

SEE ALSO AlffDiff,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1/T, 0, T+1]);
[ 0, 1, 0, T^2 + T ] / T
kash> da := AlffDiff(a);
Alff Differential
[ 0, 2*T^7 + 4, 0, 4*T^8 + T^2 ] / (T^9 + T^2) d[ T, 0, 0, 0 ]
kash> AlffDiffCartier(da, 1);
Alff Differential
[ 0, 0, 0, 0 ] d[ T, 0, 0, 0 ]
kash> gdx := (1/a)*da;
Alff Differential
[ 4*T^17 + 4*T^16 + T^11 + 4*T^9 + 2*T^7 + T^4 + T^3 + 4, 0, 3*T^9 + 2*T^8 + 2\
*T^2 + T, 0 ] / (T^19 + 2*T^18 + T^17 + 2*T^12 + 4*T^11 + 2*T^10 + T^8 + T^5 +\
 2*T^4 + T^3 + T) d[ T, 0, 0, 0 ]
kash> AlffDiffCartier(gdx, 1);
Alff Differential
[ 4*T^17 + 4*T^16 + T^11 + 4*T^9 + 2*T^7 + T^4 + T^3 + 4, 0, 3*T^9 + 2*T^8 + 2\
*T^2 + T, 0 ] / (T^19 + 2*T^18 + T^17 + 2*T^12 + 4*T^11 + 2*T^10 + T^8 + T^5 +\
 2*T^4 + T^3 + T) d[ T, 0, 0, 0 ]
```

NAME `AlffDiffCartierMatrix`

PURPOSE Computes a representation matrix of the Cartier operator.

SYNTAX `M := AlffDiffCartierMatrix(F, r);`

`matrix`	`M`	
`global function field`	`F`	
`integer`	`r`	C is r times applied.

DESCRIPTION Let F/k be a global function field, $\omega_1, \ldots, \omega_g \in \Omega(F/k)$ be a basis for the holomorphic differentials and $r \in \mathbb{Z}^{\geq 1}$. Let $M = (\lambda_{i,j})_{i,j} \in k^{g \times g}$ be the matrix such that

$$C^r(\omega_i) = \sum_{m=1}^{g} \lambda_{i,m}\omega_m$$

for all $1 \leq i \leq g$. This function returns M.

SEE ALSO `AlffDiff`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> AlffDiffCartierMatrix(F, 1);
[0 0 2 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0]
[0 2 0 0 0 0 0 0 0]
[0 0 0 0 4 0 0 0 0]
[4 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 2 0 0]
[0 0 0 0 0 0 0 0 0]
```

NAME `AlffDiffDivisor`

PURPOSE Computes the divisor of an alff differential $h\,dT$.

SYNTAX `D := AlffDiffDivisor(hdT);`

alff divisor	D
alff differential	hdT

SEE ALSO `AlffDivisor`, `AlffDiffSpace`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> dT := AlffDiff(AlffEltGenT(F));
Alff Differential
[ 1, 0, 0, 0 ] d[ T, 0, 0, 0 ]
kash> AlffDiffDivisor(dT);
Alff divisor
[ [ Alff place < [ 1/T, 0, 0, 0 ], [ 0, 1, 0, 0 ] >, -5 ],
[ Alff place < [ T + 1, 0, 0, 0 ], [ 0, 1, 0, 0 ] >, 3 ],
[ Alff place < [ T^6 + 4*T^5 + T^4 + 4*T^3 + T^2 + 4*T + 1, 0, 0, 0 ], [ 0, 1,\
 0, 0 ] >, 3 ] ]
```

Name `AlffDiffFirstKind`

Purpose Computes a basis for the differentials of the first kind (holomorphic differentials) of an algebraic function field F/k.

Syntax `B := AlffDiffFirstKind(F);`

`list`	`B`	of alff differentials
`algebraic function field`	`F`	

See Also `AlffDiffSpace`, `AlffDiffDivisor`,

Example

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> AlffDiffFirstKind(F);
[ Alff Differential
    [ 0, 1, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, T, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, T^2, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, T^3, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, T^4, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, 0, 1, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, 0, T, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, 0, T^2, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, 0, 0, 1 ] / (T^7 + 1) d[ T, 0, 0, 0 ] ]
```

NAME	`AlffDiffResiduum`
PURPOSE	Computes the residuum of a differential.
SYNTAX	`r := AlffDiffResiduum(P, hdT);`

`field element`	`r`	the residuum of hdT at P
`alff place`	`P`	
`alff differential`	`hdT`	

DESCRIPTION

Let F/k be an algebraic function field and $\omega = hdT$ be a differential of F/k. For a place P, which is currently required to be of degree one, this function returns the residuum of ω at P.

SEE ALSO

`AlffDiffDivisor`, `AlffDiffValuation`, `AlffEltResiduum`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> P := AlffPlaceSplit(F, T+3)[1];
Alff place < [ T + 3, 0, 0, 0 ], [ 1, 1, 0, 0 ] >
kash> a := AlffEltGenY(F) + 1;
[ 1, 1, 0, 0 ]
kash> AlffEltValuation(P, a);
1
kash> hdT := (1/a^2)*AlffDiff(a^2);
Alff Differential
[ T^13 + T^6, T^6, 4*T^6, T^6 ] / (T^14 + 3*T^7 + 2) d[ T, 0, 0, 0 ]
kash> AlffDiffResiduum(P, hdT);
2
```

NAME AlffDiffSpace

PURPOSE Computes a basis for the space of Weil differentials

$$\Omega(D) := \{\, \omega \in \Omega(F/k) \mid (\omega) \geq D \,\}$$

for a divisor D of an algebraic function field F/k.

SYNTAX `B := AlffDiffSpace(D);`

`list`	`B`	of alff differentials
`alff divisor`	`D`	

SEE ALSO AlffDiffFirstKind, AlffDiffDivisor,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> D := AlffDivisor(F);
Alff divisor
[  ]

kash> AlffDiffSpace(D);
[ Alff Differential
    [ 0, 1, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, T, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, T^2, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, T^3, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, T^4, 0, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, 0, 1, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, 0, T, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, 0, T^2, 0 ] / (T^7 + 1) d[ T, 0, 0, 0 ], Alff Differential
    [ 0, 0, 0, 1 ] / (T^7 + 1) d[ T, 0, 0, 0 ] ]
```

NAME AlffDiffValuation

PURPOSE Returns the order of a non zero alff differential at a place.

SYNTAX v := AlffDiffValuation(p, fdx);

integer	v
alff place	p
alff differential	fdx

SEE ALSO AlffDiffDivisor, AlffDiff,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2 + T^3 + 1);
"Defining global variables: F, o, oi, one"
kash> dT := AlffDiff(AlffEltGenT(F));
Alff Differential
[ 1, 0 ] d[ T, 0 ]
kash> infty := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> AlffDiffValuation(infty, dT);
-3
```

Name `AlffDifferent`

Purpose Computes the different of an algebraic function field extension $F/k(T)$.

Syntax `D := AlffDifferent(F);`

`alff divisor`	`D`
`algebraic function field`	`F`

Description Let F/k be an algebraic function field. Let P be a place of the rational function field $k(T)$ and P' a place of F over P. This function computes the divisor of F

$$Diff(F/k(T)) = \sum_P \sum_{P'/P} d(P'/P)P'$$

where $d(P'/P)$ is the different exponent of P' over P.

See Also `AlffDifferentDeg`, `AlffCanonicalDivisor`, `AlffDiffDivisor`,

Example

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> AlffDifferent(F);
Alff divisor
[ [ Alff place < [ T + 1, 0, 0, 0 ], [ 0, 1, 0, 0 ] >, 3 ],
[ Alff place < [ T^6 + 4*T^5 + T^4 + 4*T^3 + T^2 + 4*T + 1, 0, 0, 0 ], [ 0, 1,\
 0, 0 ] >, 3 ],
[ Alff place < [ 1/T, 0, 0, 0 ], [ 0, 1, 0, 0 ] >, 3 ] ]
```

NAME `AlffDifferentDeg`

PURPOSE Computes the degree of the different of an algebraic function field extension $F/k(T)$.

SYNTAX `d := AlffDifferentDeg(F);`

integer	d
algebraic function field	F

SEE ALSO AlffDifferent, AlffCanonicalDivisor, AlffDiffDivisor,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> AlffDifferentDeg(F);
24
```

NAME	`AlffDifferentiation`
PURPOSE	Computes higher differentiations of function field elements.
SYNTAX	`b := AlffDifferentiation(p, m, a);`

`alff element`	`b`	m-times differentiated element
`integer`	`m`	
`alff place or element`	`p`	
`alff element`	`a`	element to differentiate

DESCRIPTION

Let F/k be an algebraic function field. For a separating element p and an integer m this function returns the m-th differentiation of $a \in F$. If p is a place, a local parameter of the place as returned by `AlffPlacePrimeElt()` is taken. Remark: The implementation is currently not efficient if m is larger than the positive characteristic.

SEE ALSO

`AlffDiff`, `AlffWronskian`, `AlffRamDivisor`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2 + T^3 + 1);
"Defining global variables: F, o, oi, one"
kash> T := AlffEltGenT(F);
[ T, 0 ]
kash> AlffDifferentiation(T, 1, T);
[ 1, 0 ]
kash> AlffDifferentiation(T, 2, T);
[ 0, 0 ]
kash> AlffDifferentiation(T, 1, T^2);
[ 2*T, 0 ]
kash> AlffDifferentiation(T, 2, T^2);
[ 1, 0 ]
kash> AlffDifferentiation(T, 5, T^5);
[ 1, 0 ]
kash> AlffDifferentiation(T, 5, T^7);
[ T^2, 0 ]
```

NAME AlffDimExactConstField

PURPOSE Returns the k-dimension of the exact constant field $\widetilde{k}$ of an algebraic function field $F/k(T)$.

SYNTAX `l := AlffDimExactConstField(F);`

integer	l
algebraic function field	F

DESCRIPTION The function returns $[\widetilde{k} : k]$, where $\widetilde{k}$ is the algebraic closure of k in F.

SEE ALSO AlffConstField,

EXAMPLE

```
kash> AlffInit(FF(5,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^6 + (2*T^4 + 1)*y^3 + T^8 + T^4 + 2);
"Defining global variables: F, o, oi, one"
kash> AlffDimExactConstField(F);
2
```

Name AlffDivisor

Purpose Creation of divisors.

Syntax

```
D := AlffDivisor(F);
D := AlffDivisor(a);
D := AlffDivisor(P);
D := AlffDivisor(I, J);
D := AlffDivisor(u, v);
```

alff divisor	D	
algebraic function field	F	
alff order element	a	
alff place	P	
alff order ideal	I	of the finite maximal order
alff order ideal	J	of the infinite maximal order
alff order elements	u,v	

Description This function allows the creation of algebraic function field divisors. There are several possibilities for the arguments:

- Given the algebraic function field F the zero divisor is created,
- given an alff order element a its principal divisor is created,
- given an alff place $\mathfrak{P}$ its prime divisor is created,
- given a pair of alff order ideals I, J or alff order elements u, v in the finite or infinite maximal order, the divisor corresponding to their ideal factorizations is created.

See Also AlffPlaceSplit, AlffIdealFactor, AlffEltMove, Alff,

Example

```
kash> AlffInit(FF(5,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^2+T^3+1);
"Defining global variables: F, o, oi, one"
kash> AlffDivisor(F);
Alff divisor
[  ]

kash> a := AlffElt(o, T);
```

```
[ T, 0 ]
kash> AlffDivisor(a);
Alff divisor
[ [ Alff place < [ T, 0 ], [ 2, 1 ] >, 1 ],
[ Alff place < [ T, 0 ], [ 3, 1 ] >, 1 ],
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -2 ] ]

kash> l := AlffPlaceSplit(F, T);
[ Alff place < [ T, 0 ], [ 2, 1 ] >, Alff place < [ T, 0 ], [ 3, 1 ] > ]
kash> D := AlffDivisor(l[1]);
Alff divisor
[ [ Alff place < [ T, 0 ], [ 2, 1 ] >, 1 ] ]

kash> l := AlffPlacesDegOne(F);
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, Alff place < [ T, 0 ], [ 2, 1 ] >,
  Alff place < [ T, 0 ], [ 3, 1 ] >, Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] >,
  Alff place < [ T + 1, 0 ], [ 0, 1 ] > ]
kash> 3*Sum(l) + D;
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 3 ],
[ Alff place < [ T, 0 ], [ 2, 1 ] >, 4 ],
[ Alff place < [ T, 0 ], [ 3, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 4, 1 ] >, 3 ],
[ Alff place < [ T + 1, 0 ], [ 0, 1 ] >, 3 ] ]

kash> 2*D > D;
true
kash> 2*D > D + l[1];
false
```

Name `AlffDivisorAlff`

Purpose Given a divisor this function returns the algebraic function field it belongs to.

Syntax `F := AlffDivisorAlff(D);`

algebraic function field	F
alff divisor	D

See Also AlffOrderAlff, AlffPlaceAlff,

Example

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2)
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ] >
kash> D := AlffDivisor(P);
Alff divisor
[ [ Alff place < [ T + 1, 0, 0 ] >, 1 ] ]

kash> AlffDivisorAlff(D);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2)
Variables: T
```

NAME `AlffDivisorClassRep`

PURPOSE Find the class representation of a divisor class in given generators.

SYNTAX `L := AlffDivisorClassRep(D);`

`list`	`L`	exponents
`alff divisor`	`D`	

DESCRIPTION Let F/k be a global function field and $A_1, D_1, \ldots, D_m$ be a generating system as returned by `AlffClassGroupGens(F)`. Let d_i be the (finite) orders of $[D_i]$. This function returns a list consisting of $a, r_1, \ldots, r_m \in \mathbb{Z}$ with $0 \leq r_i < d_i$ such that

$$[D] = a[A_1] + r_1[D_1] + \cdots + r_m[D_m]$$

holds.

SEE ALSO `AlffClassGroupGens`, `AlffClassGroup`, `AlffInit`, `AlffOrders`,

EXAMPLE

```
kash> AlffInit(FF(7));;
kash> AlffOrders(y^2 + T^3 + T + 1);;
kash> L := AlffClassGroupGens(F);
[ Alff divisor
    [ [ Alff place < [ T + 1, 0 ], [ 6, 1 ] >, 1 ] ]
    , Alff divisor
    [ [ Alff place < [ T + 6, 0 ], [ 5, 1 ] >, -1 ],
    [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 1 ] ]
     ]
kash> D := -17135*L[1] + 15*L[2];
Alff divisor
[ [ Alff place < [ T + 1, 0 ], [ 6, 1 ] >, -17135 ],
[ Alff place < [ T + 6, 0 ], [ 5, 1 ] >, -15 ],
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 15 ] ]

kash> AlffDivisorClassRep(D);
[ -17135, 4 ]
```

Name `AlffDivisorDeg`

Purpose Degree of a divisor.

Syntax `d := AlffDivisorDeg(D);`

integer	d
alff divisor	D

Description This function returns the degree of an alff divisor over the constant field of definition (not the exact constant field) of the algebraic function field.

See Also AlffPlaceDeg, AlffDivisorLDim, AlffGenus,

Example

```
kash> AlffInit(FF(3,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> l := AlffPlaceSplit(F, 1/T);
[ Alff place < [ 1/T, 0, 0 ], [ 2/T, 1/T, (2*T + 2)/T ] >,
  Alff place < [ 1/T, 0, 0 ], [ 2, (2*T + 2)/T, 2 ] > ]
kash> D := Sum(l);
Alff divisor
[ [ Alff place < [ 1/T, 0, 0 ], [ 2/T, 1/T, (2*T + 2)/T ] >, 1 ],
[ Alff place < [ 1/T, 0, 0 ], [ 2, (2*T + 2)/T, 2 ] >, 1 ] ]

kash> AlffDivisorDeg(D);
2
kash> a := AlffElt(o, [0, 1, 2]);
[ 0, 1, 2 ]
kash> AlffDivisorDeg(AlffDivisor(a));
0
```

NAME `AlffDivisorDegOne`

PURPOSE Computes a divisor of degree one for a global function field.

SYNTAX `D := AlffDivisorDegOne(F);`

`alff divisor`	`D`	of degree one
`global function field`	`F`	

DESCRIPTION Given a global function field this function determines a divisor of degree one.

SEE ALSO `AlffPlaces`, `AlffPlacesDegOne`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> AlffDivisorDegOne(F);
Alff divisor
[ [ Alff place < [ T + 1, 0 ], [ 0, 1 ] >, 1 ] ]
```

Name `AlffDivisorDen`

Purpose Returns the denominator of an alff divisor.

Syntax `D2 := AlffDivisorDen(D);`

```
alff divisor  D2
alff divisor  D
```

Description This function returns the denominator of an alff divisor.

See Also AlffDivisorNum,

Example

```
kash> AlffInit(FF(3,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(3)
Variables: T

kash> l := AlffPlaceSplit(F, 1/T);
[ Alff place < [ 1/T, 0, 0 ], [ 2/T, 1/T, (2*T + 2)/T ] >,
  Alff place < [ 1/T, 0, 0 ], [ 2, (2*T + 2)/T, 2 ] > ]
kash> D := l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0, 0 ], [ 2/T, 1/T, (2*T + 2)/T ] >, 1 ],
[ Alff place < [ 1/T, 0, 0 ], [ 2, (2*T + 2)/T, 2 ] >, -1 ] ]

kash> AlffDivisorDen(D);
Alff divisor
[ [ Alff place < [ 1/T, 0, 0 ], [ 2, (2*T + 2)/T, 2 ] >, 1 ] ]
```

NAME `AlffDivisorIdeals`

PURPOSE Divisor corresponding ideals of maximal orders.

SYNTAX `L := AlffDivisorIdeals(D);`

`list`	`L`	of finite and infinite ideals
`alff divisor`	`D`	

DESCRIPTION Given an alff divisor D this function returns the ideals of the finite and infinite maximal order whose ideal factorizations define $-D$ (note the minus sign).

SEE ALSO `AlffDivisor`,

EXAMPLE

```
kash> AlffInit(FF(2,4));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^4)
Variables: T

kash> P := AlffPlaceSplit(F,T+1)[1];
Alff place < [ T + 1, 0, 0 ] >
kash> D := AlffDivisor(P);
Alff divisor
[ [ Alff place < [ T + 1, 0, 0 ] >, 1 ] ]

kash> I := AlffDivisorIdeals(D);
[ <
    [   1    0    0]
    [   0    1    0]
    [   0    0    1]
    / T + 1
     >, < [ 1, 0, 0 ] > ]
```

Name `AlffDivisorLBasis`

Purpose Computes a basis of a Riemann-Roch space

Syntax `B := AlffDivisorLBasis(D);`

`list`	`B`	of basis elements
`alff divisor`	`D`	

Description Given a divisor D of an algebraic function field F this function returns a basis for the k-vector space $\mathcal{L}(D)$, where k denotes the constant field of definition of F. The basis elements are represented in the finite maximal order.

See Also `AlffDivisorLBasisShort`, `AlffDivisorLDim`, `AlffDivisorDeg`, `AlffDivisorLIndex`, `AlffGenus`, `Alff`,

Example

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> infty := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> l := AlffPlaceSplit(F, T+3);
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] > ]
kash> D := 2*infty + 3*l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 2 ],
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 4, 1 ] >, -1 ] ]

kash> AlffDivisorLBasis(D);
[ [ 4*T^3 + 4*T^2 + 2*T + 1, T ] / (T^3 + 4*T^2 + 2*T + 2),
  [ 4*T^4 + 4*T^3 + 2*T^2 + T, T^2 ] / (T^3 + 4*T^2 + 2*T + 2),
  [ 4*T^3 + 2*T^2 + 4, 1 ] / (T^3 + 4*T^2 + 2*T + 2),
  [ 4*T^4 + 2*T^3 + 4*T, T ] / (T^3 + 4*T^2 + 2*T + 2) ]
```

NAME `AlffDivisorLBasis`

PURPOSE Computes a basis of a Riemann-Roch space

SYNTAX `B := AlffDivisorLBasis(D);`

`list`	`B`	of basis elements
`alff divisor`	`D`	

DESCRIPTION Given a divisor D of an algebraic function field F this function returns a basis for the k-vector space $\mathcal{L}(D)$, where k denotes the constant field of definition of F. The basis elements are represented in the finite maximal order.

SEE ALSO `AlffDivisorLBasisShort`, `AlffDivisorLDim`, `AlffDivisorDeg`, `AlffDivisorLIndex`, `AlffGenus`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> infty := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> l := AlffPlaceSplit(F, T+3);
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] > ]
kash> D := 2*infty + 3*l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 2 ],
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 4, 1 ] >, -1 ] ]

kash> AlffDivisorLBasis(D);
[ [ 4*T^3 + 4*T^2 + 2*T + 1, T ] / (T^3 + 4*T^2 + 2*T + 2),
  [ 4*T^4 + 4*T^3 + 2*T^2 + T, T^2 ] / (T^3 + 4*T^2 + 2*T + 2),
  [ 4*T^3 + 2*T^2 + 4, 1 ] / (T^3 + 4*T^2 + 2*T + 2),
  [ 4*T^4 + 2*T^3 + 4*T, T ] / (T^3 + 4*T^2 + 2*T + 2) ]
```

NAME AlffDivisorLBasisShort

PURPOSE Computes a basis of a Riemann-Roch space.

SYNTAX `B := AlffDivisorLBasisShort(D);`

`list`	`B`	of pairs of basis elements b_i and degree bounds d_i
`alff divisor`	`D`	

DESCRIPTION Let $F = k(T, y)$ be an algebraic function field defined by $f(T, y) = 0$ over k. Given a divisor D of F this function returns a basis of the k-vector space

$$\mathcal{L}(D) = \{a \in F^\times \mid (a) \geq -D\} \cup \{0\}$$

in the short form

$$B = [\,[b_1, d_1], \ldots, [b_k, d_k]\,]$$

with $b_i \in F^\times$ and $d_i \in \mathbb{Z}^{\geq 0}$ for all $1 \leq i \leq k$ and with $k \leq n$, where n denotes the degree in y of the defining equation f of F, such that

$$\mathcal{L}(D) = \left\{ \sum_{i=1}^{k} \lambda_i b_i \mid \lambda_i \in k[T] \text{ with } \deg \lambda_i \leq d_i \text{ for } 1 \leq i \leq k \right\}.$$

The basis elements b_i are represented in the finite maximal order.
With option `"raw"` a complete list of $n := \deg_y f$ tuples $[b_i, d_i]$ with $d_i \in \mathbb{Z}$ and the property as above is returned. The algorithm is described in [Heß99].

SEE ALSO AlffDivisorLBasis, AlffDivisorLDim, AlffDivisorLIndex, AlffDivisorDeg, AlffGenus, Alff,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> infty := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> l := AlffPlaceSplit(F, T+3);
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] > ]
kash> D := 2*infty + 3*l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 2 ],
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 4, 1 ] >, -1 ] ]
```

```
kash> AlffDivisorLBasisShort(D);
[ [ [ 4*T^3 + 4*T^2 + 2*T + 1, T ] / (T^3 + 4*T^2 + 2*T + 2), 1 ],
  [ [ 4*T^3 + 2*T^2 + 4, 1 ] / (T^3 + 4*T^2 + 2*T + 2), 1 ] ]
```

NAME `AlffDivisorLDim`

PURPOSE Dimension of a Riemann-Roch space.

SYNTAX `l := AlffDivisorLDim(D);`

```
integer       l
alff divisor  D
```

DESCRIPTION Let $F = k(T, y)$ be an algebraic function field defined by $f(T, y) = 0$ over k. Given a divisor D of F this function returns the dimension of $\mathcal{L}(D)$ as a k-vector space. Note that the field k is the constant field of definition (not the exact constant field) of F.

SEE ALSO `AlffDivisorLBasis`, `AlffDivisorLIndex`, `AlffDivisorDeg`, `AlffGenus`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> infty := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> l := AlffPlaceSplit(F, T+3);
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] > ]
kash> D := 2*infty + 3*l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 2 ],
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 4, 1 ] >, -1 ] ]

kash> AlffDivisorLDim(D);
4
```

NAME AlffDivisorLIndex

PURPOSE Index of speciality of alff divisors.

SYNTAX `i := AlffDivisorLIndex(D);`

```
integer  i
divisor  D
```

DESCRIPTION Let $F = k(T, y)$ be an algebraic function field defined by $f(T, y) = 0$ over k. Given a divisor D of F this function computes the index of speciality $i_k(D) = \dim_k \mathcal{L}(D) - \deg_k D + m * (g-1)$, where g denotes the genus of F and m the dimension of the exact constant field over the constant field of definition k, as dimension over k.

SEE ALSO AlffDivisorLBasis, AlffDivisorLDim, AlffDivisorDeg, AlffGenus, Alff,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> infty := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> l := AlffPlaceSplit(F, T+3);
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] > ]
kash> D := 2*infty + 3*l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 2 ],
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 4, 1 ] >, -1 ] ]

kash> AlffDivisorLIndex(D);
0
```

NAME AlffDivisorLargeLBasisShort

PURPOSE Computes a basis of a large divisor.

SYNTAX `B := AlffDivisorLargeLBasis(D, "raw");`

list	B	the short basis
alff divisor	D	

DESCRIPTION Given a divisor D of an algebraic function field F which exponents are very large this function returns a basis of $\mathcal{L}(D)$ similar to `AlffDivisorLBasisShort()` with option `"raw"`. If the option `"raw"` is given to `AlffDivisorLargeLBasisShort()` the basis elements are represented as power product as in `AlffDivisorReduction()`.

SEE ALSO AlffDivisorLargeLDim, AlffDivisorLBasisShort, AlffDivisorReduction, AlffDivisorLDim, AlffDivisorDeg, AlffDivisorLIndex, AlffGenus, Alff,

EXAMPLE

```
kash> AlffInit(FF(7));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3 + T*y + T^7 + T + 1);
"Defining global variables: F, o, oi, one"
kash> p1 := AlffPlaceSplit(F, T)[1];
Alff place < [ T, 0, 0 ], [ 1, 1, 0 ] >
kash> p2 := AlffPlaceSplit(F, T)[2];
Alff place < [ T, 0, 0 ], [ 2, 1, 0 ] >
kash> p3 := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0, 0 ], [ 2/T, 2, (2*T + 3)/T ] >
kash> D := -p1 - p2 + 2*p3;
Alff divisor
[ [ Alff place < [ T, 0, 0 ], [ 1, 1, 0 ] >, -1 ],
[ Alff place < [ T, 0, 0 ], [ 2, 1, 0 ] >, -1 ],
[ Alff place < [ 1/T, 0, 0 ], [ 2/T, 2, (2*T + 3)/T ] >, 2 ] ]

kash> AlffDivisorDeg(D);
0
kash> AlffDivisorLargeLBasisShort(D, "raw");
[ [ [ [ [ [ 1, 0, 0 ] ], 1 ], [ [ [ T, 0, 0 ] ], 1 ] ], -1 ],
  [ [ [ [ [ 0, 1, 0 ] ], 1 ], [ [ [ T, 0, 0 ] ], 1 ] ], -3 ],
  [ [ [ [ [ 2, 3, 1 ] / T ], 1 ], [ [ [ T, 0, 0 ] ], 1 ] ], -4 ] ]
```

Name `AlffDivisorLargeLDim`

Purpose Computes the dimension of a large divisor.

Syntax `d := AlffDivisorLargeLDim(D);`

`integer`	`d`	the dimension
`alff divisor`	`D`	

Description Given a divisor D of an algebraic function field F which exponents are very large this function returns the dimension of $\mathcal{L}(D)$ much more faster than `AlffDivisorLDim()`.

See Also `AlffDivisorLargeLBasisShort`, `AlffDivisorLBasisShort`, `AlffDivisorReduction`, `AlffDivisorLDim`, `AlffDivisorDeg`, `AlffDivisorLIndex`, `AlffGenus`, `Alff`,

Example

```
kash> AlffInit(FF(7));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3 + T*y + T^7 + T + 1);
"Defining global variables: F, o, oi, one"
kash> p1 := AlffPlaceSplit(F, T)[1];
Alff place < [ T, 0, 0 ], [ 1, 1, 0 ] >
kash> p2 := AlffPlaceSplit(F, T)[2];
Alff place < [ T, 0, 0 ], [ 2, 1, 0 ] >
kash> p3 := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0, 0 ], [ 2/T, 2, (2*T + 3)/T ] >
kash> D := -11107286187918025*p1 -271773078445978235700*p2 +
> 271784185732166153725*p3;
Alff divisor
[ [ Alff place < [ T, 0, 0 ], [ 1, 1, 0 ] >, -11107286187918025 ],
[ Alff place < [ T, 0, 0 ], [ 2, 1, 0 ] >, -271773078445978235700 ],
[ Alff place < [ 1/T, 0, 0 ], [ 2/T, 2, (2*T + 3)/T ] >, 271784185732166153725\
 ] ]

kash> AlffDivisorDeg(D);
0
kash> AlffDivisorLargeLDim(D);
1
```

NAME `AlffDivisorNorm`

PURPOSE Computes the norm of a divisor.

SYNTAX `N := AlffDivisorNorm(D);`

```
qf element    N
alff divisor  D
```

DESCRIPTION Given a divisor D of an algebraic function field F this function returns the norm of D down to the rational subfield $k(x)$ as a rational function.

SEE ALSO `AlffEltGenT`, `AlffIdealNorm`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> infty := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> l := AlffPlaceSplit(F, T+3);
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] > ]
kash> D := 2*infty + 3*l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 2 ],
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 4, 1 ] >, -1 ] ]

kash> AlffDivisorNorm(D);
(T^2 + T + 4)/T^2
```

NAME `AlffDivisorNum`

PURPOSE Returns the numerator of an alff divisor.

SYNTAX `D1 := AlffDivisorNum(D);`

```
alff divisor  D1
alff divisor  D
```

DESCRIPTION This function returns the numerator of an alff divisor.

SEE ALSO `AlffDivisorDen`,

EXAMPLE

```
kash> AlffInit(FF(3,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(3)
Variables: T

kash> l := AlffPlaceSplit(F, 1/T);
[ Alff place < [ 1/T, 0, 0 ], [ 2/T, 1/T, (2*T + 2)/T ] >,
  Alff place < [ 1/T, 0, 0 ], [ 2, (2*T + 2)/T, 2 ] > ]
kash> D := l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0, 0 ], [ 2/T, 1/T, (2*T + 2)/T ] >, 1 ],
[ Alff place < [ 1/T, 0, 0 ], [ 2, (2*T + 2)/T, 2 ] >, -1 ] ]

kash> AlffDivisorNum(D);
Alff divisor
[ [ Alff place < [ 1/T, 0, 0 ], [ 2/T, 1/T, (2*T + 2)/T ] >, 1 ] ]
```

NAME `AlffDivisorPlaces`

PURPOSE Places and exponents of an alff divisor.

SYNTAX `L := AlffDivisorPlaces(D);`

`list`	`L`	of lists of alff places and integers
`alff divisor`	`D`	

DESCRIPTION This function returns a list containing pairs of places and exponents as occuring in a given alff divisor.

SEE ALSO `AlffDivisor`,

EXAMPLE

```
kash> AlffInit(FF(2,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^2+y+T^3+T);
"Defining global variables: F, o, oi, one"
kash> D := AlffDivisor(AlffElt(o, [0, 1]));
Alff divisor
[ [ Alff place < [ T, 0 ], [ 0, 1 ] >, 1 ],
[ Alff place < [ T + 1, 0 ], [ 0, 1 ] >, 2 ],
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -3 ] ]

kash> AlffDivisorPlaces(D);
[ [ Alff place < [ T, 0 ], [ 0, 1 ] >, 1 ],
  [ Alff place < [ T + 1, 0 ], [ 0, 1 ] >, 2 ],
  [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -3 ] ]
```

NAME `AlffDivisorReduction`

PURPOSE Computes a reduced divisor.

SYNTAX
```
L := AlffDivisorReduction(D);
L := AlffDivisorReduction(D, A);
```

`list`	`L`	as above
`alff divisor`	`D`	to reduce
`alff divisor`	`D`	

DESCRIPTION Let A, D be divisors of an algebraic function field F of genus g and $\deg(A) \geq 1$. Let $r \in \mathbb{Z}$ be maximal such that $D = \widetilde{D} + rA - (a)$ with a positive divisor $\widetilde{D}$ of F and $a \in F^{\times}$. Then $\deg(\widetilde{D}) < g + \deg(A)$ and $\widetilde{D}$ is maximally reduced along A. This function returns on input of D and A a list containing $\widetilde{D}$, r, A and a list $((a_{i,j})_j, e_i)_i$, such that $a = \prod_{i,j} a_{i,j}^{e_i}$. If A is omitted, $A := (T)_\infty$ is taken. $\widetilde{D}$ is returned in ideal representation. The algorithm is described in [Heß99].

SEE ALSO `AlffDivisorLargeLDim`,

EXAMPLE

```
kash> AlffInit(FF(7));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3 + T*y + T^7 + T^2 + 1);
"Defining global variables: F, o, oi, one"
kash> p1 := AlffPlaceSplit(F, T)[1];
Alff place < [ T, 0, 0 ], [ 1, 1, 0 ] >
kash> p2 := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ], [ 5, 1, 0 ] >
kash> D := 51*p1 + 37*p2;
Alff divisor
[ [ Alff place < [ T, 0, 0 ], [ 1, 1, 0 ] >, 51 ],
[ Alff place < [ T + 1, 0, 0 ], [ 5, 1, 0 ] >, 37 ] ]

kash> A := 1*AlffPlaceSplit(F, T)[2];
Alff divisor
[ [ Alff place < [ T, 0, 0 ], [ 2, 1, 0 ] >, 1 ] ]

kash> L := AlffDivisorReduction(D, A);
[ <
```

```
    [T^6 + 4*T^5 + 6*T^4 + 5*T^3 + T^2 + 3*T + 4   0   T^5 + 6*T^4 + 5*T^3 + T\
^2 + 5*T]
    [0   T^6 + 4*T^5 + 6*T^4 + 5*T^3 + T^2 + 3*T + 4   6*T^5 + 5*T^3 + 3*T^2 +\
 6*T + 2]
    [0   0   1]
    / T^6 + 4*T^5 + 6*T^4 + 5*T^3 + T^2 + 3*T + 4
     >, <
    [1 0 0]
    [0 1 0]
    [0 0 1]
    / (1)
     > ]
    , 82, Alff divisor
    [ [ Alff place < [ T, 0, 0 ], [ 2, 1, 0 ] >, 1 ] ]
    ,
  [ [ [ [ 2*T^11 + 4*T^10 + 3*T^9 + 6*T^8 + 3*T^7 + 2*T^6 + T^5 + 4*T^4 + 4*T^\
3 + 6*T + 3, 6*T^9 + 4*T^8 + 4*T^7 + 3*T^4 + 6*T^2 + 4, 2*T^7 + T^6 + 4*T^5 + \
5*T^2 + 3 ] / (T^11 + 6*T^10 + 5*T^8 + 2*T^7 + 3*T^6 + 2*T^5 + T^4 + 4*T^2 + T\
) ], 1 ],
      [ [ [ 4*T^11 + 4*T^9 + 6*T^8 + 4*T^7 + 3*T^5 + 5*T^3 + 3*T + 4, T^8 + 5*\
T^7 + 4*T^6 + 4*T^5 + 2*T^2 + 6, 6*T^7 + 4*T^6 + 4*T^5 + 4*T^4 + T^3 + 2*T^2 +\
 4*T + 2 ] / (T^12 + 3*T^11 + T^9 + 3*T^7 + 4*T^6 + 3*T^5 + 4*T^4 + 3*T^3 + T^\
2) ], 2 ],
      [ [ [ 3*T^7 + 5*T^6 + T^5 + 2*T^4 + 3*T^3 + T^2 + 4*T + 6, T^3 + 4*T^2 +\
 2*T + 1, 5*T^2 + 3*T + 6 ] / (T^7 + 3*T^6 + 5*T^5 + 3*T^4 + 6*T^3 + 4*T^2 + 6\
*T + 1) ], 4 ],
      [ [ [ 3*T^10 + 4*T^9 + 3*T^8 + 3*T^7 + T^6 + 2*T^5 + T^4 + 3*T^3 + 1, 5*\
T^7 + 3*T^6 + 2*T^5 + 2*T^4 + 4*T^3 + T^2 + 3*T + 6, 6*T^5 + 2*T^4 + 4*T^2 + 2\
*T + 1 ] / (T^10 + 4*T^9 + 6*T^8 + 4*T^7 + T^6) ], 8 ] ] ]
kash> D - ( L[1] + L[2]*A -
> AlffDivisor(Product(L[4], x->Product(x[1])^x[2])) );
<
[1 0 0]
[0 1 0]
[0 0 1]
/ 1
 >, <
[1 0 0]
[0 1 0]
[0 0 1]
/ (1)
 > ]
```

NAME AlffDivisorSupp

PURPOSE Returns the support of a divisor as a list of places.

SYNTAX `L := AlffDivisorSupp(D);`

list	L	of places
alff divisor	D	

SEE ALSO AlffDivisorPlaces,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> infty := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> l := AlffPlaceSplit(F, T+3);
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] > ]
kash> D := 2*infty + 3*l[1] - l[2];
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 2 ],
[ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
[ Alff place < [ T + 3, 0 ], [ 4, 1 ] >, -1 ] ]

kash> AlffDivisorSupp(D);
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 4, 1 ] > ]
```

NAME `AlffDivisorsSmoothNum`

PURPOSE Compute the number of (n,m)-smooth divisors.

SYNTAX `N := AlffDivisorsSmoothNum(n, m, P);`

`integer`	`N`	number of (n,m)-smooth divisors
`integers`	`n, m`	
`list`	`P`	of integers

DESCRIPTION Given two integers $n, m \geq 0$ and a list P such that $P[i]$ is the number of places of degree $1 \leq i \leq \min\{n,m\}$ of a given global function field, return the number of divisors D which are (n,m)-smooth, i.e. $D \geq 0$ and the support of D consists of places of degree $\leq m$. A recursion from [Heß99] is used.

SEE ALSO `AlffInit`, `AlffOrders`, `AlffPlacesNum`,

EXAMPLE

```
kash> AlffInit(FF(5, 2));;
kash> AlffOrders(y^3+T^3+2);;
kash> P := List([1..3], i->AlffPlacesNum(F, i));
[ 36, 270, 5280 ]
kash> AlffDivisorsSmoothNum(7, 3, P);
2885395249
kash> Zx1 := PolyAlg(Z, "x1");;
kash> Zx1x2 := PolyAlg(Zx1, "x2");;
kash> Zx1x2x3 := PolyAlg(Zx1x2, "x3");;
kash> x1 := Poly(Zx1, [1,0]);;
kash> x2 := Poly(Zx1x2, [1,0]);;
kash> x3 := Poly(Zx1x2x3, [1,0]);;
kash> N := AlffDivisorsSmoothNum(7, 3, [ x1, x2, x3 ]);
(1/2*x1 + 1/2)*x3^2 + (1/2*x2^2 + (1/2*x1^2 + 3/2*x1 + 3/2)*x2 + (1/24*x1^4 + \
5/12*x1^3 + 35/24*x1^2 + 31/12*x1 + 3/2))*x3 + (1/6*x1 + 1/6)*x2^3 + (1/12*x1^\
3 + 1/2*x1^2 + 17/12*x1 + 1)*x2^2 + (1/120*x1^5 + 1/8*x1^4 + 19/24*x1^3 + 19/8\
*x1^2 + 53/15*x1 + 11/6)*x2 + 1/5040*x1^7 + 1/180*x1^6 + 23/360*x1^5 + 7/18*x1\
^4 + 967/720*x1^3 + 469/180*x1^2 + 363/140*x1 + 1
kash> Eval(Eval(Eval(N, P[3]), P[2]), P[1]);
2885395249
```

NAME `AlffDivisorsSmoothRatio`

PURPOSE Return a ratio of numbers of smooth divisors times a number of places.

SYNTAX `r := AlffDivisorsSmoothRatio(n, m, P);`

`real`	`r`	smoothness ratio
`integers`	`n, m`	
`list`	`P`	of integers

DESCRIPTION Given two integers $n, m \geq 0$ and a list P such that $P[i]$ is the number of places of degree $1 \leq i \leq \max\{n, m\}$ of a given global function field, return the ratio of the number of (n, m)-smooth divisors and the number of (n, n)-smooth divisors times the number of places of degree $\leq m$.

SEE ALSO `AlffDivisorsSmoothNum`, `AlffInit`, `AlffOrders`, `AlffPlacesNum`,

EXAMPLE

```
kash> AlffInit(FF(5, 2));;
kash> AlffOrders(y^3+T^3+2);;
kash> P := List([1..3], i->AlffPlacesNum(F, i));
[ 36, 270, 5280 ]
kash> P := Concatenation(P, [ (1/4)*5^8, (1/5)*5^10, (1/6)*5^12,
> (1/7)*5^14 ] );
[ 36, 270, 5280, 390625/4, 1953125, 244140625/6, 6103515625/7 ]
kash> AlffDivisorsSmoothRatio(7, 3, P);
18482.465334371076314196842291951801851740000560032
```

NAME `AlffEllipticFunField`

PURPOSE Creates an elliptic algebraic function field F/k.

SYNTAX `F := AlffEllipticFunField(k, L);`

algebraic function field	F
field	k
list	L

DESCRIPTION Let k be a field. The function creates the algebraic elliptic function field F. The field is defined by the polynomial $y^2+a_1Ty+a_3y-(T^3+a_2T^2+a_4T+a_6)$ with $a_i \in k$. In characteristic 2,3 the user should be carefull

EXAMPLE

```
kash> F := AlffEllipticFunField(FF(2,16), [0,1,1,1,1]);
Algebraic function field defined by
$.1^2 + $.1 + $.2^3 + $.2^2 + $.2 + 1
over
Univariate rational function field over GF(2^16)
Variables: T
```

Name AlffElt

Purpose Creates an element of an algebraic function field $F/k(T)$.

Syntax

```
a := AlffElt(o, s);
a := AlffElt(o, b);
a := AlffElt(o, L);
```

algebraic function field element	a	
algebraic function field order	o	
finite field element, rational or order element	s	a constant
polynomial or quotient field element	b	a rational function
list	L	of above s or b of length n

Description Let o be an alff order in F of rank n with basis $b_1, \ldots, b_n$. Invoked with s or b these arguments will be embedded into $k(T) \subseteq F$. Invoked with a list L $:= [c_1, \ldots, c_n]$ of length n the element $\sum_{i=1}^{n} c_i b_i$ will be returned. The c_i have to be of the same type as s or b.

See Also Alff, AlffOrderEqFinite, AlffOrderEqInfty, AlffOrderMaxFinite, AlffOrderMaxInfty,

Example Define an element of the finite maximal order:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> a^3+T^4+1;
[ 0, 0, 0 ]
```

NAME `AlffEltAlff`

PURPOSE Returns the algebraic function field of an alff element.

SYNTAX `F := AlffEltAlff(a);`

alff	F
alff element	a

SEE ALSO AlffEltOrder,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0,1,0] / (T+1));
[ 0, 1, 0 ] / (T + 1)
kash> AlffEltAlff(a);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2)
Variables: T
```

NAME AlffEltApprox

PURPOSE Computes an approximating element of an algebraic function field F.

SYNTAX `alpha := AlffEltApprox(S, Lambda, A [,a]);`

element	alpha	of the algebraic function field F
list	S	of places of F
alff divisor	A	
list	a	of elements of F

DESCRIPTION Let F be an algebraic function field, $S = \{P_1, \ldots, P_r\}$ a set of r places of F (r a positive integer), $\Lambda = \{n_1, \ldots, n_r\}$ a list of r integers and $a = \{a_1, \ldots, a_r\}$ a list of r elements of F (if a is not given as a parameter, then a_i is assumed be a zero for all i). Let A be a divisor of F whose support $Supp(A)$ is disjoint to S and whose degree is positive. Then this function computes an element α of F with
(1) $v_{P_i}(\alpha - a[i]) = n_i$ $(i = 1, \ldots, r)$ and
(2) $v_P(\alpha) \geq 0$ for all $P \in F$, $P \notin (S \cup Supp(A))$.

EXAMPLE

```
kash> k := FF(3,2);
Finite field of size 3^2
kash> AlffInit(k);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> f := y^3-(T+1)*y^2+2*y*T-T^5;
y^3 + (2*T + 2)*y^2 + 2*T*y + 2*T^5
kash> F:=Alff(f);
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
over
Univariate rational function field over GF(3^2)
Variables: T

kash> AlffOrders(f);
"Defining global variables: F, o, oi, one"
kash> S := [AlffPlaces(F,1)[1],AlffPlaces(F,3)[3],AlffPlaces(F,2)[5]];
[ Alff place < [ 1/T, 0, 0 ], [ 0, 1, 0 ] >, Alff place < [ T + w^5, 0, 0 ] >,
  Alff place < [ T^2 + w*T + 2, 0, 0 ], [ w*T + w^3, 1, 0 ] > ]
kash> Lambda := [-2,3,0];
[ -2, 3, 0 ]
```

```
kash> D := Sum([AlffPlaces(F,3)[2],AlffPlaces(F,3)[4]]);
Alff divisor
[ [ Alff place < [ T + w^3, 0, 0 ] >, 1 ],
[ Alff place < [ T + w^7, 0, 0 ] >, 1 ] ]

kash> Alpha := AlffEltApprox(S, Lambda, D);
[ w^6*T^12 + 2*T^11 + w^5*T^10 + T^9 + w*T^7 + T^6 + w^2*T^4 + w*T^3 + w*T^2 +\
 w^7*T + w^3, T^11 + w^3*T^9 + w^2*T^8 + T^7 + T^6 + w*T^4 + w^3*T^3 + w^6*T^2\
 + w^7*T + w^5, 2*T^10 + T^9 + w^3*T^8 + T^7 + T^6 + w*T^5 + w^5*T^4 + w^2*T^3\
 + w^3*T^2 + w^5*T + w^2 ] / (T^12 + w^2*T^6 + 2)
kash> a := [RandomElt(o,3,3),RandomElt(o,3,3),RandomElt(o,3,3)];
[ [ T^3 + w^6*T + w^7, T^3 + w^3*T + w^3, T^3 + w^6*T^2 + 1 ],
  [ T^3 + w^6*T + w, T^3 + w^6*T^2 + 2*T + w^7, T^3 + 2*T^2 + w^2*T ],
  [ T^3 + T^2 + w^3*T + w^6, T^3 + w^3*T^2 + 2*T + w^7, T^3 + 2*T^2 + w^7*T + \
2 ] ]
kash> Beta := AlffEltApprox(S, Lambda, D, a);
[ T^23 + w^5*T^20 + T^19 + w^2*T^18 + w^6*T^17 + 2*T^16 + w^6*T^15 + w^7*T^14 \
+ w^2*T^12 + w^6*T^10 + w^6*T^9 + 2*T^8 + w^3*T^7 + w^7*T^6 + T^5 + w^2*T^4 + \
w^5*T^3 + w*T^2 + w^2*T + 2, T^23 + 2*T^21 + w^3*T^20 + w^3*T^19 + w^3*T^18 + \
w^2*T^17 + w^6*T^16 + w*T^14 + 2*T^12 + w*T^10 + w*T^9 + w^6*T^8 + w*T^7 + 2*T\
^6 + T^4 + w^6*T^3 + w*T^2 + w^5*T + w^5, T^23 + w^6*T^22 + w^2*T^21 + 2*T^20 \
+ w*T^18 + w^5*T^17 + w*T^16 + T^15 + w*T^14 + w^3*T^13 + w^7*T^12 + w*T^11 + \
w^6*T^10 + w^3*T^9 + w^2*T^8 + w^3*T^7 + w^7*T^6 + w*T^4 + w^5*T^3 + 2*T^2 + w\
^7*T + w^6 ] / (T^20 + w^2*T^18 + w^2*T^2 + 2)
```

Name `AlffEltBstar`

Purpose Computes $B^*(a)$ for a global function field element a.

Syntax `q := AlffEltBstar(a);`

`rational`	`q`
`global function field element`	`a`

Description Let v_i denote the s distinct normalized valuations at the s places of F over the infinite place of $k(T)$, with ramification indices e_i. Let $e = \operatorname{lcm}\{e_1, \ldots, e_s\}$ and define

$$B^* : F \longrightarrow \{a/e \mid a \in \mathbb{Z}\} \cup \{-\infty\} : \alpha \longmapsto -\min_{i=1}^{s} v_i(\alpha)/e_i.$$

This function computes B^* for elements of a global function field F. See [Sch96] for further information.

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 1]);
[ 0, 1, 1 ]
kash> AlffEltBstar(a);
8/3
```

NAME `AlffEltCharPoly`

PURPOSE Returns the characteric polynomial of an algebraic function field element.

SYNTAX `p := AlffEltMinPoly(a);`

`polynomial`	`p`
`alff element`	`a`

DESCRIPTION Given an alff element a in F this function computes the characteristic polynomial of a over $k(T)$.

SEE ALSO `AlffEltCharPoly`,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0,1,0]);
[ 0, 1, 0 ]
kash> p := AlffEltCharPoly(a);
y^3 + T^3*y + T
```

NAME AlffEltDen

PURPOSE Returns the denominator of an algebraic function field element.

SYNTAX `d := AlffEltDen(a);`

```
quotient field element or polynomial  d
algebraic function field element      a
```

DESCRIPTION Let a be defined with respect to the R-order o in $F/k(T)$ of rank n with basis $b_1, \ldots, b_n$, R being $k[T]$ or the valuation ring of the degree valuation of $k(T)$. For $a = \sum_{i=1}^{n} c_i b_i / d$, $c_i, d \in R$ coprime, $(1 \leq i \leq n)$, the function returns d.

SEE ALSO AlffEltNum, AlffEltToList, Num, Den,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> 1/a;
[ 0, 0, 4 ] / (T^4 + 1)
kash> AlffEltDen(1/a);
T^4 + 1
```

Name `AlffEltEval`

Purpose Evaluates an algebraic function at a place.

Syntax `b := AlffEltEval(P, a);`

`field element`	`b`	value of a at P
`alff place`	`P`	
`alff element`	`a`	

Description Let F/k be an algebraic function field and P be a place of F/k. If $\mathfrak{o}_P$ is the ring of algebraic functions defined at P then $k(P) := \mathfrak{o}_P/P$ is the residue class field of P. The image $a(P)$ of an algebraic function $a \in F$ in $k(P)$ or `false` for $a \notin \mathfrak{o}_P$ is returned.

See Also `AlffResidueField`, `AlffEltLift`, `AlffEltValuation`,

Example

```
kash> AlffInit(Q);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^2 - T^4 + T + 1);
"Defining global variables: F, o, oi, one"
kash> P := AlffPlaceSplit(F, T+2)[1];
Alff place < [ T + 2, 0 ] >
kash> K := AlffResidueField(P);
Generating polynomial: x^2 - 17

kash> a := AlffEltGenT(F);
[ T, 0 ]
kash> b := AlffEltEval(P, a);
-2
```

NAME AlffEltGenT

PURPOSE Returns the indeterminate T as maximal order element.

SYNTAX `T := AlffEltGenT(F);`

algebraic function field	F
alff element	T

DESCRIPTION Returns the image of the indeterminate T with respect to the generating equation of the algebraic function field in it's finite maximal order.

SEE ALSO AlffEltGenY, Alff, AlffOrderMaxFinite,

EXAMPLE

```
kash> AlffInit(FF(7,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff((T^2+1)*y^3+y+T^4+1);
Algebraic function field defined by
$.1^3*$.2^2 + $.1^3 + $.1 + $.2^4 + 1
over
Univariate rational function field over GF(7^2)
Variables: T

kash> T := AlffEltGenT(F);
[ T, 0, 0 ]
kash> Y := AlffEltGenY(F);
[ 0, 1, 0 ] / (T^2 + 1)
kash> (T^2+1)*Y^3+Y+T^4+1;
[ 0, 0, 0 ]
```

NAME AlffEltGenY

PURPOSE Returns the indeterminate y as finite maximal order element.

SYNTAX `Y := AlffEltGenY(F);`

algebraic function field	F
alff element	Y

DESCRIPTION Returns the image of the indeterminate y with respect to the generating equation of the algebraic function field in it's finite maximal order.

SEE ALSO AlffEltGenT, Alff, AlffOrderMaxFinite,

EXAMPLE

```
kash> AlffInit(FF(7,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff((T^2+1)*y^3+y+T^4+1);
Algebraic function field defined by
$.1^3*$.2^2 + $.1^3 + $.1 + $.2^4 + 1
over
Univariate rational function field over GF(7^2)
Variables: T

kash> T := AlffEltGenT(F);
[ T, 0, 0 ]
kash> Y := AlffEltGenY(F);
[ 0, 1, 0 ] / (T^2 + 1)
kash> (T^2+1)*Y^3+Y+T^4+1;
[ 0, 0, 0 ]
```

NAME `AlffEltInftyVals`

PURPOSE Computes the inequivalent valuations of a global function field element at infinity.

SYNTAX `L := AlffEltInftyVals(a);`

`list`	`L`
`global function field element`	`a`

DESCRIPTION Denote by v_i the s inequivalent normalized valuations extending the degree valuation of $k(T)$ to F. The function returns $L := [v_1(a), \ldots, v_s(a)]$ in the tame ramified case.

SEE ALSO `AlffPlaceSplit`, `AlffSignature`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> AlffEltInftyVals(a);
[ -4 ]
```

NAME `AlffEltIsInIdeal`

PURPOSE Checks whether a function field element is in an ideal.

SYNTAX `b := AlffEltIsInIdeal(a, I);`

boolean	b
alff order element	a
alff order ideal	I

EXAMPLE Some computations:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> I := T*o;
< [ T, 0, 0 ] >
kash> AlffEltIsInIdeal(T, I);
true
kash> AlffEltIsInIdeal(1/T, I);
false
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> I := a*o;
< [ 0, 1, 0 ] >
kash> AlffEltIsInIdeal(T+w, I);
false
```

NAME AlffEltMin

PURPOSE Returns the denominator d of an alff element a from a form of the representation matrix of $da \in \mathfrak{o}$.

SYNTAX `m := AlffEltMin(a);`

coefficient ring element	m
alff element	a

DESCRIPTION If d is the denominator of an alff element a of the order $\mathfrak{o}$ this function returns the upper left corner element of the upper column Hermite normal form of the representation matrix of $da \in \mathfrak{o}$ on the basis of $\mathfrak{o}$.

SEE ALSO AlffEltRepMat,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0,1,0] / T);
[ 0, 1, 0 ] / T
kash> AlffEltMin(a);
T
```

NAME `AlffEltMinPoly`

PURPOSE Returns the minimal polynomial of an algebraic function field element.

SYNTAX `p := AlffEltMinPoly(a);`

polynomial	p
alff element	a

DESCRIPTION Given an alff element a of F this function computes the minimal polynomial of a over $k(T)$.

SEE ALSO AlffEltCharPoly,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0,1,0]);
[ 0, 1, 0 ]
kash> p := AlffEltMinPoly(a);
y^3 + T^3*y + T
```

NAME `AlffEltMove`

PURPOSE Moves an algebraic function field element between orders.

SYNTAX `a := AlffEltMove(b, o);`

`algebraic function field elements`	`a,b`
`algebraic function field order`	`o`

DESCRIPTION For given $a \in F$ this function returns $b \in F$ with $a = b$ mathematically but b is represented in the basis of o. Thus `AlffEltMove()` is a change of basis.

EXAMPLE Some transfers:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0,1,0]);
[ 0, 1, 0 ]
kash> b := AlffEltMove(a, oi);
[ 0, 1, 0 ] / 1/T^2
kash> AlffEltMove(b, o);
[ 0, 1, 0 ]
```

NAME	`AlffEltNewtonLift`
PURPOSE	Lifts an algebraic element with the Newton lifting method.
SYNTAX	`beta:=AlffEltNewtonLift(o, g, alpha, p, k, den);`

`algebraic function field element`	`beta`
`algebraic function field order`	`o`
`polynomial`	`g`
`algebraic function field element`	`alpha`
`polynomial`	`p`
`integer`	`k`
`polynomial`	`den`

DESCRIPTION Given a function field element α with $\mathtt{g}\alpha \equiv 0 \bmod \mathtt{p}$, this function calculates an algebraic element β with $\mathtt{g}(\beta) \equiv 0 \bmod \mathtt{p}^{2^{\mathtt{k}}}\mathtt{o}$.

NAME AlffEltNorm

PURPOSE Computes the norm of an element of an algebraic function field.

SYNTAX `a := AlffEltNorm(b);`

quotient field element or polynomial	a
algebraic function field element	b

DESCRIPTION Given $a \in F$ this function returns the norm of a over $k(T)$.

SEE ALSO AlffTrace, AlffEltCharPoly, Norm, Trace,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> b := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> AlffEltNorm(b);
4*T^4 + 4
```

NAME `AlffEltNum`

PURPOSE Returns the numerator of an algebraic function field element.

SYNTAX `b := AlffEltNum(a);`

```
algebraic function field element   a
algebraic function field element   b
```

DESCRIPTION Let a be defined with respect to the R-order o in $F/k(T)$ of rank n with basis $b_1, \ldots, b_n$, R being $k[T]$ or the valuation ring of the degree valuation of $k(T)$. For $a = \sum_{i=1}^n c_i b_i / d$, $c_i, d \in R$ coprime, $(1 \le i \le n)$, the function returns $\sum_{i=1}^n c_i b_i / d$.

SEE ALSO `AlffEltDen`, `AlffEltToList`, `Num`, `Den`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> 1/a;
[ 0, 0, 4 ] / (T^4 + 1)
kash> AlffEltNum(1/a);
[ 0, 0, 4 ]
```

NAME	`AlffEltOrder`
PURPOSE	Returns the order with respect to which the element has been defined.
SYNTAX	`o := AlffEltOrder(a);`

algebraic function field order	o
algebraic function field element	a

SEE ALSO AlffElt, AlffEltMove, AlffOrderAlff,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> AlffEltOrder(a);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
```

NAME `AlffEltPthRoot`

PURPOSE Computes the $p^r th$ root of an element of a global function field.

SYNTAX `b := AlffEltPthRoot(a, r);`

alff element	b
global function field	F
positive integer	r

DESCRIPTION Let F/k be a global function field of characteristic p. This function computes the unique p^rth root b of an alff element a in F, or returns false if it does not exist.

SEE ALSO `AlffDivisor`, `AlffDivisorLDim`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^7 + 1);
"Defining global variables: F, o, oi, one"
kash> c := AlffEltGenY(F);
[ 0, 1, 0, 0 ]
kash> a := c^25;
[ 0, T^42 + T^35 + T^7 + 1, 0, 0 ]
kash> AlffEltPthRoot(a, 1);
[ 0, 4*T^7 + 4, 0, 0 ]
kash> AlffEltPthRoot(a, 2);
[ 0, 1, 0, 0 ]
kash> AlffEltPthRoot(a, 3);
false
```

NAME `AlffEltRepMat`

PURPOSE Returns a representation matrix of an algebraic function field element.

SYNTAX `M := AlffEltRepMat(a);`

matrix	M
alff element	a

DESCRIPTION Given an alff element a represented in its order $\mathfrak{o}$ this function computes the representation matrix of a on the basis of $\mathfrak{o}$.

SEE ALSO `AlffEltCharPoly`,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0,1,0] / (T+1));
[ 0, 1, 0 ] / (T + 1)
kash> AlffEltRepMat(a);
[          0           0   T/(T + 1)]
[  1/(T + 1)           0 T^3/(T + 1)]
[          0   1/(T + 1)           0]
```

Name `AlffEltResiduum`

Purpose Computes the residuum of an algebraic function.

Syntax `r := AlffEltResiduum(P, t, a);`

`field element`	`r`	the residuum of a at P for t
`alff place`	`P`	
`alff element`	`t`	a local paramter at P
`alff element`	`a`	an algebraic function

Description Let F/k be an algebraic function field and P be a place of F/k. It is currently required that $\deg(P) = 1$ holds. For a local parameter $t \in F$ at P this function computes the residuum of an algebraic function $a \in F$.

See Also `AlffPlacePrimeElt`, `AlffResidueField`, `AlffEltEval`, `AlffEltLift`, `AlffEltValuation`, `AlffDiffResiduum`,

Example

```
kash> AlffInit(Q);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^2 - T^4 + T + 1);
"Defining global variables: F, o, oi, one"
kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0 ], [ -1, 1 ] >
kash> t := AlffPlacePrimeElt(P);
[ T + 1, 0 ]
kash> AlffEltResiduum(P, t, 1/t^2 + 3 + t);
0
kash> AlffEltResiduum(P, t, 1/t^2 + 2/t + 3 + t);
2
```

NAME AlffEltRoot

PURPOSE Computes an n-th root of a function field element a.

SYNTAX `b := AlffEltRoot(a,n );`

alff element	b
alff element	a
integer	n

SEE ALSO AlffEltPthRoot,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0,1,0] / T);
[ 0, 1, 0 ] / T
kash> AlffEltRoot(a,2);
false
```

NAME AlffEltToList

PURPOSE Returns the coefficients and the denominator of an algebraic function field element.

SYNTAX L := AlffEltToList(a);

list	L
algebraic function field element	a

DESCRIPTION Let a be defined with respect to the R-order o in $F/k(T)$ of rank n with basis $b_1, \ldots, b_n$, R being $k[T]$ or the valuation ring of the degree valuation of $k(T)$. For $a = \sum_{i=1}^{n} c_i b_i / d$, $c_i, d \in R$ coprime, $(1 \leq i \leq n)$, the function returns $L := [[c_1, \ldots, c_n], d]$.

SEE ALSO AlffEltNum, AlffEltDen, AlffElt, Num, Den,

EXAMPLE Decompose an element into a list:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]) / T^7;
[ 0, 1, 0 ] / T^7
kash> AlffEltToList(a);
[ [ 0, 1, 0 ], T^7 ]
```

NAME AlffEltToResField

PURPOSE Returns a representative of the class of an algebraic function in the residue class field of a place.

SYNTAX `b := AlffEltToResField(a, P);`

field element	b	value of a at P
alff element	a	
alff place	P	

DESCRIPTION Let F/k be an algebraic function field and P be a place of F/k. If $\mathfrak{o}_P$ is the ring of algebraic functions defined at P then $k(P) := \mathfrak{o}_P/P$ is the residue class field of P. The image $a(P)$ of an algebraic function $a \in F$ in $k(P)$ or `false` for $a \notin \mathfrak{o}_P$ is returned.

SEE ALSO AlffResFieldEltLift, AlffELtToResField, AlffEltValuation,

EXAMPLE

```
kash> AlffInit(FF(5,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(5^3)
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ], [ 3, 1, 0 ] >
kash> AlffPlaceResField(P);
Finite field of size 5^3
kash> of := AlffOrderMaxFinite(F);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(5^3)
Variables: T

kash> a := AlffEltToResField(AlffElt(of,T),P);
```

```
4
kash> AlffResFieldEltLift(a,P);
[ 4, 0, 0 ]
kash> p := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0, 0 ], [ w^48/T, (w^102*T + w^5)/T, (w^18*T + w^82)/T ] >
kash> AlffPlaceResField(p);
Finite field of size 5^3
kash> oi := AlffOrderMaxInfty(F);
Infinite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(5^3)
Variables: T
given by transformation matrix
[1/T   0   0]
[  0 1/T   0]
[  0   0   1]
with denominator 1/T
kash> b := AlffEltToResField(AlffElt(oi,1/T),p);
0
kash> AlffResFieldEltLift(b,p);
[ 0, 0, 0 ]

kash> AlffInit(Q);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over Rational Field
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ] >
kash> AlffPlaceResField(P);
Generating polynomial: x^3 - x - 1

kash> of := AlffOrderMaxFinite(F);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
```

```
Univariate rational function field over Rational Field
Variables: T

kash> a := AlffEltToResField(AlffElt(of,2),P);
2
kash> AlffResFieldEltLift(a,P);
[ 2, 0, 0 ]
kash> p := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0, 0 ], [ -13/T, (11/3*T - 35)/T, (-3/2*T + 32/3)/T ] >
kash> AlffPlaceResField(p);
Rational Field

kash> oi := AlffOrderMaxInfty(F);
Infinite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over Rational Field
Variables: T
given by transformation matrix
[1/T   0   0]
[  0 1/T   0]
[  0   0   1]
with denominator 1/T
kash> b := AlffEltToResField(AlffElt(oi,1/T),p);
0
kash> AlffResFieldEltLift(b,p);
[ 0, 0, 0 ]
```

NAME AlffEltTrace

PURPOSE Computes the trace of an element of an algebraic function field.

SYNTAX
```
a := AlffEltTrace(b);
```

quotient field element or polynomial	a
algebraic function field element	b

DESCRIPTION Given $a \in F$ this function returns the trace of a over $k(T)$.

SEE ALSO AlffEltNorm,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> b := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> AlffEltTrace(b);
0
```

Name AlffEltValuation

Purpose The valuation of an algebraic function field element at a place.

Syntax `v := AlffEltValuation(P, a);`

integer	v
alff place	P
alff order element	a

Description This function returns the valuation $\nu_{\mathfrak{P}}(a)$ of an algebraic function field order element a at a given place $\mathfrak{P}$.

See Also AlffPlaceSplit, AlffDivisor,

Example Compute a valuation:

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> P := AlffPlaceSplit(F, T)[1];
Alff place < [ T, 0, 0 ], [ 0, 1, 0 ] >
kash> AlffEltValuation(P, a);
1
kash> AlffEltValuation(P, a^(-2));
-2
```

NAME AlffGapNumbers

PURPOSE Returns the gap numbers of a divisor.

SYNTAX

```
L := AlffGapNumbers( D [, P] );
L := AlffGapNumbers( F [, P] );
```

list	L	containing the gap numbers
alff divisor	D	
alff	F	equivalent to taking $D = 0$

DESCRIPTION Let F/k be an algebraic function field, D a divisor and P a place of degree one. An integer $m \geq 1$ is a D-gap number of P if $\dim\bigl(D+(m-1)P\bigr) = \dim(D+mP)$ holds. The D-gap numbers m satisfy $1 \leq m \leq 2g-1-\deg(D)$ and their cardinality equals the index of speciality $i(D)$. If P of degree one is given as argument this function returns the sequence of its D-gap numbers. The sequences of D-gap numbers are independent of constant field extensions for perfect k and are the same for all but a finite number of places P of degree one (consider e.g. k algebraically closed). If P is omitted in the function call, this uniform sequence is returned. The places P which have different sequences of D-gap numbers are called D-Weierstraß places. The constant field k is required to be exact.

SEE ALSO AlffWeierstrassPlaces, AlffWronskian, AlffWronskianOrders, AlffDiff, AlffDifferentiation,

EXAMPLE

```
kash> AlffInit(Q);;
kash> AlffOrders(y^2 - (T-1)*(T-2)*(T-3)*(T-4)*(T-5)*(T-6)*(T-7));
"Defining global variables: F, o, oi, one"
kash> P := AlffPlaceSplit(F, T-1)[1];
Alff place < [ T - 1, 0 ], [ 0, 1 ] >
kash> AlffGapNumbers(F);
[ 1, 2, 3 ]
kash> AlffGapNumbers(F, P);
[ 1, 3, 5 ]
```

NAME	`AlffGenus`
PURPOSE	missing shortdoc
SYNTAX	`g := AlffGenus(F);`

`algebraic function field`	`F`	
`integer`	`g`	the genus

DESCRIPTION	Computes the genus of the algebraic function field.
SEE ALSO	`AlffDimExactConstField`, `Alff`, `AlffDivisorLBasis`,
EXAMPLE	

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffGenus(F);
3
```

NAME AlffHasseWittInvariant

PURPOSE Computes the Hasse-Witt invariant of a global function field.

SYNTAX

```
s := AlffHasseWittInvariant(F);
```

integer	s
global function field	F

DESCRIPTION Let F/k be a global function field of characteristic p. Let $F\bar{k}/\bar{k}$ be the constant field extension by the algebraic closure $\bar{k}$ of k within an algebraic closure $\bar{F}$ of F. Consider the subgroup $Cl^0(F\bar{k}/\bar{k})[p]$ of p-torsion elements of the group of divisor classes of degree zero. This function returns its dimension as an $\mathbb{F}_p$-vector space. Possible values range from 0 to g, where g is the genus of F/k.

SEE ALSO AlffClassGroupPRank,

EXAMPLE

```
kash> AlffInit(FF(2, 1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(T*y^3 + y + T^3);
Algebraic function field defined by
$.1^3*$.2 + $.1 + $.2^3
over
Univariate rational function field over GF(2)
Variables: T

kash> AlffHasseWittInvariant(F);
3
```

NAME `AlffHermitianFunField`

PURPOSE Creates a Hermitian global function field $F/\mathbb{F}_{q^2}(T)$.

SYNTAX `F := AlffHermitianFunField(p, d);`

`global function field`	`F`
`integer`	`p, d`

DESCRIPTION Creates a Hermitian global function field defined by the polynomial $y^q + y = T^{q+1} \in \mathbb{F}_{q^2}[T, y]$ with $q = p^d$.

SEE ALSO `AlffInit`,

EXAMPLE The definition of a Hermitian global function field:

```
kash> F := AlffHermitianFunField(2, 3);
Algebraic function field defined by
$.1^8 + $.1 + $.2^9
over
Univariate rational function field over GF(2^6)
Variables: T
```

Name `AlffHermitianFunField`

Purpose Creates a Hermitian global function field $F/\mathbb{F}_{q^2}(T)$.

Syntax `F := AlffHermitianFunField(p, d);`

`global function field`	`F`
`integer`	`p, d`

Description Creates a Hermitian global function field defined by the polynomial $y^q + y = T^{q+1} \in \mathbb{F}_{q^2}[T, y]$ with $q = p^d$.

See Also `AlffInit`,

Example The definition of a Hermitian global function field:

```
kash> F := AlffHermitianFunField(2, 3);
Algebraic function field defined by
$.1^8 + $.1 + $.2^9
over
Univariate rational function field over GF(2^6)
Variables: T
```

NAME AlffIdeal2EltAssure

PURPOSE Compute a two element representation of an ideal.

SYNTAX `AlffIdeal2EltAssure(I);`

`alff order ideal I`

DESCRIPTION This function tries to represent a given ideal as the sum of two principal ideals (currently only for global function fields).

SEE ALSO AlffIdealBasisUpperHNF,

EXAMPLE

```
kash> AlffInit(FF(5,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+(2*T+4)*y^2+(3*T^2+2*T+2)*y+T^7+T^6+2*T^5+3*T^4);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [ 3, 0, 1 ]);
[ 3, 0, 1 ]
kash> I := (T+2)*o + a*o;
<
[    1     0     0]
[    0 T + 2     0]
[    2     0 T + 2]
/ 1
 >
kash> AlffIdeal2EltAssure(I);
kash> I;
< [ T + 2, 0, 0 ], [ 4*T + 1, 2*T + 4, 3*T + 2 ] >
kash> AlffIdeal2EltAssure(I);
kash> I;
< [ T + 2, 0, 0 ], [ 4*T + 1, 2*T + 4, 3*T + 2 ] >
```

NAME AlffIdealAlff

PURPOSE Returns the algebraic function field of an alff order ideal.

SYNTAX `F := AlffIdealAlff(I);`

alff	F
alff order ideal	I

SEE ALSO AlffIdealOrder,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> I := AlffElt(o, [0,1,0] / (T+1)) * o;
< [ 0, 1, 0 ] / (T + 1) >
kash> AlffIdealAlff(I);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2)
Variables: T
```

NAME	`AlffIdealBasis`
PURPOSE	Returns a list containing the basis elements of an alff order ideal.
SYNTAX	`B := AlffIdealBasis(I);`

`list`	`B`	of basis elements
`alff order ideal`	`I`	

SEE ALSO `AlffOrderBasis`, `AlffIdealBasisUpperHNF`,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> I := AlffElt(o, [0,1,0] / (T+1)) * o;
< [ 0, 1, 0 ] / (T + 1) >
kash> AlffIdealBasis(I);
[ [ T, 0, 0 ] / (T + 1), [ 0, 1, 0 ] / (T + 1), [ 0, 0, 1 ] / (T + 1) ]
```

NAME AlffIdealBasisUpperHNF

PURPOSE Return an ideal basis in upper Hermite normal form.

SYNTAX L := AlffIdealBasisUpperHNF(I);

list	L
alff order ideal	I

DESCRIPTION This function returns a list of two entries. The first entry is a matrix in upper Hermite normal form representing an ideal basis with respect to the order basis. The second entry is the denominator of the ideal.

SEE ALSO AlffIdealBasis, AlffIdeal2EltAssure,

EXAMPLE Some computations:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> I := T*o;
< [ T, 0, 0 ] >
kash> AlffIdealBasisUpperHNF(I);
[ [T 0 0]
    [0 T 0]
    [0 0 T], 1 ]
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> I := a*o;
< [ 0, 1, 0 ] >
kash> AlffIdealBasisUpperHNF(I);
[ [   T^4 + 1          0          0]
    [         0          1          0]
    [         0          0          1], 1 ]
```

NAME `AlffIdealClassGroupUnitsInfty`

PURPOSE Compute the unit group and the ideal class group of the finite maximal order $\mathrm{Cl}(k[T], F)$.

SYNTAX `L := AlffIdealClassGroupUnitsInfty(F);`

`list`	`L`	units and ideal class group info
`alff`	`F`	global function field

DESCRIPTION Let F/k be a global function field and $\mathfrak{o}$ be the finite maximal order $\mathrm{Cl}(k[T], F)$. This function returns a list containing the regulator and the structure of the ideal class group of $\mathfrak{o}$ in the form group order, list of cyclic factors.

SEE ALSO `AlffClassGroup`, `AlffSUnits`, `AlffInit`, `AlffOrders`,

EXAMPLE

```
kash> AlffInit(FF(7));;
kash> AlffOrders(y^2 + T^3 + T + 1);;
kash> AlffPlaceSplitType(F, 1/T);
[ [ 2, 1 ] ]
kash> AlffIdealClassGroupUnitsInfty(F);
[ 1, [ 11, [ 11 ] ] ]
kash> AlffClassGroup(F);
[ 11, [ 11 ] ]
```

NAME `AlffIdealFactor`

PURPOSE Return the factorization of an ideal.

SYNTAX `L := AlffIdealFactor(I);`

`list`	`L`	
`alff order ideal`	`I`	in maximal order

DESCRIPTION This function computes the factorization of an ideal I in its order which must be maximal. The return value is a list of pairs consisting of prime ideals $\mathfrak{P}_i$ and exponents e_i such that $I = \prod \mathfrak{P}_i^{e_i}$.

SEE ALSO AlffIdealValuation, AlffIdealIsPrime,

EXAMPLE Factor some ideals:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffIdealFactor((T+w^3)*o);
[ [ < [ T + w^3, 0, 0 ], [ 0, 1, 0 ] >, 3 ] ]
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> I := (T + w^3)*o + a*o;
<
[ T + w^3        0        0]
[       0        1        0]
[       0        0        1]
/ 1
 >
kash> AlffIdealFactor(I);
[ [ < [ T + w^3, 0, 0 ], [ 0, 1, 0 ] >, 1 ] ]
```

NAME AlffIdealGenerators

PURPOSE Returns a list of generators of an ideal.

SYNTAX `L := AlffIdealGenerators(I);`

ideal	I	
list	L	of two algebraic numbers

SEE ALSO AlffIdealBasis,

EXAMPLE

```
kash> k := FF(3);
Finite field of size 3
kash> AlffInit(k);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> f := y^3-y^2*(T+1)+2*y*T-T^5;
y^3 + (2*T + 2)*y^2 + 2*T*y + 2*T^5
kash> F:=Alff(f);
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
over
Univariate rational function field over GF(3)
Variables: T

kash> AlffOrders(f);
"Defining global variables: F, o, oi, one"
kash> AlffIdealGenerators((1+T^2)*o);
[ [ T^2 + 1, 0, 0 ] ]
```

NAME	`AlffIdealIsPrime`
PURPOSE	missing shortdoc
SYNTAX	`b := AlffIdealIsPrime(I);`

`boolean`	`b`	
`alff order ideal`	`i`	in maximal order

DESCRIPTION	Perform a primality test for an ideal which is defined in a maximal order.
SEE ALSO	`AlffIdealFactor`, `AlffIdealIsPrimeKnown`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3+1);
"Defining global variables: F, o, oi, one"
kash> I := T*o;
< [ T, 0, 0 ] >
kash> AlffIdealIsPrime(I);
false
kash> l := AlffIdealFactor(I);
[ [ < [ T, 0, 0 ], [ 1, 1, 0 ] >, 1 ], [ < [ T, 0, 0 ], [ w^8, 1, 0 ] >, 1 ],
  [ < [ T, 0, 0 ], [ w^16, 1, 0 ] >, 1 ] ]
kash> AlffIdealIsPrime(l[1][1]);
true
```

NAME `AlffIdealIsPrimeKnown`

PURPOSE missing shortdoc

SYNTAX `b := AlffIdealIsPrimeKnown(I);`

`boolean`	`b`	
`alff order ideal`	`I`	in maximal order

DESCRIPTION Return whether an ideal defined in a maximal order is already known to be prime.

SEE ALSO `AlffIdealIsPrime`, `AlffIdealFactor`,

EXAMPLE

```
kash> AlffInit(Q);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> I := (T + 2)*o;
< [ T + 2, 0, 0 ] >
kash> AlffIdealIsPrimeKnown(I);
false
kash> l := AlffIdealFactor(I);
[ [ < [ T + 2, 0, 0 ] >, 1 ] ]
kash> AlffIdealIsPrimeKnown(I);
false
```

NAME `AlffIdealNorm`

PURPOSE missing shortdoc

SYNTAX `a := AlffIdealNorm(I);`

rational function or polynomial	a
alff order ideal	I

DESCRIPTION Compute the norm of an ideal over the rational function field $k(T)$.

SEE ALSO AlffIdealBasisUpperHNF,

EXAMPLE

```
kash> AlffInit(FF(7,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^2+2);
"Defining global variables: F, o, oi, one"
kash> I := T*o;
< [ T, 0, 0 ] >
kash> AlffIdealNorm(I);
T^3
```

NAME `AlffIdealOrder`

PURPOSE missing shortdoc

SYNTAX `o := AlffIdealOrder(a);`

```
alff order        o
alff order ideal  a
```

DESCRIPTION Return the order in which the ideal is defined.

SEE ALSO `AlffOrderAlff`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> I := (T + w^3)*o + a*o;
<
[ T + w^3        0        0]
[       0        1        0]
[       0        0        1]
/ 1
 >
kash> AlffIdealOrder(I);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
```

NAME `AlffIdealPlace`

PURPOSE Given a prime ideal this function returns the corresponding place.

SYNTAX `P := AlffIdealPlace(I);`

prime ideal	I
alff place	P

SEE ALSO AlffPlaceIdeal, AlffPlaces,

EXAMPLE

```
kash> k := FF(3);
Finite field of size 3
kash> AlffInit(k);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> f := y^3-y^2*(T+1)+2*y*T-T^5;
y^3 + (2*T + 2)*y^2 + 2*T*y + 2*T^5
kash> F:=Alff(f);
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
over
Univariate rational function field over GF(3)
Variables: T

kash> AlffOrders(f);
"Defining global variables: F, o, oi, one"
kash> u := AlffElt(o,T);
[ T, 0, 0 ]
kash> ud:=AlffIdealFactor(u*o);
[ [ < [ T, 0, 0 ], [ T, 1, 2*T + 2 ] >, 1 ],
  [ < [ T, 0, 0 ], [ 2*T + 1, T + 2, T ] >, 1 ],
  [ < [ T, 0, 0 ], [ T + 1, 2*T, 2 ] >, 1 ] ]
kash> AlffIdealPlace(ud[1][1]);
Alff place < [ T, 0, 0 ], [ 0, 0, 1 ] >
```

Name	`AlffIdealValuation`
Purpose	Return the valuation of an ideal at a prime ideal.
Syntax	`v := AlffIdealValuation(P, I);`

`integer`	`v`	
`alff order ideals`	`P, I`	in maximal order o

Description	This function returns the valuation of an ideal at a prime ideal. They have to be defined in the same order which must be maximal (i. e. a Dedekind ring).
See Also	`AlffIdealFactor`, `AlffIdealIsPrime`,

Example Compute a valuation:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a := AlffElt(o, [0, 1, 0]);
[ 0, 1, 0 ]
kash> I := (T + w^3)*o + a*o;
<
[ T + w^3        0        0]
[       0        1        0]
[       0        0        1]
/ 1
 >
kash> P := AlffIdealFactor(I)[1][1];
< [ T + w^3, 0, 0 ], [ 0, 1, 0 ] >
kash> AlffIdealValuation(P, I^(-2));
-2
```

NAME AlffIharaBound

PURPOSE missing shortdoc

SYNTAX
```
b := AlffIharaBound(F);
b := AlffIharaBound(q, g);
```

global function field	F
integer	q, g
integer	b

DESCRIPTION Computes the Ihara bound for a global function field over the exact constant field $\mathbb{F}_q$ for the number of places of degree one from the function field or for given q and g, where q is the size of the finite field and g denotes the genus.

SEE ALSO AlffSerreBound, AlffPlacesDegOneNumBound,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3 + T^3 * y + T);
"Defining global variables: F, o, oi, one"
kash> AlffGenus(F);
3
kash> AlffIharaBound(F);
8
kash> AlffIharaBound(2, 1);
5
```

NAME `AlffInit`

PURPOSE Initializes some useful variables for an algebraic function field session.

SYNTAX
```
AlffInit(k);
AlffInit(k, T);
AlffInit(k, T, y);
```

`field`	`k`	
`strings`	`T, y`	the variable names

DESCRIPTION This function is used for convenience to initialize some useful variables for an algebraic function field session and is not recommended for use in programming. It initializes:

- `k`, the constant field,
- `w`, a generator of the multiplicative group of k if finite,
- `kT`, the polynomial ring $k[T]$,
- `kTf`, the field $k(T)$,
- `kTy`, the polynomial ring $k[T][y]$,
- `T`, the variable T,
- `y`, the variable y,
- `AlffGlobals`, a record containing all the above values.

The optional two parameter strings define the names of variables T and y.

SEE ALSO `AlffOrders`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(5, 2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> k;
Finite field of size 5^2
kash> kTy;
Univariate Polynomial Ring in y over Univariate Polynomial Ring in T over GF(5\
^2)

kash> y^4+T^3+2;
y^4 + T^3 + 2
```

NAME `AlffInit`

PURPOSE Initializes some useful variables for an algebraic function field session.

SYNTAX
```
AlffInit(k);
AlffInit(k, T);
AlffInit(k, T, y);
```

`field`	`k`	
`strings`	`T, y`	the variable names

DESCRIPTION This function is used for convenience to initialize some useful variables for an algebraic function field session and is not recommended for use in programming. It initializes:

- `k`, the constant field,
- `w`, a generator of k over its base field if finite,
- `kT`, the polynomial ring $k[T]$,
- `kTf`, the field $k(T)$,
- `kTy`, the polynomial ring $k[T][y]$,
- `T`, the variable T,
- `y`, the variable y,
- `AlffGlobals`, a record containing all the above values.

The optional two parameter strings define the names of variables T and y.

SEE ALSO `AlffOrders`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(5, 2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> k;
Finite field of size 5^2
kash> kTy;
Univariate Polynomial Ring in y over Univariate Polynomial Ring in T over GF(5\
^2)

kash> y^4+T^3+2;
y^4 + T^3 + 2
```

NAME `AlffIsAbs`

PURPOSE Returns whether a function field is an absolute extension.

SYNTAX `b := AlffIsAbs(F);`

boolean	b
function field	F

DESCRIPTION

EXAMPLE

```
kash> k := FF(3);
Finite field of size 3
kash> AlffInit(k);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> f := y^3-y^2*(T+1)+2*y*T-T^5;
y^3 + (2*T + 2)*y^2 + 2*T*y + 2*T^5
kash> F:=Alff(f);
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
over
Univariate rational function field over GF(3)
Variables: T

kash> AlffIsAbs(F);
true
kash> AlffOrders(f);
"Defining global variables: F, o, oi, one"
kash> a1:=AlffElt(o,[1,3,T]);
[ 1, 0, T ]
kash> a2:=AlffElt(o,[T,T^2,0]);
[ T, T^2, 0 ]
kash> a3:=AlffElt(o,[1,3,0]);
[ 1, 0, 0 ]
kash> L:=[a3,a2,a1];
[ [ 1, 0, 0 ], [ T, T^2, 0 ], [ 1, 0, T ] ]
kash> A:=PolyAlg(o);
Univariate Polynomial Ring in x over Finite maximal order of
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
```

```
over
Univariate rational function field over GF(3)
Variables: T
given by transformation matrix
[T 0 0]
[0 T 2]
[0 0 1]
with denominator T

kash> g:=Poly(A,L);
x^2 + [ T, T^2, 0 ]*x + [ 1, 0, T ]
kash> G:=Alff(g);
Algebraic function field defined by
$.1^2 + $.1*$.2*$.3^2 + $.1*$.3 + $.2^2 + 2*$.2 + 1
over
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
over
Univariate rational function field over GF(3)
Variables: T

kash> AlffIsAbs(G);
false
```

Name AlffIsGlobal

Purpose Returns whether an algebraic function field F is global.

Syntax `b := AlffIsGlobal(F);`

boolean	b	whether global or not
algebraic function field	F	

See Also AlffIsGlobalAssert, AlffConstField, Characteristic,

Example

```
kash> AlffInit(Q);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over Rational Field
Variables: T

kash> AlffIsGlobal(F);
false
```

NAME `AlffIsGlobalAssert`

PURPOSE Signals an error if an algebraic function field F is not global.

SYNTAX `AlffIsGlobalAssert(F);`

algebraic function field F

SEE ALSO AlffIsGlobal, AlffConstField, Characteristic,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2)
Variables: T

kash> AlffIsGlobalAssert(F);
```

Name AlffLPoly

Purpose Computes the L-polynomial of a global function field.

Syntax `L := AlffLPoly(F);`

polynomial	L
global function field	F

Description This function computes the L-polynomial of a global function field F/k. The algorithm used is exponential in the genus of F/k. The constant field of definition of F/k has to be exact.

See Also AlffPlacesDegOneNum, AlffPlacesDegOne,

Example

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^4 + 1);
"Defining global variables: F, o, oi, one"
kash> AlffLPoly(F);
125*x^6 - 150*x^5 + 135*x^4 - 68*x^3 + 27*x^2 - 6*x + 1
```

NAME AlffLPolyLift

PURPOSE Lifts the L-polynomial of a global function field to constant field extensions.

SYNTAX `Lr := AlffLPolyLift(L, r);`

polynomial	Lr, L
integer	r

DESCRIPTION This function computes the L-polynomial of a constant field extension of a global function field F/k of degree r from the L-polynomial L of F/k.

SEE ALSO AlffLPoly, AlffPlacesDegOne,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^4 + 1);
"Defining global variables: F, o, oi, one"
kash> L := AlffLPoly(F);
125*x^6 - 150*x^5 + 135*x^4 - 68*x^3 + 27*x^2 - 6*x + 1
kash> AlffLPolyLift(L, 2);
15625*x^6 + 11250*x^5 + 4575*x^4 + 1116*x^3 + 183*x^2 + 18*x + 1
kash> AlffInit(FF(5,2));;
kash> AlffOrders(y^4 + T^4 + 1);
"Defining global variables: F, o, oi, one"
kash> AlffLPoly(F);
15625*x^6 + 11250*x^5 + 4575*x^4 + 1116*x^3 + 183*x^2 + 18*x + 1
```

NAME	`AlffLPolyRed`
PURPOSE	Computes the L-polynomial of a global function field.
SYNTAX	`L := AlffLPolyRed(F);` `L := AlffLPolyRed(F, "nocheck");`

polynomial	L
F global function field	F

DESCRIPTION This function computes the L-polynomial of a global function field of characteristic p modulo p. The constant field of definition of F/k has to be exact and F/k must contain a non special system of places. If the parameter `"nocheck"` is given, the existence of the non special system is not checked (hence the resulting L-polynomial may be false).

SEE ALSO `AlffLPolyRed`,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^4 + T^4 + T + 1);
"Defining global variables: F, o, oi, one"
kash> AlffLPolyRed(F);
3*x^3 + 4*x^2 + 4*x + 1
kash> AlffLPoly(F) mod 5;
-2*x^3 - x^2 - x + 1
```

NAME AlffLinearSeriesEnumElt

PURPOSE Return the current divisor in the series enumeration.

SYNTAX `E := AlffLinearSeriesEnumElt(env);`

alff divisor	E	current divisor
record	env	of enumeration data

DESCRIPTION Let F/k be a global function field and D a divisor. Let $v_1, \ldots, v_l$ be k-linearly independent elements of F. This function returns the (according to an internal counter) current divisor $D + \left(\sum_{i=1}^{l} \lambda_i v_i\right)$ in the linear series enumeration.

SEE ALSO AlffLinearSeriesEnumNext, AlffLinearSeriesEnumEnv,

EXAMPLE

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> D := 1*AlffPlaceRandom(F, 3);
Alff divisor
[ [ Alff place < [ T^3 + 4*T^2 + T + 2, 0 ], [ 3*T + 1, 1 ] >, 1 ] ]

kash> B := AlffDivisorLBasis(D);
[ [ 1, 0 ], [ 2*T + 4, 1 ] / (T^3 + 4*T^2 + T + 2),
  [ 2*T^2 + 4*T, T ] / (T^3 + 4*T^2 + T + 2) ]
kash> env := AlffLinearSeriesEnumEnv(D, B);
rec(
  ffelts := [ 0, 2, 4, 3, 1 ],
  D := Alff divisor
    [ [ Alff place < [ T^3 + 4*T^2 + T + 2, 0 ], [ 3*T + 1, 1 ] >, 1 ] ]
    ,
  basis := [ [ 1, 0 ], [ 2*T + 4, 1 ] / (T^3 + 4*T^2 + T + 2),
      [ 2*T^2 + 4*T, T ] / (T^3 + 4*T^2 + T + 2) ],
  dim := 3 )
kash> AlffLinearSeriesEnumNext(env);
true
kash> AlffLinearSeriesEnumElt(env);
<
[1 0]
[0 1]
/ T
```

```
 >, <
[1/T   0]
[  0   1]
/ (1/T)
 > ]

kash> AlffLinearSeriesEnumNext(env);
true
kash> AlffLinearSeriesEnumElt(env);
<
[T^3 + 2       4]
[      0       1]
/ T^3 + 2
 >, <
[1 0]
[0 1]
/ (1)
 > ]
```

Name `AlffLinearSeriesEnumEnv`

Purpose Prepare for enumeration of a linear series over a finite field.

Syntax `env := AlffLinearSeriesEnumEnv(D, B);`

`record`	`env`	of enumeration data
`alff divisor`	`D`	
`list`	`B`	of $v_1, \ldots, v_l$

Description Let F/k be a global function field and D a divisor. Let $v_1, \ldots, v_l$ be k-linearly independent elements of F. This function returns a record which is used by `AlffLinearSeriesEnumNext()` and `AlffLinearSeriesEnumElt()` to enumerate successively the divisors $D + \left(\sum_{i=1}^{l} \lambda_i v_i\right)$ (each once).

See Also `AlffLinearSeriesEnumNext`, `AlffLinearSeriesEnumElt`,

Example

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> D := 1*AlffPlaceRandom(F, 3);
Alff divisor
[ [ Alff place < [ T^3 + 4*T^2 + T + 2, 0 ], [ 3*T + 1, 1 ] >, 1 ] ]

kash> B := AlffDivisorLBasis(D);
[ [ 1, 0 ], [ 2*T + 4, 1 ] / (T^3 + 4*T^2 + T + 2),
  [ 2*T^2 + 4*T, T ] / (T^3 + 4*T^2 + T + 2) ]
kash> env := AlffLinearSeriesEnumEnv(D, B);
rec(
  ffelts := [ 0, 2, 4, 3, 1 ],
  D := Alff divisor
    [ [ Alff place < [ T^3 + 4*T^2 + T + 2, 0 ], [ 3*T + 1, 1 ] >, 1 ] ]
    ,
  basis := [ [ 1, 0 ], [ 2*T + 4, 1 ] / (T^3 + 4*T^2 + T + 2),
      [ 2*T^2 + 4*T, T ] / (T^3 + 4*T^2 + T + 2) ],
  dim := 3 )
```

Name `AlffLinearSeriesEnumNext`

Purpose Whether there is a next divisor in the linear series enumeration.

Syntax `b := AlffLinearSeriesEnumNext(env);`

`bool`	`b`	whether there is a next divisor
`record`	`env`	of enumeration data

Description Let F/k be a global function field and D a divisor. Let $v_1, \ldots, v_l$ be k-linearly independent elements of F. This function returns `true` or `false` with respect to whether there is a not yet considered divisor in a linear series enumeration of the form $D + \left(\sum_{i=1}^{l} \lambda_i v_i\right)$ and increases an internal counter.

See Also AlffLinearSeriesEnumEnv, AlffLinearSeriesEnumElt,

Example

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2+T^3+1);;
kash> D := 1*AlffPlaceRandom(F, 3);
Alff divisor
[ [ Alff place < [ T^3 + 4*T^2 + T + 2, 0 ], [ 3*T + 1, 1 ] >, 1 ] ]

kash> B := AlffDivisorLBasis(D);
[ [ 1, 0 ], [ 2*T + 4, 1 ] / (T^3 + 4*T^2 + T + 2),
  [ 2*T^2 + 4*T, T ] / (T^3 + 4*T^2 + T + 2) ]
kash> env := AlffLinearSeriesEnumEnv(D, B);
rec(
  ffelts := [ 0, 2, 4, 3, 1 ],
  D := Alff divisor
    [ [ Alff place < [ T^3 + 4*T^2 + T + 2, 0 ], [ 3*T + 1, 1 ] >, 1 ] ]
    ,
  basis := [ [ 1, 0 ], [ 2*T + 4, 1 ] / (T^3 + 4*T^2 + T + 2),
      [ 2*T^2 + 4*T, T ] / (T^3 + 4*T^2 + T + 2) ],
  dim := 3 )
kash> AlffLinearSeriesEnumNext(env);
true
kash> AlffLinearSeriesEnumNext(env);
true
```

NAME `AlffOesterleBound`

PURPOSE Computes the Oesterle bound of a global function field.

SYNTAX `b := AlffOesterleBound(q,g);`

```
integer  b
integer  q, g
```

DESCRIPTION Let g be the genus of a function field F over the exact constant field $\mathbb{F}_q$. This function computes the Oesterle bound for the number of places of degree one of F. This function is the optimization of the explicit formulae. To use if the genus is large.

SEE ALSO `AlffSerreBound`, `AlffIharaBound`, `AlffPlacesDegOneNum`,

EXAMPLE

```
kash> AlffInit(FF(7));;
kash> F:=Alff(y^9 + y + 6*T^16);
Algebraic function field defined by
$.1^9 + $.1 + 6*$.2^16
over
Univariate rational function field over GF(7)
Variables: T

kash> g:=AlffGenus(F);
60
kash> AlffOesterleBound(7,g);
176
```

Name AlffOrderAlff

Purpose Returns the algebraic function field of an algebraic function field order.

Syntax `F := AlffOrderAlff(o);`

algebraic function field	F
algebraic function field order	o

See Also AlffOrderPoly, Alff,

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffOrderAlff(o);
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
```

NAME `AlffOrderBasis`

PURPOSE Returns a basis as elements of the given alff order with a denominator.

SYNTAX `L := AlffOrderBasis(o);`

list of algebraic elements	L
alff order	o

DESCRIPTION The basis is returned as elements of the equation order.

SEE ALSO AlffOrder,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> Bf := AlffOrderBasis(o);
[ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ]
kash> Bi := AlffOrderBasis(oi);
[ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ]
kash> Bf = Bi;
false
```

NAME	`AlffOrderBasis`
PURPOSE	Returns a list containing the basis elements of an alff order.
SYNTAX	`B := AlffOrderBasis(o);`

`list`	`B`	of basis elements
`alff order`	`o`	

SEE ALSO `AlffIdealBasis`,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> Bf := AlffOrderBasis(o);
[ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ]
kash> Bi := AlffOrderBasis(oi);
[ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ]
kash> Bf = Bi;
false
```

NAME AlffOrderBasisValues

PURPOSE Returns some values depending on the B^*-values of the basis of an order of a global function field.

SYNTAX `L := AlffOrderBasisValues(o);`

list	L
global function field order	o

DESCRIPTION Denote by n the $\mathbb{F}_q[T]$-rank of the order o and by $b_1, \ldots, b_n \in o$ its basis. Then the function returns

$$L := [[eB^*(b_1), \ldots, eB^*(b_n)], \max_{i=1}^{n} B^*(b_i), \sum_{i=1}^{n} B^*(b_i), e].$$

The infinite place of $k(T)$ has to be tamely ramified in F. See `AlffEltBstar()` for the definition of B^*, e and [Sch96] for further definitions and algorithms.

SEE ALSO Alff, AlffOrderL0, AlffOrderReduce,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffOrderBasisValues(o);
[ [ 0, 4, 8 ], 8, 12, 3 ]
```

NAME `AlffOrderDedekindTest`

PURPOSE Performs the Dedekind-Test on an equation order in an algebraic function field.

SYNTAX
```
b := AlffOrderDedekindTest(o);
b := AlffOrderDedekindTest(o, g);
```

boolean	b
algebraic function field order	o
polynomial	g

DESCRIPTION Let o be an equation order in an algebraic function field $F/k(T)$. Invoked with one argument, the function returns whether o is $1/T$-maximal or not. Invoked with a prime polynomial $g \in k[T]$, the function returns whether o is g-maximal or not.

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^4+1);
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T

kash> AlffOrderDedekindTest(AlffOrderEqFinite(F), T+1);
true
kash> AlffOrderDedekindTest(AlffOrderEqInfty(F));
false
```

NAME `AlffOrderDeg`

PURPOSE Returns the degree of an alff order over its coefficient ring.

SYNTAX `d := AlffOrderDeg(o);`

`integer`	`d`	degree
`alff order`	`o`	

SEE ALSO `AlffDeg`,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffOrderDeg(o);
3
```

Name `AlffOrderDisc`

Purpose Returns the discriminant of an algebraic function field order up to a unit.

Syntax `d := AlffOrderDisc(o);`

polynomial or quotient field element	d
algebraic function field order	o

Description The discriminant d is an element of the order's coefficient ring, i.e. d is in $k[T]$ or in the valuation ring of the degree valuation of $k(T)$. It is determined up to a unit.

See Also Alff, AlffOrderMaxFinite, AlffOrderMaxInfty,

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffOrderDisc(o);
3*T^8 + T^4 + 3
kash> AlffOrderDisc(oi);
(3*T^8 + T^4 + 3)/T^10
```

Name `AlffOrderEqFinite`

Purpose Computes an equation order of the given algebraic function field.

Syntax `o := AlffOrderEqFinite(F);`

algebraic function field order	o
algebraic function field	F

Description Computes the $k[T]$-equation order of an algebraic function field $F/k(T)$, that is $k[T,\rho]$ where $f(T,\rho) = 0$ and $F/k(T)$ is defined by f.

See Also AlffOrderEqInfty, Alff, AlffElt,

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^4+1);
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T

kash> AlffOrderEqFinite(F);
Finite equation order of
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
```

NAME `AlffOrderEqInfty`

PURPOSE Computes an equation order of the given algebraic function field.

SYNTAX `o := AlffOrderEqInfty(F);`

`algebraic function field order`	`o`
`algebraic function field`	`F`

DESCRIPTION Let R be the valuation ring of the degree valuation of $k(T)$, that is

$$R = \{g/h \mid g, h \in k[T], h \neq 0, \deg(g) \leq \deg(h)\},$$

and let $F/k(T)$ be an algebraic function field. This function computes an R-equation order $R[\rho]$ where $\rho \in F$ generates F over $k(T)$ and is integral over R.

SEE ALSO `AlffOrderEqFinite`, `Alff`, `AlffElt`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^4+1);
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T

kash> AlffOrderEqInfty(F);
Infinite equation order of
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
```

NAME	`AlffOrderIndex`
PURPOSE	Returns the index of an order.
SYNTAX	`d := AlffOrderIndex(o);`

`element`	`d`	of the coefficient ring
`algebraic function field order`	`o`	

DESCRIPTION This function returns the index of the equation order in the given order (as a module over its coefficient ring).

SEE ALSO `AlffOrderMaxFinite`, `AlffOrderMaxInfty`,

EXAMPLE

```
kash> AlffInit(FF(7,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> o := AlffOrderMaxFinite(F);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(7)
Variables: T
given by transformation matrix
[T^3 + T^2 + 5*T + 6                   0     4*T^2 + 6*T + 3]
[                  0 T^3 + T^2 + 5*T + 6     3*T^2 + 5*T + 3]
[                  0                   0                   1]
with denominator T^3 + T^2 + 5*T + 6
kash> p := AlffOrderIndex(o);
1
```

NAME AlffOrderIsFinite

PURPOSE Returns whether a function field order is finite.

SYNTAX `b := AlffOrderIsFinite(o);`

boolean	b
alff order	P

DESCRIPTION

EXAMPLE

```
kash> k := FF(3);
Finite field of size 3
kash> AlffInit(k);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> f := y^3-(T+1)*y^2+2*y*T-T^5;
y^3 + (2*T + 2)*y^2 + 2*T*y + 2*T^5
kash> F:=Alff(f);
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
over
Univariate rational function field over GF(3)
Variables: T

kash> AlffOrders(f);
"Defining global variables: F, o, oi, one"
kash> oe := AlffOrderEqFinite(F);
Finite equation order of
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
over
Univariate rational function field over GF(3)
Variables: T

kash> oie:=AlffOrderEqInfty(F);
Infinite maximal order of
Algebraic function field defined by
$.1^3 + 2*$.1^2*$.2 + 2*$.1^2 + 2*$.1*$.2 + 2*$.2^5
over
Univariate rational function field over GF(3)
```

```
Variables: T

kash> AlffOrderIsFinite(o);
true
kash> AlffOrderIsFinite(oi);
false
kash> AlffOrderIsFinite(oe);
true
kash> AlffOrderIsFinite(oie);
false
```

Name `AlffOrderL0`

Purpose Returns a list of 0-reduced basis elements of an order of a global function field with their B^*-values.

Syntax `L := AlffOrderL0(o);`

`list`	`L`
`global function field order`	`o`

Description For the $\mathbb{F}_q[T]$-order o, the function returns a 0-reduced $\mathbb{F}_q[T]$-basis $b_1, \dots b_n \in o$ with their B^*-values, i.e.

$$L := [b_1, B^*(b_1), \dots, b_n, B^*(b_n)].$$

Here n denotes the $\mathbb{F}_q[T]$-rank of o. The infinite place of $k(T)$ has to be tamely ramified in F. See `AlffEltBstar()` for the definition of B^* and [Sch96] for the definition of 0-reduced and for algorithms.

See Also `AlffOrderBasisValues`, `AlffOrderReduce`,

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffOrderL0(o);
[ [ 1, 0, 0 ], 0, [ 0, 1, 0 ], 4/3, [ 0, 0, 1 ], 8/3 ]
```

NAME `AlffOrderMaxFinite`

PURPOSE Computes a maximal order of an algebraic function field.

SYNTAX `o := AlffOrderMaxFinite(F);`

`algebraic function field order`	`o`
`algebraic function field`	`F`

DESCRIPTION Computes the $k[T]$-maximal order of an algebraic function field $F/k(T)$. It is the integral closure of $k[T]$ in F.

SEE ALSO `AlffOrderMaxInfty`, `AlffOrderMaximal`, `Alff`, `AlffElt`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^4+1);
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T

kash> AlffOrderMaxFinite(F);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
```

NAME AlffOrderMaxInfty

PURPOSE Computes a maximal order of an algebraic function field.

SYNTAX `o := AlffOrderMaxInfty(F);`

algebraic function field order	o
algebraic function field	F

DESCRIPTION Let R be the valuation ring of the degree valuation of $k(T)$, that is

$$R = \{g/h \mid g, h \in k[T], h \neq 0, \deg(g) \leq \deg(h)\},$$

and let $F/k(T)$ be an algebraic function field. This function computes the R-maximal order of F. It is the integral closure of R in F.

SEE ALSO AlffOrderMaxFinite, AlffOrderMaximal, Alff, AlffElt,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^4+1);
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T

kash> AlffOrderMaxInfty(F);
Infinite maximal order of
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
given by transformation matrix
[1/T   0   0]
[  0 1/T   0]
[  0   0   1]
with denominator 1/T
```

NAME `AlffOrderMaximal`

PURPOSE missing shortdoc

SYNTAX `O := AlffOrderMaximal(o);`

```
algebraic function field order  O
algebraic function field order  o
```

DESCRIPTION Returns the maximal overorder of a given order o of an algebraic function field $F/k(T)$. It is the integral closure of o in F.

SEE ALSO `AlffOrderMaxFinite`, `AlffOrderMaxInfty`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^4+1);
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T

kash> o := AlffOrderEqInfty(F);
Infinite equation order of
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T

kash> AlffOrderMaximal(o);
Infinite maximal order of
Algebraic function field defined by
$.1^3 + $.2^4 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
given by transformation matrix
[1/T   0   0]
```

```
[  0 1/T   0]
[  0   0   1]
with denominator 1/T
```

NAME `AlffOrderPoly`

PURPOSE Returns the defining polynomial of the associated equation order of an algebraic function field.

SYNTAX `f := AlffOrderPoly(o);`

polynomial	f
algebraic function field order	o

DESCRIPTION The polynomial f is an element of $k[T, y]$ or $R[y]$. See `AlffOrderEqInfty()` for the definition of R.

SEE ALSO `AlffOrderDeg`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffOrderPoly(o);
y^3 + T^4 + 1
kash> AlffOrderPoly(oi);
x^3 + (T^4 + 1)/T^6
```

Name AlffOrderReduce

Purpose Given an order of a global function field, the function returns an order with a 0-reduced basis.

Syntax `o2 := AlffOrderReduce(o1);`

global function field order	o2
global function field order	o1

Description The infinite place of $k(T)$ has to be tamely ramified in F. See [Sch96] for definitions and algorithms.

See Also AlffOrderL0, AlffOrderBasisValues,

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^13*y^2+T^4+T+1);
"Defining global variables: F, o, oi, one"
kash> o;
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.1^2*$.2^13 + $.2^4 + $.2 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
given by transformation matrix
[   T + 3       0       3]
[       0   T + 3       4]
[       0       0       1]
with denominator T + 3
kash> AlffOrderReduce(o);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.1^2*$.2^13 + $.2^4 + $.2 + 1
over
Univariate rational function field over GF(5^2)
Variables: T
given by transformation matrix
[              T + 3             3*T + 4              4*T + 3]
[                  0        T^14 + 3*T^13    3*T^14 + T^13 + 4]
```

```
[                   0                T + 3                3*T + 1]
with denominator T + 3
```

NAME `AlffOrderTransformationMatrix`

PURPOSE Returns a list containing information about the transformation from the alff-suborder of the given alfforder to the given alfforder.

SYNTAX `L := AlffOrderTransformationMatrix(O);`

`list`	`L`
`alfforder`	`O`

DESCRIPTION The list L contains the denominator d and the $n \times n$-matrix T with

$$(\alpha|1, \ldots, \alpha|n) = \frac{1}{d}(a|1, \ldots, a|n)T,$$

where $\alpha|1, \ldots, \alpha|n$ is a basis of the given alfforder $\mathcal{O}$ and $a|1, \ldots, a|n$ is a basis of the alffsuborder of the given order. If the order $\mathcal{O}$ is the equation order, simply $d = 1$ and $T = \mathrm{id}$ are returned.

NAME `AlffOrders`

PURPOSE Initializes variables for some structures of an algebraic function field.

SYNTAX `AlffOrders(f);`

`polynomial in` $k[T][y]$ `f`

DESCRIPTION This function is used for convenience to initialize variables for some structures of an algebraic function field and is not recommended for use in programming. It initializes:

- `F`, the algebraic function field generated by $f(T, y) = 0$,
- `o`, the integral closure of $k[T]$ in F,
- `oi`, the integral closure of the degree valuation ring,
- `one`, the $1 \in F$.

The polynomial f is checked to be irreducible and separable.

SEE ALSO `AlffInit`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3+2);
"Defining global variables: F, o, oi, one"
kash> F;
Algebraic function field defined by
$.1^3 + $.2^3 + 2
over
Univariate rational function field over GF(5^2)
Variables: T

kash> o;
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.2^3 + 2
over
Univariate rational function field over GF(5^2)
Variables: T
```

```
kash> AlffGlobals.oi;
Infinite maximal order of
Algebraic function field defined by
$.1^3 + $.2^3 + 2
over
Univariate rational function field over GF(5^2)
Variables: T

kash> one;
[ 1, 0, 0 ]
```

NAME `AlffOrders`

PURPOSE Initializes variables for some structures of an algebraic function field.

SYNTAX `AlffOrders(f);`

`polynomial in` $k[T][y]$ `f`

DESCRIPTION This function is used for convenience to initialize variables for some structures of an algebraic function field and is not recommended for use in programming. It initializes:

- `F`, the algebraic function field generated by $f(T, y) = 0$,
- `o`, the integral closure of $k[T]$ in F,
- `oi`, the integral closure of the degree valuation ring,
- `one`, the $1 \in F$.

The polynomial f is checked to be irreducible and separable.

SEE ALSO `AlffInit`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3+2);
"Defining global variables: F, o, oi, one"
kash> F;
Algebraic function field defined by
$.1^3 + $.2^3 + 2
over
Univariate rational function field over GF(5^2)
Variables: T

kash> o;
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.2^3 + 2
over
Univariate rational function field over GF(5^2)
Variables: T
```

```
kash> AlffGlobals.oi;
Infinite maximal order of
Algebraic function field defined by
$.1^3 + $.2^3 + 2
over
Univariate rational function field over GF(5^2)
Variables: T

kash> one;
[ 1, 0, 0 ]
```

Name `AlffPlaceAlff`

Purpose Given a place this function returns the algebraic function field it is belonging to.

Syntax `F := AlffPlaceAlff(P);`

algebraic function field	F
alff place	P

See Also AlffOrderAlff,

Example

```
kash> AlffInit(FF(2,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^3)
Variables: T

kash> P := AlffPlaceSplit(F, T);
[ Alff place < [ T, 0, 0 ], [ 0, 1, 0 ] > ]
kash> AlffPlaceAlff(P[1]);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^3)
Variables: T
```

NAME `AlffPlaceBeta`

PURPOSE Return an inverse prime element for a place.

SYNTAX `b := AlffPlaceBeta(P);`

```
alff element  b
alff place    P
```

DESCRIPTION Let a finite (infinite) place $\mathfrak{P}$ of the algebraic function field F/k be given and let p be the minimum of $\mathfrak{P}$. This function returns an element $\beta \in F$ such that β/p is integral at all finite (infinite) places different from $\mathfrak{P}$ and satisfies $v_{\mathfrak{P}}(\beta/p) = -1$.

SEE ALSO `AlffEltValuation`, `AlffPlacePrimeElt`, `AlffPlaceMin`,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> E := Alff(y^2+y+T^3+T);
Algebraic function field defined by
$.1^2 + $.1 + $.2^3 + $.2
over
FunctionField(GF(2))

kash> P := AlffPlacesDegOne(E)[2];
Alff place < [ T, 0 ], [ 0, 1 ] >
kash> p := AlffPlaceMin(P);
T
kash> b := AlffPlaceBeta(P);
[ 1, 1 ]
kash> AlffDivisor(b/p);
Alff divisor
[ [ Alff place < [ T, 0 ], [ 0, 1 ] >, -1 ],
[ Alff place < [ T + 1, 0 ], [ 1, 1 ] >, 2 ],
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -1 ] ]
```

NAME AlffPlaceDeg

PURPOSE Computes the degree of a place.

SYNTAX `d := AlffPlaceDeg(P);`

```
integer    d
alff place P
```

DESCRIPTION This function returns the degree d of a place $\mathfrak{P}$ over the constant field of definition (not the exact constant field) of the algebraic function field.

SEE ALSO AlffPlaceSplit, AlffPlaceRam, AlffPlaceResDeg,

EXAMPLE

```
kash> AlffInit(FF(3,4));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^7+y+T^9+T+1);
Algebraic function field defined by
$.1^7 + $.1 + $.2^9 + $.2 + 1
over
Univariate rational function field over GF(3^4)
Variables: T

kash> l := AlffPlaceSplit(F, T);
[ Alff place < [ T, 0, 0, 0, 0, 0, 0 ], [ 2, 1, 0, 0, 0, 0, 0 ] >,
  Alff place < [ T, 0, 0, 0, 0, 0, 0 ], [ w^50, w^50, w^10, 1, 0, 0, 0 ] >,
  Alff place < [ T, 0, 0, 0, 0, 0, 0 ], [ w^70, w^70, w^30, 1, 0, 0, 0 ] > ]
kash> d := AlffPlaceDeg(l[2]);
3
```

NAME `AlffPlaceIdeal`

PURPOSE missing shortdoc

SYNTAX `I := AlffPlaceIdeal(P);`

`alff order ideal`	`I`
`alff place`	`P`

DESCRIPTION Given a place $\mathfrak{P}$ this function returns a prime ideal corresponding to $\mathfrak{P}$.

SEE ALSO `AlffPlaceOrder`, `AlffPlaceSplit`, `AlffIdealFactor`,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> E := Alff(y^2+y+T^3+T);
Algebraic function field defined by
$.1^2 + $.1 + $.2^3 + $.2
over
FunctionField(GF(2))

kash> P := AlffPlacesDegOne(E)[2];
Alff place < [ T, 0 ], [ 0, 1 ] >
kash> I := AlffPlaceIdeal(P);
< [ T, 0 ], [ 0, 1 ] >
```

NAME AlffPlaceIsFinite

PURPOSE Returns whether a place is finite.

SYNTAX `b := AlffPlaceIsFinite(P);`

boolean	b
alff place	P

DESCRIPTION Let $F = k(T, y)$ be an algebraic function field defined by $f(T, y) = 0$ over k and $\mathfrak{P} \mid \mathfrak{p}$ be a place of F over a place $\mathfrak{p}$ of the rational function field $k(T)$. The function returns true if and only if $\mathfrak{p}$ is not the place at infinity (degree valuation).

SEE ALSO AlffPlaceSplit,

EXAMPLE

```
kash> H := AlffHermitianFunField(2,1);
Algebraic function field defined by
$.1^2 + $.1 + $.2^3
over
Univariate rational function field over GF(2^2)
Variables: T

kash> P := AlffPlacesDegOne(H)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> AlffPlaceIsFinite(P);
false
```

NAME AlffPlaceMin

PURPOSE Returns the "minimum" of a place.

SYNTAX `p := AlffPlaceMin(P);`

prime polynomial or $1/T$	p
alff place	P

DESCRIPTION Let $F = k(T, y)$ be an algebraic function field defined by $f(T, y) = 0$ over k and $\mathfrak{P} \,|\, \mathfrak{p}$ be a place of F over a place $\mathfrak{p}$ of the rational function field $k(T)$. This function returns a monic prime polynomial in $k[T]$ or $1/T$, corresponding to $\mathfrak{p}$.

SEE ALSO AlffPlaceSplit,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3+1);
"Defining global variables: F, o, oi, one"
kash> P := AlffPlaceSplit(F, T+3)[1];
Alff place < [ T + 3, 0, 0 ], [ w^4, 1, 0 ] >
kash> AlffPlaceMin(P);
T + 3
```

NAME `AlffPlaceOrder`

PURPOSE missing shortdoc

SYNTAX `o := AlffPlaceOrder(P);`

```
alff order  o
alff place  P
```

DESCRIPTION This function returns the finite or infinite maximal order according to whether the given place is finite or infinite.

SEE ALSO `AlffPlaceIdeal`, `AlffPlaceIsFinite`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> E := Alff(y^2+y+T^3+T);
Algebraic function field defined by
$.1^2 + $.1 + $.2^3 + $.2
over
FunctionField(GF(2))

kash> P := AlffPlacesDegOne(E)[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> o := AlffPlaceOrder(P);
Infinite maximal order of
Algebraic function field defined by
$.1^2 + $.1 + $.2^3 + $.2
over
FunctionField(GF(2))
```

NAME AlffPlacePrimeElt

PURPOSE missing shortdoc

SYNTAX `u := AlffPlacePrimeElt(P);`

```
alff element  u
alff place    P
```

DESCRIPTION Given a place $\mathfrak{P}$ this function returns a prime element (local uniformizer) for $\mathfrak{P}$.

SEE ALSO AlffEltValuation, AlffPlaceBeta,

EXAMPLE

```
kash> AlffInit(FF(2,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> E := Alff(y^2+y+T^3+T);
Algebraic function field defined by
$.1^2 + $.1 + $.2^3 + $.2
over
FunctionField(GF(2))

kash> P := AlffPlacesDegOne(E)[2];
Alff place < [ T, 0 ], [ 0, 1 ] >
kash> AlffPlacePrimeElt(P);
[ T, 0 ]
```

NAME `AlffPlaceRam`

PURPOSE Ramification of a place.

SYNTAX `r := AlffPlaceRam(P);`

```
integer     r
alff place  P
```

DESCRIPTION Let $F = k(T, y)$ be an algebraic function field defined by $f(T, y) = 0$ over k and $\mathfrak{P} \,|\, \mathfrak{p}$ be a place of F over a place $\mathfrak{p}$ of the rational function field $k(T)$. This function returns the ramification index r of $\mathfrak{P}$ over $\mathfrak{p}$.

SEE ALSO `AlffPlaceSplit`, `AlffPlaceDeg`, `AlffPlaceResDeg`,

EXAMPLE

```
kash> AlffInit(FF(2,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^3)
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ], [ w, 1, 0 ] >
kash> r := AlffPlaceRam(P);
1
```

Name	`AlffPlaceRandom`
Purpose	Returns a random place of degree d.
Syntax	`p := AlffPlaceRandom(F,d);`

`alff place`	`p`	of degree d
`global function field`	`F`	
`integer`	`d`	

Description This function returns a random place of degree d of a global function field F. It may return false though there exist such a place (try calling again).

See Also `AlffPlaces`, `AlffPlacesDegOne`,

Example

```
kash> AlffInit(FF(2, 3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffPlaceRandom(F, 2);
Alff place < [ T^2 + w^4*T + w, 0, 0 ], [ w^2*T + w^6, 1, 0 ] >
```

NAME `AlffPlaceResDeg`

PURPOSE Residue class field extension degree of a place.

SYNTAX `f := AlffPlaceResDeg(P);`

integer	f
alff place	P

DESCRIPTION Let $F = k(T, y)$ be an algebraic function field defined by $f(T, y) = 0$ over k and $\mathfrak{P} \mid \mathfrak{p}$ be a place of F over a place $\mathfrak{p}$ of the rational function field $k(T)$. This function returns the degree f of the extension of the residue class fields of $\mathfrak{P}$ and $\mathfrak{p}$ (k is the constant field of definition, not the exact constant field).

SEE ALSO `AlffPlaceSplit`, `AlffPlaceRam`, `AlffPlaceDeg`,

EXAMPLE

```
kash> AlffInit(FF(2,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^3)
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ], [ w, 1, 0 ] >
kash> f := AlffPlaceResDeg(P);
1
```

Name `AlffPlaceResField`

Purpose Return the residue field of a place.

Syntax `K := AlffPlaceResField(P);`

`field`	`K`	residue field of P
`alff place`	`P`	

Description Let F/k be an algebraic function field and P be a place of F/k. If $\mathfrak{o}_P$ is the ring of algebraic functions defined at P then $k(P) := \mathfrak{o}_P/P$ is the residue class field of P. It is returned by this function.

See Also `AlffResFieldEltLift`, `AlffELtToResField`, `AlffEltValuation`,

Example

```
kash> AlffInit(FF(2,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^3)
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ], [ w, 1, 0 ] >
kash> f := AlffPlaceResField(P);
Finite field of size 2^3
```

NAME `AlffPlaceSplit`

PURPOSE Decompose a rational function field place.

SYNTAX `L := AlffPlaceSplit(F, p);`

`list`	`L`	of places $\mathfrak{P}$ above $\mathfrak{p}$
`algebraic function field`	`F`	
`prime polynomial or` $1/T$	`p`	

DESCRIPTION Let $F = k(T, y)$ be an algebraic function field defined by $f(T, y) = 0$ over k. This function computes the places $\mathfrak{P}$ of F lying above a place $\mathfrak{p}$ of $k(T)$ given by a prime polynomial of $k[T]$ or $1/T$, the later specifying the place at infinity (degree valuation). A list containing the places is returned.

SEE ALSO `AlffPlaceRam`, `AlffPlaceDeg`, `AlffPlaceResDeg`, `AlffEltValuation`,

EXAMPLE

```
kash> AlffInit(FF(2,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^3)
Variables: T

kash> L := AlffPlaceSplit(F, 1/T);
[ Alff place < [ 1/T, 0, 0 ], [ w^4/T, w^3/T, (w^4*T + w^4)/T ] >,
  Alff place < [ 1/T, 0, 0 ], [ (w^4*T + 1)/T, (w*T + w^2)/T, (w^4*T + w^6)/T \
] > ]
kash> L := AlffPlaceSplit(F, T+1);
[ Alff place < [ T + 1, 0, 0 ], [ w, 1, 0 ] >,
  Alff place < [ T + 1, 0, 0 ], [ w^2, 1, 0 ] >,
  Alff place < [ T + 1, 0, 0 ], [ w^4, 1, 0 ] > ]
```

NAME `AlffPlaceSplitType`

PURPOSE Returns a list with ramification indices and relative degrees of all pairwise distinct places lying over a rational function field place.

SYNTAX `L := AlffPlaceSplitType(F, p);`

`list`	`L`
`algebraic function field`	`F`
`polynomial or` $1/T$	`p`

SEE ALSO `AlffPlaceSplit`, `AlffPlaceRam`, `AlffPlaceResDeg`, `AlffSignature`, `AlffIdealFactor`,

EXAMPLE

NAME `AlffPlaceSplitType`

PURPOSE Returns a list with ramification indices and relative degrees of all pairwise distinct places lying over a rational function field place.

SYNTAX `L := AlffPlaceSplitType(F, p);`

list	L
algebraic function field	F
polynomial or $1/T$	p

SEE ALSO `AlffPlaceSplit`, `AlffPlaceRam`, `AlffPlaceResDeg`, `AlffSignature`, `AlffIdealFactor`,

EXAMPLE

NAME `AlffPlaces`

PURPOSE Computes places of a global function field.

SYNTAX `L := AlffPlaces(F,d);`

`list`	`L`	of alff places
`global function field`	`F`	
`integer`	`d`	

DESCRIPTION This function returns a list of all places of a degree d of a global function field F. The constant field of definition should be the exact constant field of F.

SEE ALSO `AlffPlacesNum`, `AlffPlacesDegOne`, `AlffPlacesDegOneNum`,

EXAMPLE

```
kash> AlffInit(FF(2, 3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffPlaces(F, 2);
[ Alff place < [ T^2 + w^5*T + w^3, 0, 0 ], [ w^6*T + w^4, 1, 0 ] >,
  Alff place < [ T^2 + w^3*T + w^6, 0, 0 ], [ w^5*T + w, 1, 0 ] >,
  Alff place < [ T^2 + w^6*T + w^5, 0, 0 ], [ w^3*T + w^2, 1, 0 ] >,
  Alff place < [ T^2 + w^4*T + w, 0, 0 ], [ w^2*T + w^6, 1, 0 ] >,
  Alff place < [ T^2 + T + 1, 0, 0 ], [ T + 1, 1, 0 ] >,
  Alff place < [ T^2 + w*T + w^2, 0, 0 ], [ w^4*T + w^5, 1, 0 ] >,
  Alff place < [ T^2 + w^2*T + w^4, 0, 0 ], [ w*T + w^3, 1, 0 ] > ]
```

NAME `AlffPlacesDegOne`

PURPOSE Compute all places of degree one of a global function field.

SYNTAX `L := AlffPlacesDegOne(F);`

`global function field`	`F`	
`list`	`L`	of alff places

DESCRIPTION This function returns a list of all places of degree one of a global function field F. The constant field of definition must be the exact constant field of F.

SEE ALSO `AlffPlacesDegMNum`,

EXAMPLE

```
kash> AlffInit(FF(2, 3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffPlacesDegOne(F);
[ Alff place < [ 1/T, 0, 0 ], [ w^3/T, w, (w^2*T + w^6)/T ] >,
  Alff place < [ 1/T, 0, 0 ], [ (T + w)/T, (w*T + w^4)/T, (T + w^6)/T ] >,
  Alff place < [ T, 0, 0 ], [ 0, 1, 0 ] >,
  Alff place < [ T + w, 0, 0 ], [ 1, 1, 0 ] >,
  Alff place < [ T + w, 0, 0 ], [ w^2, 1, 0 ] >,
  Alff place < [ T + w, 0, 0 ], [ w^6, 1, 0 ] >,
  Alff place < [ T + w^2, 0, 0 ], [ 1, 1, 0 ] >,
  Alff place < [ T + w^2, 0, 0 ], [ w^4, 1, 0 ] >,
  Alff place < [ T + w^2, 0, 0 ], [ w^5, 1, 0 ] >,
  Alff place < [ T + w^3, 0, 0 ], [ w^2, 1, 0 ] >,
  Alff place < [ T + w^3, 0, 0 ], [ w^3, 1, 0 ] >,
  Alff place < [ T + w^3, 0, 0 ], [ w^5, 1, 0 ] >,
  Alff place < [ T + w^4, 0, 0 ], [ 1, 1, 0 ] >,
  Alff place < [ T + w^4, 0, 0 ], [ w, 1, 0 ] >,
  Alff place < [ T + w^4, 0, 0 ], [ w^3, 1, 0 ] >,
  Alff place < [ T + w^5, 0, 0 ], [ w, 1, 0 ] >,
  Alff place < [ T + w^5, 0, 0 ], [ w^5, 1, 0 ] >,
  Alff place < [ T + w^5, 0, 0 ], [ w^6, 1, 0 ] >,
  Alff place < [ T + w^6, 0, 0 ], [ w^3, 1, 0 ] >,
  Alff place < [ T + w^6, 0, 0 ], [ w^4, 1, 0 ] >,
  Alff place < [ T + w^6, 0, 0 ], [ w^6, 1, 0 ] >,
```

```
Alff place < [ T + 1, 0, 0 ], [ w, 1, 0 ] >,
Alff place < [ T + 1, 0, 0 ], [ w^2, 1, 0 ] >,
Alff place < [ T + 1, 0, 0 ], [ w^4, 1, 0 ] > ]
```

NAME `AlffPlacesDegOne`

PURPOSE Computes all places of degree one of a global function field.

SYNTAX `L := AlffPlacesDegOne(F);`

`global function field`	`F`	
`list`	`L`	of alff places of degree 1

DESCRIPTION This function returns a list of all places of degree one of a global function field F. The constant field of definition should be the exact constant field of F since the degree is taken over the former.

SEE ALSO `AlffPlacesDegOneNum`, `AlffPlaces`, `AlffPlacesNum`, `AlffPlaceRandom`,

EXAMPLE

```
kash> AlffInit(FF(2, 3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffPlacesDegOne(F);
[ Alff place < [ 1/T, 0, 0 ], [ w^3/T, w, (w^2*T + w^6)/T ] >,
  Alff place < [ 1/T, 0, 0 ], [ (T + w)/T, (w*T + w^4)/T, (T + w^6)/T ] >,
  Alff place < [ T, 0, 0 ], [ 0, 1, 0 ] >,
  Alff place < [ T + w, 0, 0 ], [ 1, 1, 0 ] >,
  Alff place < [ T + w, 0, 0 ], [ w^2, 1, 0 ] >,
  Alff place < [ T + w, 0, 0 ], [ w^6, 1, 0 ] >,
  Alff place < [ T + w^2, 0, 0 ], [ 1, 1, 0 ] >,
  Alff place < [ T + w^2, 0, 0 ], [ w^4, 1, 0 ] >,
  Alff place < [ T + w^2, 0, 0 ], [ w^5, 1, 0 ] >,
  Alff place < [ T + w^3, 0, 0 ], [ w^2, 1, 0 ] >,
  Alff place < [ T + w^3, 0, 0 ], [ w^3, 1, 0 ] >,
  Alff place < [ T + w^3, 0, 0 ], [ w^5, 1, 0 ] >,
  Alff place < [ T + w^4, 0, 0 ], [ 1, 1, 0 ] >,
  Alff place < [ T + w^4, 0, 0 ], [ w, 1, 0 ] >,
  Alff place < [ T + w^4, 0, 0 ], [ w^3, 1, 0 ] >,
  Alff place < [ T + w^5, 0, 0 ], [ w, 1, 0 ] >,
  Alff place < [ T + w^5, 0, 0 ], [ w^5, 1, 0 ] >,
  Alff place < [ T + w^5, 0, 0 ], [ w^6, 1, 0 ] >,
  Alff place < [ T + w^6, 0, 0 ], [ w^3, 1, 0 ] >,
  Alff place < [ T + w^6, 0, 0 ], [ w^4, 1, 0 ] >,
```

```
Alff place < [ T + w^6, 0, 0 ], [ w^6, 1, 0 ] >,
Alff place < [ T + 1, 0, 0 ], [ w, 1, 0 ] >,
Alff place < [ T + 1, 0, 0 ], [ w^2, 1, 0 ] >,
Alff place < [ T + 1, 0, 0 ], [ w^4, 1, 0 ] > ]
```

NAME `AlffPlacesDegOneNonSingFiniteNum`

PURPOSE Computes the number of finite non singular places of degree one of a global function field's constant field extension of degree m.

SYNTAX `b := AlffPlacesDegOneNonSingFiniteNum(F, m);`

global function field	F
integers	b,m

SEE ALSO AlffPlacesDegOneNumBound,

EXAMPLE

```
kash> AlffInit(FF(7,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^7+y+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffPlacesDegOneNonSingFiniteNum(F, 1);
175
```

Name `AlffPlacesDegOneNum`

Purpose Computes the number of places of degree one of a global function field's constant field extension.

Syntax `N := AlffPlacesDegOneNum(F,m);`

`global function field`	`F`
`integer`	`N, m`

Description Let $N_1(F')$ be the number of places of degree one of the constant field extension F' of degree m of a global function field F. This function calculates $N_1(F')$ (using a simple algorithm).

See Also `AlffPlacesDegOneNumBound`, `AlffPlacesDegOne`,

Example

```
kash> AlffInit(FF(2, 3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffPlacesDegOneNum(F,1);
24
```

NAME `AlffPlacesDegOneNum`

PURPOSE Computes the number of places of degree one of a global function field's constant field extension.

SYNTAX `N := AlffPlacesDegOneNum(F,m);`

`global function field`	`F`
`integer`	`N, m`

DESCRIPTION Let $N_1(F')$ be the number of places of degree one of the constant field extension F' of degree m of a global function field F. This function calculates $N_1(F')$ (using a simple algorithm).

SEE ALSO `AlffPlacesDegOneNumBound`, `AlffPlacesDegOne`, `AlffPlaces`, `AlffPlacesNum`,

EXAMPLE

```
kash> AlffInit(FF(2, 3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffPlacesDegOneNum(F,1);
24
```

Name `AlffPlacesDegOneNumBound`

Purpose Computes a bound for the number of places of degree one of a global function field.

Syntax

```
b := AlffPlacesDegOneNumBound(F);
b := AlffPlacesDegOneNumBound(q, g);
```

global function field	F
integer	q, g
integer	b

Description Let $N_q(g)$ be the maximal number of places of degree one of a global function field F with genus g over $\mathbb{F}_q$. This function calculates the minimum of the Serre bound and the Ihara bound as an upper bound for $N_q(g)$ from the given special function field or for given q and g.

See Also AlffSerreBound, AlffIharaBound,

Example

```
kash> AlffInit(FF(7,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^7+y+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffGenus(F);
9
kash> AlffPlacesDegOneNumBound(F);
176
kash> AlffPlacesDegOneNumBound(5, 6);
25
```

NAME `AlffPlacesNonSpecial`

PURPOSE Computes a non special system of g distinct places for a global function field F/k of genus g.

SYNTAX

```
S := AlffPlacesNonSpecial(F);
S := AlffPlacesNonSpecial(F, d);
```

`list`	`S`	of alff places
`global function field`	`F`	
`integer`	`d`	≥ 1

DESCRIPTION Let g be the genus of the global function field F/k and c be the dimension of the exact constant field over the constant field of definition. This function returns a list of distinct places $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of F/k such that $\sum_{i=1}^{m} \deg(\mathfrak{p}_i) = cg$ and $\max_{i=1}^{m} \deg(\mathfrak{p}_i)$ is minimal among any such places. If a second parameter $d \in \mathbb{Z}^{\geq 1}$ is given, the maximum is forced to be $\leq d$. If there do not exist such places in either case, false is returned.

SEE ALSO `AlffPlaces`, `AlffDivisorLDim`,

EXAMPLE

```
kash> AlffInit(FF(5));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^2 + T^17 + T + 1);
"Defining global variables: F, o, oi, one"
kash> AlffPlacesNonSpecial(F);
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, Alff place < [ T, 0 ], [ 2, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 0, 1 ] >, Alff place < [ T + 1, 0 ], [ 1, 1 ] >
    , Alff place < [ T^2 + 3*T + 4, 0 ], [ T + 4, 1 ] >,
  Alff place < [ T^2 + 4*T + 1, 0 ], [ T + 2, 1 ] > ]
kash> AlffPlacesNonSpecial(F, 1);
false
```

Name	`AlffPlacesNum`
Purpose	Computes the number of places of a given degree of a global function field.
Syntax	`N := AlffPlacesNum(F, m);`

global function field	F
integers	N, m

See Also	`AlffPlacesDegMNum`, `AlffPlacesDegOne`,
Example	

```
kash> AlffInit(FF(2, 3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffPlacesNum(F, 2);
7
```

NAME `AlffPlacesNum`

PURPOSE Computes the number of places of a given degree of a global function field.

SYNTAX `N := AlffPlacesNum(F, m);`

global function field	F
integers	N, m

SEE ALSO AlffPlacesDegOneNum, AlffPlaces, AlffPlacesDegOne,

EXAMPLE

```
kash> AlffInit(FF(2, 3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffPlacesNum(F, 2);
7
```

NAME AlffPoly

PURPOSE missing shortdoc

SYNTAX `f := AlffPoly(F);`

polynomial	f
algebraic function field	F

DESCRIPTION Returns defining polynomial of an algebraic function field.

SEE ALSO Alff,

EXAMPLE

```
kash> AlffInit(FF(7,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff((T^2+1)*y^3+y+T^4+1);
Algebraic function field defined by
$.1^3*$.2^2 + $.1^3 + $.1 + $.2^4 + 1
over
Univariate rational function field over GF(7^2)
Variables: T

kash> AlffPoly(F);
(T^2 + 1)*y^3 + y + T^4 + 1
```

NAME `AlffPolyIrrIsSep`

PURPOSE missing shortdoc

SYNTAX `b := AlffPolyIrrIsSep(f);`

polynomial in T and y over field	f
boolean	b

DESCRIPTION Returns whether an irreducible polynomial $f(T, y)$ over a field is separable in y.

SEE ALSO AlffPolyIsIrreducible, AlffPolyIsIrrSep, Alff,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffPolyIrrIsSep(y+T);
true
kash> AlffPolyIrrIsSep(y^5+T);
false
```

NAME `AlffPolyIsIrrSep`

PURPOSE missing shortdoc

SYNTAX `b := AlffPolyIsIrrSep(f);`

polynomial in T and y over field	f
boolean	b

DESCRIPTION Returns whether a polynomial $f(T, y)$ over a field is irreducible and separable in y.

SEE ALSO AlffPolyIsIrreducible, Alff,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffPolyIsIrrSep(y+T);
true
kash> AlffPolyIsIrrSep(y^5+T);
false
```

Name `AlffPolyIsIrreducible`

Purpose missing shortdoc

Syntax `b := AlffPolyIsIrreducible(f);`

polynomial in T and y over field	f
boolean	b

Description Returns whether a polynomial $f(T, y)$ over a field is irreducible or not.

See Also `AlffPolyIsIrrSep`, `Alff`,

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffPolyIsIrreducible(y+T);
true
kash> AlffPolyIsIrreducible(y^5+T);
true
```

Name `AlffPuiseuxCoeff`

Purpose Returns the finite constant field of the Puiseux series field into which all completions of a global function field with respect to the infinite valuations can be embedded.

Syntax `k := AlffPuiseuxCoeff(F);`

`finite field`	`k`
`global function field`	`F`

Description Let $f(T,y)$ be a generating irreducible equation for a global function field F, of degree n and separable in y. Let d be a minimal positive integer such that all n distinct roots of $f(T,y)$ in y can be represented in $\mathbb{F}_{q^d}((T^{-1/e}))$, where e is the least common multiple of the ramification indices of the infinite places and $p \nmid e$. The function returns $\mathbb{F}_{q^d}$.

See Also AlffRoots,

Example

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffRoots(F);;
kash> AlffPuiseuxCoeff(F);
Finite field of size 5^2
```

NAME `AlffRamDivisor`

PURPOSE Returns the ramification divisor of a divisor.

SYNTAX

```
A := AlffRamDivisor(D);
A := AlffRamDivisor(F);
```

`alff divisor`	`A`	the ramification divisor of D
`alff divisor`	`D`	
`alff`	`F`	equivalent to taking $D = 0$

DESCRIPTION Let F/k be an algebraic function field, x a separating variable and D a divisor. The ramification divisor of D is defined to be

$$i(D)\,(W-D)+\big(W_x(D)\big)+\nu\,(dx),$$

where W is a canonical divisor of F/k, $W_x(D)$ is the determinant of the Wronksi matrix of D with respect to x and ν is the sum of the Wronski orders of D with respect to x. It is positive and consists of the D-Weierstraß places. The constant field k is required to be exact.

SEE ALSO `AlffWronskian`, `AlffWronskianOrders`, `AlffWeierstrassPlaces`, `AlffGapNumbers`, `AlffDiff`, `AlffDifferentiation`,

EXAMPLE

```
kash> AlffInit(Q);;
kash> AlffOrders(y^2 - (T-1)*(T-2)*(T-3)*(T-4)*(T-5)*(T-6)*(T-7));
"Defining global variables: F, o, oi, one"
kash> AlffRamDivisor(F);
Alff divisor
[ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, 3 ],
[ Alff place < [ T - 7, 0 ], [ 0, 1 ] >, 3 ],
[ Alff place < [ T - 6, 0 ], [ 0, 1 ] >, 3 ],
[ Alff place < [ T - 5, 0 ], [ 0, 1 ] >, 3 ],
[ Alff place < [ T - 4, 0 ], [ 0, 1 ] >, 3 ],
[ Alff place < [ T - 3, 0 ], [ 0, 1 ] >, 3 ],
[ Alff place < [ T - 2, 0 ], [ 0, 1 ] >, 3 ],
[ Alff place < [ T - 1, 0 ], [ 0, 1 ] >, 3 ] ]
```

NAME	`AlffRegulator`
PURPOSE	Computes the regulator of a global function field's finite maximal order's unit group.
SYNTAX	`R := AlffRegulator(o);`

`integer`	`R`
`finite maximal order`	`o`

DESCRIPTION

For $1 \leq i \leq s-1$ let ν_i denote distinct normalized valuations of F at the infinite places with degrees f_i over the degree valuation of $k(T)$. Let $M \in \mathbb{Z}^{(s-1)\times(s-1)}$ be the image of a basis of the finite maximal order's unit group in the "logarithm space" obtained by applying the ν_i for $1 \leq i \leq s-1$ on each basis element. The function returns $\det(M)\prod_{i=1}^{r} f_i$.

SEE ALSO

`AlffSignature`, `AlffUnitRank`, `AlffUnitsFund`,

EXAMPLE

```
kash> AlffInit(FF(5,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^4+(2*T+3)*y^3+y^2+(3*T+2)*y+1);
"Defining global variables: F, o, oi, one"
kash> AlffUnitsFund(o);
[ [ 1, 1, 0, 0 ], [ 1, 4*T, 4*T + 3, 2 ], [ 3*T + 1, 1, 2*T + 3, 1 ] ]
kash> AlffRegulator(o);
1
```

NAME `AlffResFieldEltLift`

PURPOSE Lifts a residue field element.

SYNTAX `a := AlffResFieldEltLift(b, P);`

`alff element`	`a`	in $\mathfrak{o}_P$
`field element`	`b`	value of a at P to be lifted
`alff place`	`P`	

DESCRIPTION Let F/k be an algebraic function field and P be a place of F/k. If $\mathfrak{o}_P$ is the ring of algebraic functions defined at P then $k(P) := \mathfrak{o}_P/P$ is the residue class field of P. For $b \in k(P)$ this function returns $a \in \mathfrak{o}_P$ such that $a(P) = b$. Constants are lifted to constants.

SEE ALSO `AlffResFieldEltLift`, `AlffELtToResField`, `AlffEltValuation`,

EXAMPLE

```
kash> AlffInit(FF(5,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(5^3)
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ], [ 3, 1, 0 ] >
kash> AlffPlaceResField(P);
Finite field of size 5^3
kash> of := AlffOrderMaxFinite(F);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(5^3)
Variables: T

kash> a := AlffEltToResField(AlffElt(of,T),P);
4
```

```
kash> AlffResFieldEltLift(a,P);
[ 4, 0, 0 ]
kash> p := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0, 0 ], [ w^48/T, (w^102*T + w^5)/T, (w^18*T + w^82)/T ] >
kash> AlffPlaceResField(p);
Finite field of size 5^3
kash> oi := AlffOrderMaxInfty(F);
Infinite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(5^3)
Variables: T
given by transformation matrix
[1/T   0   0]
[  0 1/T   0]
[  0   0   1]
with denominator 1/T
kash> b := AlffEltToResField(AlffElt(oi,1/T),p);
0
kash> AlffResFieldEltLift(b,p);
[ 0, 0, 0 ]

kash> AlffInit(Q);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over Rational Field
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ] >
kash> AlffPlaceResField(P);
Generating polynomial: x^3 - x - 1

kash> of := AlffOrderMaxFinite(F);
Finite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over Rational Field
```

```
Variables: T

kash> a := AlffEltToResField(AlffElt(of,2),P);
2
kash> AlffResFieldEltLift(a,P);
[ 2, 0, 0 ]
kash> p := AlffPlaceSplit(F, 1/T)[1];
Alff place < [ 1/T, 0, 0 ], [ -13/T, (11/3*T - 35)/T, (-3/2*T + 32/3)/T ] >
kash> AlffPlaceResField(p);
Rational Field

kash> oi := AlffOrderMaxInfty(F);
Infinite maximal order of
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over Rational Field
Variables: T
given by transformation matrix
[1/T   0   0]
[  0 1/T   0]
[  0   0   1]
with denominator 1/T
kash> b := AlffEltToResField(AlffElt(oi,1/T),p);
0
kash> AlffResFieldEltLift(b,p);
[ 0, 0, 0 ]
```

NAME `AlffRootParams`

PURPOSE Sets/Displays the iteration depth of root expansions and the precision for series operations.

SYNTAX

```
L := AlffRootParams(F);
L := AlffRootParams(F, n, m);
```

`list`	`L`
`global function field`	`F`
`integer`	`n`
`integer`	`m`

DESCRIPTION If the infinite place of $k(T)$ is tamely ramified in F, all roots of the defining polynomial f of a global function field $F/\mathbb{F}_q(T)$ can be expanded into Puiseux series in $1/T$. The root expansions are obtained via the Newton–Puiseux method which computes the roots up to a given iteration depth n. The roots are represented as a truncated Puiseux series, and all series operations in F are performed with precision at least m. If $n < 25$ ($m < 50$) it will be set to $n = 25$ ($m = 50$). These parameters for the root computation can only be changed before the roots are computed (`AlffRoots()`). For details see [Sch96].

SEE ALSO `AlffRoots`,

EXAMPLE Change iteration depth and precision:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+T+1);
"Defining global variables: F, o, oi, one"
kash> AlffRootParams(F);
[ 25, 50 ]
kash> AlffRootParams(F, 50, 100);
[ 50, 100 ]
```

NAME `AlffRoots`

PURPOSE Computes the Puiseux expansions in $T^{-1/e}$ of the roots of the defining polynomial of a global function field $F/\mathbb{F}_q(T)$.

SYNTAX `L := AlffRoots(F);`

list	L
global function field	F

DESCRIPTION If the infinite place ∞ of $k(T)$ is tamely ramified in F, then all roots of the defining polynomial f of $F/\mathbb{F}_q(T)$ can be expanded in $T^{-1/e}$. The function returns a list

$$L := [\rho_{1,1}, \ldots, \rho_{1,e_1 f_1}, \rho_{2,1}, \ldots, \rho_{s,e_s f_s}],$$

where $\rho_{i,j} \in \mathbb{F}_{q^d}\langle T^{-1/e_i}\rangle \subset \mathbb{F}_{q^d}\langle T^{-1/e}\rangle$, $(1 \leq j \leq e_i f_i)$, are the root expansions at the s infinite places above ∞ (the e_i and f_i are the ramification indices and residue degrees of these places, e is the lcm of the e_i).

SEE ALSO AlffRootParams, InftyVal,

EXAMPLE Compute all roots of $y^3 + T^4 + 1$ over $\mathbb{F}_25$:

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffRoots(F);
[ 4*(T^(-1/3))^-4 + 3*(T^(-1/3))^8 + 4*(T^(-1/3))^20 + 2*(T^(-1/3))^56 + 4*(T^\
(-1/3))^68 + 2*(T^(-1/3))^80 + 2*(T^(-1/3))^116 + 4*(T^(-1/3))^128 + 2*(T^(-1/\
3))^140 + 4*(T^(-1/3))^176 + 3*(T^(-1/3))^188 + 4*(T^(-1/3))^200 + 4*(T^(-1/3)\
)^296 + 3*(T^(-1/3))^308 + 4*(T^(-1/3))^320 + 2*(T^(-1/3))^356 + 4*(T^(-1/3))^\
368 + 2*(T^(-1/3))^380 + 2*(T^(-1/3))^416 + 4*(T^(-1/3))^428 + 2*(T^(-1/3))^44\
0 + 4*(T^(-1/3))^476 + 3*(T^(-1/3))^488 + 4*(T^(-1/3))^500 + 2*(T^(-1/3))^1496\
 + O((T^(-1/3))^1498),
  w^20*(T^(-1/3))^-4 + w^2*(T^(-1/3))^8 + w^20*(T^(-1/3))^20 + w^14*(T^(-1/3))\
^56 + w^20*(T^(-1/3))^68 + w^14*(T^(-1/3))^80 + w^14*(T^(-1/3))^116 + w^20*(T^\
(-1/3))^128 + w^14*(T^(-1/3))^140 + w^20*(T^(-1/3))^176 + w^2*(T^(-1/3))^188 +\
 w^20*(T^(-1/3))^200 + w^20*(T^(-1/3))^296 + w^2*(T^(-1/3))^308 + w^20*(T^(-1/\
3))^320 + w^14*(T^(-1/3))^356 + w^20*(T^(-1/3))^368 + w^14*(T^(-1/3))^380 + w^\
14*(T^(-1/3))^416 + w^20*(T^(-1/3))^428 + w^14*(T^(-1/3))^440 + w^20*(T^(-1/3)\
)^476 + w^2*(T^(-1/3))^488 + w^20*(T^(-1/3))^500 + w^14*(T^(-1/3))^1496 + O((T\
^(-1/3))^1498),
```

```
  w^4*(T^(-1/3))^-4 + w^10*(T^(-1/3))^8 + w^4*(T^(-1/3))^20 + w^22*(T^(-1/3))^\
56 + w^4*(T^(-1/3))^68 + w^22*(T^(-1/3))^80 + w^22*(T^(-1/3))^116 + w^4*(T^(-1\
/3))^128 + w^22*(T^(-1/3))^140 + w^4*(T^(-1/3))^176 + w^10*(T^(-1/3))^188 + w^\
4*(T^(-1/3))^200 + w^4*(T^(-1/3))^296 + w^10*(T^(-1/3))^308 + w^4*(T^(-1/3))^3\
20 + w^22*(T^(-1/3))^356 + w^4*(T^(-1/3))^368 + w^22*(T^(-1/3))^380 + w^22*(T^\
(-1/3))^416 + w^4*(T^(-1/3))^428 + w^22*(T^(-1/3))^440 + w^4*(T^(-1/3))^476 + \
w^10*(T^(-1/3))^488 + w^4*(T^(-1/3))^500 + w^22*(T^(-1/3))^1496 + O((T^(-1/3))\
^1498) ]
```

NAME AlffSUnits

PURPOSE Find the S-regulator and a basis for the free part of the S-units.

SYNTAX `L := AlffSUnits( S [, "raw" ] );`

list	L	the S-regulator and a basis of S-units
list	S	a list of places

DESCRIPTION Let F/k be a global function field and S be a set of places of F/k. This function returns a list L such that $L[1]$ is the S-regulator and $L[2]$ is a basis for the free part of the S-unit subgroup of $F^\times$, consisting of principal divisors if the option `"raw"` is given.

SEE ALSO AlffClassGroupGens, AlffClassGroup, AlffInit, AlffOrdersAlffDivisorReduction, AlffDivisorLargeLDim, AlffPlacesDegOne,

EXAMPLE

```
kash> AlffInit(FF(7));;
kash> AlffOrders(y^2 + T^3 + T + 1);;
kash> S := AlffPlacesDegOne(F);
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, Alff place < [ T + 3, 0 ], [ 1, 1 ] >,
  Alff place < [ T + 3, 0 ], [ 6, 1 ] >, Alff place < [ T + 2, 0 ], [ 3, 1 ] >
    , Alff place < [ T + 2, 0 ], [ 4, 1 ] >,
  Alff place < [ T + 6, 0 ], [ 2, 1 ] >, Alff place < [ T + 6, 0 ], [ 5, 1 ] >
    , Alff place < [ T + 4, 0 ], [ 2, 1 ] >,
  Alff place < [ T + 4, 0 ], [ 5, 1 ] >, Alff place < [ T + 1, 0 ], [ 1, 1 ] >
    , Alff place < [ T + 1, 0 ], [ 6, 1 ] > ]
kash> AlffSUnits(S);
[ 11,
  [ [ 3*T^5 + 4*T^4 + 2*T^3 + 5*T^2 + 6*T + 4, 2*T^4 + T^3 + 3*T^2 + 3*T + 2 ]\
 / (T^11 + 5*T^10 + 5*T^9 + 3*T^8 + 4*T^7 + 3*T^4 + T^3 + T^2 + 2*T + 5),
      [ 1, 0 ] / (T + 3),
      [ 4*T^4 + 4*T^3 + 5*T^2 + 1, 4*T + 2 ] / (T^8 + 2*T^7 + 3*T + 6),
      [ 2*T^2 + 6*T + 2, 6*T + 2 ] / (T^5 + T^3 + 6*T^2 + 3*T + 1),
      [ 5*T^3 + 2*T^2 + 4*T + 2, 6*T + 5 ] / (T^6 + 5*T^4 + 5*T^3 + 2*T^2 + 6*\
T + 2),
      [ 5*T^3 + 2*T^2 + 3, 6*T^2 + 4*T + 2 ] / (T^7 + 3*T^6 + 5*T^5 + 6*T^4 + \
3*T^3 + 5*T^2 + 6*T + 6), [ 6*T^2 + T + 4, 1 ] / (T^4 + 6*T^3 + 2*T + 3),
      [ 2*T^4 + 3*T^3 + 2*T + 1, 4*T^3 + 3*T^2 + T + 1 ] / (T^9 + 5*T^7 + 3*T^\
```

```
2 + 1), [ 6, 1 ] / (T^3 + T + 2),
      [ 2*T^5 + 3*T^4 + T^3 + 4*T^2 + 6*T + 6, 4*T^3 + 5*T^2 + T + 4 ] / (T^10\
 + T^8 + 2*T^7 + 3*T^3 + 3*T + 6) ] ]
kash> AlffSUnits(S, "raw");
[ 11, [ Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -11 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 11 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -2 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 1 ],
        [ Alff place < [ T + 3, 0 ], [ 6, 1 ] >, 1 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -8 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 7 ],
        [ Alff place < [ T + 2, 0 ], [ 3, 1 ] >, 1 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -5 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 4 ],
        [ Alff place < [ T + 2, 0 ], [ 4, 1 ] >, 1 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -6 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 5 ],
        [ Alff place < [ T + 6, 0 ], [ 2, 1 ] >, 1 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -7 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 6 ],
        [ Alff place < [ T + 6, 0 ], [ 5, 1 ] >, 1 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -4 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 3 ],
        [ Alff place < [ T + 4, 0 ], [ 2, 1 ] >, 1 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -9 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 8 ],
        [ Alff place < [ T + 4, 0 ], [ 5, 1 ] >, 1 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -3 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 2 ],
        [ Alff place < [ T + 1, 0 ], [ 1, 1 ] >, 1 ] ]
        , Alff divisor
        [ [ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, -10 ],
        [ Alff place < [ T + 3, 0 ], [ 1, 1 ] >, 9 ],
        [ Alff place < [ T + 1, 0 ], [ 6, 1 ] >, 1 ] ]
```

```
] ]
```

Name `AlffSerreBound`

Purpose Computes the Serre bound of a global function field.

Syntax

```
b := AlffSerreBound(F);
b := AlffSerreBound(q, g);
```

global function field	F
integer	q, g
integer	b

Description Let g be the genus of a function field F over the exact constant field $\mathbb{F}_q$. This function computes the Serre bound (improvement of the Hasse-Weil bound) for the number of places of degree one of F using the function field or for given q and g.

See Also AlffIharaBound, AlffPlacesDegOneNumBound,

Example

```
kash> AlffInit(FF(2,3));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^3*y+T);
"Defining global variables: F, o, oi, one"
kash> AlffGenus(F);
3
kash> AlffSerreBound(F);
24
kash> AlffSerreBound(5, 7);
34
```

NAME `AlffSignature`

PURPOSE Returns the signature of a global function field extension $F/\mathbb{F}_q(T)$.

SYNTAX `L := AlffSignature(F);`

`list`	`L`
`global function field`	`F`

DESCRIPTION The function returns the list $L := [[e_1, f_1], \ldots, [e_s, f_s]]$ consisting of the ramification and relative degrees of the inequivalent valuations of F extending the degree valuation of $\mathbb{F}_q(T)$ ordered according to

$$(e_i < e_j) \text{ or } (e_i = e_j \text{ and } f_i \leq f_j) \text{ for all } i < j.$$

SEE ALSO AlffPlaceSplit, AlffUnitRank,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffSignature(F);
[ [ 3, 1 ] ]
```

NAME AlffTameInftyPlace

PURPOSE Returns whether the infinite place of $k(T)$ is tamely ramified in the algebraic field extension $F/k(T)$, or not.

SYNTAX `b := AlffTameInftyPlace(F);`

boolean	b
algebraic function field	F

DESCRIPTION See `Alff` for definitions.

SEE ALSO AlffPlaceSplit,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffTameInftyPlace(F);
true
```

NAME AlffUnitRank

PURPOSE Returns the unit rank of a global function field extension $F/\mathbb{F}_q(T)$.

SYNTAX `r := AlffUnitRank(F);`

integer	r
global function field	F

DESCRIPTION The number $s-1$, where s denotes the number of inequivalent valuations extending the degree valuation of $\mathbb{F}_q(T)$, is returned.

SEE ALSO AlffSignature,

EXAMPLE

```
kash> AlffInit(FF(5,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> AlffUnitRank(F);
0
```

NAME AlffUnitsFund

PURPOSE Computes $s-1$ fundamental units of a global function fields's finite maximal order.

SYNTAX `L := AlffUnitsFund(o);`

list	L	
finite maximal order	o	of a global function field

DESCRIPTION This function computes $s-1$ fundamental units of the given finite maximal order of a global function field F (the integral closure of $k[T]$ in F), where s is the number of places of F at infinity. A list containing the units is returned. The infinite place of $k(T)$ must have tame ramification in F. See [Sch96] for further definitions and algorithms.

SEE ALSO AlffRegulator,

EXAMPLE

```
kash> AlffInit(FF(5,1));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^4+(2*T+3)*y^3+y^2+(3*T+2)*y+1);
"Defining global variables: F, o, oi, one"
kash> AlffUnitsFund(o);
[ [ 1, 1, 0, 0 ], [ 1, 4*T, 4*T + 3, 2 ], [ 3*T + 1, 1, 2*T + 3, 1 ] ]
```

NAME `AlffVarT`

PURPOSE missing shortdoc

SYNTAX `T := AlffVarT(F);`

algebraic function field	F
polynomial	T

DESCRIPTION Returns the indeterminate T of the generating equation of the algebraic function field.

SEE ALSO `AlffEltGenT`, `Alff`,

EXAMPLE

```
kash> AlffInit(FF(7,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff((T^2+1)*y^3+y+T^4+1);
Algebraic function field defined by
$.1^3*$.2^2 + $.1^3 + $.1 + $.2^4 + 1
over
Univariate rational function field over GF(7^2)
Variables: T

kash> AlffVarT(F);
T
```

NAME AlffVarY

PURPOSE missing shortdoc

SYNTAX `y := AlffVarY(F);`

algebraic function field	F
polynomial	y

DESCRIPTION Returns the indeterminate y of the generating equation of the algebraic function field.

SEE ALSO AlffEltGenY, Alff,

EXAMPLE

```
kash> AlffInit(FF(7,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff((T^2+1)*y^3+y+T^4+1);
Algebraic function field defined by
$.1^3*$.2^2 + $.1^3 + $.1 + $.2^4 + 1
over
Univariate rational function field over GF(7^2)
Variables: T

kash> AlffVarY(F);
y
```

NAME `AlffWeierstrassPlaces`

PURPOSE Returns the Weierstraß places of a divisor.

SYNTAX
```
L := AlffWeierstrassPlaces( D );
L := AlffWeierstrassPlaces( F );
```

`list`	`L`	containing the Weierstraß places
`alff divisor`	`D`	
`alff`	`F`	equivalent to taking $D = 0$

DESCRIPTION Let F/k be an algebraic function field, D a divisor and P a place of degree one. An integer $m \geq 1$ is a D-gap number of P if $\dim\big(D+(m-1)P\big) = \dim(D+mP)$ holds. The D-gap numbers m of P satisfy $1 \leq m \leq 2g-1-\deg(D)$ and their cardinality equals the index of speciality $i(D)$. The sequences of D-gap numbers are independent of constant field extensions for perfect k and are the same for all but a finite number of places P of degree one (consider e.g. k algebraically closed). The places P of degree one which have different sequences of D-gap numbers are called D-Weierstraß places.
This function returns a list of all places of F/k (having not necessarily degree one) which are lying below D-Weierstraß places of $F\bar{k}/\bar{k}$ (k perfect). The constant field k is required to be exact. Remark: If the characteristic of F is positive this function is currently very slow for large genus because of `AlffDifferentiation()`.

SEE ALSO `AlffGapNumbers`, `AlffRamDivisor`, `AlffWronskian`, `AlffWronskianOrders`, `AlffDiff`, `AlffDifferentiation`,

EXAMPLE

```
kash> AlffInit(Q);;
kash> AlffOrders(y^2 - (T-1)*(T-2)*(T-3)*(T-4)*(T-5)*(T-6)*(T-7));
"Defining global variables: F, o, oi, one"
kash> AlffWeierstrassPlaces(F);
[ Alff place < [ 1/T, 0 ], [ 0, 1 ] >, Alff place < [ T - 7, 0 ], [ 0, 1 ] >,
  Alff place < [ T - 6, 0 ], [ 0, 1 ] >, Alff place < [ T - 5, 0 ], [ 0, 1 ] >
    , Alff place < [ T - 4, 0 ], [ 0, 1 ] >,
  Alff place < [ T - 3, 0 ], [ 0, 1 ] >, Alff place < [ T - 2, 0 ], [ 0, 1 ] >
    , Alff place < [ T - 1, 0 ], [ 0, 1 ] > ]
```

NAME `AlffWronskian`

PURPOSE Returns the Wronski matrix of a Riemann-Roch space.

SYNTAX
```
W := AlffWronskian(D);
W := AlffWronskian(F);
```

`list`	`W`	list of rows
`alff divisor`	`D`	$\mathcal{L}(D)$ is computed
`alff`	`F`	equivalent to taking $D = 0$

DESCRIPTION Let F/k be an algebraic function field with separating element x and let $v_1, \dots v_l$ be a basis of $\mathcal{L}(D)$. For the differentiation D_x with respect to x consider the successively smallest $\nu_1 \le \cdots \le \nu_l \in \mathbb{Z}^{\ge 0}$ such that the rows $D_x^{(\nu_i)}(v_1), \dots, D_x^{(\nu_i)}(v_l)$, $1 \le i \le l$ are F-linearly independent. These rows form the Wronski matrix of D with respect to x and are returned as a list of lists W. If D has dimension zero, `false` is returned. The constant field k is required to be exact.

SEE ALSO AlffWronskianOrders, AlffDiff, AlffDifferentiation, AlffRamDivisor,

EXAMPLE

```
kash> AlffInit(Q);;
kash> AlffOrders(y^2 - (T-1)*(T-2)*(T-3)*(T-4)*(T-5)*(T-6)*(T-7));
"Defining global variables: F, o, oi, one"
kash> AlffWronskian(AlffCanonicalDivisor(F));
[ [ [ 0, 1 ] / (T^7 - 28*T^6 + 322*T^5 - 1960*T^4 + 6769*T^3 - 13132*T^2 + 130\
68*T - 5040),
      [ 0, T ] / (T^7 - 28*T^6 + 322*T^5 - 1960*T^4 + 6769*T^3 - 13132*T^2 + 1\
3068*T - 5040),
      [ 0, T^2 ] / (T^7 - 28*T^6 + 322*T^5 - 1960*T^4 + 6769*T^3 - 13132*T^2 +\
 13068*T - 5040) ],
  [ [ 0, -7/2*T^6 + 84*T^5 - 805*T^4 + 3920*T^3 - 20307/2*T^2 + 13132*T - 6534\
 ] / (T^14 - 56*T^13 + 1428*T^12 - 21952*T^11 + 226982*T^10 - 1667568*T^9 + 89\
62364*T^8 - 35733376*T^7 + 105994833*T^6 - 232253336*T^5 + 369120808*T^4 - 411\
449472*T^3 + 303143184*T^2 - 131725440*T + 25401600),
      [ 0, -5/2*T^7 + 56*T^6 - 483*T^5 + 1960*T^4 - 6769/2*T^3 + 6534*T - 5040\
 ] / (T^14 - 56*T^13 + 1428*T^12 - 21952*T^11 + 226982*T^10 - 1667568*T^9 + 89\
62364*T^8 - 35733376*T^7 + 105994833*T^6 - 232253336*T^5 + 369120808*T^4 - 411\
449472*T^3 + 303143184*T^2 - 131725440*T + 25401600),
      [ 0, -3/2*T^8 + 28*T^7 - 161*T^6 + 6769/2*T^4 - 13132*T^3 + 19602*T^2 - \
```

```
10080*T ] / (T^14 - 56*T^13 + 1428*T^12 - 21952*T^11 + 226982*T^10 - 1667568*T\
^9 + 8962364*T^8 - 35733376*T^7 + 105994833*T^6 - 232253336*T^5 + 369120808*T^\
4 - 411449472*T^3 + 303143184*T^2 - 131725440*T + 25401600) ],
  [ [ 0, 63/8*T^12 - 378*T^11 + 16331/2*T^10 - 104860*T^9 + 3562405/4*T^8 - 52\
64168*T^7 + 44354149/2*T^6 - 67008956*T^5 + 1151518655/8*T^4 - 214111870*T^3 +\
 209157193*T^2 - 120435336*T + 30947094 ] / (T^21 - 84*T^20 + 3318*T^19 - 8192\
8*T^18 + 1417983*T^17 - 18282684*T^16 + 182182984*T^15 - 1436603616*T^14 + 910\
4502015*T^13 - 46835050444*T^12 + 196681551150*T^11 - 675791033064*T^10 + 1898\
151600817*T^9 - 4340764298724*T^8 + 8021533034676*T^7 - 11838983956912*T^6 + 1\
3715719388784*T^5 - 12159503504064*T^4 + 7936942375872*T^3 - 3582803508480*T^2\
 + 995844326400*T - 128024064000),
      [ 0, 35/8*T^13 - 196*T^12 + 7763/2*T^11 - 44492*T^10 + 1292585/4*T^9 - 1\
512140*T^8 + 8504693/2*T^7 - 4475548*T^6 - 120419341/8*T^5 + 76204800*T^4 - 15\
9963615*T^3 + 188151768*T^2 - 120624498*T + 32931360 ] / (T^21 - 84*T^20 + 331\
8*T^19 - 81928*T^18 + 1417983*T^17 - 18282684*T^16 + 182182984*T^15 - 14366036\
16*T^14 + 9104502015*T^13 - 46835050444*T^12 + 196681551150*T^11 - 67579103306\
4*T^10 + 1898151600817*T^9 - 4340764298724*T^8 + 8021533034676*T^7 - 118389839\
56912*T^6 + 13715719388784*T^5 - 12159503504064*T^4 + 7936942375872*T^3 - 3582\
803508480*T^2 + 995844326400*T - 128024064000),
      [ 0, 15/8*T^14 - 70*T^13 + 2051/2*T^12 - 6076*T^11 - 69307/4*T^10 + 5723\
20*T^9 - 9420035/2*T^8 + 22324484*T^7 - 544398673/8*T^6 + 134268134*T^5 - 1599\
63615*T^4 + 85289400*T^3 + 30947094*T^2 - 65862720*T + 25401600 ] / (T^21 - 84\
*T^20 + 3318*T^19 - 81928*T^18 + 1417983*T^17 - 18282684*T^16 + 182182984*T^15\
 - 1436603616*T^14 + 9104502015*T^13 - 46835050444*T^12 + 196681551150*T^11 - \
675791033064*T^10 + 1898151600817*T^9 - 4340764298724*T^8 + 8021533034676*T^7 \
- 11838983956912*T^6 + 13715719388784*T^5 - 12159503504064*T^4 + 7936942375872\
*T^3 - 3582803508480*T^2 + 995844326400*T - 128024064000) ] ]
```

Name	`AlffWronskianOrders`
Purpose	Returns the Wronski orders of a Riemann-Roch space.
Syntax	`W := AlffWronskianOrders(D);` `W := AlffWronskianOrders(F);`

`list`	`W`	list of rows
`alff divisor`	`D`	$\mathcal{L}(D)$ is computed
`alff`	`F`	equivalent to taking $D = 0$

Description

Let F/k be an algebraic function field with separating element x and let $v_1, \ldots v_l$ be a basis of $\mathcal{L}(D)$. For the differentiation D_x with respect to x consider the successively smallest $\nu_1 \leq \cdots \leq \nu_l \in \mathbb{Z}^{\geq 0}$ such that the rows $D_x^{(\nu_i)}(v_1), \ldots, D_x^{(\nu_i)}(v_l)$, $1 \leq i \leq l$ are F-linearly independent. The numbers $\nu_1, \ldots, \nu_l$ are the Wronski orders of D with respect to x and are returned. If D has dimension zero, the empty list is returned. The constant field k is required to be exact.

See Also

`AlffWronskian`, `AlffDiff`, `AlffDifferentiation`, `AlffRamDivisor`,

Example

```
kash> AlffInit(Q);;
kash> AlffOrders(y^2 - (T-1)*(T-2)*(T-3)*(T-4)*(T-5)*(T-6)*(T-7));
"Defining global variables: F, o, oi, one"
kash> AlffWronskianOrders(AlffCanonicalDivisor(F));
[ 0, 1, 2 ]
```

NAME ArcCos

PURPOSE Returns the inverse cosine of a number.

SYNTAX `y := ArcCos(x);`

```
complex  y
complex  x
```

DESCRIPTION Given an `x` the function returns the inverse cosine (arccos) of `x`. The computation is done in the current precision of the real field.

SEE ALSO `Cos`, `Tan`,

EXAMPLE

```
kash> ArcCos(.5);
1.047197551196597746154214461093167628065723133126
```

NAME ArcSin

PURPOSE Returns the inverse sine of a number.

SYNTAX `y := ArcSin(x);`

```
complex  y
complex  x
```

DESCRIPTION Given an `x` the function returns the inverse sine (arcsin) of `x`. The computation is done in the current precision of the real field.

SEE ALSO Sin, Cos, Tan,

EXAMPLE

```
kash> ArcSin(.5);
0.5235987755982988730771072305465838140328615665624224
```

NAME	`ArcTan`
PURPOSE	Returns the inverse tangens of a number.
SYNTAX	`y := ArcTan(x);`
	`complex y` `complex x`
DESCRIPTION	Given an `x` the function returns the inverse tangens (arctan) of `x`. The computation is done in the current precision of the real field.
SEE ALSO	`Sin`, `Cos`, `Tan`,
EXAMPLE	

```
kash> ArcTan(.5);
0.4636476090008061162142562314612144020285370542861202
```

Arg

NAME	`Arg`
PURPOSE	Returns the argument of a complex number.
SYNTAX	`phi := Arg(z);`

`real`	`phi`
`complex`	`z`

DESCRIPTION Returns $\phi \in (-\pi, \pi]$ for $z = |z|e^{i\phi} \in \mathbb{C}$.

EXAMPLE Compute the argument of i

```
kash> z := Comp(0, 1);
1*i
kash> Arg(z);
1.570796326794896619231321691639751442098584699688
```

Compute the argument of -1

```
kash> z := Comp(-1, 0);
-1
kash> Arg(z);
3.141592653589793238462643383279502884197169399375
```

NAME `BagRead`

PURPOSE Reads a bag from an open file.

SYNTAX `BagRead(file); BagRead("filename");`

`File file`

DESCRIPTION

SEE ALSO BagWrite,

EXAMPLE See `BagWrite` for an example.

BagWrite

NAME	`BagWrite`
PURPOSE	Writes a bag to an open file.
SYNTAX	`BagWrite (file, arg [,arg]); BagWrite("filename", arg [,arg]);`

`File`	`file`
`any expression`	`arg`

DESCRIPTION

SEE ALSO `BagRead`,

EXAMPLE Open new file for writing.

```
kash> file:=Open("/tmp/dump","w");
Filename: /tmp/dump / Mode: w / Open (fid): 5
kash> f:=Poly(Zx,[1,0,-5]);;
```

Write data to file.

```
kash> BagWrite(file,f);
```

Close file.

```
kash> Close(file);
true
```

Open existing file for appending.

```
kash> file:=Open("/tmp/dump","a");
Filename: /tmp/dump / Mode: a / Open (fid): 5
```

Append data to file.

```
kash> o:=Order(f);;
kash> BagWrite(file,o);
kash> e1:=Elt(o,[1,0]/1);;
kash> e2:=Elt(o,1/1);;
kash> e3:=Elt(o,[-5,0]/1);;
kash> e4:=Elt(o,[1,1]/1);;
kash> BagWrite(file,e1,e2,e3,e4);
kash> M:=OrderMaximal(Order (o,[e1,e2,e3,e4]));;
kash> BagWrite(file,M);
```

Close file.

```
kash> Close(file);
true
```

Open existing file for reading.

```
kash> file:=Open("/tmp/dump","r");
Filename: /tmp/dump / Mode: r / Open (fid): 5
```

Now, reread the data from this file.

```
kash> BagRead(file);
[ x^2 - 5 ]
kash> BagRead(file);
[ Generating polynomial: x^2 - 5
     ]
kash> BagRead(file);
[ 1, 1, -5, [1, 1] ]
kash> BagRead(file);
[    F[1]
        |
       F[2]
      /
     /
    Q
```

```
    F  [ 1]     Given by transformation matrix
    F  [ 2]     x^2 - 5
    Discriminant: 5
     ]
kash> BagRead(file);
false
```

Close file.

```
kash> Close(file);
true
```

NAME Bell

PURPOSE Sounds the terminal bell.

SYNTAX `Bell();`

DESCRIPTION

EXAMPLE

```
Bell();
```

Bernoulli

NAME Bernoulli

PURPOSE Returns a list of Bernoulli numbers.

SYNTAX `bern := Bernoulli(n);`

integer	n
list	bern

DESCRIPTION Given an integer n, the list of the rational Bernoulli numbers $0, 1, 2, 4, 6, ..., 2n-2$ is returned. For small n BernoulliMagma is much faster, but not exact and for $n \geq 130$ not computable.

SEE ALSO BernoulliMagma,

EXAMPLE

```
kash> bern := Bernoulli(6);
[ 1/1, -1/2, 1/6, -1/30, 1/42, -1/30 ]
```

Name `BernoulliMagma`

Purpose Returns a list of Bernoulli numbers.

Syntax `bern := BernoulliMagma(n);`

```
integer  n
list     bern
```

Description Given an integer n, the list of Bernoulli numbers $2, ..., 2n$ is returned. For $n \geq 130$ this function does not work, see Bernoulli instead.

See Also `Bernoulli`,

Example

```
kash> bern := BernoulliMagma(6);
[ 0.1666666666666666651508421637117862701416015625,
  -0.03333333333333333371228945907205343246459960937 5,
  0.023809523809523807358345948159694671630859375,
  -0.03333333333333333371228945907205343246459960937 5,
  0.07575757575757576205432997085154056549072265625,
  -0.2531135531135530982282944023609161376953125 ]
```

NAME C

PURPOSE Predefined constant: Complex field $\mathbb{C}$.

SYNTAX `C;`

```
ring  C
```

DESCRIPTION This is a predefined constant for the complex field $\mathbb{C}$ and is referenced by the variable `C`. The precision can be set or displayed with `Prec`. Default precision is 50.

SEE ALSO `Z`, `Q`, `R`, `Prec`,

EXAMPLE

```
kash> C;
Complex Field of precision 52
```

NAME `Ceil`

PURPOSE Returns the minimal integer greater than or equal to a number.

SYNTAX `y := Ceil(x);`

`integer`	`y`
`real`	`x`

DESCRIPTION Given an `x` in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ the function returns

$$\min\{k \in \mathbb{Z} : k \geq \mathtt{x}\}.$$

The computation is done in the current precision of the real (complex) field.

SEE ALSO `Trunc`, `Round`, `Floor`,

EXAMPLE

```
kash> Ceil(3);
3
kash> Ceil(-3);
-3
kash> Ceil(3.5);
4
kash> Ceil(-3.5);
-3
```

CharPoly

Name `CharPoly`

Purpose Computes the characteristic polynomial of an algebraic element, a matrix or an alff element

Syntax

```
p := CharPoly (a [,PA]);
p := CharPoly (a [,O]);
p := CharPoly( M );
```

polynomial	p
polynomial algebra	PA
suborder	O
algebraic element alff order element	a
matrix	M

Description For an algebraic element a in an Order O and a polynomial algebra PA over another order o or this order, this function computes the characteristic polynomial of a over o i.e. the returned polynomial is contained in PA.
Given a matrix this function returns its characteristic polynomial
For an alff element a in an order o in the alff F this function computes the characteristic polynomial of a over o

See Also EltCharPolyAlffEltCharPolyMatCharPoly,

Example

```
kash> o := Order(Z,2,2);
Generating polynomial: x^2 - 2

kash> a := Elt(o,[0,1]);
[0, 1]
kash> p := CharPoly(a);
x^2 - 2
kash> M := Mat(o,[[1,a],[1,-a]]);
[1 [0, 1]]
[1 [0, -1]]
kash> p := CharPoly(M);
x^2 + [-1, 1]*x + [0, -2]
```

NAME `CharToInt`

PURPOSE Returns the ASCII code of a character.

SYNTAX `i := CharToInt(c);`

```
character  c
integer    i
```

SEE ALSO IntToChar,

EXAMPLE

```
kash> CharToInt('a');
97
```

Characteristic

NAME Characteristic

PURPOSE Return the characteristic of a ring.

SYNTAX `m := Characteristic(R);`

integer	m
ring	R

EXAMPLE

```
kash> Characteristic(Z);
0
kash> Characteristic(C);
0
kash> Characteristic(FF(7, 2));
7
```

NAME `CheckArgus`

PURPOSE Handy tool to check the input to a kash function.

SYNTAX `r = CheckArgus (hdCall,string);`

`r`	`int`	number of type matching format string
`hdCall`	`TypHandle`	all arguments
`string of format strings`	`t_string`	

DESCRIPTION

Handle–Type	Format	Description
T_BOOL	%B	
T_INT	%d	
T_KantELT	%E	
T_KantFF	%FF	
T_KantFFE	%FFE	
T_KantFIELD	%FLD	
T_KantFFO	%FFO	
T_KantFFOE	%FFOE	
T_KantFFOI	%FFOI	
T_KantFFS	%FFS	
T_KantFFSE	%FFSE	
T_KantFFPL	%FFPL	
T_KantFFDI	%FFDI	
T_KantIDEAL	%I	
T_LIST, T_VECTOR	%L	
T_KantLATELT	%LE	
T_KantLAT	%LT	
T_KantMAT	%M	
T_KantMODULE	%MO	
T_KantORDER	%O	
T_KantPOLY	%P	
T_KantPOLYALG	%PA	
T_PERM16, T_PERM32	%PERM	
T_RAT	%Q	
T_KantQF	%QF	
T_KantQFE	%QFE	
T_KantREAL	%R	
T_KantRING	%r	
T_STRING	%s	
T_KantSIMP	%SI	
T_KantTHUE	%TH	
T_KantVR	%VR	
T_KantVRE	%VRE	
T_KantINT	%Z	

NAME	`ChineseRemainder`
PURPOSE	Chinese remainder for integers, ideals, and polynomials.

SYNTAX

```
elt := ChineseRemainder(M,L);
elt := ChineseRemiander(LM);
elt := ChineseRemainder(m1, m2, a1, a2);
```

integer, polynommial, or order element	elt
list of elements (integers, polynomials, or order elements)	L
list of modules (integers, polynomials, or ideals)	M
list of pairs of elements and modules	LM
modules (integers, polynomials, or ideals)	m1,m2
elements (integers, polynomials, or order elements)	a1,a2

SEE ALSO `IdealChineseRemainder`,

Name	`Close`
Purpose	Closes a file.
Syntax	`closed := Close(f);`

```
File     f
boolean  closed
```

Description	
See Also	`BagRead`, `BagWrite`, `ECHOon`, `ECHOoff`, `FLDin`, `FLDout`, `LOFILES`, `Open`,
Example	See `Open` for an example.

NAME Coeff

PURPOSE

SYNTAX `a := Coeff(poly, i [,"x"]);`

ring element	a
polynomial	poly
positive integer	i

DESCRIPTION

SEE ALSO Coef,

EXAMPLE

```
kash> a := Coeff(2*x^3+4*x^2-7*x+18,2);
4
```

NAME ColorString

PURPOSE Defines a string for use in `Print` to get colorful output.

SYNTAX `s := ColorString(l);`

`string`	`s`
`list of strings`	`l`

DESCRIPTION This function generates a *VT100/ANSI* compatibel escape sequence to change the output color. Regardless of the names used here, the actual outcome is influenced by the terminal itself: the terminal may have a visible bell, different colors. The terminal can ignore certain combinations, refuse to blink, ...
The `l` argument contains strings of the following form:

- `"f_`*`color`*`"` with *`color`* one of `black, red, green, brown, blue, magenta, cyan, white`. This defines the *foreground* color.
- `"b_`*`color`*`"` for the background color.
- `"bright", "blink", "underscore", "reverse", "blank"` to set additional attributes.
- `"bell"` to include a terminal-bell.

For example to produce red letters on a blue background, one should use `ColorString("f_red", "b_blue")`. Depending on the *termcap* entries and (eventually) the `.Xresources`.
To get yellow, one should use `"brown"` together with `"bright"`.

EXAMPLE

```
Print(ColorString("f_red", "b_black", "bright"),
"Hallo", ColorString("normal"), "\n");
```

NAME	`Colors`
PURPOSE	Activate or deactivate color mode and customize colors.

SYNTAX

```
Colors(flag);
Colors(s1, ..., sn);
Colors(L);
```

`boolean`	`flag`	switch color mode on or off
`string`	`s1, ..., sn`	resource strings
`list`	`L`	list of lists of resource strings

DESCRIPTION

This function can be used to switch on or off a color mode in KASH compatible with `ANSI/ISO 6429` standards. The resource strings consist first of one of the strings `"normal"`, `"prompt"`, `"command"`, `"result"` or `"error"` to indicate for which type of output a color should be defined. The other strings can be `"black"`, `"red"`, `"green"`, `"yellow"`, `"blue"`, `"magenta"`, `"cyan"` or `"white"` and additionally `"bold"` or `"light"`. The color mode works only for terminals which can display colored letters like ”`color_xterm`” or ”`rxvt`”. See the instructions for your terminal.
The default colors can be customized via the `_ColorTable` array.
`_ColorTable` is a list with exacly 5 entries defining (in this order) the ”normal”, ”prompt”, ”command”, ”result”, ”error” colors. Each entry is a list as described in `ColorString`.

EXAMPLE

```
kash> Colors(true);
kash> Colors("error", "red", "bold");
kash> Colors([ [ "error", "red" ], [ "result", "green", "bold" ] ]);
kash> Colors(false);
```

Comp

NAME Comp

PURPOSE Creates a complex number.

SYNTAX `z := Comp(a, b);`

complex	z
real	a
real	b

DESCRIPTION Creates the complex number $z = a + ib$.

EXAMPLE Create the complex number $1 + 2i$:

```
kash> z := Comp(1, 2);
1 + 2*i
```

NAME `ComplexGamma`

PURPOSE Returns .

SYNTAX `c := ComplexGamma(z);`

```
complex  z
complex  c
```

DESCRIPTION Given .

EXAMPLE

```
kash> c := ComplexGamma(Comp(2.4,3.1));
-0.1751454757439141626699741825902816647895320549295366 + 0.052332890056926068\
5054381992239370308373178721307248*i
```

NAME `Conj`

PURPOSE Returns the conjugate complex number.

SYNTAX `w := Conj(z);`

```
complex  w
complex  z
```

EXAMPLE Compute the conjugate complex number of $1 + 2i$:

```
kash> z := Comp(1, 2);
1 + 2*i
kash> w := Conj(z);
1 - 2*i
```

NAME Cos

PURPOSE Returns the cosine of a number.

SYNTAX `y := Cos(x);`

```
complex  y
complex  x
```

DESCRIPTION Given an `x` (in radians) the function returns the cosine of `x`. The computation is done in the current precision of the real (complex) field.

SEE ALSO Sin, Tan,

EXAMPLE

```
kash> Cos(.5);
0.877582561890372716116281582603829651991645197109744
kash> i := Comp(0, 1);
1*i
kash> Cos(i);
1.54308063481524377847790562075706168260152911236
```

Date

NAME Date

PURPOSE Returns the current date and time in a record

SYNTAX `t := Date( );`

`record t`

EXAMPLE An example:

```
kash> t := Date();
22.2.2004 15:55:17
```

NAME DbQuery

PURPOSE Performs a query in the database "kate"

SYNTAX `DbQuery(query [, num])`

string	query
integer	num

DESCRIPTION DbQuery(query) performs a SQL-Query in the KANT-database "kate". SQL means Structured-Query-Language, a language for executing a query in the database.
You can also access the database through a web interface at

http://www.math.tu-berlin.de/cgi-bin/kant/database.cgi

The following table lists all possible names of table-names in the database:

1. degree
2. class_number
3. galois_grno
4. sig_real
5. sig_im
6. discriminant
7. disc_dec_len
8. disc_ascii
9. regulator
10. reg_ascii disc_known
11. reg_known
12. generator_roots_unity
13. poly_coeff
14. number_roots_unity
15. number_cyclic_factors_clgr
16. list_of_orders_cyclic_factors_clgr

If you use this table-names in your query, it is not case sensitive and the same holds for using underscore, minus and space, this means for example their is no difference

between sig_real, sig-real or sig real.
Most of the table-names are self-explanatory:

degree	degree of a number field
class_number	class number of a number field
galois_grno	every galois-group has a galois-number, which is to be the same an in the Atlas for groups. If you know this number, you can use this in your query
sig_real	real signature of a number field
sig_im	imaginary signature of a number field
discriminant	discriminant of a number field, but if it is too big, so see disc_ascii
disc_dec_len	decimal length of discriminant
disc_ascii	discriminant as ascii, here you have the exact value for the discriminant, but as text(ascii) and you have to convert,if possible, into longinteger
regulator	regulator of an number field
reg_ascii	regulator as ascii, the exact value for the regulator as text
reg_known	is an integer, which means regulator is known, if reg_known = 1, regulator unknown otherwise
generator_roots_unity	generator of the roots of unity
poly_coeff polynomial	list of coefficients of the defining
number_roots_unity	number roots of unity
number_cyclic_factors_clgr	number of cyclic factors of the class group
list_of_orders_cyclic_factor_clgr	list of orders of the cyclic factors of classgroup

Example For example, if you want fields with degree=3, class number=2 discriminant > 0 and discrimiant < 10000 ordered by discriminant:

```
kash> DbQuery("deg=3 and class number=2 and disc> 0 and disc<10000 order by disc");
```

and for a descending order

```
kash> DbQuery("deg=3 and class number=2 and disc>0 and disc<10000 order by -disc");
```

So you will get a list up to 1000 orders in the database with this properties. If you want the next 1000 orders, you have to change your estimate for the discriminant: if the last order of the list L has discriminant=9876, then you query should be

```
kash> DbQuery("deg=3 and class number=2 and disc>=9876 order by disc");
```

If you want to know how many fields are in the database with this porperties, you have to type

```
kash> DbQuery("count deg=3 and class number=2 and disc>0 and disc<10000 order by disc");
```

You will get a statistic foro the database, if you type

```
kash> DbQuery("Statistic");
```

NAME DedekindEta

PURPOSE Calculates the value of Dedekind's η–function.

SYNTAX `u := DedekindEta(z);`

complex	u	
complex	z	complex number with $\mathrm{Im}\,(z) > 0$

DESCRIPTION Let z be a complex number with $\mathrm{Im}\,(z) > 0$. The value

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

of the Dedekind's η–function (with $q = e^{2\cdot\pi\cdot i\cdot z}$) is then a well–defined complex number. It is determined by writing the infinite product as a (quickly converging) infinite sum and calculating this sum up to a certain precision, that depends on the precision of the defining complex field. For further information about the algorithm see [Sch90b].

EXAMPLE Compute $\eta(1 + \sqrt{-5})$:

```
kash> DedekindEta(1+Sqrt(-5));
0.537906647812710452043726684037109359674831806688O552 + 0.1441316518847481027\
8966577928836599098451545457093O1*i
```

NAME Den

PURPOSE Returns the denominator of an object.

SYNTAX

```
d := Den( a );
d := Den(a, "rep");
d := Den( q );
I := Den( I );
```

integer	d
quotient field element or polynomial	d
rational	q
algebraic element	a
algebraic function field order element	a
quotient field element or polynomial	q
ideal	I

DESCRIPTION For a rational number or a rational function the function returns the denominator of the rational argument.
The denominator of the algebraic number a is the smallest natural number d such that $d \cdot a$ is an algebraic integer. If you use `Den(a, "rep")` the denominator in the representation of a is returned.
For an algebraic function field order element see `AlffEltDen`.
The denominator of a fractional ideal $\mathfrak{a}$ is the smallest positive integer d that $d \cdot \mathfrak{a}$ is an integral ideal.

EXAMPLE An absolute example for an algebraic element:

```
kash> o := Order (x^3 - 23*x^2 + 146*x - 244);;
kash> a := Elt (o, [1/2,3/5,45/5657567]);
[28287835, 33945402, 450] / 56575670
kash> Den ( a );
56575670
kash> Den ( a, "rep" );
56575670
```

NAME `Disc`

PURPOSE Computes the discriminant of a polynomial, an order, an algebraic element, an alff order or a lattice.

SYNTAX

```
d := Disc(f);
d := Disc(o);
d := Disc(l);
```

ring element	d
polynomial	f
order alff order	o
lattice	l

DESCRIPTION

EXAMPLE We will compute some indices:

```
kash> o := OrderMaximal(Order(x^6 - 9*x^4 - 4*x^3 + 27*x^2 - 36*x - 23));;
kash> O := OrderShort(o);;
kash> Disc(OrderPoly(Zx, o)) / Disc(o);
121352256
kash> Disc(OrderPoly(Zx, O)) / Disc(O);
25
```

NAME `DrinfMPhi`

PURPOSE Definition of a Drinfeld Modul

SYNTAX `D:= DrinfMPhi(f,L);`

finite field polynomial	D
finite field polynomial	f
list	L

DESCRIPTION Consider the ring $A := \mathbb{F}_q[T]$ as $\mathbb{F}_q$ vector space and the ring $A\{\tau\}$, defined by the Frobenius map. This function is an implementation of the homomorphism $\phi : A \to A\{\tau\}$ defined by

$$\phi(T) = T + a_1\tau + a_2\tau^2 + \vdots + a_r\tau^r$$

with $L := [a_1, .., a_r]$, $a_i \in \mathbb{F}_q$. r is the rank of the modul. The output is given by

$$z \mapsto Tz + a_1 z^q + a_2 z^{q^2} + \vdots + a_r z^{q^r}$$

This is a preliminary version. To compute $\phi(T^2)$ the user must calculate using the homomorphism $\phi(T^2) = \phi(T)\phi(T)$

EXAMPLE

```
kash> k:=FF(3);
Finite field of size 3
kash> kT:=PolyAlg(k,"T");
Univariate Polynomial Ring in T over GF(3)

kash> kTz:=PolyAlg(kT,"z");
Univariate Polynomial Ring in z over Univariate Polynomial Ring in T over GF(3\
)

kash> L:=[1,1];
[ 1, 1 ]
kash> DrinfMPhi(T,L);
Drinfeld module of rank 2
z^9 + z^3 + T*z
```

NAME `DrinfMProduct`

PURPOSE Computes a non-commutative product of polynomials

SYNTAX `az := DrinMProduct(a,z);`

```
finite field polynomial  az
finite field polynomial  a
finite field polynomial  z
```

DESCRIPTION Let $a, z \in A = \mathbb{F}_q[T]$ be polynomials and τ the Frobenius map. This function computes the product of a and z given by $\tau a = a^q \tau$ in the non-commutative subring of all $\mathbb{F}_q$-linear maps of A.

SEE ALSO `DrinTau`,

EXAMPLE

```
kash> k:=FF(3);
Finite field of size 3
kash> kT:=PolyAlg(k,"T");
Univariate Polynomial Ring in T over GF(3)

kash> kTz:=PolyAlg(kT,"z");
Univariate Polynomial Ring in z over Univariate Polynomial Ring in T over GF(3\
)

kash> DrinfMProduct(T,z);
T*z^3
```

NAME `DrinfMTau`

PURPOSE Computes the Frobenius map for polynomials.

SYNTAX `f := DrinfMTau(g);`

`finite field polynomial`	`f`
`finite field polynomial`	`g`

DESCRIPTION For given $g \in A = \mathbb{F}_q[T]$ this function computes the Frobenius map $g \mapsto g^q$.

EXAMPLE

```
kash> k:=FF(3);
Finite field of size 3
kash> kT:=PolyAlg(k,"T");
Univariate Polynomial Ring in T over GF(3)

kash> DrinfMTau(T);
T^3
```

ECHOoff

Name	`ECHOoff`
Purpose	Switches back to normal behaviour after `ECHOon`.
Syntax	`ECHOoff();`
Description	
See Also	`ECHOon`,
Example	See `ECHOon` for an example.

NAME `ECHOon`

PURPOSE Switches the output of stdout to a file.

SYNTAX `ECHOon ( name` ‖ `file` [, `arg`] `);`

`string`	`name`	filename to write to
`file`	`file`	open for writing
`expression`	`arg`	should produce an output to stdout

DESCRIPTION Changes the output stream to output in a file. Note that it is dangerous if an error occurs, because you will not be able to see most of the KASHmessages. It is advisable to use it only in the form:

`ECHOon(file, arg); ECHOoff();`

because now KASHwill change back to normal behaviour even if an error occurs. `ECHOoff` switches the indirection off again. Note, that a call like

`ECHOon(file);`

forces you to type all commands, including

`ECHOoff();`

without any echo on the screen. All output is redirected to file.

SEE ALSO `ECHOoff`,

EXAMPLE Debug information of an order maximal call.

```
kash> PRINTLEVEL ("ROUND2", 2);
2
kash> f := Open ("/tmp/dump","w");
Filename: /tmp/dump / Mode: w / Open (fid): 5
kash> ECHOon (f, OrderMaximal (Order (x^5 + 5*x^3 + 16*x^2 - 6*x + 38)));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^5 + 5*x^3 + 16*x^2 - 6*x + 38
```

```
Discriminant: 3828384

kash> Exec ("cat /tmp/dump");
kash> Close (f);
true
```

NAME `EccDecrypt`

PURPOSE Decrypt a message that was encrypted using the ElGamal public key cryptosystem and a subgroup of the group of an elliptic curve.

SYNTAX `P := EccDecrypt(K,E,a,M);`

`finite field`	`K`
`list`	`E`
`integer`	`a`
`list`	`M`
`list`	`P`

DESCRIPTION Let E be an elliptic curve over a finite field K, let B be a basepoint of a subgroup of the group of E. Bob chooses an integer a, as his secret key. The message $M = [P_1, P_2]$ consisting of two points on E is decrypted to a point P by setting $P = P_1 - a \cdot P_2$. The elliptic curve E is either given by a list of two or five elements of K or integers. If the equation of E is $y^2 = x^3 + a_4x + a_6$ the curve is represented by $[a_4, a_6]$, if the equation of E is $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ then the representation is $[a_1, a_2, a_3, a_4, a_6]$. Points on the curve are given by a pair of elements of K or integers; the point at infinity is represented by the empty list [ ].

SEE ALSO `EccPointsAdd`, `EccIntPointMult`, `EccPointIsOnCurve`, `EccEncrypt`, `FF`,

EXAMPLE

```
kash> field := FF(751);
Finite field of size 751
kash> ec := [0,0,1,-1,0];
[ 0, 0, 1, -1, 0 ]
kash> base_point := [0,0];
[ 0, 0 ]
kash> secret_key := 58;
58
kash> public_key := EccIntPointMult(field,ec,secret_key,base_point);
[ 201, 380 ]
kash> plaintext := [[562,576],[581,395],[484,214],[501,220],[1,0]];
[ [ 562, 576 ], [ 581, 395 ], [ 484, 214 ], [ 501, 220 ], [ 1, 0 ] ]
kash> k := [254,180,99,472,275];
[ 254, 180, 99, 472, 275 ]
kash> ciphertext := List([1..5], \
```

```
> x-> EccEncrypt(field,ec,base_point,public_key,k[x],plaintext[x]));
[ [ [ 268, 146 ], [ 378, 547 ] ], [ [ 680, 469 ], [ 409, 94 ] ],
  [ [ 710, 395 ], [ 195, 432 ] ], [ [ 747, 222 ], [ 101, 371 ] ],
  [ [ 13, 246 ], [ 386, 303 ] ] ]
kash> List(ciphertext, x-> EccDecrypt(field,ec,secret_key,x));
[ [ 562, 576 ], [ 581, 395 ], [ 484, 214 ], [ 501, 220 ], [ 1, 0 ] ]
```

NAME `EccEncrypt`

PURPOSE Encrypt a message (point) using the ElGamal public key cryptosystem and a subgroup of the group of an elliptic curve.

SYNTAX `M := EccEncrypt(K,E,B,aB,k,P);`

finite field	K
list	E
list	B
list	aB
integer	k
list	P
list	M

DESCRIPTION Let E be an elliptic curve over a finite field K, let B be a basepoint of a subgroup of the group of E. Bob chooses an integer a, as his secret key. He publishes the public key $a_B := a \cdot B$. Before sending a message to Bob Alice picks a random integer k, then she sends him the point pair $[k \cdot B, P + k \cdot a_B]$. The elliptic curve E is either given by a list of two or five elements of K or integers. If the equation of E is $y^2 = x^3 + a_4x + a_6$ the curve is represented by $[a_4, a_6]$, if the equation of E is $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ then the representation is $[a_1, a_2, a_3, a_4, a_6]$. Points on the curve are given by a pair of elements of K or integers; the point at infinity is represented by the empty list [ ].

SEE ALSO EccPointsAdd, EccIntPointMult, EccPointIsOnCurve, EccDecrypt, FF,

EXAMPLE

```
kash> K := FF(11);
Finite field of size 11
kash> E := [0,0,0,1,6];
[ 0, 0, 0, 1, 6 ]
kash> B := [2,7];
[ 2, 7 ]
kash> a := 8;
8
kash> aB := EccIntPointMult(K,E,a,B);
[ 3, 5 ]
kash> k := 7;
7
```

```
kash> M := EccEncrypt(K,E,B,aB,k,[3,6]);
[ [ 7, 2 ], [ 10, 9 ] ]
kash> EccDecrypt(K,E,a,M);
[ 3, 6 ]
kash> k := 5;
5
kash> M := EccEncrypt(K,E,B,aB,k,[3,6]);
[ [ 3, 6 ], [ 7, 9 ] ]
kash> EccDecrypt(K,E,a,M);
[ 3, 6 ]
```

NAME `EccInit`

PURPOSE Initializes a representation of an elliptic curve for the Ecc-package.

SYNTAX `ec := EccInit(p,a,b); ec := EccInit(p,li); ec := EccInit(fli);`

elliptic curve	ec
integer	a
integer	b
integer	p
list of integers	li
list of finite field elements	fli

DESCRIPTION

EXAMPLE

```
kash> p:=65112*2^144-1;
1452046121366725933991673688168680114377396846591
kash> IsPrime(p);
true
kash> EccInit(p,-3,49);
[ 1452046121366725933991673688168680114377396846588, 49 ]
```

NAME EccIntPointMult

PURPOSE Scalar multiplication of a point on an elliptic curve.

SYNTAX `nP := EccIntPointMult(K,E,n,P);`

finite field	K
list	E
integer	n
list	P
list	nP

DESCRIPTION Using the group law on an elliptic curve E over a finite field K a point $P = [x, y]$ on E is multiplied by the scalar n. E is either given by a list of two or five elements of K or integers. If the equation of E is $y^2 = x^3 + a_4x + a_6$ the curve is represented by $[a_4, a_6]$, if the equation of E is $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ then the representation is $[a_1, a_2, a_3, a_4, a_6]$. Points on the curve are given by a pair of elements of K or integers; the point at infinity is represented by the empty list [ ].

SEE ALSO EccPointsAdd, EccPointIsOnCurve, EccEncryptEccDecrypt, FF,

EXAMPLE

```
kash> K := FF(11);
Finite field of size 11
kash> E := [0,0,0,1,6];
[ 0, 0, 0, 1, 6 ]
kash> EccIntPointMult(K,E,-1,[2,7]);
[ 2, 4 ]
kash> EccIntPointMult(K,E,5,[2,7]);
[ 3, 6 ]
kash> EccIntPointMult(K,E,12,[2,7]);
[ 2, 4 ]
kash> EccIntPointMult(K,E,13,[2,7]);
[  ]
kash> EccIntPointMult(K,E,14,[2,7]);
[ 2, 7 ]
```

NAME `EccKangaroo`

PURPOSE Computes the number of places of an elliptic curve, given by the Weierstrass form over a prime finite field $\mathcal{F}_p$.

SYNTAX `NP:=EccNumberOfPoints(p|F,a,b); NP:=EccNumberOfPoints(ec);`

`int or finite field element`	`a`
`int or finite field element`	`b`
`int`	`p`
`finite field`	`F`
`list`	`ec`

DESCRIPTION

NAME `EccPointIsOnCurve`

PURPOSE Checks whether a point is on a given elliptic curve.

SYNTAX `b := EccPointIsOnCurve(K,ec,point);`

`finite field`	`K`
`list`	`ec`
`list`	`point`
`boolean`	`b`

DESCRIPTION Checks whether a point $P = [x, y]$ is on the elliptic curve E over a finite field K. E is either given by a list of two or five elements of K or integers. If the equation of E is $y^2 = x^3 + a_4x + a_6$ the curve is represented by $[a_4, a_6]$, if the equation of E is $y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6$ then the representation is $[a_1, a_2, a_3, a_4, a_6]$. Points on the curve are given by a pair of elements of K or integers; the point at infinity is represented by the empty list [ ].

SEE ALSO `EccPointsAdd`, `EccIntPointMult`, `EccEncrypt`, `EccDecrypt`, `FF`,

EXAMPLE

```
kash> K := FF(11);
Finite field of size 11
kash> ec := [0,0,0,1,6];
[ 0, 0, 0, 1, 6 ]
kash> EccPointIsOnCurve(K,ec,[2,7]);
true
kash> EccPointIsOnCurve(K,ec,[1,1]);
false
kash> EccPointIsOnCurve(K,ec,[]);
true
```

NAME EccPointsAdd

PURPOSE Returns the sum of two points on an elliptic curve over a finite field.

SYNTAX `sum := EccPointsAdd(K,ec,P1,P2);`

finite field	K
list	E
list	P1
list	P2
list	sum

DESCRIPTION Using the group law on an elliptic curve E over a finite field K two points $P_1 = [x_1, y_1]$ and $P_2 = [x_2, y_2]$ are added. E is either given by a list of two or five elements of K or integers. If the equation of E is $y^2 = x^3 + a_4x + a_6$ the curve is represented by $[a_4, a_6]$, if the equation of E is $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ then the representation is $[a_1, a_2, a_3, a_4, a_6]$. Points on the curve are given by a pair of elements of K or integers; the point at infinity is represented by the empty list [ ].

SEE ALSO EccIntPointMult, EccPointIsOnCurve, EccEncrypt, EccDecrypt, FF,

EXAMPLE

```
kash> K := FF(11);
Finite field of size 11
kash> E := [0,0,0,1,6];
[ 0, 0, 0, 1, 6 ]
kash> EccPointsAdd(K,E,[2,7],[5,9]);
[ 2, 4 ]
kash> EccPointsAdd(K,E,[2,7],[]);
[ 2, 7 ]
```

NAME `EccRandomPoint`

PURPOSE Returns a random point on a given elliptic curve.

SYNTAX `P := EccRandomPoint(F,E);`

point on the elliptic curve E	P
finite field	F
elliptic curve	E

DESCRIPTION

EXAMPLE

```
kash> p:=65112*2^144-1;
1452046121366725933991673688168680114377396846591
kash> IsPrime(p);
true
kash> E := EccInit(p,-3,49);
[ 1452046121366725933991673688168680114377396846588, 49 ]
kash> EccRandomPoint(FF(p),E);
[ 279321685036220943951873246284411164161011401467,
  278535878987595364269277267695870885018052411927 ]
```

NAME Ei

PURPOSE This function computes an approximation of the integral from -∞ to x of $exp(t)/t$.

SYNTAX `a := Ei(b);`

```
real  a
real  b
```

DESCRIPTION

$$\int_{-\infty}^{x} \frac{exp(t)}{t}\, dt$$

EXAMPLE

```
kash> Ei(1.0);
1.895117816355936755466520934331634269017060581732
```

NAME `EisensteinSeries`

PURPOSE Returns the value of the non-holomorphic Eisensteinseries.

SYNTAX `c := EisensteinSeries(tau,s,a1,a2,q,N);`

complex	tau
real	s
integer	a1
integer	a2
integer	q
positive integer	N

DESCRIPTION Given $E(\tau, s, \mathfrak{a}_1, a_2, q) := \sum_{n \equiv a_1 \mod q, m \equiv a_2 \mod q} |n\tau + m|^s$ for $s \in \mathbb{C}$ and $Re(s) > 1$. This function has a meromorphic continuation to $\mathbb{C}$. We use the Fourier-series- expansion to calculate this function. [Maa49].

EXAMPLE

```
kash> c := EisensteinSeries(Comp(2.4,3.1),0.4,2,3,6,5);
-0.0464001568398209806672836216520773803072859195067551 + 1.016212292896410126\
673369727459856258569024111299e-52*i
```

NAME `Elt`

PURPOSE Creates an algebraic number.

SYNTAX

```
a := Elt(O,L);
a := Elt(O,L/d);
a := Elt(O,h/d);
a := Elt(O,h);
```

algebraic element	a	
order	O	
list	L	of coefficients
integer	h	
integer	d	

DESCRIPTION Let $\mathfrak{o}$ be an order with coefficient order $\mathfrak{c}$. Denote by $\mathrm{Q}(\mathfrak{o})$ and $\mathrm{Q}(\mathfrak{c})$ the quotient field of $\mathfrak{o}$, respectively $\mathfrak{c}$. Let $\omega_1, \ldots, \omega_n$ be the $\mathrm{Q}(\mathfrak{c})$ basis of $\mathrm{Q}(\mathfrak{o})$ which belongs to the order $\mathfrak{o}$. Each algebraic number a in $\mathfrak{o}$ is of a form

$$a = a_1\omega_1 + \cdots + a_n\omega_n$$

with coefficients $a_1, \ldots, a_n \in \mathrm{Q}(\mathfrak{c})$.

To create a in KASH the coefficients of a must be passed to the `Elt` function as a list `L` of length n. The entries of `L` correspond to the coefficients a_i of a. Each entry `l` of `L` is either an algebraic number in $\mathfrak{c}$ or it has to be a valid argument for the function call `Elt(c,l)`. In the special case that $\mathfrak{c}$ equals $\mathbb{Z}$ the `Elt` function requires that the entries of `L` are rational integers.

`Elt(o,L)`
creates the element $a = a_1\,\omega_1 + \cdots + a_n\,\omega_n$.

`Elt(O,L/d)`
creates the element $a = \frac{a_1}{d}\,\omega_1 + \cdots + \frac{a_n}{d}\,\omega_n$.

`Elt(O,h)`
creates the algebraic number which corresponds to the rational integer h.

`Elt(O,h/d)`
creates the algebraic number which corresponds to the rational $\frac{h}{d}$.

EXAMPLE Create some algebraic elements:

```
kash> o1 := Order (Poly(Zx,[1,0,73,-280,-2399]));;
kash> u := Elt (o1,[1,2,3,4]/2);
[1, 2, 3, 4] / 2
```

```
kash> v := Elt (o1,2);
2
kash> u*v;
[1, 2, 3, 4]

kash> o2 := Order (o1,2,2);;
kash> o3 := Order (o2,2,3);
         F[1]
           /
          /
      E2[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 3
E 2[ 1]     x^2 - 2
E 1[ 1]     x^4 + 73*x^2 - 280*x - 2399

kash> a := Elt (o3,[1,2]);
[1, 2]
kash> b := Elt (o3,[[[1,1,1,1],1/2],2]);
[[[2, 2, 2, 2], 1], 4] / 2
kash> c := Elt (o3,[[[1,1,1,1],1/2],[1/3,[4,5,6,8]/4]]);
[[[12, 12, 12, 12], 6], [4, [12, 15, 18, 24]]] / 12
kash> d := Elt (o3,1/2);
1 / 2
```

NAME `EltAbs`

PURPOSE missing shortdoc

SYNTAX `v := EltAbs (a);`

real matrix	v
algebraic element	a

DESCRIPTION Let a be an algebraic number whose underlying order is over $\mathbb{Z}$ having r_1 real and r_2 complex places. The `EltAbs` function returns the real matrix $(|a^{(1)}|, \ldots, |a^{(r_1+r_2)}|)$.

SEE ALSO `OrderSig`, `EltCon`,

EXAMPLE

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> a := Elt(o,[0,1,0,0]);
[0, 1, 0, 0]
kash> EltAbs (a);
[6.104004643496482052536282272258995385473953052355 3.867936665996692356127108\
603527719150033334692743 10.080173776775582528154588754110436634147502950471]
```

NAME `EltAbsLogHeight`

PURPOSE Computes the absolute logarithmic height of an algebraic number.

SYNTAX `h := EltAbsLogHeight(a);`

real	h
algebraic element	a

DESCRIPTION Let α be an algebraic number. Denote by

$$m_\alpha(t) = a_0 t^n + \cdots + a_n = a_0 \prod_{j=1}^{n} (t - \alpha^{(j)})$$

its minimal polynomial over $\mathbb{Z}$. The absolute logarithmic height of α is defined by

$$h(\alpha) = \frac{1}{n} \log \left(a_0 \prod_{j=1}^{n} \max(1, |\alpha^{(j)}|) \right).$$

EXAMPLE

```
kash> o := Order(Z,4,7);
Generating polynomial: x^4 - 7

kash> a := Elt(o,[1,1,1,1]);
[1, 1, 1, 1]
kash> EltAbsLogHeight(a);
1.343819601921041250609358018785526704542243019136
kash> EltAbsLogHeight(a/100);
4.085309800568132385973058818275095989145577876487
```

NAME EltApproximation

PURPOSE Returns an element with certain valuations at prime ideals.

SYNTAX `E := EltApproximation(P, L);`

list	P	of distinct prime ideals over the same order
list	L	of small integers
algebraic element	E	

DESCRIPTION This is a version of the approximation theorem for algebraic numbers over number rings.

SEE ALSO IdealChineseRemainder, RayCantoneseRemainder,

EXAMPLE Weak approximation theorem in $\mathbb{Q}(\rho)$ with $\rho^3 - 12\rho + 10 = 0$.

```
kash> k := OrderMaximal (Order (x^3 - 12*x + 10));
Generating polynomial: x^3 - 12*x + 10
Discriminant: 4212

kash> P := Filtered (Flat (Factor (56*k)), IsIdeal);
[ <2, [0, 1, 0]>, <7, [1, 1, 0]>, <7, [3, 6, 1]> ]
kash> L := [3,6,7];
[ 3, 6, 7 ]
kash> mu := EltApproximation (P,L);
[10117814, -4470662, -470596]
kash> for I in P do Print (I," --> ", Valuation (I, mu),"\n"); od;
<2, [0, 1, 0]> --> 3
<7, [1, 1, 0]> --> 6
<7, [3, 6, 1]> --> 7
```

NAME `EltAutomorphism`

PURPOSE Applies an automorphism to an algebraic number.

SYNTAX
```
L := EltAutomorphism (a);
c := EltAutomorphism (a,i);
```

list	L
algebraic element	c
algebraic element	a
integer	i

DESCRIPTION Let $\mathcal{F}$ be a normal number field with $\mathbb{Q}$-automorphisms $\sigma_1, \ldots, \sigma_n$. Let a be an algebraic number in $\mathcal{F}$.

`EltAutomorphism(a)`
returns the list $\mathtt{L} = \{\sigma_1(a), \ldots, \sigma_n(a)\}$. Each entry of `L` is an algebraic number whose underlying order is the same order as a is defined over.

`EltAutomorphism(a,i)`
returns the algebraic number $\sigma_i(a)$. Note that σ_1 is always the identity mapping of $\mathcal{F}$.

Before calling `EltAutomorphism`, the automorphisms $\sigma_1, \ldots, \sigma_n$ of $\mathcal{F}$ must be computed by the `OrderAutomorphisms` routine.

SEE ALSO `OrderAutomorphisms`,

EXAMPLE Consider the normal field $\mathbb{Q}(\sqrt[6]{-108})$:

```
kash> o := Order(Z,6,-108);
Generating polynomial: x^6 + 108

kash> OrderAutomorphisms(o);;
kash> a := Elt(o,[1,2,3,4,5,6]);
[1, 2, 3, 4, 5, 6]
kash> A := EltAutomorphism(a);
[ [1, 2, 3, 4, 5, 6], [12, -552, 630, 48, -28, -39] / 12,
  [12, 528, -666, 48, -32, -33] / 12, [1, -2, 3, -4, 5, -6],
  [12, 552, 630, -48, -28, 39] / 12, [12, -528, -666, -48, -32, 33] / 12 ]
kash> A[1]+A[2]+A[3]+A[4]+A[5]+A[6];
6
kash> EltTrace(a);
6
kash> EltAutomorphism(a,1);
[1, 2, 3, 4, 5, 6]
```

NAME `EltCharPoly`

PURPOSE Characteristic polynomial of an algebraic element over a subfield.

SYNTAX

```
p := EltCharPoly (a [, PA]);
p := EltCharPoly (a [, O]);
```

polynomial	p
polynomial algebra	PA
suborder	O
algebraic element	a

DESCRIPTION Given an algebraic element a in an Order O and a polynomial algebra PA over another order o or this order, this function computes the characteristic polynomial of a over o i.e. the returned polynomial is contained in PA.

EXAMPLE Characteristic polynomials of elements in $\mathbb{Q}(\sqrt{10}, \sqrt{5}, \sqrt{3})$.

```
kash> E1 := Order (Z,2,10);
Generating polynomial: x^2 - 10

kash> E2 := Order (E1, 2, 5);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> F := Order (E2, 2, 3);
         F[1]
           /
          /
      E2[1]
        /
       /
   E1[1]
  /
 /
Q
```

```
F  [ 1]     x^2 - 3
E 2[ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> E1x := PolyAlg (E1);
Univariate Polynomial Ring in x over Generating polynomial: x^2 - 10

kash> Fx := PolyAlg (F);
Univariate Polynomial Ring in x over          F[1]
           /
          /
      E2[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 3
E 2[ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> a := Elt (F, [[[1,2],[3,4]],[[3/5, 5],[7,9]]]);
[[[5, 10], [15, 20]], [[3, 25], [35, 45]]] / 5
kash> g := EltCharPoly (a, E1x);
x^4 + [-4, -8]*x^3 + [-717904, -100800] / 25*x^2 + [-3443792, -2723584] / 25*x\
 + [99481555504, 24875390400] / 625
kash> f := PolyMove (g, Fx);
x^4 + [[[-4, -8], 0], 0]*x^3 + [[[-717904, -100800], 0], 0] / 25*x^2 + [[[-344\
3792, -2723584], 0], 0] / 25*x + [[[99481555504, 24875390400], 0], 0] / 625
kash> Eval (f, a);
0
```

NAME EltCon

PURPOSE Computes the conjugates of an algebraic number.

SYNTAX

```
v := EltCon(alpha);
c := EltCon(alpha,i);
```

complex matrix	v
complex number	c
algebraic element	alpha
integer	i

DESCRIPTION Let a be an algebraic number whose underlying order is of degree n over $\mathbb{Z}$.

`EltCon(a)`
returns the complex matrix $(a^{(1)}, \ldots, a^{(n)})$.

`EltCon(a,i)`
returns the complex number $a^{(i)}$.

SEE ALSO EltAbs, OrderSig,

EXAMPLE

```
kash> O := Order (Z,3,2);
Generating polynomial: x^3 - 2

kash> rho := Elt(O,[0,1,0]);
[0, 1, 0]
kash> EltCon(rho);
[1.259921049894873164767210607278228350570251464701 -0.62996052494743658238360\
530363911417528512573235O9776 + 1.091123635971721403560072614189808881325873333\
874*i -0.629960524947436582383605303639114175285125732350 9776 - 1.091123635971\
7214035600726141898088813258733 3874*i]
kash> a1 := EltCon (rho,1);
1.259921049894873164767210607278228350570251464701
kash> a2 := EltCon (rho,2);
-0.62996052494743658238360530363911417528512573235O9776 + 1.091123635971721403\
56007261418980888132587333874*i
kash> a3 := EltCon (rho,3);
-0.62996052494743658238360530363911417528512573235O9776 - 1.091123635971721403\
56007261418980888132587333874*i
kash> a1^3;
1.999999999999999999999999999999999999999999999998
```

NAME EltDen

PURPOSE Returns the denominator of an algebraic element.

SYNTAX
```
d := EltDen(a);
d := EltDen(a, "rep");
```

integer	d
algebraic element	a

DESCRIPTION The denominator of the algebraic number a is the smallest natural number d such that $d \cdot a$ is an algebraic integer. If you use `EltDen(a, "rep")` the denominator in the representation of a is returned.

EXAMPLE An absolute example.

```
kash> o := Order (x^3 - 23*x^2 + 146*x - 244);;
kash> a := Elt (o, [1/2,3/5,45/5657567]);
[28287835, 33945402, 450] / 56575670
kash> EltDen (a);
56575670
kash> EltDen (a, "rep");
56575670
```

Name `EltDivisors`

Purpose Computes all divisors of an algebraic integer.

Syntax `L := EltDivisors (a);`

list	L
algebraic integer	a

Description The `EltDivisors` function computes a list containing all divisors of an algebraic integer a up to multiplication by a unit. The underlying order of a must be known to be maximal.

Example

```
kash> o := OrderMaximal(Z,2,-5);
Generating polynomial: x^2 + 5
Discriminant: -20

kash> a := Elt(o,6);
6
kash> EltDivisors(a);
[ 1, 2, [1, 1], [1, -1], 3, 6 ]
```

NAME `EltExcepUnitOrbit`

PURPOSE Computes the orbit of an exceptional unit.

SYNTAX `L := EltExcepUnitOrbit(epsilon);`

`list`	`L`
`algebraic element`	`epsilon`

DESCRIPTION Computes the orbit of an exceptional unit ε. The orbit is given by

$$\left\{ \varepsilon, \quad \frac{1}{\varepsilon}, \quad 1-\varepsilon, \quad \frac{1}{1-\varepsilon}, \quad \frac{\varepsilon-1}{\varepsilon}, \quad \frac{\varepsilon}{\varepsilon-1} \right\}.$$

If ε is not an exceptional unit, an error message is returned.

SEE ALSO `OrderUnitsExcep`,

EXAMPLE Compute the orbit of $1+\zeta_7 \in \mathbb{Z}[\zeta_7]$:

```
kash> o := Order(Poly(Zx,[1,1,1,1,1,1,1]));
Generating polynomial: x^6 + x^5 + x^4 + x^3 + x^2 + x + 1

kash> EltExcepUnitOrbit(Elt(o,[1,1,0,0,0,0]));
[ [0, -1, 0, 0, 0, 0], [1, 1, 0, 0, 0, 0], [0, -1, -1, -1, -1, -1],
  [0, -1, 0, -1, 0, -1], [1, 1, 0, 1, 0, 1], [1, 1, 1, 1, 1, 1] ]
```

NAME EltFactor

PURPOSE Computes the prime ideal factorization of an algebraic integer.

SYNTAX `L := EltFactor(a);`

list	L
algebraic element	a

DESCRIPTION The `EltFactor` function returns the prime ideal decomposition of the principal ideal generated by the algebraic element a. The element must be defined over a maximal order or its norm must be coprime to the discriminant of the equation order.

SEE ALSO Factor,

EXAMPLE

```
kash> o := Order(Z,2,-110);
Generating polynomial: x^2 + 110

kash> O := OrderMaximal(o);
Generating polynomial: x^2 + 110
Discriminant: -440

kash> alpha := Elt(O, [0,1]);
[0, 1]
kash> EltFactor(alpha);
[ [ <11, [0, 1]>, 1 ], [ <5, [0, 1]>, 1 ], [ <2, [0, 1]>, 1 ] ]
kash> IdealFactor(alpha*O);
[ [ <2, [0, 1]>, 1 ], [ <5, [0, 1]>, 1 ], [ <11, [0, 1]>, 1 ] ]
```

NAME EltIdealReduce

PURPOSE Returns a canonical representative modulo an ideal.

SYNTAX

```
b := EltIdealReduce(a,I); same as b := EltIdealReduce(a,I,HNF);
b := EltIdealReduce(a,I,LLL);
b := EltIdealReduce(a,I,HNF_POS);
```

algebraic element	b
algebraic element	a
Ideal	I
interpreted as strings	HNF, LLL, HNF_POS

DESCRIPTION This function returns a representation of $a \mod I$ using either the usual ideal base, a LLL–reduced basis or yet another basis.

LLL–reducing results in smaller elements in terms of the norm of the element but takes longer — firstly to compute another basis of the ideal (which is then stored for this ideal) and secondly to compute the representative is more difficult.

The HNF_POS method reduces in the same way as HNF, but the coefficients of the returned element are positive.

A fractional ideal is treated as follows: the element α is reduced to β with the ideal $\frac{\mathfrak{a}}{n}$ if the element $n\alpha$ is reduced to $n\beta$ with $\mathfrak{a}$.

A fractional elements is treated as follows: the element $\frac{\alpha}{n}$ is reduced to $\frac{\beta}{n}$ with the ideal $\mathfrak{a}$ if the element α is reduced to β with $n\mathfrak{a}$.

There is a different generalization to fractional elements where the denominator (as a principal ideal) is inverted in the factor ring.

This function can only handle $\mathbb{Z}$-orders.

EXAMPLE

```
kash> o := OrderMaximal(Order(Z,6,2));
Generating polynomial: x^6 - 2
Discriminant: 1492992

kash> I := 2*o;
<2>
kash> elt := Elt(o,[214124,2144,3245,3252354545,436436436436,564643544365]);
[214124, 2144, 3245, 3252354545, 436436436436, 564643544365]
kash> b := EltIdealReduce(elt,I);
[0, 0, 1, 1, 0, 1]
```

NAME `EltIndex`

PURPOSE Computes the index of an equation suborder of a given order

SYNTAX `index := EltIndex (alpha [,Z]);`

`integer`	`index`
`algebraic integer`	`alpha`
`ring of integers`	`Z`

DESCRIPTION Let $\mathfrak{o}$ be an arbitrary order over $\mathbb{Z}$ and let α be an algebraic integer from $\mathfrak{o}$. The `EltIndex` function returns the module index $(\mathfrak{o} : \mathbb{Z}[\alpha])$. If the index is infinite, 0 is returned. In the relative case, $\mathbb{Z}$ has to be the argument that determines the ring of the index.

SEE ALSO `OrderIndexFormEquation`,

EXAMPLE Compute the index $(\mathbb{Z}[\sqrt[4]{5}] : \mathbb{Z}[\sqrt[4]{5} + \sqrt{5}])$:

```
kash> o := Order(Z,4,5);
Generating polynomial: x^4 - 5

kash> a := Elt(o,[0,1,1,0]);
[0, 1, 1, 0]
kash> EltIndex(a);
21
```

NAME EltIsInIdeal

PURPOSE missing shortdoc

SYNTAX `b := EltIsInIdeal (alpha,a);`

boolean	b
algebraic element	alpha
ideal	a
	INDES test on; element of an ideal
	check for; element of an ideal
	algebraic number; element of an ideal test
	element of an ideal test

DESCRIPTION The `EltIsInIdeal` function returns `true` if the algebraic number α lies in the ideal $\mathfrak{a}$. Otherwise `false` is returned.

EXAMPLE

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));;
kash> O := OrderMaximal (o);;
kash> a := 2*O + Elt (O,[0, 1, -1, -2])*O;
<2, [0, 1, -1, -2]>
kash> EltIsInIdeal (2,a);
true
kash> EltIsInIdeal (1,a^-1);
true
kash> EltIsInIdeal (Elt(O,1/3),a^-1);
false
```

NAME `EltIsInt`

PURPOSE missing shortdoc

SYNTAX `b := EltIsInt(a);`

boolean	b
algebraic element	a

DESCRIPTION The `EltIsInt` function returns `true`, if the algebraic number a is a rational integer. Otherwise `false` is returned.

EXAMPLE

```
kash> o := Order(Z,2,2);
Generating polynomial: x^2 - 2

kash> a := Elt(o,[0,1]);
[0, 1]
kash> EltIsInt (a);
false
kash> EltIsInt (a^2);
true
kash> EltIsInt (a^2/4);
false
```

NAME EltIsIntegral

PURPOSE Tests whether the element is an algebraic integer resp. an element of the order. The second parameter "order" checks, if the element is integral in the mathematical sence, e.g. if it has a denominator.

SYNTAX
```
b := EltIsIntegral(a);
b := EltIsIntegral(a, "order");
```

boolean	b
algebraic element	a

EXAMPLE

```
kash> o := Order(Z,4,3);
Generating polynomial: x^4 - 3

kash> a := Elt(o,[1,2,3,4]);
[1, 2, 3, 4]
kash> EltIsIntegral(a);
true
kash> EltIsIntegral(a/13);
false

kash> o:= Order(Z,2,5);
Generating polynomial: x^2 - 5

kash> a := Elt(o, [1,1]/2);
[1, 1] / 2
kash> EltIsIntegral(a);
true
kash> EltIsIntegral(a, "order");
false
```

NAME	`EltIsPrimitive`
PURPOSE	Checks whether an algebraic element is primitive.
SYNTAX	`b := EltIsPrimitive (a);`

`boolean`	`b`
`algebraic element`	`a`

SEE ALSO `EltMinPoly`,

EXAMPLE

```
kash> O := Order(Z,4,2);
Generating polynomial: x^4 - 2

kash> alpha := Elt(O, [0,0,1,0]);
[0, 0, 1, 0]
kash> EltIsPrimitive(alpha);
false
kash> beta := Elt(O, [0,0,0,1]);
[0, 0, 0, 1]
kash> EltIsPrimitive(beta);
true
```

NAME EltListAbsDisc

PURPOSE Discriminant of a module.

SYNTAX `d := EltListAbsDisc (L);`

list	L	of algebraic elements
integer	d	

DESCRIPTION Computes the (absolute) discriminant of the module $M = \mathfrak{o} \cdot L[1] + \cdots + \mathfrak{o} \cdot L[n]$, where $\mathfrak{o}$ is the defining order of the elements of L and n is the number of elements in L. The discriminant is computed as the determinant of $(\mathrm{Tr}_{\mathfrak{o}/\mathbb{Q}}(L[i]L[j]))$.

EXAMPLE Discriminant of $\mathbb{Q}(\sqrt{10}, \sqrt{5})$.

```
kash> F := OrderMaximal (Order (Z,2,10));;
kash> E := Order (F,2,5);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> EltListAbsDisc (OrderKextGenAbs (E));
0
kash> OrderDisc (OrderMaximal (OrderAbs (E)));
1600
```

Name `EltListToMat`

Purpose Converts a list of algebraic numbers into a matrix of their coefficients.

Syntax `M := EltListToMat(L);`

`matrix`	`M`	matrix over the coefficient order of the elements of L
`list`	`L`	of algebraic numbers

Description This procedure cares for the denominators of the elements.

See Also EltMatToList,

Example

```
kash> o := OrderMaximal(Order(Z,2,3));;
kash> O := Order(o,2,7);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 7
E 1[ 1]     x^2 - 3

kash> alpha := Elt(O,[1,Elt(o,[0,1])]);
[1, [0, 1]]
kash> beta := Elt(O,[Elt(o,[0,2]),Elt(o,[1,1])]);
[[0, 2], [1, 1]]
kash> EltListToMat([alpha,beta]);
[1 [0, 2]]
[[0, 1] [1, 1]]
```

NAME EltLogs

PURPOSE Returns a matrix v with the logarithms of the absolute values of the conjugates the algebraic number a. Matrix v has length $r|1+r|2$.

SYNTAX `v := EltLogs(a);`

`real matrix`	`v`
`algebraic element`	`a`

DESCRIPTION Let $\tau|1,\ldots,\tau|s$ be all archimedian valuations of the the number field containing a. Then v is defined as

$$v = [\tau|1(a),\ldots,\tau|s(a)].$$

SEE ALSO OrderSig, EltCon,

EXAMPLE Matrix of logarithms of archimedean valuations of an algebraic element.

```
kash> O := Order (Poly(Zx,[1,0,73,-280,-2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> rho := Elt(O,[0,1,0,0]);
[0, 1, 0, 0]
kash> EltLogs (rho);
[1.80894505471195860607076588169642554084529697289 1.3527212036195319561389981\
23553287717215552888911 2.310570502254102005606434026695873075937323698075]
```

NAME EltMatToList

PURPOSE Converts a matrix of algebraic numbers over a maximal order o into a dynamic array of elements of the relative order O.

SYNTAX `L := EltMatToList(O, M);`

list	L	of algebraic numbers over O
matrix	M	of algebraic numbers
order	O	relative order over the coefficient order of the elements of the matrix

DESCRIPTION Denominators are collected to a single denominator (lcm) and the coefficients are multiplied accordingly.

SEE ALSO EltListToMat,

EXAMPLE

```
kash> o := OrderMaximal(Order(Z,2,3));;
kash> O := Order(o,2,7);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 7
E 1[ 1]     x^2 - 3

kash> M := Mat(o, [ [1, Elt(o,[0,2])], [Elt(o,[0,1]), Elt(o,[1,1])]]);
[1 [0, 2]]
[[0, 1] [1, 1]]
kash> EltMatToList(O, M);
[ [1, [0, 1]], [[0, 2], [1, 1]] ]
```

NAME `EltMinPoly`

PURPOSE Minimal polynomial of an algebraic element over a subfield.

SYNTAX

```
p := EltMinPoly (a [, PA]);
p := EltMinPoly (a [, O]);
```

polynomial	p
polynomial algebra	PA
suborder	O
algebraic element	a

DESCRIPTION Given an algebraic element a in an Order O and a polynomial algebra PA over another order o or this order, this function computes the minimal polynomial of a over o i.e. the returned polynomial is contained in PA.

EXAMPLE Minimal polynomials of elements in $\mathbb{Q}(\sqrt{10}, \sqrt{5}, \sqrt{3})$.

```
kash> E1 := Order (Z,2,10);
Generating polynomial: x^2 - 10

kash> E2 := Order (E1, 2, 5);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> F := Order (E2, 2, 3);
         F[1]
           /
          /
      E2[1]
        /
       /
   E1[1]
  /
 /
Q
```

```
F  [ 1]     x^2 - 3
E 2[ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> E1x := PolyAlg (E1);
Univariate Polynomial Ring in x over Generating polynomial: x^2 - 10

kash> Fx := PolyAlg (F);
Univariate Polynomial Ring in x over          F[1]
           /
          /
      E2[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 3
E 2[ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> a := Elt (F, [[[1,2],[3,4]],[[3/5, 5],[7,9]]]);
[[[5, 10], [15, 20]], [[3, 25], [35, 45]]] / 5
kash> g := EltMinPoly (a, E1x);
x^4 + [-4, -8]*x^3 + [-717904, -100800] / 25*x^2 + [-3443792, -2723584] / 25*x\
 + [99481555504, 24875390400] / 625
kash> f := PolyMove (g, Fx);
x^4 + [[[-4, -8], 0], 0]*x^3 + [[[-717904, -100800], 0], 0] / 25*x^2 + [[[-344\
3792, -2723584], 0], 0] / 25*x + [[[99481555504, 24875390400], 0], 0] / 625
kash> Eval (f, a);
0
```

NAME EltMinkowski

PURPOSE Returns the image of an algebraic element under the Minkowski map.

SYNTAX `v := EltMinkowski(a);`

real matrix	v
algebraic element	a

DESCRIPTION Let $F = \mathrm{Q}(O)$ be the field of fractions of the order O, which contains the algebraic element a and let $\sigma|1, \ldots, \sigma|n$ be the Galois isomorphisms of F, where $\sigma|1, \ldots, \sigma|r|1$ are precisely the real isomorphisms and $\overline{\sigma|j + r|2} = \sigma|j$ for $r|1 < j \leq r|1 + r|2$ are the complex ones. Then the image of a under the Minkoswski map $\psi : O \mapsto \mathbb{R}^{r|1+2r|2}$ is defined as

$$[\sigma|1(a), \ldots, \sigma|r|1(a), \sqrt{2}\Re(\sigma|r|1+1(a)), \quad \ldots, \sqrt{2}\Re(\sigma|r|1+r|2(a)),$$
$$\sqrt{2}\Im(\sigma|r|1+1(a)), \quad \ldots, \sqrt{2}\Im(\sigma|r|1+r|2(a))].$$

EXAMPLE Matrix of logarithms of archimedean valuations of an algebraic element.

```
kash> O := Order (Z,2,-1);
Generating polynomial: x^2 + 1

kash> rho := Elt (O,[0,1]);
[0, 1]
kash> EltMinkowski (rho);
[ 0, 1.41421356237309504880168872420969807856967187537 7]
kash> O := Order (Poly(Zx,[1,0,73,-280,-2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> rho := Elt (O,[1,1,0,1]);
[1, 1, 0, 1]
kash> EltMinkowski (rho);
[ 234.53233654143439091592226693369060767991676415540 3, -60.735882043911047217\
340544162862269813825402072 25, 473.9054718000681082230949125242621731996360759\
3719, -1354.55921260507532919765645642599201591001449969379 6]
```

NAME `EltMove`

PURPOSE Returns an algebraic element in a different representation.

SYNTAX `b := EltMove(a,S);`

`algebraic element or lattice element`	`b`
`algebraic element or lattice element`	`a`
`order or lattice`	`S`

DESCRIPTION The element a is contained in a certain order or lattice and is therefore given in a specific representation. This function tries to give a representation of a in the structure S.

SEE ALSO `PolyMove`, `IdealMove`,

EXAMPLE Moving elements in relative extensions.

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));;
kash> O := OrderMaximal (o);;
kash> oo := Order (O,2,3);;
kash> a := Elt (O,[0,0,0,1]);;
kash> OrderBasis (O);
[ 1, [0, 1, 0, 0], [1, 1, 1, 0] / 2, [414, 857, 795, 1] / 1646 ]
kash> EltMove (a,o);
[414, 857, 795, 1] / 1646
kash> EltMove (a,oo);
[[0, 0, 0, 1], 0]
```

NAME `EltNewtonLift`

PURPOSE Lifts an algebraic element with the Newton lifting method.

SYNTAX `alpha := EltNewtonLift(o, a, f, p, k);`

algebraic element	alpha
order	o
algebraic element	a
polynomial	f
integer	p
integer	k

DESCRIPTION Given an algebraic element `a` with $\mathtt{f}(\mathtt{a}) \equiv 0 \bmod \mathtt{p}\,\mathtt{o}$, this function calculates an algebraic element α with $\mathtt{f}(\alpha) \equiv 0 \bmod \mathtt{p}^{2^{\mathtt{k}}}\mathtt{o}$.

EXAMPLE

```
kash> o:=Order(Poly(Zx,[1,0,-4,0,1]));
Generating polynomial: x^4 - 4*x^2 + 1

kash> f:=Poly(Zx,[1,40,596,3920,9601]);
x^4 + 40*x^3 + 596*x^2 + 3920*x + 9601
kash> a:=Elt(o,[0,4,0,0]);
[0, 4, 0, 0]
kash> b:=EltNewtonLift(o,a,f,5,2);
[-10, -1, 0, 0]
kash> Eval(f, b);
0
```

NAME EltNorm

PURPOSE Returns the norm of an algebraic element in a given order.

SYNTAX `n := EltNorm(a [,o]);`

rational number or algebraic element	n
algebraic element	a
order	o

SEE ALSO EltTrace,

EXAMPLE Computation of a norm of an algebraic element.

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));;
kash> O := OrderMaximal (o);;
kash> oo := Order (O,2,3);
      F[1]
        /
       /
   E1[1]
     |
   E1[2]
  /
 /
Q
F  [ 1]     x^2 - 3
E 1[ 1]     Given by transformation matrix
E 1[ 2]     x^4 + 73*x^2 - 280*x - 2399

kash> a := Elt (oo,[[0,0,0,1],4]/2);;
kash> EltNorm (a);
[705, 55, -765, 758] / 4
```

NAME EltNumberReduce

PURPOSE Returns a canonical representative modulo an integer or rational.

SYNTAX `b := EltNumberReduce(a,m);`

integer or rational number	m
algebraic element	a
algebraic element	b

DESCRIPTION Returns a canonical representation of an algebraic element modulo an integer or a rational.

EXAMPLE

```
kash> o := OrderMaximal(Order(Z,6,2));
Generating polynomial: x^6 - 2
Discriminant: 1492992

kash> elt := Elt(o,[214124,2144,3245,3252354545,436436436436,564643544365]);
[214124, 2144, 3245, 3252354545, 436436436436, 564643544365]
kash> b := EltNumberReduce(elt,2);
[0, 0, 1, 1, 0, 1]
```

NAME `EltOrder`

PURPOSE Returns the order of an algebraic element.

SYNTAX `o := EltOrder(alpha);`

`order`	`o`
`algebraic element or integer`	`alpha`

DESCRIPTION The `EltOrder` function returns the order in which the algebraic element α is defined.

EXAMPLE

```
kash> O:=Order(Z,2,-3);
Generating polynomial: x^2 + 3

kash> alpha:=Elt(O,[1,2]);
[1, 2]
kash> EltOrder(alpha);
Generating polynomial: x^2 + 3
```

NAME EltPowerMod

PURPOSE Computes $a^{\mathtt{pow}} \bmod M$ for an algebraic element.

SYNTAX `e := EltPowerMod(a, pow, M);`

algebraic element	e, a
integer	pow
integer	M

EXAMPLE

```
kash> o := Order(Z,6,2);
Generating polynomial: x^6 - 2

kash> a := Elt(o,[1,2,3,4,5,6]);
[1, 2, 3, 4, 5, 6]
kash> EltPowerMod(a, 10, 5);
[1, 1, 1, 0, 2, 1]
```

Name `EltPowerProduct`

Purpose Returns an algeraic element β with the power product of the algebraic elements in Alpha and the corresponding exponents in Expons. The matrices Alpha and Expons must have the same length and only one row.

Syntax `beta := EltPowerProduct(o,Alpha,Expons);`

`algebraic element`	`beta`
`order`	`o`
`matrix of algebraic elements`	`Alpha`
`matrix of integers`	`Expons`

Description Returns an algeraic element β with the power product of the algebraic elements in Alpha and the corresponding exponents in Expons. The matrices Alpha and Expons must have the same length and only one row.

Example Power product of algeraic elements.

NAME `EltRayResidueRingRep`

PURPOSE Returns a representive of the class of an algebraic element in the multiplicative group of the ray residue ring.

SYNTAX `r := EltRayResidueRingRep(elt,m0,minf);`

`matrix`	`r`	
`algebraic element`	`elt`	
`ideal`	`m0`	
`list`	`minf`	of integers/infinite primes

DESCRIPTION Calculates a representative of an algebraic element `elt` which is represented by a matrix $[a_0, \ldots, a_k]$. If $\eta_0, \ldots, \eta_k$ is a basis of the residue ring as given by `RayResidueRingCyclicFactors`, `elt` $\equiv \eta_0^{a_0} \cdots \eta_k^{a_k} \operatorname{mod}^*(\mathfrak{m}_0\mathfrak{m}_\infty)$ holds.
If the residue ring is not known yet, it will be computed. Infinite primes are represented in the same way as for RayResidueRing.
This function uses a mixture of the discrete logarithm algorithms developed in [Pau96, PP98] and [CDO96, CDO97].

SEE ALSO `EltCon`, `RayResidueRing`, `RayResidueRingRepToElt`, `RayResidueRingCyclicFactors`,

EXAMPLE

```
kash> O := OrderMaximal(Order(x^2-2*x-5));
Generating polynomial: x^2 - 2*x - 5
Discriminant: 24

kash> m0 := 27*O;;
kash> minf := [2];;
kash> L := RayResidueRingCyclicFactors(m0,minf);
[ [ [-1, -27], 2 ], [ [1344, 388], 3 ], [ [5, 2], 9 ], [ [1612, 444], 9 ],
  [ [1, 27], 2 ] ]
kash> elt := L[1][1];
[-1, -27]
kash> EltRayResidueRingRep(elt,m0,minf);
[1 0 0 0 0]
```

NAME `EltReconstruct`

PURPOSE Lifts an element from a modulo m approximation to an element with rational coefficients

SYNTAX `alpha := EltReconstruct (gamma, m);`

`false or algebraic element`	`alpha`
`algebraic element`	`gamma`
`integer`	`m`

DESCRIPTION Given an element $\gamma = c_1\omega_1 + \ldots + c_n\omega_n$ in an order over $\mathbb{Z}$ with coefficients less than the positive integer m, the function `EltReconstruct` computes an element α in the same order which is equal to the element $q_1\omega_1 + \ldots + q_n\omega_n$, where the q_i are the reconstructed rationals from c_i and m ($q_i = \frac{a_i}{b_i}$ such that $a_i \equiv b_i c_i \bmod m$ and $0 \le |a_i|, b_i < \sqrt{\frac{m}{2}}, b_i \ne 0$, if such a pair exists). Otherwise `false` is returned.

SEE ALSO `RationalReconstruct`,

EXAMPLE

```
kash> O := Order(Z,2,3);
Generating polynomial: x^2 - 3

kash> gamma := Elt(O,[17,4]);
[17, 4]
kash> EltReconstruct(gamma,49);
[2, 12] / 3
```

NAME EltRepMat

PURPOSE A representation matrix of an algebraic element over o together with a suitable denominator.

SYNTAX `L := EltRepMat(a[,o]);`

list	L	
algebraic element	a	
order	o	must be a direct suborder of EltOrder (a)

DESCRIPTION Given an algebraic element a in an order O and another order o, such that o is a direct suborder of O, this function computes a list consisting of an integer den and a matrix M, such that $1/den \cdot M$ is a representation matrix of a with coefficients in o.

EXAMPLE The representation matrix of certain algebraic elements.

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> O1 := Order (o,2,2);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 2
E 1[ 1]     x^4 + 73*x^2 - 280*x - 2399

kash> O2 := Order (O1, 2, 3);
         F[1]
           /
          /
      E2[1]
        /
       /
   E1[1]
  /
 /
Q
```

```
F  [ 1]     x^2 - 3
E 2[ 1]     x^2 - 2
E 1[ 1]     x^4 + 73*x^2 - 280*x - 2399

kash> alpha := Elt (O2,[[[1,2,3,4],1],[2,[7,8,9,0]]]/ 34);
[[[1, 2, 3, 4], 1], [2, [7, 8, 9, 0]]] / 34
kash> EltRepMat (alpha,o);
[ 34, [[1, 2, 3, 4] 2 6 [42, 48, 54, 0]]
    [1 [1, 2, 3, 4] [21, 24, 27, 0] 6]
    [2 [14, 16, 18, 0] [1, 2, 3, 4] 2]
    [[7, 8, 9, 0] 2 1 [1, 2, 3, 4]] ]
```

NAME `EltRoot`

PURPOSE Computes a root of an algebraic element.

SYNTAX

```
beta := EltRoot(alpha,m);
beta := EltRoot(alpha,m [,"enum"|"mode"]);
beta := EltRoot(alpha,m ,"enum", mode);
```

false or algebraic element	beta
algebraic element	alpha
small integer	m
small integer or string	mode
string	enum

DESCRIPTION Let α be an algebraic element from a maximal order $\mathcal{O}$.

`EltRoot(alpha,m)` (or `EltRoot(alpha,m,"mode")`)
The `EltRoot` function checks whether an algebraic number $\beta \in \mathcal{O}$ exists such that $\beta^m = \alpha$. The `EltRoot` uses a generalization of a method described in P. L. Montgomery, *Square roots of products of algebraic numbers*, Proceedings of Symposia in Applied Mathematics, Vol. **48** (1994), 567 – 571.

`EltRoot(alpha,m,"enum")` (or `EltRoot(alpha,m,"enum",0)`)
The `EltRoot` function does the same as above, but a method is used which bases on the enumeration of a weighted positive definite quadratic form.

`EltRoot(alpha,m,"enum",1)`
The `EltRoot` function checks whether an algebraic number $\beta \in \mathcal{O}$ and a torsion unit $\zeta \in \mathcal{O}$ exist such that $\beta^m = \zeta\,\alpha$.

`EltRoot(alpha,m,"enum",2)`
The `EltRoot` function checks whether an algebraic number $\beta \in \mathcal{O}$ and a unit $\varepsilon \in \mathcal{O}$ exist such that $\beta^m = \varepsilon\,\alpha$.

EXAMPLE Computation of roots of some algebraic elements:

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));;
kash> o := OrderMaximal (Order (Poly(Zx,[1,0,73,-280,-2399])));;
kash> a := Elt (o,[48427150173, 2658413621, -99500340384, 103801743979]);
[48427150173, 2658413621, -99500340384, 103801743979]
kash> EltRoot (a, 2);
[156369, 8584, -321281, 335170]
kash> o := OrderMaximal (Order (Z,7,2));;
kash> a := Elt (o, 2);
```

```
2
kash> EltRoot (a,2);
false
kash> EltRoot (a,7);
[0, 1, 0, 0, 0, 0, 0]
```

NAME EltSimplify

PURPOSE Returns a in simplified representation.

SYNTAX

```
e := EltSimplify(a);
```

```
integer or algebaric element  e
integer or algebraic element  a
```

DESCRIPTION This function returns an improved representation of the algebraic element a. It might be the case that a has a simpler basis representation than given. EltSimplify will find the simplest representation possible. This function is important for programs written in KASH, since the output of an algebraic number is always simplified, although the number itself can have a bad representation.

EXAMPLE There is no example demonstrating the effect, since output is always simplified.

Name EltToFFE

Purpose Returns the class of an algebraic element viewed as an element of a finite field.

Syntax `f := EltToFFE( a, p);`

`integer or algebraic element`	`a`	
`prime ideal`	`p`	
`finite field element or integer`	`f`	interpreted as finite field element

Description This is the canonical homomorphism

$$\mathcal{O} \to \mathcal{O}/\mathfrak{p}.$$

The prime ideal must not lie over a number which divides the discriminant of the generating polynomial of $\mathcal{O}$.

See Also IdealResidueField, IdealResidueFieldIsomorphism, FFEToElt,

Example

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> p := Factor(7*O)[1][1];
<7, [0, 1, 0, 0, 0]>
kash> b := Elt(O,[3, 1,5,1,8]);
[3, 1, 5, 1, 8]
kash> EltToFFE(b, p);
3
```

NAME `EltToList`

PURPOSE Returns the coefficient list of the algebraic element a.

SYNTAX `L := EltToList(a);`

`algebraic element`	`a`	
`list`	`L`	of integers or algebraic elements

EXAMPLE

```
kash> o := Order(Z,6,2);
Generating polynomial: x^6 - 2

kash> a := Elt(o,[1,2,3,4,5,6]);
[1, 2, 3, 4, 5, 6]
kash> EltToList(a);
[ 1, 2, 3, 4, 5, 6 ]
kash> a := a/2;
[1, 2, 3, 4, 5, 6] / 2
kash> EltToList(a);
[ 1/2, 1, 3/2, 2, 5/2, 3 ]
```

Name EltTrace

Purpose Returns the trace of an algebraic element. The trace is contained in the coefficient ring of the defining order or in the given order *o*.

Syntax `t := EltTrace (a [,o]);`

rational number or algebraic element	t
algebraic element	a
order	o

See Also EltNorm,

Example Computation of several traces.

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));;
kash> O := OrderMaximal (o);;
kash> oo := Order (O,2,3);
      F[1]
        /
       /
   E1[1]
    |
   E1[2]
  /
 /
Q
F  [ 1]     x^2 - 3
E 1[ 1]     Given by transformation matrix
E 1[ 2]     x^4 + 73*x^2 - 280*x - 2399

kash> a := Elt (O,[0,0,0,1]);;
kash> b := EltMove (a,oo);
[[0, 0, 0, 1], 0]
kash> EltTrace (b);
[0, 0, 0, 2]
kash> EltTrace (EltTrace (b));
-138
```

NAME `EltUnitDecompose`

PURPOSE Returns the decomposition of an unit with respect to the computed system of fundamental units of a maximal order.

SYNTAX
```
F := EltUnitDecompose(u);
L := EltUnitDecompose(u,"expons");
```

list	F
list	L
algebraic element	u

DESCRIPTION Let $\mathfrak{o}$ be an order over $\mathbb{Z}$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be the system of fundamental units of $\mathfrak{o}$ computed by the `OrderUnitsFund` function and let $\zeta \in \mathfrak{o}$ be the generator of the cyclic torsion subgroup which is accessible by the `OrderTorsionUnit` function. Each unit $u \in \mathfrak{o}$ is of a form

$$u = \zeta^{m_0} \varepsilon^{m_1} \cdots \varepsilon^{m_1}$$

with rational integers $m_0, m_1, \ldots, m_r$.

`F := EltUnitDecompose(u)`
returns the above decomposition of u.

`L := EltUnitDecompose(u,"expons")`
returns the exponents $m_0, m_1, \ldots, m_r$ of the above decomposition of u.

EXAMPLE Decompose certain units in $\mathbb{Z}[\sqrt[4]{2}]$:

```
kash> o := Order(Z,4,2);
Generating polynomial: x^4 - 2

kash> L := OrderUnitsFund(o);
[ [1, 0, 1, 0], [1, -1, 0, 0] ]
kash> u := L[1]; v:= L[2];
[1, 0, 1, 0]
[1, -1, 0, 0]
kash> EltUnitDecompose(u*v);
[ [ [1, 0, 1, 0], 1 ], [ [1, -1, 0, 0], 1 ] ]
kash> EltUnitDecompose(u*v,"expons");
[ 0, 1, 1 ]
kash> EltUnitDecompose(-u*v);
[ [ -1, 1 ], [ [1, 0, 1, 0], 1 ], [ [1, -1, 0, 0], 1 ] ]
kash> EltUnitDecompose(-u*v,"expons");
```

```
[ 1, 1, 1 ]
kash> EltUnitDecompose(u^3*v^2);
[ [ [1, 0, 1, 0], 3 ], [ [1, -1, 0, 0], 2 ] ]
kash> EltUnitDecompose(u^3*v^2,"expons");
[ 0, 3, 2 ]
```

NAME EltValuation

PURPOSE Compute the valuation of an algebraic element or an integer.

SYNTAX `val := EltValuation(P, alpha);`

integer	val	returned value.
ideal	P	a prime ideal.
algebraic element or integer	alpha	number to evaluate.

DESCRIPTION This function returns the valuation of an algebraic element or an integer at a prime ideal.

SEE ALSO IdealValuation, IntValuation,

EXAMPLE

```
kash> o := OrderMaximal(Order(Z,2,3));
Generating polynomial: x^2 - 3
Discriminant: 12

kash> L:=Factor(2*o);
[ [ <2, [1, 1]>, 2 ] ]
kash> P := L[1][1];
<2, [1, 1]>
kash> EltValuation(P, Elt(o,2));
2
kash> EltValuation(P, Elt(o,[1,1]));
1
kash> o := OrderMaximal (Order (Poly (Zx,[1,6,6,6])));
Generating polynomial: x^3 + 6*x^2 + 6*x + 6
Discriminant: -1836

kash> P := Factor(2*o)[1][1];
<2, [0, 1, 0]>
kash> EltValuation (P, 3);
0
kash> EltValuation (P, 2);
3
kash> EltValuation (P, 2^7);
21
kash> EltValuation (P, 2^40);
120
```

EulerGamma

NAME EulerGamma

PURPOSE Returns the value of the Euler constant γ.

SYNTAX `y := EulerGamma();`

```
real  y
```

DESCRIPTION

EXAMPLE

```
kash> EulerGamma();
0.577215664901532860606512090082402431042159335939923 6
```

NAME EutacticCoef

PURPOSE Returns a list of eutactic coefficents of a Humbert form in two variables

SYNTAX `E := EutacticCoef(HF,M,n);`

list	L
Humbert form with $S_i \in \overline{Q}^{2\times 2}$	HF=[S1,S2]
List of minimal vectors of HF	M
integers,	n

DESCRIPTION Given a Humbert form in two variables with algebraic entries and its minimal vectors, it will return a list of positive (in the n-th conjugate field) algebraic elements if the given Humbert form is eutactic. This means that the given humbert form is in the linear convex hull of matrix tuples, which will be received by its minimal vectors. All computations would be made in the n-th conjugate field from which the entries of the humbert form are.
If the form is not eutactic a list of zeros is returned.

SEE ALSO MinVec,

EXAMPLE

```
kash> o := OrderMaximal(x^2-3);
Generating polynomial: x^2 - 3
Discriminant: 12

kash> u := Elt(o,[2, 1]);
[2, 1]
kash> us := Elt(o,[2,-1]);
[2, -1]
kash> ur := Re(EltCon(Elt(o, [2, 1]))[1][1]);
3.73205080756887729352744634150587236694280525381
kash> urs := Re(EltCon(Elt(o, [2, 1]))[1][2]);
0.26794919243112270647255365849412763305719474619

kash> S1 := Mat(R, [ [ 1, ur/2 ], [ur/2, ur ] ]);
[1 1.866025403784438646763723170752936183471402626905]
[1.866025403784438646763723170752936183471402626905 3.7320508075688772935274463\
4150587236694280525381]
kash> S2 := Mat(R, [ [ 1, urs/2], [urs/2, urs] ]);
[1 0.133974596215561353236276829247063816528597373095]
```

```
[0.133974596215561353236276829247063816528597373095 0.267949192431122706472553\
65849412763305719474619]
kash> SA1 := Mat(o, [ [ 1, u/2 ], [u/2, u ] ]);
[1 [2, 1] / 2]
[[2, 1] / 2 [2, 1]]
kash> SA2 := Mat(o, [ [ 1, us/2 ], [ us/2,us ] ]);
[1 [2, -1] / 2]
[[2, -1] / 2 [2, -1]]
kash> HF := [S1,S2];
[ [1 1.866025403784438646763723170752936183471402626905]
    [1.866025403784438646763723170752936183471402626905 3.73205080756887729352\
744634150587236694280525381],
  [1 0.133974596215561353236276829247063816528597373095]
    [0.133974596215561353236276829247063816528597373095 0.26794919243112270647\
255365849412763305719474619] ]
kash> HFA := [SA1,SA2];
[ [1 [2, 1] / 2]
    [[2, 1] / 2 [2, 1]], [1 [2, -1] / 2]
    [[2, -1] / 2 [2, -1]] ]
kash> L := MinVec(HF,o,0)[1];
[ [ [0, 1], [1, -1] ], [ [0, 1], -1 ], [ [1, 1], [0, -1] ], [ [1, 1], -1 ],
  [ [2, 1], [0, -1] ], [ [2, 1], -2 ], [ 0, 1 ], [ 1, [1, -1] ],
  [ 1, [-2, 1] ], [ 1, -1 ], [ 1, 0 ], [ 2, -1 ] ]
kash> EutacticCoef(HFA,L,1);
[ 20 / 243, 55 / 486, 8 / 243, 8 / 243, 44 / 243, 56 / 243, 77 / 243, 1 / 9,
  23 / 162, 44 / 243, 56 / 243, 28 / 81 ]
```

NAME Eval

PURPOSE Evaluates a polynomial at a value.

SYNTAX `y := Eval(f, s);`

	y
polynomial	f
	s

DESCRIPTION This function assumes that the arithmetic with elements of the coefficient ring of the polynomial algebra of `f` and `s` is defined.

EXAMPLE First we create a polynomial (over $\mathbb{Z}$):

```
kash> f := x^2+2*x+1;
x^2 + 2*x + 1
```

Now we want to evalute it at 2 and π:

```
kash> Eval(f, 2);
9
kash> Eval(f, pi);
17.1527897082689450957597776643515690370803820599
```

Finally, we want to substitute x by x+1:

```
kash> Eval(f, x+1);
x^2 + 4*x + 4
```

NAME Exp

PURPOSE Returns the exponential of a number.

SYNTAX `y := Exp(x);`

```
complex  y
complex  x
```

DESCRIPTION Given a number `x` the function returns $e^{\mathtt{x}}$. The computation is done in the current precision of the real (complex) field.

SEE ALSO Log,

EXAMPLE

```
kash> Exp(1);
2.718281828459045235360287471352662497757247093699
kash> i := Comp(0, 1);
1*i
kash> Exp(i);
0.5403023058681397174009366074429766037323104206179222 + 0.8414709848078965066\
525023216302989996225630607983710*i
```

NAME FF

PURPOSE Creates a finite field.

SYNTAX
```
F := FF(p);
F := FF(p, d);
F := FF(f);
```

finite field	F
integer	p
integer	d
polynomial	f

DESCRIPTION A finite field can be created in three different ways:

- `FF(p)` returns the finite field $\mathbb{F}_p$ where p is a prime.
- `FF(p, d)` returns the finite field $\mathbb{F}_{p^d}$ where p is a prime and d a positive integer.
- `FF(f)` returns an extension field of a finite field: f is a monic, irreducible polynomial over a finite field $\mathbb{F}_q$ (q a prime power) of degree $l > 1$; $\mathbb{F}_{q^l}$ is returned.

SEE ALSO FFElt, FFGenerator, FFPrimitiveElt,

EXAMPLE Creation of finite fields:

```
kash> F2 := FF(2);
Finite field of size 2
kash> F2x := PolyAlg(F2);
Univariate Polynomial Ring in x over GF(2)

kash> f := Poly(F2x, [1, 1, 1]);
x^2 + x + 1
kash> F4 := FF(f);
Finite field of size 2^2
kash> F9 := FF(3, 3);
Finite field of size 3^3
```

NAME FFCreate

PURPOSE Creates a finite field.

SYNTAX FFCreate(O,p,k)

order	O
integer	p
integer	k

DESCRIPTION The finite field with p^k elements is created and stored in O. This routine differs from FunFF in that it only allows one way of generating the field; however, it has the advantage that one can work relatively.

SEE ALSO kantff,

EXAMPLE

```
kash> O:=OrderMaximal(Poly(Zx,[1,0,-4,0,1]));
Generating polynomial: x^4 - 4*x^2 + 1
Discriminant: 2304

kash> FFCreate(O, 3, 4);
Finite field of size 3^4
```

Name	`FFEToElt`
Purpose	Returns a canonical representative of a finite field element viewed as a representative of a class of algebraic numbers.
Syntax	`a := FFEToElt( f, p);`

finite field element	f	
ideal	p	must be prime
algebraic number or integer	a	interpreted as algebraic number

Description

This is a inverse function of EltToFFE, and a embedding

$$\mathcal{O}/\mathfrak{p} \to \mathcal{O}.$$

The prime ideal must not lie over a number which divides the degree of the order.

See Also

IdealResidueField, IdealResidueFieldIsomorphism, EltToFFE,

Example

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> p := Factor(7*O)[1][1];
<7, [0, 1, 0, 0, 0]>
kash> b := Elt(O,[3, 1,5,1,8]);
[3, 1, 5, 1, 8]
kash> n := EltToFFE(b, p);
3
kash> FFEToElt(n, p);
3
```

NAME `FFElt`

PURPOSE Creates an element of a finite field.

SYNTAX `a := FFElt(F, n);`

`finite field element`	`a`
`finite field`	`F`
`integer`	`n`

DESCRIPTION Embeds the integer n into a finite field $\mathbb{F}_q$.

SEE ALSO `FFPrimitiveElt`, `FFGenerator`, `FFEltToList`,

EXAMPLE Creation of a finite field element and some arithmetic:

```
kash> F5 := FF(5);
Finite field of size 5
kash> 1 + 4;
5
kash> a := FFElt(F5, 1) + 4;
0
```

NAME `FFEltFF`

PURPOSE Gives the finite field corresponding to the element.

SYNTAX `F := FFEltFF(a);`

finite field element	a
finite field	F

DESCRIPTION This function returns the finite field in which the given element is defined.

SEE ALSO FFElt, FF,

EXAMPLE

```
kash> F4 := FF(2, 2);
Finite field of size 2^2
kash> a := FFPrimitiveElt(F4);
w
kash> F := FFEltFF(a);
Finite field of size 2^2
```

NAME `FFEltIsZero`

PURPOSE Returns true iff the argument is of type "KANT finite field elt" AND zero.

SYNTAX `b := FFEltIsZero(a);`

```
boolean  b
         a
```

EXAMPLE

```
kash> k := FF(2,4);
Finite field of size 2^4
kash> a := FFElt(k,0);
0
kash> FFEltIsZero(a);
true
```

NAME FFEltLog

PURPOSE Discrete logarithm for finite fields.

SYNTAX `d := FFEltLog(a);`

finite field element	a
integer	d

DESCRIPTION This function computes the discrete logarithm of an element a with respect to a primitive element ω, that means a non negative integer d with $a = \omega^d$. The primitive element ω is that returned by `FFPrimitiveElt()`.

SEE ALSO FFPrimitiveElt,

EXAMPLE Compute some discrete logarithms:

```
kash> F125 := FF(5, 3);
Finite field of size 5^3
kash> a := FFElt(F125, 3);
3
kash> d := FFEltLog(a);
93
kash> w := FFPrimitiveElt(F125);
w
kash> w^d;
3
kash> FFEltLog(w^47);
47
```

Name `FFEltMinPoly`

Purpose Minimal polynomial of finite field elements.

Syntax
```
p := FFEltMinPoly(a);
p := FFEltMinPoly(a, J);
p := FFEltMinPoly(a, Jx);
```

`polynomial`	`p`	
`finite field element`	`a`	
`finite field`	`J`	subfield of the finite field of a
`polynomial algebra in x over J`	`Jx`	

Description This function computes the minimal polynomial of a finite field element over the base field of the finite field or over the finite field given as second parameter. It is also possible to specify the polynomial algebra in which the minimal polynomial should be.

See Also `FFPrimitiveElt`,

Example

NAME `FFEltMove`

PURPOSE Move finite field element into another finite field if possible.

SYNTAX `b := FFEltMove(a, J);`

`finite field element`	`b`
`finite field element`	`a`
`finite field`	`J`

DESCRIPTION

SEE ALSO `FFPrimitiveElt`,

EXAMPLE

NAME	`FFEltNorm`
PURPOSE	Norm of finite field elements.
SYNTAX	`p := FFEltNorm(a);` `p := FFEltNorm(a, J);`

`polynomial`	`p`	
`finite field element`	`a`	
`finite field`	`J`	subfield of the finite field of a

DESCRIPTION This function computes the Norm of a finite field element over the base field of the finite field or over the finite field given as second parameter.

SEE ALSO `FFPrimitiveElt`,

EXAMPLE

Name FFEltRoot

Purpose

Syntax `b := FFEltRoot(a, n);`

finite field element	b
finite field element	a
small integer	n

Description

Example

NAME `FFEltToInt`

PURPOSE Converts an element of a finite prime field to the corresponding integer.

SYNTAX `i := FFEltToInt(b);`

`finite field element`	`b`
`integer`	`i`

NAME FFEltToList

PURPOSE Returns a basis representation of a finite field element.

SYNTAX
```
L := FFEltToList(b);
L := FFEltToList(b, k);
```

finite field element	b
subfield	k
list	L

DESCRIPTION Let F/k be an extension of finite fields of degree d and a be a generator as returned by `FFGenerator()`. Then $1, a, \ldots, a^{d-1}$ is a k-basis for F. The function returns a list of the coefficients of an element $b \in F$ with respect to the powers of a. If k is omitted in the function call, it is assumed to be the base field.

SEE ALSO FFGenerator,

EXAMPLE

```
kash> F125 := FF(5, 3);
Finite field of size 5^3
kash> a := FFGenerator(F125);
w
kash> FFEltToList(1 + 2*a + 3*a^2);
[ 1, 2, 3 ]
```

Name	`FFEltTrace`
Purpose	Trace of finite field elements.
Syntax	`p := FFEltTrace(a);` `p := FFEltTrace(a, J);`

`polynomial`	`p`	
`finite field element`	`a`	
`finite field`	`J`	subfield of the finite field of a

Description	This function computes the trace of a finite field element over the base field of the finite field or over the finite field given as second parameter.
See Also	`FFPrimitiveElt`,
Example	

NAME FFEmbed

PURPOSE Embeds one finite field into another.

SYNTAX
```
FFEmbed(F1, F2);
FFEmbed(F1, F2, b);
```

finite field	F1
finite field	F2
finite field element	b

DESCRIPTION Embeds one finite field into another, if possible. The optional third parameter specifies the element of F_2 to which the generator of F_1 over its prime field should be mapped to.

SEE ALSO FF,

EXAMPLE

```
kash> FFEmbed(FF(2,3), FF(2,15));
true
```

NAME FFGenerator

PURPOSE Returns a generator of a finite field.

SYNTAX
```
a := FFGenerator(F);
a := FFGenerator(F, k);
```

finite field element	a
finite field	F
subfield of F	k

DESCRIPTION Given a finite field extension F/k of degree d the function returns $a \in F$ such that $F = k \cdot 1 + k \cdot a + \ldots + k \cdot a^{d-1}$. If k is omitted in the function call, it is assumed to be the base field.

SEE ALSO FFPrimitiveElt, FFEltToList,

EXAMPLE

```
kash> F25 := FF(5, 2);
Finite field of size 5^2
kash> a := FFGenerator(F25);
w
kash> y := FFPrimitiveElt(F25);
w
kash> F5 := FF(5);
Finite field of size 5
kash> b := FFGenerator(F5);
1
kash> z := FFPrimitiveElt(F5);
2
kash> F64 := FF(2, 6);
Finite field of size 2^6
kash> FFGenerator(F64, F64);
w
kash> FFGenerator(F64, FF(2,2));
w
```

Name `FFPrimitiveElt`

Purpose Returns a primitive element of a finite field.

Syntax `w := FFPrimitiveElt(F);`

`finite field element`	`w`
`finite field`	`F`

Description For a finite field $\mathbb{F}_q$ ($q = p^d$ a prime power) the function returns a generator ω of $\mathbb{F}_q^\times$.

See Also `FFElt`, `FFGenerator`,

Example

```
kash> F25 := FF(5, 2);
Finite field of size 5^2
kash> w := FFPrimitiveElt(F25);
w
kash> w + w;
w^7
kash> w^0;
1
kash> w^-1;
w^23
```

NAME `FFSize`

PURPOSE Gives the size of the finite field $\mathbb{F}_q$.

SYNTAX `L := FFSize(F);`

`list`	`L`	of p and d with $p^d = q$.
`finite field`	`F`	

DESCRIPTION This function returns a list consisting of the characteristic p and the degree d of $\mathbb{F}_q$ over $\mathbb{F}_p$.

EXAMPLE

```
kash> F9 := FF(3, 2);
Finite field of size 3^2
kash> L := FFSize(F9);
[ 3, 2 ]
```

NAME FLDin

PURPOSE Reads an order in the FLD format.

SYNTAX
```
o := FLDin(name [, n]);
o := FLDin();
o := FLDin(f [, n]);
```

`file`	`f`	open for reading
`string`	`name`	filename
`order`	`o`	
`integer`	`b`	the number of fields to skip. If given, the number of fields is returned.

If the function is called with a string the order is read from a file with this name. If no argument is given the order is read from the standard input where twice the return key finishes the input.

The last option is a call with an already opened file. When reading multiple orders from the same file it has to be opened with modus `"R"` (see `Open`).

Now some Information about the structure of the FLD format:

The routine `FLDout` produces an output which can be read again by `FLDin`. Example:

Consider the following output:

```
(FLD=) 6, 2, 2, 0, 6, 2, 0, 18, 0, 0, 0, * -182099043 19.183962221%45404527676
1 10 7 89 72 64
3272 0 0 0 0 0 3272
0 3272 0 0 0 0 3272
0 0 3272 0 0 0 3272
0 0 0 3272 0 0 3272
0 1636 0 0 1636 0 3272
720 2117 2543 2554 1237 1 3272
-9 -11 -32 -32 -31 41
-27 -33 -102 -102 -99 131
10 11 35 35 34 -45
18 2 3 6
2
0 2 -4 -3 -3 -1 -3 -1
0 2 -5 -4 -1 -1 -1 0
3 0
0 6
```

First line: (FLD=): command for FLDin

4: field degree N

EXFLAG(1),...,EXFLAG(10): Existence flags for field data: Meaning:

No. 1:	=	0:	no integral basis known
	=	1:	integral basis is power basis
	=	2:	integral basis is not power basis
	=	-1:	integral basis is not power basis but not known
	=	-2:	the basis given is only probably an integral basis
No. 2:	>	0:	number of independent or fundamental units
No. 4:	>	0:	number of roots of unity (if computed)
No. 5:	>	0:	The units are fundamental units
No. 6:	>	0:	The Galois group structure is known
No. 7:	>	0:	Information about the class number.
No. 10	>	0:	Information about the signature

after the asterisk:

first number = field discriminant

second number = regulator of unit system (if computed already)

Second line: coefficients of defining polynomial, decreasing and without leading coefficient (= 1)

Then: Beginning in the third line: If EXFLAG(1)=2 you find the transformation matrix of the integral basis referring to the power basis. Every line corresponds to an integral basis element where the denominator stands in the last position.

After the integral basis matrix: If computed, coefficients of the unit system, corresponding to the integral basis

After the unit system: If already computed and rank of torsion unit subgroup > 2: coefficient of a torsion unit subgroup generator, corresponding to the integral basis

After this: If known: The structure of the Galois group (six characters)

After this: If the class number is known and larger than one in the following four (or more) lines describe the class group structure:

First line: class number, number of cyclic factors, orders

Second line: number of prime ideals needed for description

Then: Prime ideals in two-element-representation, starting with the degree of inertia, the first generator, then the coefficients of the second generator (Each prime ideal in a row)

Then: Exponent vectors for the generators of the cyclic factors, corresponding to the ideals (Each generator in a row)

The above output corresponds to the totally complex sextic number field $\mathcal{F}$. A generating polynomial of $\mathcal{F}$ is:

$$f(x) = x^6 + x^5 + 10x^4 + 7x^3 + 89x^2 + 72x + 64.$$

An integral basis $\omega_1, \ldots, \omega_6$ (referring to a root ρ of f) is given by:

$$\begin{aligned}
\omega_1 &= & 3272/3272 \\
\omega_2 &= & 3272\rho/3272 \\
\omega_3 &= & 3272\rho^2/3272 \\
\omega_4 &= & 3272\rho^3/3272 \\
\omega_5 &= & (1636\rho + 1636\rho^4)/3272 \\
\omega_6 &= & (720 + 2117\rho + 2543\rho^2 + 2554\rho^3 + 1237\rho^4 + \rho^5)/3272
\end{aligned}$$

The field discriminant is:

$$-182099043.$$

A system of fundamental units is:

$$\begin{aligned}
\eta_1 &= -9\omega_1 - 11\omega_2 - 32\omega_3 - 32\omega_4 - 31\omega_5 + 41\omega_6 \\
\eta_2 &= -27\omega_1 - 33\omega_2 - 102\omega_3 - 102\omega_4 - 99\omega_5 + 131\omega_6.
\end{aligned}$$

The regulator of $\mathcal{F}$ is 19.18396.

$\mathcal{F}$ contains 6 roots of unity. A primitive 6th root of unity is

$$10\omega_1 + 11\omega_2 + 35\omega_3 + 35\omega_4 + 34\omega_5 - 45\omega_6.$$

The class number of $\mathcal{F}$ is 18. The class group is isomorphic to

$$\mathbb{Z}_3 \times \mathbb{Z}_6.$$

Let

$$\begin{aligned}
\wp_1 &= 2o_\mathcal{F} + (-4\omega_1 - 3\omega_2 - 3\omega_3 - \omega_4 - 3\omega_5 - \omega_6)o_\mathcal{F} \qquad \text{(norm=2)} \\
\wp_2 &= 2o_\mathcal{F} + (-5\omega_1 - 4\omega_2 - \omega_3 - \omega_4 - \omega_5)o_\mathcal{F} \qquad \text{(norm=2)}
\end{aligned}$$

Then the cyclic factors of the class group are generated by the classes of

$$\begin{aligned}
&\wp_1^3 \qquad \text{(order: 3)} \\
&\wp_2^6 \qquad \text{(order: 6)}
\end{aligned}$$

SEE ALSO FLDout, Open,

EXAMPLE We give an example on how to compute tables using KASH: First, we will compute a series of orders using a small program written to check fields generated by polynomials of the form

$$f = x^3 + p * x^2 + p * x + p$$

for non-trivial class groups. Of course this is only an example. In practice you should define a function taking two arguments (`p` and the bound `c` for p) and the file to write to. Initialization:

```
kash> \# Open a file for printing the output
kash> outfile := Open("poly.tbl","w");
Filename: poly.tbl / Mode: w / Open (fid): 5
kash> \#
kash> p := 3; \# start
3
kash> c := 3; \# how many fields
3
```

Now do a loop over the first c primes:

```
kash> for j in [1..c] do
> p:=NextPrime(p);
> \#
> \# Assign f and create the corresponding maximal order
> \#
> f := Poly(Zx,[1,p,p,p]);
> O := OrderMaximal(Order(f));
> \#
> \# Compute the class group of o
> \#
> clg := OrderClassGroup(O);
> \#
> \# check if the class group is trivial
> \#
> if IsList(clg) then
> Print ("field generated by ", f," has class number: ", clg[1],"\n");
> \# print the field in the output file
> FLDout(O, outfile);
> fi;
> od;
field generated by x^3 + 5*x^2 + 5*x + 5 has class number: 1
field generated by x^3 + 7*x^2 + 7*x + 7 has class number: 3
field generated by x^3 + 11*x^2 + 11*x + 11 has class number: 6
kash> \# closing the output file
kash> Close(outfile);
true
```

The file should look this:

```
(FLD=) 3, 2, 0, 0, 2, 0, 0, 1, 0, 0, 1, -200, *
5 5 5
8 0 0 8
0 8 0 8
4 0 4 8
(FLD=) 3, 1, 1, 0, 2, 0, 0, 3, 0, 0, 1, -3724, *
7 7 7
1 1 1
3 1 3
1
1 2 1 1 0
1
(FLD=) 3, 1, 1, 0, 2, 0, 0, 6, 0, 0, 1, -28556, *
11 11 11
1 1 1
6 1 6
2
1 5 8 1 0
1 2 1 1 0
1 1
```

Use the output as input to compute fundamental units:

```
kash> \#
kash> \# Open file for input. Use mode "R" (not "r") to loop
kash> \# over the whole file
kash> \#
kash> infile := Open("poly.tbl", "R");
Filename: poly.tbl / Mode: R / Open (fid): 5
kash> \#
kash> \# Read first order
kash> o1 := FLDin(infile);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^3 + 5*x^2 + 5*x + 5
Discriminant: -200
```

```
Class Number 1

kash> OrderUnitsFund(o1);
[ [0, 1, 2] ]
kash> \#
kash> \# Read second order
kash> o2 := FLDin(infile);
Generating polynomial: x^3 + 7*x^2 + 7*x + 7
Discriminant: -3724
Class Number 3
Class Group Structure C3
Cyclic Factors of the Class Group:
<2, 2>

kash> OrderUnitsFund(o2);
[ [1, 1, 1] ]
kash> \#
kash> \# Read third order
kash> o3 := FLDin(infile);
Generating polynomial: x^3 + 11*x^2 + 11*x + 11
Discriminant: -28556
Class Number 6
Class Group Structure C6
Cyclic Factors of the Class Group:
<10, 10>

kash> OrderUnitsFund(o3);
[ [1, 1, 1] ]
kash> Close(infile);
true
```

Name `FLDin_DB`

Purpose Reads an order in the FLD format for database "kate" without fundamental units and only number of generators of the class group and their orders

Syntax

```
o := FLDin(name [, n]);
o := FLDin();
o := FLDin(f [, n]);
```

`file`	`f`	open for reading
`string`	`name`	filename
`order`	`o`	
`integer`	`b`	the number of fields to skip. If given, the number
	`of fields is returned.`	

See Also FLDout, Open,

NAME `FLDout`

PURPOSE Writes an order using the FLD format.

SYNTAX `FLDout(o [, name | file]);`

`Order`	`o`	
`String`	`name`	filename to use
`File`	`file`	opened for writing

DESCRIPTION Dumps an order using the FLD format. Depending on the (optional) second parameter it is written to the standard output stream (your terminal) or to a file.
Be careful when giving a file name: the file will be opened for writing and possibly its contents destroyed.
If you want to append your order to an existing file you have to open it using mode="a" to append (see `Open`).
Combined with `FLDin` this function may be used to produce tables of orders. Using the FLD format you are able to store information like an integral basis, units and the class group, and to retrieve it in a later KASH session.
For information about the FLD format see `FLDin`.

SEE ALSO `Open`, `Close`, `LOFILES`, `FLDin`, `ECHOon`, `ECHOoff`,

EXAMPLE See `FLDin` for an example.

NAME Factor

PURPOSE Returns the factorization of the given argument.

SYNTAX

```
F := Factor(d);
F := Factor(f);
F := Factor(f, p);
F := Factor(a);
F := Factor(alpha);
F := Factor(O, d);
F := Factor(O, a);
F := Factor(O, alpha);
```

list	F
prime	p
integer	d
polynomial	f
ideal	a
algebraic element	alpha
order	O

DESCRIPTION

At the moment 4 different objects can be factored:

Factor(d) Returns the factorization of the integer $d \in \mathbb{N}$. d must be greater 1.

Factor(f) Returns the factorization of the polynomial $f \in \mathbb{Z}[x], \mathbb{Q}[x], o[x], FF[x]$.

Factor(f, p) Returns the factorization of the polynomial $f \in \mathbb{Z}[x] \bmod p\mathbb{Z}[x]$.

Factor(a) Let $\mathcal{O}$ be the maximal order where $\mathfrak{a}$ has been defined within. The function returns the factorization of $\mathfrak{a}$ in $\mathcal{O}$.

Factor(alpha) Let $\mathcal{O}$ be the maximal order where α has been defined within. The function returns the factorization of the principal ideal $\alpha\mathcal{O}$ in $\mathcal{O}$.

Factor(O,d) Returns the factorization of the principal ideal $d\mathcal{O}$ in the maximal order $\mathcal{O}$.

Factor(O,alpha) Returns the factorization of the principal ideal $\alpha\mathcal{O}$ in the maximal order $\mathcal{O}$. α must be defined within $\mathcal{O}$. Same as `Factor(alpha)`.

Factor(O,a) Returns the factorization of the ideal $\mathfrak{a}$ in the maximal order $\mathcal{O}$. $\mathfrak{a}$ must be defined within $\mathcal{O}$. Same as `Factor(a)`.

Factor

The factorization of the argument $A = u \prod |i = 1^k \pi|i^{e|i}$ with $A \in \{d, f, \mathfrak{a}, \alpha\}$, a unit u, $k \in \mathbb{N}|0$ and prime elements $\pi|i, 1 \leq i \leq k$ is returned in a list of the following form: $[[\pi|1, e|1], \ldots, [\pi|k, e|k]]$.

SEE ALSO IntFactor, EltFactor, PolyFactor, IdealFactor,

EXAMPLE Factorization of 827462647264826482642764872364 8276:

```
kash> Factor(8274626472648264826427648723648276);
[ [ 2, 2 ], [ 11, 1 ], [ 41, 1 ], [ 86423, 1 ],
  [ 53074086409412759917474753, 1 ] ]
```

NAME `Factorial`

PURPOSE The factorial of an integer

SYNTAX `a := Factorial(b);`

`integer`	`a`
`positive integer`	`b`

SEE ALSO `Gamma`,

EXAMPLE

```
kash> a := Factorial(5);
120
```

FilePosition

NAME	`FilePosition`
PURPOSE	Sets or reads the file position, allowing thus random access of `FLD` files.
SYNTAX	`o := FilePosition(file [, pos ]);`

`file`	`f`	open for reading

SEE ALSO `FLDout`, `Open`,

NAME `FindMaximalCentralField`

PURPOSE Finds the maximal factorgroup where a list of automorphisms acts trivial.

SYNTAX `sg := FindMaximalCentralField(o | I [, inf][, aut]);`
`sg := FindMaximalCentralField(G [, aut]);`

`Matrix`	`sg`	describing the aditional relations
`order`	`o`	
`integral ideal`	`I`	
`list`	`inf`	of infinite places
`AbelianGroup`	`G`	must be from RayClassGroupToAbelianGroup
`list`	`aut`	of automorphisms of `o`, if omitted `OrderAutomorphisms(o, [])` is used.

DESCRIPTION

EXAMPLE We'll investigate the ray class field mod $m := (6)\mathfrak{p}|1^\infty\mathfrak{p}|2^\infty$ in $k := \mathbb{Q}(\sqrt{10})$:

```
kash> o := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> OrderAutomorphisms(o);
[ [0, 1], [0, -1] ]
kash> G := RayClassGroupToAbelianGroup(6*o, [1, 2]);
RayClassGroupToAbelianGroup(<6>, [ 1, 2 ])
Group with relations:
[2 0 0]
[0 4 0]
[0 0 2]
kash> rcg := FindMaximalCentralField(G);
[2 0 0]
[0 2 0]
[0 0 2]
[0 0 0]
[0 0 0]
[0 0 0]
```

Since `rcg` corresponds not to the whole group, we can construct two fields that are normal over $\mathbb{Q}$: the ray class field modulo m and the maximal central extension of k:

```
kash> cf1 := RayClassField(6*o, [1, 2], rcg);
[ x^2 - 6, x^2 - 2, x^2 + 2 ]
kash> cf2 := RayClassField(6*o, [1, 2]);
[ x^2 - 6, x^2 + 2, x^4 + [-8, 4]*x^2 + [28, -8] ]
```

Finally we'll compute defining equations and the automorphisms:

```
kash> O1 := RayClassFieldAuto(cf1);
[       F[1]
            /
           /
       E1[1]
      /
     /
    Q
    F  [ 1]     x^8 - 24*x^6 + 248*x^4 + 288*x^2 + 2704
    E 1[ 1]     x^2 - 10
    Discriminant: <17352869066524232277088370178392064>
    , [ a, a*b, a*b*c, a*b*c*d, a*b*d, a*c, a*c*d, a*d, b, b*c, b*c*d, b*d,
      c, c*d, d, e ] ]
```

...and see that O1 is abelian over $\mathbb{Q}$ whereas O2 is non-abelian. Since k is cyclic over $\mathbb{Q}$, central extensions coincide with absolute abelian ones.

```
kash> a := O1[2][Position(O1[2], "a")];
a
kash> b := O1[2][Position(O1[2], "b")];;
kash> c := O1[2][Position(O1[2], "c")];;
kash> d := O1[2][Position(O1[2], "d")];;
kash> a*b = b*a; a*c = c*a; a*d = d*a;
true
true
true
kash> b*c = c*b; b*d = d*b; c*d = d*c;true
true
> true
```

NAME `FindQuotientOfShapeEnumInit`

PURPOSE Initializes an environment to enumerate certain subgroups

SYNTAX `s := FindQuotientOfShapeEnumInit(G, L);`

`record`	`s`	
`AbelianGroup`	`G`	
`list`	`L`	of integers describing the shape of the subgroup

DESCRIPTION

SEE ALSO `FindQuotientOfShapeEnumNext`,

EXAMPLE Let $G := C^2|8 \times C|6^2 \times C|2$. We'll enumerate all quotients $Q = G/U$ of G such that $Q \cong C|4 \times C|2$:

```
kash> G := AbelianGroupCreate(MatDiag(Z, [8, 8, 6, 6, 2]));
Group with relations:
[8 0 0 0 0]
[0 8 0 0 0]
[0 0 6 0 0]
[0 0 0 6 0]
[0 0 0 0 2]
kash> s := FindQuotientOfShapeEnumInit(G, [4, 2]);
Record of type FindQuotientOfShapeEnum

kash> while FindQuotientOfShapeEnumNext(s) and s.no <3 do
> Print("Number ", s.no, "\n", s.elt, "\n");
> l := AbelianQuotientGroup(G, AbelianSubGroup(G, s.elt));
> gs := AbelianGroupEnumInit(l);
> l := [];
> while AbelianGroupEnumNext(gs) do Add(l, gs.elt); od;
> Apply(l, x->AbelianGroupEltMove(x, G));
> Print("Containing: ", l, "\n");
> od;
Number 1
[2 0 4 0 1]
[6 0 1 0 0]
[2 4 4 3 0]
[0 2 0 4 0]
[4 0 2 0 0]
```

```
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
Containing: [ [0 0 0 0 0], [6 7 0 5 0], [1 0 0 0 0], [7 7 0 5 0],
  [2 0 0 0 0], [0 7 0 5 0], [3 0 0 0 0], [1 7 0 5 0] ]
Number 2
[0 0 0 0 1]
[6 0 1 0 0]
[2 4 4 3 0]
[0 2 0 4 0]
[4 0 2 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
[0 0 0 0 0]
Containing: [ [0 0 0 0 0], [6 7 0 5 0], [1 0 0 0 0], [7 7 0 5 0],
  [2 0 0 0 0], [0 7 0 5 0], [3 0 0 0 0], [1 7 0 5 0] ]
kash> while FindQuotientOfShapeEnumNext(s) do
> i := 1;
> od;
kash> Print("Total: ", s.no, "\n");
Total: 360
```

NAME `FindQuotientOfShapeEnumNext`

PURPOSE Steps through the subgroups.

SYNTAX `flag := FindQuotientOfShapeEnumNext(s [, 1]);`

`boolean`	`flag`	
`record`	`s`	generated by `FindQuotientOfShapeEnumInit`
`if present find subgroups instead of quotients.`	`1`	

DESCRIPTION

SEE ALSO `FindQuotientOfShapeEnumInit`,

EXAMPLE See `FindQuotientOfShapeEnumInit` for an example.

Floor

NAME Floor

PURPOSE Returns the maximal integer less than or equal to a given number.

SYNTAX `y := Floor(x);`

integer	y
real	x

DESCRIPTION Given an `x` in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ the function returns

$$\max\{k \in \mathbb{Z} : k \leq \mathtt{x}\}.$$

The computation is done in the current precision of the real (complex) field.

SEE ALSO Trunc, Round, Ceil,

EXAMPLE

```
kash> Floor(3);
3
kash> Floor(-3);
-3
kash> Floor(3.5);
3
kash> Floor(-3.5);
-4
```

NAME	`GPin`
PURPOSE	Reads an order in the pari/gp format.

SYNTAX

```
o := GPin(name);
o := GPin();
o := GPin(f);
```

`file`	`f`	file opened for reading
`string`	`name`	filename
`order`	`o`	
	`**`	

DESCRIPTION If the function is called with a string the order is read from a file with this name. If no argument is given the order is read from the standard input where twice the return key finishes the input.
The last option is a call with an already opened file. When reading multiple order from the same file it has to be opened with modus `"R"` (see `Open`).

EXAMPLE We read from the standard input:

```
kash> o:=GPin();
false
```

NAME `Galois`

PURPOSE Computation of Galois groups.

SYNTAX

```
Galois ( f [,p] [, "fast"] );
Galois ( o [,p] [, "fast"] );

Galois ( f , "complex" [, n] );
Galois ( o , "complex" [, n] );

GaloisT ( f [,p] [, "fast"] );
GaloisT ( o [,p] [, "fast"] );

GaloisT ( f , "complex" [, n] );
GaloisT ( o , "complex" [, n] );
```

`polynomial`	`f`	
`order`	`o`	
`prime number`	`p`	
`positive integer`	`n`	precision

DESCRIPTION This function computes the Galois group of an irreducible polynomial with coefficients in $\mathbb{Q}$ and with degree up to 23.
The first two callings are a p-adic version of the method of Stauduhar. The optional integer p gives the prime number to be used for the p-adic computations. Not using the parameter `"fast"` the p-adic version returns a proven result. The optional parameter `"fast"` is able to speed up the computation considerably, but incorrect results may occure. The option `"fast"` makes use of lower bounds during the inclusion test of a Galois group computation. The next two callings are the complex version of the method of Stauduhar. The optional integer n gives the precision to be used. The minimal precision is 100.

`Galois()` returns the name of the Galois group whereas `GaloisT()` returns its number in T-notation as classified in GAP [S+97].

The implementation of the `Galois` function [Gei97] in KASH bases on the algorithm of R.P. Stauduhar [Sta73].

SEE ALSO `GaloisGlobals`, `GaloisGroupsPossible`, `GaloisModulo`, `GaloisTree`, `GaloisRoots`, `GaloisNumberToName`, `GaloisBlocks`,

EXAMPLE

```
kash> Galois(x^12-2);
"D(4)[x]S(3)"
kash> o := Order(Z, 12, 2);
Generating polynomial: x^12 - 2

kash> GaloisModulo(o, 100);
[ 28, 81, 83, 86, 125, 134, 141, 143, 156, 185, 186, 193, 208, 209, 213, 217,
  222, 239, 240, 248, 250, 258, 260, 267, 268, 270, 274, 281, 283, 288, 289,
  292, 293, 294, 299, 301 ]
kash> f := x^15 + 2*x^9 - 5*x^6 + 2*x^3 - 1;
x^15 + 2*x^9 - 5*x^6 + 2*x^3 - 1
kash> GaloisT(f);
61
```

NAME `GaloisBlocks`

PURPOSE Excluding unpossible Galois groups by blocks.

SYNTAX
```
GaloisBlocks(o);
GaloisBlocks(o, true);
```

```
order  o
```

DESCRIPTION This function computes the lengths of the blocks of the Galois group of the order using subfields. Transitive groups with other block systems are excluded from the list of possible Galois groups. Is the second parameter used, this function computes the Galois groups of the subfields and excludes transitive groups with other Galois groups of the subfields from the list of possible Galois groups.

SEE ALSO `GaloisGroupsPossible`, `GaloisModulo`, `Galois`,

EXAMPLE

```
kash> o := Order(x^15+5*x^14+8*x^13+7*x^12+8*x^11-3*x^9+7*x^8-2*x^7+7*x^6-2*x^5-3*x^4+5*x^3-2*x+1);
Generating polynomial: x^15 + 5*x^14 + 8*x^13 + 7*x^12 + 8*x^11 - 3*x^9 + 7*x^\
8 - 2*x^7 + 7*x^6 - 2*x^5 - 3*x^4 + 5*x^3 - 2*x + 1

kash> GaloisGroupsPossible(o);
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,
  22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40,
  41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59,
  60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78,
  79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97,
  98, 99, 100, 101, 102, 103, 104 ]
kash> GaloisBlocks(o);
[ 1, 2, 3, 4, 6, 7, 8, 11, 16, 22, 23, 24, 29 ]
kash> GaloisBlocks(o, true);
[ 23 ]
```

NAME `GaloisGlobals`

PURPOSE Prints the current settings in the Galois group computation.

SYNTAX `GaloisGlobals(o);`

`order o`

DESCRIPTION This function prints some settings for the current Galois group computation, for example the used padic or complex fields, the number of transitive permutation groups and the number of computed Tschirnhausen transformations. It is intended for internal use.

SEE ALSO `Galois`, `GaloisTree`, `GaloisRoots`,

EXAMPLE

```
kash> o := Order(Z, 12, 2);
Generating polynomial: x^12 - 2

kash> GaloisT(o, "complex", 150);
28
kash> GaloisGlobals(o);
******* galois globals begin *******
degree is: 12
polynomial is: x^12 - 2
Fields are:
Galois ring of type real over
Complex Field of precision 152

gg_p_bound is: 100
gg_tschirn is: 0
gg_num_groups is: 301
gg_prec is: 150
******* galois globals end   *******
kash> o := Order(Z, 12, 2);
Generating polynomial: x^12 - 2

kash> GaloisT(o);
28
kash> GaloisGlobals(o);
******* galois globals begin *******
```

```
degree is: 12
polynomial is: x^12 - 2
Fields are:
Galois ring of type padic over
Generating polynomial: x^2 - 3

for p = 17,
    k = 4,
    m = 48661191875666868481

gg_p_bound is: 100
gg_tschirn is: 0
gg_num_groups is: 301
gg_prec is: 0
******* galois globals end   *******
```

NAME GaloisGroupKnown

PURPOSE This function returns the Galois group if it is already set in the order or the algebraic function field. If the Galois group is not set `false` will be returned.

SYNTAX `b:= GaloisGroupKnown(K);`

int or false	b
order or algebraic function field	K

DESCRIPTION

SEE ALSO GaloisGlobals, GaloisGroupsPossible, GaloisModulo, GaloisTree, GaloisRoots, GaloisNumberToName, GaloisBlocks,

EXAMPLE

```
kash> o1 := Order(x^15 - 240*x + 224);
Generating polynomial: x^15 - 240*x + 224

kash> o2 := Order(x^15+2*x^9-2*x^6-x^3+2);
Generating polynomial: x^15 + 2*x^9 - 2*x^6 - x^3 + 2

kash> g := Galois(o1);
"A15"
kash> b := GaloisGroupKnown(o2);
false
kash> b := GaloisGroupKnown(o1);
103
```

NAME `GaloisGroupOrder`

PURPOSE Returns the order of the transitive permutation group.

SYNTAX `GaloisGroupOrder(n, k);`

`integer`	`n`	representing degree
`integer`	`k`	representing group in T-notation

DESCRIPTION This function returns the order of a transitive permutation group up to degree 20.

SEE ALSO `Galois`,

EXAMPLE

```
kash> GaloisGroupOrder(18, 354);
1944
```

NAME `GaloisGroupSet`

PURPOSE Sets the number of the Galois group in the order or algebraic function field.

SYNTAX `GaloisGroupSet(K, G);`

`order or algebraic function field`	`K`
`int`	`G`

DESCRIPTION

SEE ALSO `GaloisGlobals`, `GaloisGroupsPossible`, `GaloisModulo`, `GaloisTree`, `GaloisRoots`, `GaloisNumberToName`, `GaloisBlocks`,

EXAMPLE

```
kash> o := Order(x^13-2);
Generating polynomial: x^13 - 2

kash> GaloisGroupSet(o,6);
```

NAME `GaloisGroupsPossible`

PURPOSE Handling of possible Galois groups.

SYNTAX

```
L := GaloisGroupsPossible(o);
GaloisGroupsPossible(o, G, flag);
GaloisGroupsPossible(o, L, flag);
```

`order`	`o`	
`integer`	`G`	representing transitive group in T-notation
`list`	`L`	of integers G
`boolean`	`flag`	whether to add or remove groups

DESCRIPTION This function is intended for use before calling `Galois()`. At the beginning of an Galois group computation there is a set of possible Galois groups which will be reduced by computing possible cycle types or block systems for the Galois group. This function either returns the list of current possible Galois groups or is able to remove from or add groups to the list.

SEE ALSO `Galois`, `GaloisTree`, `GaloisNumberToName`,

EXAMPLE

```
kash> o := Order(Z, 8, 2);
Generating polynomial: x^8 - 2

kash> GaloisGroupsPossible(o);
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,
  22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40,
  41, 42, 43, 44, 45, 46, 47, 48, 49, 50 ]
kash> GaloisGroupsPossible(o, 49, false);
kash> GaloisGroupsPossible(o, [1, 2, 3, 4, 5], false);
kash> GaloisGroupsPossible(o);
[ 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25,
  26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44,
  45, 46, 47, 48, 50 ]
```

NAME GaloisMSetPol

PURPOSE Computing of a polynomial of degree $\binom{n}{k}$ which roots are products of k distinct roots of f.

SYNTAX `g:= GaloisMSetPol(f, k);`

int	k	must be positive
polynomial	f, g	

DESCRIPTION Let f be a monic polynomial of degree n in $\mathbb{Q}$, $\mathbb{Q}(x)$ or a simple relative order.This function computes a primitive polynomial of degree $\binom{n}{k}$. The roots of `GaloisMSetPol` are the products of k distinct roots of f.

SEE ALSO GaloisGlobals, GaloisGroupsPossible, GaloisModulo, GaloisTree, GaloisRoots, GaloisNumberToName,

EXAMPLE

```
kash> Zxy := PolyAlg(Zx);;
kash> y := Poly(Zxy, [1,0]);;
kash> f := y^5-x*(5*y-4);
y^5 - 5*x*y + 4*x
kash> g := GaloisMSetPol(f,3);
y^10 + 20*x^2*y^7 - 125*x^3*y^6 + 128*x^3*y^5 - 400*x^4*y^4 + 1280*x^5*y^2 + 4\
096*x^6
```

Name	`GaloisMSumPol`
Purpose	Computing of a polynomial of degree $\binom{n}{k}$ which roots are sums of k distinct roots of f.
Syntax	`g:= GaloisMSumPol(f, k);`

`int`	`k`	must be positive
`polynomial`	`f,g`	

Description Let f be a monic polynomial of degree n in $\mathbb{Q}$, $\mathbb{Q}(x)$ or a simple relative order.This function computes a primitive polynomial of degree $\binom{n}{k}$. The roots of `GaloisMSumPol` are the sums of k distinct roots of f.

See Also `GaloisGlobals`, `GaloisGroupsPossible`, `GaloisModulo`, `GaloisTree`, `GaloisRoots`, `GaloisNumberToName`, `GaloisBlocks`,

Example

```
kash> o := OrderMaximal(Z, 2, 2);
Generating polynomial: x^2 - 2
Discriminant: 8

kash> ox := PolyAlg(o);
Univariate Polynomial Ring in x over Generating polynomial: x^2 - 2
Discriminant: 8

kash> f := Poly(ox, [Elt(o,1), Elt(o,[74,-16] / 73), Elt(o,[5988,-2368]
> / 5329), Elt(o, [160667739, -69612720] / 133432831), Elt(o,
> [98807254537, -54032137568] / 68184176641)]);
x^4 + [74, -16] / 73*x^3 + [5988, -2368] / 5329*x^2 + [160667739, -69612720] /\
 133432831*x + [98807254537, -54032137568] / 68184176641
kash> g := GaloisMSumPol(f,2);
x^6 + (222/73*$.1 - 48/73*$.2)*x^5 + (29940/5329*$.1 - 11840/5329*$.2)*x^4 + (\
2594440/389017*$.1 - 1355200/389017*$.2)*x^3 + (42642475310/68184176641*$.1 - \
42196712560/68184176641*$.2)*x^2 + (-16578528081908/4977444894793*$.1 + 988909\
5753408/4977444894793*$.2)*x - 35497382888701141/17804320388674561*$.1 + 22361\
191523984800/17804320388674561*$.2
```

NAME `GaloisMissionS`

PURPOSE Returns all non-trivial subfields of given degree `m`. If no `m` is specified, all subfields are calculated.

SYNTAX
```
L := GaloisMissionS(o);
L := GaloisMissionS(o, m);
```

`list`	`L`	list of suborders
`order`	`O`	the given order
`small integer`	`m`	the prescribed degree of subfields

DESCRIPTION This function calculates all non-trivial subfields of given degree `m`. If no `m` is specified, all non-trivial subfields are calculated. The result is a list `L`, which contains the calculated subfields as orders. The function uses the algorithms described in [Klü95, KP97, Klü97]
It is possible that this function returns the same order for two subfields. In this case, these subfields are isomorphic, but not identical. In the example below, the three subfields are isomorphic If the running time for this function seems too long, the user may want to try OrderSubfieldSub.

SEE ALSO `GaloisMissionS`,

EXAMPLE Computation of subfields of given degree.

```
kash> O:=Order(Poly(Zx,[1,-3,5,-5,5,-3,1]));
Generating polynomial: x^6 - 3*x^5 + 5*x^4 - 5*x^3 + 5*x^2 - 3*x + 1

kash> L:=GaloisMissionS(O,3);
[ Generating polynomial: x^3 - 3*x^2 + 2*x - 1
    , Generating polynomial: x^3 - 2*x^2 + 3*x - 1
    , Generating polynomial: x^3 - 3*x^2 + 2*x - 1
     ]
kash> elt:=Elt(L[1],[1,1,1]);
[1, 1, 1]
kash> EltMove(elt,O);
[2, -3, 3, -5, 3, -2]
```

NAME `GaloisModulo`

PURPOSE Excluding impossible Galois groups by cycle types.

SYNTAX `GaloisModulo(o, b);`

`order`	`o`	
`integer`	`b`	bound

DESCRIPTION This function computes cycle types of the Galois group by factoring the generating polynomial modulo some prime numbers up to a given bound. Transitive groups without the cycle types are excluded from the list of possible Galois groups and the list is returned.

SEE ALSO `Galois`, `GaloisBlocks`, `GaloisGroupsPossible`, `GaloisTree`,

EXAMPLE

```
kash> o := Order(Z, 8, 2);
Generating polynomial: x^8 - 2

kash> GaloisGroupsPossible(o);
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,
  22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40,
  41, 42, 43, 44, 45, 46, 47, 48, 49, 50 ]
kash> GaloisModulo(o, 100);
[ 6, 8, 15, 23, 26, 27, 35, 40, 43, 44, 47, 50 ]
```

NAME `GaloisNumberToName`

PURPOSE Returns name of transitive permutation group.

SYNTAX `GaloisNumberToName(n, k);`

`integer`	`n`	representing degree
`integer`	`k`	representing group in T-notation

DESCRIPTION This function returns the name of the transitive permutation group which is given by an integer according to its T-notation as in GAP [S+97]. For the names of the permutation groups see also "On transitive permutation groups" by John H. Conway, Alexander Hulpke and John McKay [CHM98].

SEE ALSO `Galois`,

EXAMPLE

```
kash> GaloisNumberToName(8, 50);
"S8"
```

NAME GaloisRing

PURPOSE missing shortdoc

SYNTAX `L := GaloisRing(o);`

order	o	
list	L	of either the p-adic order, the prime (ideal) p and the exponent k or the complex field and the precesion

DESCRIPTION Returns the p-adic order, the prime p and the exponent k of the precesion p^k or in the complex case the complex field with the current precision.

SEE ALSO Galois,

EXAMPLE

```
kash> o := Order(Z, 8, 2);
Generating polynomial: x^8 - 2

kash> Galois(o);
"2D_8(8)"
kash> GaloisRing(o);
[ Generating polynomial: x^2 - 5
    , 23, 3 ]
```

NAME `GaloisRoots`

PURPOSE Returns current root ordering.

SYNTAX
```
L := GaloisRoots(o);
L := GaloisRoots(o,k);
```

`order`	`o`	
`int`	`k`	
`list`	`L`	of list of roots and permutation

DESCRIPTION This function returns a list of the roots of the generating polynomial in the current ordering and a permutation giving this ordering in view of the standard ordering as returned by `Solve()`. After computation by `Galois()` the action of the Galois group on this root ordering is equivalent to the computed transitive permutation group as classified in GAP [S+97]. Is the second parameter k used, the roots are returned with precision k for the complex case and p^k in p-adic case.

SEE ALSO `Galois`,

EXAMPLE

```
kash> o := Order(Z, 8, 2);
Generating polynomial: x^8 - 2

kash> Galois(o);
"2D_8(8)"
kash> GaloisRoots(o);
[ [ -171, [1171, 5618], [0, 384], [1171, -5618], 171, [-1171, -5618],
      [0, -384], [-1171, 5618] ], (2,5)(4,6,7) ]
```

NAME GaloisSymb

PURPOSE Computation of Galois groups

SYNTAX

```
G := GaloisSymb(o);
G := GaloisSymb(F);
G := GaloisSymb(f);
```

string	G
order	o
algebraic function field	F
polynomial	f

DESCRIPTION This function computes the Galois group of an irreducible polynomial with coefficients in $\mathbb{Q}$ or Q(x) and with degree up to 7. This function uses factorization of absolute resolvents as described in the articles [CM94] resp. [SM85], so the results are unconditional. The name of the Galois group is returned.

SEE ALSO Galois,

EXAMPLE

```
kash> Zxy := PolyAlg(Zx);;
kash> y := Poly(Zxy, [1,0]);;
kash> f := y^6 - 9/(x^2 + 1)*y^2 + 12/(x^2 + 1)*y - 4/(x^2 + 1);
y^6 - 9/(x^2 + 1)*y^2 + 12/(x^2 + 1)*y - 4/(x^2 + 1)
kash> GaloisSymb(f);
"F_36(6)"
```

NAME `GaloisSymb`

PURPOSE Computation of the Galois group.

SYNTAX `G := GaloisSymb(o|F|f);`

string	G
order	o
algebraic function field	F
polynomial	f

DESCRIPTION This function computes the Galois group of an irreducible polynomial with coefficients in $\mathbb{Q}$ or $\mathbb{Q}(x)$ and simple relative orders with degree up to 7. This function uses factorization of absolute resolvents as described in the articles [CM94] resp. [SM85], so the results are unconditional. The name of the Galois group is returned.

SEE ALSO `Galois`,

EXAMPLE

```
kash> Zxy := PolyAlg(Zx);;
kash> y := Poly(Zxy, [1,0]);;
kash> f := y^6 - 9/(x^2 + 1)*y^2 + 12/(x^2 + 1)*y - 4/(x^2 + 1);
y^6 - 9/(x^2 + 1)*y^2 + 12/(x^2 + 1)*y - 4/(x^2 + 1)
kash> GaloisSymb(f);
"F_36(6)"
```

```
kash> E1 := OrderMaximal(Z, 2, 2);;
kash> E1x:= PolyAlg(E1);;
kash> F1:= Order(Poly(E1x,[Elt(E1,1),0,-4,0, Elt(E1,[2,-1])]));
      F[1]
        /
       /
   E1[1]
  /
 /
Q
```

```
F  [ 1]     x^4 - 4*x^2 + [2, -1]
E 1[ 1]     x^2 - 2

kash> GaloisSymb(F1);
"C(4)"
```

NAME `GaloisSymbT`

PURPOSE Computation of Galois groups

SYNTAX
```
G := GaloisSymbT(o);
G := GaloisSymbT(F);
G := GaloisSymbT(f);
```

int	G
order	o
algebraic function field	F
polynomial	f

DESCRIPTION This function computes the Galois group of an irreducible polynomial with coefficients in $\mathbb{Q}$ or Q(x) and with degree up to 7. This function uses factorization of absolute resolvents as described in the articles [CM94] resp. [SM85], so the results are unconditional. The number of the Galois group is returned.

SEE ALSO `Galois`,

EXAMPLE

```
kash> Qx := PolyAlg(Q);
Univariate Polynomial Ring in x over Rational Field

kash> Qxy := PolyAlg(Qx);
Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Rati\
onal Field

kash> y:= Poly(Qxy, [1,0]);;
kash> f := y^7-3*y^6-y^5+3*y^4+(-x + 1)*y^3+(x + 1)*y^2-5*y+4;
y^7 - 3*y^6 - y^5 + 3*y^4 + (-x + 1)*y^3 + (x + 1)*y^2 - 5*y + 4
kash> F := Alff(f);
Algebraic function field defined by
$.1^7 - 3*$.1^6 - $.1^5 + 3*$.1^4 - $.1^3*$.2 + $.1^3 + $.1^2*$.2 + $.1^2 - 5*\
$.1 + 4
over
Univariate rational function field over Rational Field
Variables: x

kash> GaloisSymbT(F);
5
```

NAME `GaloisSymbT`

PURPOSE Computation of the Galois group.

SYNTAX `G := GaloisSymbT(o|F|f);`

`int`	`G`
`order`	`o`
`algebraic function field`	`F`
`polynomial`	`f`

DESCRIPTION This function computes the Galois group of an irreducible polynomial with coefficients in $\mathbb{Q}$ or $Q(x)$ and simple relative orders with degree up to 7. This function uses factorization of absolute resolvents as described in the articles [CM94] resp. [SM85], so the results are unconditional. The number of the Galois group is returned.

SEE ALSO `Galois`,

EXAMPLE

```
kash> Qx := PolyAlg(Q);
Univariate Polynomial Ring in x over Rational Field

kash> Qxy := PolyAlg(Qx);
Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Rati\
onal Field

kash> y:= Poly(Qxy, [1,0]);;
kash> f := y^7-3*y^6-y^5+3*y^4+(-x + 1)*y^3+(x + 1)*y^2-5*y+4;
y^7 - 3*y^6 - y^5 + 3*y^4 + (-x + 1)*y^3 + (x + 1)*y^2 - 5*y + 4
kash> F := Alff(f);
Algebraic function field defined by
$.1^7 - 3*$.1^6 - $.1^5 + 3*$.1^4 - $.1^3*$.2 + $.1^3 + $.1^2*$.2 + $.1^2 - 5*\
$.1 + 4
over
Univariate rational function field over Rational Field
Variables: x

kash> GaloisSymbT(F);
5
```

NAME `GaloisT`

PURPOSE Computation of Galois groups.

SYNTAX

DESCRIPTION This function computes the Galois group of an irreducible polynomial with coefficients in $\mathbb{Q}$ and with degree up to 15. The number of the Galois group is returned in T-notation as classified in GAP [S^+97]. For description see `Galois()`.

SEE ALSO `Galois`, `GaloisGlobals`, `GaloisGroupsPossible`, `GaloisModulo`, `GaloisTree`, `GaloisRoots`, `GaloisNumberToName`, `GaloisBlock`, NAME `GaloisTree`

PURPOSE Output of possible Galois groups as subgroup lattice.

SYNTAX `L := GaloisTree(o);`

`order`	`o`	
`list`	`L`	list of two lists of integers

DESCRIPTION This function prints the subgroup lattice of possible Galois groups as non-connected and directed tree and returns a list with two entries which contain the roots of the tree of odd respectively even transitive groups.

SEE ALSO `Galois`, `GaloisTreeRoots`, `GaloisGroupsPossible`, `GaloisNumberToName`,

EXAMPLE

```
kash> o := Order(Z, 6, 2);
Generating polynomial: x^6 - 2

kash> GaloisTree(o);
******* tree of possible groups *******

rootSn:
16: 13 11 14
13: 9
9: 5 3
5: 2 1
2:
1:
3: 2 1
11: 8 6 3
8: 2
6: 1
14:

rootAn:
15: 10 7 12
10:
7: 4
4:
```

```
12:

**************************************
[ [ 16 ], [ 15 ] ]
```

Name `GaloisTreeRoots`

Purpose Return the roots of the tree of possible Galois groups.

Syntax `L := GaloisTreeRoots(o);`

`order`	`o`	
`list`	`L`	list of two lists of integers

Description This function returns a list with two entries which contain the roots of the tree of odd respectively even transitive groups. The groups are given by their number in T-notation.

See Also `GaloisTree`, `Galois`, `GaloisGroupsPossible`, `GaloisNumberToName`,

Example

```
kash> o := Order(Z, 8, 2);
Generating polynomial: x^8 - 2

kash> GaloisTreeRoots(o);
[ [ 50 ], [ 49 ] ]
kash> GaloisGroupsPossible(o, [49, 50], false);
kash> GaloisTreeRoots(o);
[ [ 47, 44, 43 ], [ 45, 39, 48 ] ]
```

NAME `GaloisTwoSequencePol`

PURPOSE Computing of a polynomial of degree $n \cdot (n-1)$ which roots are products of two distinct roots of f.

SYNTAX `g:= GaloisTwoSequencePol(f);`

```
polynomial  f, g
```

DESCRIPTION Let f be a monic polynomial of degree n in $\mathbb{Q}$, $\mathbb{Q}(x)$ or a simple relative order.This function computes a primitive polynomial of degree $n(n-1)$. The roots of `GaloisTwoSequencePol` are the products of the form $x|i+2x|j$, where $x|i$ and $x|j$ are distinct roots of f.

SEE ALSO `GaloisGlobals`, `GaloisGroupsPossible`, `GaloisModulo`, `GaloisTree`, `GaloisRoots`, `GaloisNumberToName`, `GaloisBlocks`,

EXAMPLE

```
kash> f := x^6-3*x^5+6*x^4-7*x^3+2*x^2+x-4;
x^6 - 3*x^5 + 6*x^4 - 7*x^3 + 2*x^2 + x - 4
kash> g := GaloisTwoSequencePol(f);
x^30 - 45*x^29 + 1026*x^28 - 15687*x^27 + 179817*x^26 - 1639404*x^25 + 1232376\
0*x^24 - 78227505*x^23 + 426372238*x^22 - 2019887529*x^21 + 8394234972*x^20 - \
30824981910*x^19 + 100623894335*x^18 - 293550986223*x^17 + 769208456704*x^16 -\
 1819602767877*x^15 + 3905435727190*x^14 - 7640849227485*x^13 + 13675751675898\
*x^12 - 22435288075854*x^11 + 33744397630017*x^10 - 46517472600975*x^9 + 58815\
885814020*x^8 - 68378015808513*x^7 + 73157522800460*x^6 - 71284825965465*x^5 +\
 61112931042866*x^4 - 43100727314520*x^3 + 22477430709080*x^2 - 7271427323616*\
x + 984235529312
```

NAME `Gamma`

PURPOSE Returns the value of the Γ-function.

SYNTAX `y := Gamma(x);`

real	y
real	x

EXAMPLE

```
kash> Gamma(3);
2
kash> Gamma(4);
6
kash> Gamma(5);
24
kash> Gamma(5.5);
52.342777784553520181149008492418193679490132376102
kash> Gamma(6);
120
```

NAME Gcd

PURPOSE Returns greatest common divisor of list of arguments.

SYNTAX
```
gcd := Gcd(L);
gcd := Gcd(a, b);
```

integer or polynomial or ideal	gcd
list of integers of polynomials or ideals	L
integer or polynomial or ideal	a
integer or polynomial or ideal	b

DESCRIPTION Returns the greatest common divisor of list of arguments or simply the greatest common divisor of two arguments. Supported are integers, ideals and polynomials over S where S is a field or $\mathbb{Z}$.

SEE ALSO IntGcd, PolyGcd,

EXAMPLE Gcd of integers:

```
kash> Gcd([18, 12, 4, 6]);
2
kash> Gcd(18, 12);
6
kash> Gcd([18, 12]);
6
```

GetEnvironment

NAME `GetEnvironment`

PURPOSE Returns the current value of the environment variable name ($name).

SYNTAX `st := GetEnvironment(name);`

```
string  st
string  name
```

EXAMPLE

```
kash> GetEnvironment(HOSTNAME);
"kantorowitsch"
```

NAME HermiteUpperBound

PURPOSE Returns a upper bound for Hermite's constant γ_n^n (according to Blichfeldt [Bli14]).

SYNTAX `y := HermiteUpperBound(n);`

real	y
integer	n

EXAMPLE

```
kash> HermiteUpperBound(4);
4
kash> HermiteUpperBound(5);
8
kash> HermiteUpperBound(6);
21.333333333333333333333333333333333333333333333333
kash> HermiteUpperBound(10);
5668.4748300339395076115877442889065093708002744182 14
```

HurwitzZeta

Name	`HurwitzZeta`
Purpose	Returns the value of the Hurwitz Zeta-function.
Syntax	`c := HurwitzZeta(s,v,m0);`

```
complex  s
real     v
integer  m0
complex  c
```

Description	Given $s \in \mathbb{C} \setminus \{1\}$ and $v \in \mathbb{R}$. Then HurwitzZeta(s,v) returns the value of the Hurwitz Zeta-function at s.
Example	

```
kash> c := HurwitzZeta(Comp(2.4,3.1),0.4,6);
-8.511982 + 2.304635*i
kash> HurwitzZeta(Comp(2.4,3.1),0.4);
-8.51198187508935436273191718773675923621648069199 + 2.30463482701114878654193\
3445673918516011077291233*i
```

NAME `Ideal`

PURPOSE Creates an ideal defined by the arguments.

SYNTAX
```
I := Ideal(e);
I := Ideal(o, n);
I := Ideal(e, f);
I := Ideal(n, e);
I := Ideal(o, M, d);
I := Ideal(o, MO);
```

`Ideal`	`I`	
`algebraic element`	`e`	
`algebraic element`	`f`	
`order over Z`	`o`	
`matrix`	`M`	of integers
`integer`	`d`	denominator of M
`integer`	`n`	
`module`	`MO`	over the coefficient order of o

DESCRIPTION

EXAMPLE All examples will be in the equation order of $x^3 + 6x^2 + 6x + 6$. Here we will create `alpha*o` using different methods:

```
kash> o := Order(Poly(Zx, [1, 6, 6, 6]));;
kash> alpha := Elt(o, [0,1,0]);
[0, 1, 0]
kash> I := Ideal(alpha);
<[0, 1, 0]>
kash> I := alpha*o;
<[0, 1, 0]>
```

NAME `Ideal2EltAssure`

PURPOSE The 2-element representation of the ideal is computed if it is not given yet.

SYNTAX `Ideal2EltAssure(I);`

`ideal I`

DESCRIPTION This procedure is for clarity, IdealGenerators serves for the same purpose but returns the generators. This procedure returns nothing but adds a representation of the ideal.

SEE ALSO `IdealGenerators`, `IdealBasis`,

EXAMPLE We consider an ideal in the equation order of $x^3 + 6x^2 + 6x + 6$ which is the result of a multiplication which usually yields an ideal in basis representation. We obtain the 2-element representation

```
kash> O := Order(Poly(Zx, [1, 6, 6, 6]));;
kash> I := Ideal(6,Elt(O,[0,0,1]))*Ideal(3,Elt(O,[0,1,1]));
<
[6 0 0]
[0 6 0]
[0 0 3]
>

kash> Ideal2EltAssure(I);
kash> I;
<6, [24, 12, 33]>
```

NAME `Ideal2EltIntAssure`

PURPOSE The 2-element representation where the first generator is a rational integer of the ideal is computed if it is not given yet.

SYNTAX `Ideal2EltIntAssure(I);`

`ideal  I`

DESCRIPTION

SEE ALSO Ideal2EltAssure, Ideal2EltNormalAssure,

EXAMPLE

```
kash> O := Order(Poly(Zx, [1, 6, 6, 6]));;
kash> alpha := Elt(O, [0, 1, 0]);;
kash> beta := Elt(O, [2, 2, 2]);;
kash> I := O*alpha + O*beta;
<[0, 1, 0], [2, 2, 2]>
kash> Ideal2EltIntAssure(I);
kash> I;
<2, [0, 3, 0]>
```

NAME `Ideal2EltKnown`

PURPOSE Returns `true` if the ideal is given in two element representation.

SYNTAX `b := Ideal2EltKnown(I);`

Boolean	b
ideal	I

DESCRIPTION The representation is not computed, this is only a check. This function is useful when programming in KASHif an algorithm can handle both basis and 2-element-representation but prefers the 2-element-representation. The computation of the 2-element-representation can be cumbersome in large orders because it involves a trial-and-error algorithm. So it can be handy just to check if the 2-element-representation is already present.

SEE ALSO `IdealGenerators`,

EXAMPLE

```
kash> O := OrderMaximal(Order(Poly(Zx, [1, 6, 6, 6])));;
kash> Lp := Factor(6*O);
[ [ <2, [0, 1, 0]>, 3 ], [ <3, [0, 1, 0]>, 3 ] ]
kash> I := Lp[1][1]*Lp[2][1];
<
[6 0 0]
[0 1 0]
[0 0 1]
>

kash> Ideal2EltKnown(I);
false
kash> IdealGenerators(I);
[ 6, [0, 1, 35] ]
kash> Ideal2EltKnown(I);
true
```

NAME `Ideal2EltNormalAssure`

PURPOSE Calculates a 2 element normal representation of an ideal.

SYNTAX `Ideal2EltNormalAssure(I);`

`ideal I`

DESCRIPTION This is a special represenation to speed up multipication and inverting an ideal. The two generators are computed if not already present and then transformed to a normal representation via a propabilistic algorithm. Multiplying and inverting use normal representations on appropriate occasions.

SEE ALSO `IdealGenerators`,

EXAMPLE

```
kash> O := OrderMaximal (Order (Poly (Zx, [1,0,0,0,657])));;
kash> I := 5*Elt (O,[345,76,345,46])*O;
<[1725, 380, 1725, 230]>
kash> fac := Filtered (Flat (Factor (I)), IsIdeal);
[ <2, [1, 1, 0, 0]>, <3, [0, 7, 3, 7]>, <5>, <7, [1, 1, 0, 0]>,
  <41, [3, 1, 0, 0]>, <41, [27, 1, 0, 0]>, <197603671, [39148097, 1, 0, 0]> ]
kash> for I in fac do
> Ideal2EltNormalAssure (I);
> Print (I,"\n");
> od;
<2, [1, 1, 0, 0]>
<3, [0, 7, 3, 7]>
<5>
<7, [1, 1, 0, 0]>
<41, [3, 1, 0, 0]>
<41, [27, 1, 0, 0]>
<197603671, [39148097, 1, 0, 0]>
```

NAME	`Ideal2EltNormalKnown`
PURPOSE	True if the ideal is given with normal two elements representation.
SYNTAX	`b := Ideal2EltNormalKnown(I);`

```
Boolean   b
ideal     I
```

DESCRIPTION	The representation is not computed, this is only a check.
SEE ALSO	`IdealGenerators`,
EXAMPLE	See function Ideal2EltNormalAssure

NAME	`IdealAutomorphism`
PURPOSE	Applies an automorphism to an ideal.
SYNTAX	`B := IdealAutomorphism (A,i);`

```
ideal    B
ideal    A
integer  i
```

DESCRIPTION

Let $\mathcal{F}$ be a normal number field with $\mathbb{Q}$-automorphisms $\sigma_1, \ldots, \sigma_n$ and let A be an ideal in $\mathcal{F}$. The `IdealAutomorphism` function returns the ideal $\sigma_i(A)$.

Before calling `IdealAutomorphism` the automorphisms $\sigma_1, \ldots, \sigma_n$ of $\mathcal{F}$ must be computed by the `OrderAutomorphisms` routine.

SEE ALSO

`EltAutomorphism`, `OrderAutomorphisms`, NAME `IdealBasis`

PURPOSE Returns the basis of an ideal.

SYNTAX `M := IdealBasis(I);`

`ideal`	`I`	
`list`	`M`	two elements, the denominator and the representation matrix

DESCRIPTION Ideals are given either in Z-basis representation or 2 element representation. If the ideal is given in basis representation this representation is simply returned. Otherwise it will be computed. In this case the ideal has got 2 equivalent representations stored afterwards. This can be of advantage because some algorithms prefer 2 element representations, others prefer Z-basis representations. Once a basis is fixed (entered or calculated) it will never change even if simplification is possible.

SEE ALSO `IdealGenerators`,

EXAMPLE

```
kash> o := Order(Poly(Zx, [1, 6, 6, 6]));;
kash> alpha := Elt(o, [0, 1, 0]);;
kash> I := Ideal(6, alpha);
<6, [0, 1, 0]>
kash> IdealBasis(I);
[ 1, [6 0 0]
    [0 1 0]
    [0 0 1] ]
```

NAME	`IdealBasisKnown`
PURPOSE	Returns `true` if the ideal is given via a $\mathbb{Z}$ basis.
SYNTAX	`b := IdealBasisKnown(I);`

`Boolean`	`b`
`ideal`	`I`

DESCRIPTION	The basis representation is not computed, this is only a check.
SEE ALSO	`IdealBasis`,
EXAMPLE	

```
kash> o := Order(Poly(Zx, [1, 6, 6, 6]));;
kash> alpha := Elt(o, [0, 1, 0]);;
kash> I := Ideal(6, alpha);
<6, [0, 1, 0]>
kash> IdealBasisKnown(I);
false
kash> IdealBasis(I);
[ 1, [6 0 0]
    [0 1 0]
    [0 0 1] ]
kash> IdealBasisKnown(I);
true
```

Name	`IdealBasisLowerHNF`
Purpose	The basis of the ideal transformed in lower HNF
Syntax	`M:=IdealBasisLowerHNF(I);`

`list`	`M`	two elements, the denominator and the basis matrix
`ideal`	`I`	

Description

This returns the basis of the ideal transformed in lower HNF. The basis is computed from the actual (which can be seen with IdealBasis). Once computed the basis in lower HNF is stored in the ideal data structure so it has not to be computed again.

See Also

`IdealBasisUpperHNF`, `IdealLowerHNFTrans`,

Example

```
kash> O:=OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> I:=Ideal(O,Mat(Z,[[1,2,1,4,6],[3,8,4,6,2],[6,4,3,3,8],
> [3,2,7,9,5],[0,2,8,8,4]]),1);
<
[1 2 1 4 6]
[3 8 4 6 2]
[6 4 3 3 8]
[3 2 7 9 5]
[0 2 8 8 4]
>

kash> IdealBasisLowerHNF(I);
[ 1, [   1    0    0    0    0]
    [   0    1    0    0    0]
    [   0    0    1    0    0]
    [   0    1    0    3    0]
    [ 512  910  858 1672 1678] ]
```

NAME `IdealBasisUpperHNF`

PURPOSE The basis of the ideal transformed in upper HNF.

SYNTAX `M := IdealBasisUpperHNF(I);`

`list`	`M`	two elements, the denominator and the basis matrix
`ideal`	`I`	

DESCRIPTION This returns the basis of the ideal transformed in (right) upper HNF. Either it is computed from a plain basis (which can be seen with IdealBasis) or from a 2-element representation. Once computed the basis in upper HNF is stored in the ideal data structure so it has not to be computed again. In most cases IdealBasis and IdealBasisUpperHNF are identical.

SEE ALSO `IdealBasisLowerHNF`, `IdealUpperHNFTrans`,

EXAMPLE

```
kash> o := Order(Poly(Zx, [1, 6, 6, 6]));;
kash> alpha := Elt(o, [0, 1, 0]);;
kash> I := Ideal(6, alpha);
<6, [0, 1, 0]>
kash> IdealBasisUpperHNF(I);
[ 1, [6 0 0]
    [0 1 0]
    [0 0 1] ]
```

NAME `IdealChineseRemainder`

PURPOSE Chinese Remainder Theorem for number fields

SYNTAX `beta := IdealChineseRemainder(a1, a2, alpha1, alpha2)`

ideals	a1, a2
algebraic numbers	alpha1, alpha2
algebraic number	beta

DESCRIPTION Computes an algebraic number β in the order $\mathfrak{o}$, such that $\alpha|i$ - β is contained in the ideal $\mathfrak{a}|i$, $i = 1, 2$.

SEE ALSO EltApproximation, RayCantoneseRemainder,

EXAMPLE

```
kash> O := OrderMaximal (Order (Poly(Zx,[1,0,73,-280,-2399])));;
kash> a1 := Factor (2*O)[1][1];
<2, [1, 2, 0, 1]>
kash> a2 := Factor (NextPrime (4343343)*O)[1][1];
<4343357, [1353610, 1860090, 2, 0]>
kash> beta := IdealChineseRemainder (a1, a2, Elt (O,[1,2,3,5]), Elt (O, 4));
[4, 4343357, 0, 0]
kash> Valuation (a1, beta - Elt (O,[1,2,3,5]));
2
kash> Valuation (a2, beta - Elt (O,4));
1
```

NAME `IdealClassRep`

PURPOSE Computes a representation of the ideal class over the cyclic generators of the class group.

SYNTAX
```
L := IdealClassRep(I);
L := IdealClassRep(I, "gen");
```

```
list   L
ideal  I
```

DESCRIPTION If $\mathfrak{a}|1,\dots,\mathfrak{a}|k$ are the cyclic generators of the class group as returned by `OrderClassGroupCyclicFactors`, for an ideal $\mathfrak{b}$ this function computes a representation of the form

$$\mathfrak{b} = \alpha \prod |i = 1^k \mathfrak{a}|i^{s|i}$$

with α an algebraic element. The result is stored in a list of the form $\{\alpha, \{\mathfrak{a}|1, s|1\}, \dots, \{\mathfrak{a}|k, s|k\}\}$, where α is omitted when the second parameter "gen" is not given.

SEE ALSO `OrderClassGroup`, `IdealIsPrincipal`, `IdealRayClassRep`,

EXAMPLE

```
kash> o := OrderMaximal(Order(x^3-117));;
kash> OrderClassGroup(o, euler);
[ 3, [ 3 ] ]
kash> OrderClassGroupCyclicFactors(o);
[ [ <2, [1, 1, 3]>, 3 ] ]
kash> p := Factor(1021*o)[1][1];
<1021, [499, 1, 0]>
kash> L := IdealClassRep(p, "gen");
[ [287, 59, 36] / 2, [ <2, [1, 1, 3]>, 1 ] ]
kash> alpha := L[1];
[287, 59, 36] / 2
kash> a1 := L[2][1];
<2, [1, 1, 3]>
kash> s1 := L[2][2];
1
kash> alpha*a1^s1/p;
<
[1 0 0]
[0 1 0]
```

```
[0 0 1]
>
```

NAME `IdealCollection`

PURPOSE Solves a special equation.

SYNTAX `M := IdealCollection(I1,I2);`

`list`	`M`	two lists of two elements of the order
`ideals`	`I1,I2`	integral, over a maximal order

DESCRIPTION Let $\alpha|1, \alpha|2$ be two algebraic numbers and $\mathfrak{I}|1, \mathfrak{I}|2$ two ideals over the same order $\mathcal{O}$. The result of this function is a matrix M which satisfies

$$\alpha|1\mathfrak{I}|1 + \alpha|2\mathfrak{I}|2 = \beta|1\mathcal{O} + \beta|2\mathfrak{I}|1\mathfrak{I}|2$$

where

$$\begin{pmatrix}\beta|1\\ \beta|2\end{pmatrix} = \begin{pmatrix}\alpha|1\\ \alpha|2\end{pmatrix} M.$$

SEE ALSO ModuleSteinitz,

EXAMPLE

```
kash> O := OrderMaximal(Poly(Zx, [1, 6, 6, 6]));;
kash> I1:=Ideal(6,Elt(O,[0,0,1]));
<6, [0, 0, 1]>
kash> I2:=Ideal(3,Elt(O,[0,1,1]));
<3, [0, 1, 1]>
kash> M := IdealCollection(I1,I2);
[ [ [0, 0, 1], 0 ], [ -1, [0, 5, 1] / 6 ] ]
kash> M[1][1]*M[2][2]-M[1][2]*M[2][1];
1
kash> a1 := Elt(O,[2,1,0]);
[2, 1, 0]
kash> a2 := Elt(O,[0,1,-1]);
[0, 1, -1]
kash> b1 := a1 * M[1][1] + a2 * M[1][2];;
kash> b2 := a1 * M[2][1] + a2 * M[2][2];;
kash> a1*I1+a2*I2 = b1*O+b2*I1*I2;
true
```

NAME `IdealDegree`

PURPOSE Calculates the degree of inertia of a prime ideal.

SYNTAX `d := IdealDegree(I);`

`integer`	`d`	
`ideal`	`I`	must be prime

DESCRIPTION Let $\mathcal{O}$ be a maximal order. Let p be a prime number and $\mathfrak{p}$ a prime $\mathcal{O}$-ideal over this prime number p. The index

$$[\mathcal{O}/\mathfrak{p} : \mathbb{Z}|p]$$

is called the degree of inertia of $\mathfrak{p}$ over p.

SEE ALSO `IdealRamIndex`,

EXAMPLE

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> I := 6*O;
<6>
kash> p := Factor(I)[2][1];
<2, [1, 0, 0, 1, 1]>
kash> IdealDegree(p);
4
```

Name `IdealDen`

Purpose The denominator of the ideal.

Syntax `d := IdealDen(I);`

ideal	I
integer	d

Description The denominator of a fractional ideal $\mathfrak{a}$ is the smallest positive integer d that $d\mathfrak{a}$ is an integral ideal. Integral ideals are just fractional ideals with denominator 1 and have no separate internal data structure.

Once computed the denominator is stored in the ideal data structure so it has not to be computed again.

See Also `IdealIsIntegral`,

Example

```
kash> O := OrderMaximal(Poly(Zx,[1,4,1,-4,-3,7]));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> a := Elt(O,[1,5,1,8,0]);
[1, 5, 1, 8, 0]
kash> b := Elt(O,[3, 1,5,1,8]);
[3, 1, 5, 1, 8]
kash> I := Ideal(a, b);
<[1, 5, 1, 8, 0], [3, 1, 5, 1, 8]>
kash> IdealDen(I^-1);
3
```

NAME IdealDivisors

PURPOSE Computes all divisors of an ideal.

SYNTAX `L := IdealDivisors (A);`

list	L
ideal	A

DESCRIPTION The `IdealDivisors` function computes a list containing all divisors of an integral ideal. The underlying order of A must be known to be maximal.

EXAMPLE

```
kash> o := OrderMaximal(Z,2,-5);
Generating polynomial: x^2 + 5
Discriminant: -20

kash> A := 6*o;
<6>
kash> IdealDivisors(A);
[ <1>, <2, [1, 1]>, <4, [0, 2]>, <3, [1, 1]>, <
    [6 1]
    [0 1]
    >
    , <6, [2, 2]>, <3, [2, 1]>, <
    [6 5]
    [0 1]
    >
    , <6, [4, 2]>, <3>, <
    [6 3]
    [0 3]
    >
    , <6> ]
```

NAME `IdealFactor`

PURPOSE Returns the factorization of the given ideal over a maximal order.

SYNTAX `F := IdealFactor(a);`

list	F
ideal	a

DESCRIPTION Because a maximal order $\mathcal{O}$ is a Dedekind ring every integral $\mathcal{O}$-ideal has a unique factorization in prime ideals. More generally, it is possible to factor ideals in prime ideals over arbitrary orders above the equation order at least in case they do not divide the discriminant of the equation order.
The factorization can simply be extended to fractional ideals: if d is the denominator of $\mathfrak{a}$ and $\mathfrak{b}$, the integral ideal $d\mathfrak{a}$, the ideal $\mathfrak{b}$ and the ideal $d\mathcal{O}$ is factorized. The exponents of the factorization of $d\mathcal{O}$ are negated and multiplied to the factorization of $\mathfrak{b}$ simplifying exponents of identical prime ideals.

SEE ALSO Factor,

EXAMPLE Factorization of the principal ideal (3) in the maximal order of $\mathbb{Q}(\sqrt{-110})$:

```
kash> o := Order(Z,2,-110);
Generating polynomial: x^2 + 110

kash> O := OrderMaximal(o);
Generating polynomial: x^2 + 110
Discriminant: -440

kash> IdealFactor(3*O);
[ [ <3, [1, 1]>, 1 ], [ <3, [2, 1]>, 1 ] ]
kash> a:=Ideal(O,3);
<3>
kash> IdealFactor(a);
[ [ <3, [1, 1]>, 1 ], [ <3, [2, 1]>, 1 ] ]
```

NAME IdealGen

PURPOSE missing shortdoc

SYNTAX `g := IdealGen(I, i);`

algebraic element	g	
ideal	I	
small integer	i	$\in \{1, 2\}$

DESCRIPTION Any ideal I of a maximal order $\mathfrak{o}$ may be generated using two elements g_1 and g_2 (i.e. $\mathtt{I} = g_1\mathfrak{o} + g_2\mathfrak{o}$). This functions returns g_i ($i \in \{1, 2\}$) after computing suitable generators if necessary.
The restriction to maximal orders is only necessary to compute the generators. Note that the generators are not unique! In fact since the algorithm is probabilistic, they may change between two sessions.

SEE ALSO IdealGenerators, IdealBasis,

EXAMPLE

```
kash> O := Order(Poly(Zx, [1, 6, 6, 6]));;
kash> alpha := Elt(O, [0, 1, 0]);;
kash> I := Ideal(6, alpha);
<6, [0, 1, 0]>
kash> IdealGen(I, 1);
6
kash> IdealGen(I, 2);
[0, 1, 0]
```

NAME `IdealGenerators`

PURPOSE Returns the 2 generators of the 2-element-representation of the ideal.

SYNTAX `L := IdealGenerators(I);`

`ideal`	`I`	
`list`	`L`	of two algebraic numbers

DESCRIPTION Ideals are given either in Z-basis representation or 2 element representation. If the ideal is given in 2 element representation this representation is simply returned. Otherwise it will be computed. In this case the ideal has got 2 equivalent representations stored afterwards. This can be of advantage because some algorithms prefer 2 element representations, others prefer Z-basis representations.

SEE ALSO `IdealBasis`,

EXAMPLE

```
kash> O := OrderMaximal(Order(Poly(Zx, [1, 6, 6, 6])));;
kash> Lp:=Factor(6*O);
[ [ <2, [0, 1, 0]>, 3 ], [ <3, [0, 1, 0]>, 3 ] ]
kash> I := Lp[1][1]*Lp[2][1];
<
[6 0 0]
[0 1 0]
[0 0 1]
>

kash> IdealGenerators(I);
[ 6, [0, 1, 35] ]
```

NAME `IdealIdempotents`

PURPOSE Returns elements of the comaximal ideals which sum to 1.

SYNTAX `E := IdealIdempotents(L)`

`list`	`L`	of integral ideals over the same order
`false or list`	`E`	of algebraic elements over the same order

DESCRIPTION Let L be the list of ideals $[I|1, \ldots, I|n]$.
This function tries to find algebraic elements $a|i$ in $I|i$ such that: $\sum a|i = 1$.
The function will return false if this in not possible (iff $I|1, \ldots, I|n$ are NOT comaximal).
Otherwise it returns the list $[a|1, \ldots, a|n]$ of elements which sum to one. Each entry of E is an element of the ideal in L with the same index.
Fractional ideals are allowed and considered as the biggest integral ideal which contains the fractional ideal, i.e. disregarding the denominator.

EXAMPLE

```
kash> O := OrderMaximal(Order(Z,2,- 45));;
kash> L := List( Factor(210*O), x->x[1]);
[ <2, [1, 3]>, <3, [1, 2]>, <3, [7, 7]>, <5, [0, 3]>, <7, [2, 3]>,
  <7, [5, 3]> ]
kash> E := IdealIdempotents([L[2]*L[6]*L[3],L[2]*L[4],L[1]*L[4]*L[6]]);
[ 966, -1035, 70 ]
```

NAME `IdealImprove`

PURPOSE Improves the generators of the ideal given in two element representation.

SYNTAX `I1:= IdealImprove(I2);`

```
ideal  I1, I2
```

DESCRIPTION The ideal can be improved if

- one of the generators is an rational integer. Then every basis coefficient of the other generator can be reduced modulo this integer.
- the minimum is given. Coefficients of both generators can be reduced modulo this integer. This procedure is fast and simple therefore it does not **compute** the minimum of the ideal. For more possible improvement call `IdealMin` first.

SEE ALSO `IdealMin`,

EXAMPLE

```
kash> O:=Order(Poly(Zx,[1,0,2,4,58]));
Generating polynomial: x^4 + 2*x^2 + 4*x + 58

kash> E:=Elt(O,[1,2,-1,-2]);
[1, 2, -1, -2]
kash> I:=Ideal(3,E^7);
<3, [-470899145529, -57698441418, 37053141409, 29584796218]>
kash> IdealImprove(I);
<3, [-470899145529, -57698441418, 37053141409, 29584796218]>
```

NAME `IdealIntegrity`

PURPOSE Checks the ideals for various inconsistencies

SYNTAX `errcount = IdealIntegrity(I);`

`ideal`	`I`	
`small integer`	`errcount`	the number of detected errors

DESCRIPTION The most important application of this function is this: Not all matrices (square and with the correct dimension) over the coefficient order of the ideal form an valid ideal. If an ideal is created with a basis matrix it is not checked if this matrix forms a correct ideal basis (because it is much too expensive to check this for every ideal creation by default.) Such an *wrong* ideal can cause all sorts of follow-up errors which might be difficult to detect. If You are not sure if the ideal is correct apply this function.

The ideal invariants which are stored in the ideal data structure are checked for their validity as well, including factorization, norm, minimum, principal generator, prime element, primality.

If an inconsistency is detected a warning message is printed to the screen but the break loop is not entered (except if it is not possible to find a 2 element representation where the break loop is entered). If the return value is 0 the ideal is (very likely) correct.

EXAMPLE Correct ideal

```
kash> O := OrderMaximal(Order(Poly(Zx, [1, 6, 6, 6])));;
kash> I := Ideal(O, Mat(Z, [[2,0,0],[0,1,0],[0,0,1]]),1);
<
[2 0 0]
[0 1 0]
[0 0 1]
>

kash> IdealIntegrity(I);
0
```

NAME `IdealIsIntegral`

PURPOSE Checks if a (fractional) ideal is an integral (or true) ideal.

SYNTAX `b := IdealIsIntegral(I);`

boolean	b
Ideal	I

DESCRIPTION A fractional ideal $\mathfrak{a}$ of $\mathcal{O}$ is an $\mathcal{O}$-module $\mathfrak{a} \subset Q(\mathcal{O})$. An integral ideal is an $\mathcal{O}$-ideal which is an $\mathcal{O}$-module $\mathfrak{a} \subset \mathcal{O}$.

SEE ALSO IdealDen,

EXAMPLE

```
kash> o := Order (Poly (Zx,[1,6,6,6]));
Generating polynomial: x^3 + 6*x^2 + 6*x + 6

kash> I := Elt (o,[1,0,1])*o + 5*o;
<5, [1, 0, 1]>
kash> IdealIsIntegral (I);
true
kash> I := Elt (o,[1,0,1]/5)*o + 5*o;
<5, [1, 0, 1] / 5>
kash> IdealIsIntegral (I);
false
```

NAME IdealIsPrime

PURPOSE Checks whether an ideal is a prime ideal. Returns `true` or `false`.

SYNTAX `b := IdealIsPrime(I);`

`boolean`	`b`	
`ideal`	`I`	integral

DESCRIPTION The order over which the prime ideal is defined must be maximal.

SEE ALSO orderidealisprimeideal,

EXAMPLE

```
kash> o := OrderMaximal (Order (Poly (Zx,[1,6,6,6])));
Generating polynomial: x^3 + 6*x^2 + 6*x + 6
Discriminant: -1836

kash> IdealIsPrime (5*o);
false
kash> Factor (5*o);
[ [ <5, [1, 1, 0]>, 1 ], [ <5, [2, 1, 0]>, 1 ], [ <5, [3, 1, 0]>, 1 ] ]
kash> IdealIsPrime (11*o);
true
kash> Factor (11*o);
[ [ <11>, 1 ] ]
```

NAME `IdealIsPrincipal`

PURPOSE Returns a principal generator of an ideal, if it is a principal ideal and `false` otherwise.

SYNTAX
```
g := IdealIsPrincipal(I);
g := IdealIsPrincipal(I, classgroup);
```

`ideal`	`I`	
`false or algebraic number`	`g`	principal generator

DESCRIPTION There are two different algorithms used by this function. The standard procedure is not an easy check and can include expensive enumerations. If a second argument "classgroup" is given, it uses precomputed data from class group calculations. In this case it is often quite fast. Once computed, the result is stored and the ideal arithmetic uses the principal generator. A warning: The standard procedure for number fields uses real arithmetic and the correctness of the results depends on the precision defined by `OrderPrec()` whereas the second method works unconditional.

EXAMPLE We will test the prime ideal lying over 2 in the maximal order of $x^3 + 6x^2 + 6x + 6$.

```
kash> O := OrderMaximal(Order(Poly(Zx, [1, 6, 6, 6])));
Generating polynomial: x^3 + 6*x^2 + 6*x + 6
Discriminant: -1836

kash> p := Factor(O, 2*O)[1][1];
<2, [0, 1, 0]>
kash> IdealIsPrincipal(p);
false
kash> IdealIsPrincipal(p^3);
2
```

NAME `IdealLLL`

PURPOSE Creates an ideal with LLL-reduced real basis

SYNTAX `I := IdealLLL(a);`

`ideal I, a`

DESCRIPTION The function calculates a real basis for a given ideal $\mathfrak{a}$ and performs a LLL reduction on it.

SEE ALSO IdealBasisIdealGenerators,

EXAMPLE

```
kash> o:=OrderMaximal(Z,2,2);
Generating polynomial: x^2 - 2
Discriminant: 8

kash> id:=Factor(7*o)[1][1]^44;
<152867006319425761937651857692768264O1, [1181726564066674541521448323З, 83560\
6866959824669429500778 8]>
kash> IdealLLL(id);
<
[ 1985257763218632743  4216024203981766227]
[ 3100640983600199485 -1115383220381566742]
>
```

NAME `IdealLcm`

PURPOSE Least common multiplier of 2 ideals.

SYNTAX `I:=IdealLcm(I1, I2);`

```
Ideals  I, I1, I2
```

DESCRIPTION The lcm of $\mathfrak{a}$ and $\mathfrak{b}$ is computed as $\frac{\mathfrak{a}\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}$. The order over which the ideals are defined must be maximal.

EXAMPLE

```
kash> O:=OrderMaximal(Order(Poly(Zx,[1,6,6,6])));
Generating polynomial: x^3 + 6*x^2 + 6*x + 6
Discriminant: -1836

kash> I1:=Ideal(18,Elt(O,[0,0,1]));
<18, [0, 0, 1]>
kash> I2:=Ideal(24,Elt(O,[0,0,1]));
<24, [0, 0, 1]>
kash> IdealLcm(I1,I2);
<
[6 0 0]
[0 6 0]
[0 0 1]
>
```

NAME IdealLowerHNFTrans

PURPOSE the transformation matrix from the original basis to the basis in lower HNF

SYNTAX `M:=IdealLowerHNFTrans(I);`

```
matrix  M
ideal   I
```

DESCRIPTION This returns the transformation matrix when transforming the original basis (which can bee seen with IdealBasis) to the basis in lower HNF (which can bee seen with IdealBasisLowerHNF). Once computed the transformation matrix is stored in the ideal data structure so it has not to be computed again.

SEE ALSO IdealBasisLowerHNF,

EXAMPLE

```
kash> O:=OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> I:=Ideal(O,Mat(Z,[[1,2,1,4,6],[3,8,4,6,2],[6,4,3,3,8],
> [3,2,7,9,5],[0,2,8,8,4]]),1);
<
[1 2 1 4 6]
[3 8 4 6 2]
[6 4 3 3 8]
[3 2 7 9 5]
[0 2 8 8 4]
>

kash> IdealBasis(I)[2]*IdealLowerHNFTrans(I);
[   1    0    0    0    0]
[   0    1    0    0    0]
[   0    0    1    0    0]
[   0    1    0    3    0]
[ 512  910  858 1672 1678]
kash> IdealBasisLowerHNF(I);
[ 1, [   1    0    0    0    0]
    [   0    1    0    0    0]
    [   0    0    1    0    0]
    [   0    1    0    3    0]
    [ 512  910  858 1672 1678] ]
```

NAME `IdealMakeCoprime`

PURPOSE missing shortdoc

SYNTAX `c := IdealMakeCoprime(A,B));`

`ideal`	`A,B`
`algebraic number`	`c`

DESCRIPTION Let $\mathfrak{a}$ and $\mathfrak{b}$ integral ideals. The procedure finds c so that $c\mathfrak{a}$ is an integral ideal coprime to $\mathfrak{b}$. This algorithm is due to a paper by H. Cohen [Coh96].

SEE ALSO `IdealMakeInvCoprime`, `IdealClassRep`, `OrderClassGroup`,

EXAMPLE

```
kash> O := OrderMaximal(x^2-10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> OrderClassGroup(O,500,"euler","fast");
[ 2, [ 2 ] ]
kash> A := OrderClassGroupCyclicFactors(O)[1][1];
<2, [0, 1]>
kash> B := 2*O;
<2>
kash> c := IdealMakeCoprime(A,B);
[0, 1] / 2
kash> D := c*A;
<5, [0, 1]>
kash> IdealClassRep(D);
[ [ <2, [0, 1]>, 1 ] ]
```

Name IdealMakeInvCoprime

Purpose Subroutine of OrderRelNfColl

Syntax `a := IdealMakeInvCoprime(I1,I2);`

ideals	A, B	must be integral
algebraic number	a	

Description Let $\mathfrak{a}, \mathfrak{b}$ be integral ideals. The procedure finds an algebraic number c so that $c * \mathfrak{a}^{-1}$ is an integral ideal coprime to $\mathfrak{b}$.

See Also IdealMakeCoprime,

Example

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,-2,57,-56,602])));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 - 2*x^3 + 57*x^2 - 56*x + 602
Discriminant: 529984

kash> LP := List(Factor(210*O), x->x[1]);
[ <2, [0, 1, 0, 0]>, <2, [1, 1, 0, 0]>, <3, [1, 0, 1, 0]>, <3, [2, 1, 1, 0]>,
  <5, [3, 0, 1, 0]>, <5, [4, 3, 1, 0]>, <7, [0, 1, 0, 0]>, <7, [6, 1, 0, 0]> ]
kash> A := LP[1]^2*LP[3]*LP[6];
<
[30  0  4 17]
[ 0 30 18  2]
[ 0  0  1  0]
[ 0  0  0  1]
>

kash> B := LP[1]^2*LP[4]*LP[5];
<
[30  0  8 17]
[ 0 30 10 28]
```

```
[ 0  0  1  0]
[ 0  0  0  1]
>

kash> c := IdealMakeInvCoprime(A,B);
[-195, -206, 11, -7]
kash> D := c/A;
<
[-195 -382  -67  -26]
[-206 -442  -88 -103]
[  11 -555   99  -92]
[  -7  724 -376   -9]
>

kash> IdealIdempotents([D,B]);
[ 1623181771, -1623181770 ]
```

NAME `IdealMin`

PURPOSE The minimum of the ideal.

SYNTAX `m:=IdealMin(I);`

```
integer  m
ideal    I
```

DESCRIPTION The intersection of the ideal with the coefficient ring (which is $\mathbb{Z}$ for absolute ideals and the coefficient order for relative ideals) defines an ideal over the coefficient ring.
In case of absolute ideals the principal generator of this ideal is returned (which is the smallest positive rational integer in the ideal), in case of relative ideals the ideal itself is returned.
Iterative application of `IdealMin` to a relative ideal eventually leads to the smallest positive rational integer in the ideal.
Once computed the minimum is stored in the ideal data structure so it has not to be computed again.

SEE ALSO `IdealGen`,

EXAMPLE for absolute ideals

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> a := Elt(O,[1,5,1,8,0]);
[1, 5, 1, 8, 0]
kash> I := Ideal(a);
<[1, 5, 1, 8, 0]>
kash> IdealMin(I);
3563523
```

NAME `IdealMove`

PURPOSE Embeds one or more ideals into a given order.

SYNTAX `I2 := IdealMove(I1, o);`

`same type as I1`	`I2`
`ideal` | `list of ideals`	`I1`
`order`	`o`

DESCRIPTION If `I1` is an ideal, it is moved into `o`. This is done via a 2-element representation of the ideal. We assume that KASH already knows an embedding of the coeficient order of `I1` into `o`.
If `I1` is a (factor) list of ideals all ideals contained are moved.

SEE ALSO `EltMove`,

EXAMPLE We will create an order $\mathcal{O}$, search (and find) an order $\mathfrak{o}$ created by a better polynomial (smaller index or coefficients), factor 2 in $\mathfrak{o}$ where it is no index divisor and move it to $\mathcal{O}$.

```
kash> O := OrderMaximal(Order(x^6 - 9*x^4 - 4*x^3 + 27*x^2 - 36*x - 23));;
kash> o := OrderShort(O);;
kash> IdealMove(2*o, O);
<2>
kash> IdealMove(Factor(2*o), O);
[ [ <2, [1, -2, 2, 1, 4, -4]>, 6 ] ]
```

NAME `IdealNorm`

PURPOSE Returns the norm of an ideal.

SYNTAX `n := IdealNorm(I);`

ideal	I
rational number	n

DESCRIPTION If the ideal $\mathfrak{a}$ is an ideal over an absolute order $\mathcal{O}$ the norm is the number of elements of the finite ring $\mathcal{O}/\mathfrak{a}$.
If the ideal $\mathfrak{a}$ is an ideal over a relative order the relative norm of this ideal is returned which is the ideal generated by the norms of all elements of the ideal $\mathfrak{a}$.
Iterative application of `IdealNorm` eventually leads to the absolute norm of an relative ideal.
Once computed the minimum is stored in the ideal data structure so it has not to be computed again.

SEE ALSO `Norm`, `EltNorm`,

EXAMPLE Computing the norm of $2 * O + (1 + \rho) * O$ where O is the maximal order of $x^2 - 2$ and $\rho^2 = 2$.

```
kash> O := OrderMaximal(Order(Poly(Zx, [1, 0, -2])));
Generating polynomial: x^2 - 2
Discriminant: 8

kash> IdealNorm(Elt(O, [1 ,4])*O);
31
kash> IdealNorm (Elt (O,[3,6]/2)*O);
63/4
```

Name `IdealOrder`

Purpose This function returns the order of an ideal.

Syntax `O := IdealOrder(I);`

`Order`	`O`
`Ideal`	`I`

Description This function returns the order of an ideal under which it is defined. The order is included in the KASH data structure **ideal**, no calculation occurs.

Example

```
kash> O:=Order(Z,2,-3);
Generating polynomial: x^2 + 3

kash> I:=3*O;
<3>
kash> IdealOrder(I);
Generating polynomial: x^2 + 3
```

NAME `IdealPrimeCountInit`

PURPOSE Initializes an environment for enumerating prime ideals

SYNTAX `s := IdealPrimeCountInit(o[, m);`

`record`	`s`	
`order`	`o`	
`integer`	`m`	only primes coprime to `m` are used.

DESCRIPTION

SEE ALSO `IdealPrimeCountNext`,

EXAMPLE See `IdealPrimeCountNext` for examples.

NAME	`IdealPrimeCountNext`
PURPOSE	Enumerates the next prime ideal.
SYNTAX	`p := IdealPrimeCountNext(s [, 1]);`

`prime ideal`	`p`	
`record`	`s`	from IdealPrimeCountInit
`if present, p will be a pair` $[p, f]$ `with p a prime of the coefficient order and f the degree of p`	`1`	

DESCRIPTION Be careful with relative extensions: index divisors won't work here!

SEE ALSO `IdealPrimeCountInit`,

EXAMPLE We'll work in the field $\mathbb{Q}(\sqrt{2})$:

```
kash> o := OrderMaximal(Z, 2, 2);
Generating polynomial: x^2 - 2
Discriminant: 8

kash> s := IdealPrimeCountInit(o);
Record of type IdealPrimeCount

kash> repeat
> p := IdealPrimeCountNext(s);
> Print(p, "\n");
> until IdealMin(p)>20;
<2, [0, 1]>
<3>
<5>
<7, [3, 1]>
<7, [4, 1]>
<11>
<13>
<17, [6, 1]>
<17, [11, 1]>
<19>
<23, [5, 1]>
```

Next, only the factorization shape of ideals coprime to 2 is investigated:

```
kash> s := IdealPrimeCountInit(o, 2);
Record of type IdealPrimeCount

kash> repeat
> fs := IdealPrimeCountNext(s, 1);
> Print(fs, "\n");
> until fs[1]>20;
[ 3, 2 ]
[ 5, 2 ]
[ 7, 1 ]
[ 11, 2 ]
[ 13, 2 ]
[ 17, 1 ]
[ 19, 2 ]
[ 23, 1 ]
```

NAME	`IdealPrimeElt`
PURPOSE	primitive element of an ideal
SYNTAX	`e:=IdealPrimeElt(I);`

algebraic number	e
Ideal	I

DESCRIPTION An element of an ideal is called primitive if it is not contained in the square of the ideal. Once computed, the primitve element is stored in the ideal data structure so it will not be recomputed.
This function is implemented only for absolute ideals.

SEE ALSO `IdealMin`, `IdealGen`,

EXAMPLE

```
kash> O := OrderMaximal(Order(Poly(Zx, [1, 6, 6, 6])));;
kash> Lp := Factor(6*O);
[ [ <2, [0, 1, 0]>, 3 ], [ <3, [0, 1, 0]>, 3 ] ]
kash> I := Lp[1][1]*Lp[2][1];
<
[6 0 0]
[0 1 0]
[0 0 1]
>

kash> IdealPrimeElt(I);
[0, 1, 35]
```

NAME `IdealRadical`

PURPOSE Computes a radical.

SYNTAX `r := IdealRadical (A, O);`

```
order  O
ideal  A
ideal  r
```

DESCRIPTION

EXAMPLE

```
kash> o := OrderMaximal(Z,2,-5);
Generating polynomial: x^2 + 5
Discriminant: -20

kash> A := 6*o;
<6>
kash> IdealDivisors(A);
[ <1>, <2, [1, 1]>, <4, [0, 2]>, <3, [1, 1]>, <
    [6 1]
    [0 1]
    >
    , <6, [2, 2]>, <3, [2, 1]>, <
    [6 5]
    [0 1]
    >
    , <6, [4, 2]>, <3>, <
    [6 3]
    [0 3]
    >
    , <6> ]
```

Name `IdealRamIndex`

Purpose Calculates the ramification index of a prime ideal.

Syntax `d := IdealRamIndex(I);`

`ideal`	`I`	must be prime
`integer`	`d`	

Description Let $\mathcal{O}$ a maximal order. Let p be a prime number and $\mathfrak{p}$ a prime $\mathcal{O}$-ideal over this prime number p. The natural number e which satisfies $\mathfrak{p}^e \mid p\mathcal{O}$ and $\mathfrak{p}^{e+1} \nmid p\mathcal{O}$ is called the ramification index of $\mathfrak{p}$ over p.

See Also IdealDegree,

Example

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> P := Factor(6*O)[2][1];
<2, [1, 0, 0, 1, 1]>
kash> IdealDegree(P);
4
kash> IdealRamIndex(P);
1
```

Name `IdealRayClassRep`

Purpose Returns a representive of the class of an ideal in the ray class group.

Syntax `r := IdealRayClassRep(I,m0,minf);`

`matrix`	`r`	
`ideal`	`I`	
`ideal`	`m0`	
`list`	`minf`	of integers/infinite primes

Description Decomposes an ideal, or more exactly, the class of this ideal (that need to be coprime to $\mathfrak{m}_0$) into a power product of the classes generating the whole ray class group.
Based on the solutions for the discrete logarithm problem for class groups ([Heß96]) and for ray residue rings (see `EltRayResidueRingRep`), an exponent vector relative to the generators is computed.

See Also `EltCon`, `OrderClassGroup`, `RayClassGroup`, `RayClassGroupCyclicFactors`,

Example

```
kash> O := OrderMaximal(Order(x^2-2*x-5));
Generating polynomial: x^2 - 2*x - 5
Discriminant: 24

kash> OrderClassGroup(O,500,"euler","fast");
[ 1, [ 1 ] ]
kash> m0 := 27*O;;
kash> minf := [2];;
kash> L := RayClassGroupCyclicFactors(m0,minf);
[ [ <[1344, 388]>, 3 ], [ <[1612, 444]>, 9 ] ]
kash> I := L[1][1];
<[1344, 388]>
kash> IdealRayClassRep(I,m0,minf);
[1 0]
```

NAME `IdealRemainderSet`

PURPOSE Computes a remainder system of an ideal

SYNTAX `L:= IdealRemainderSet(I);`

list	L
ideal	I

DESCRIPTION This function returns a list with the complete remainder set of an ideal in an order

SEE ALSO `IdealNorm`,

EXAMPLE

```
kash> O:=Order(Poly(Zx,[1,0,2,4,58]));
Generating polynomial: x^4 + 2*x^2 + 4*x + 58

kash> E:=Elt(O,[1,2,-1,-2]);
[1, 2, -1, -2]
kash> I:=Ideal(3,E^7);
<3, [-470899145529, -57698441418, 37053141409, 29584796218]>
kash> IdealRemainderSet(I);
[ -1, 0, 1 ]
```

NAME `IdealResidueField`

PURPOSE Returns the finite field defined by a prime ideal.

SYNTAX `K := IdealResidueField(p);`

ideal	p
finite field	K

DESCRIPTION The field can be written as

$$\mathcal{O}/\mathfrak{p},$$

where $\mathfrak{p}$ is the prime ideal and $\mathcal{O}$ the defining order.

SEE ALSO IdealResidueFieldIsomorphism, EltToFFE, FFEToElt, RayResidueRing,

EXAMPLE

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> p := Factor(7*O)[1][1];
<7, [0, 1, 0, 0, 0]>
kash> IdealResidueField(p);
Finite field of size 7
kash> a := Elt (O, [1,234,54,57,4]);
[1, 234, 54, 57, 4]
kash> EltToFFE (a, p);
1
kash> a := Elt (O, [4,234,54,57,4]);
[4, 234, 54, 57, 4]
kash> b:=EltToFFE (a, p);
4
kash> TYPE (b);
"KANT finite field elt"
```

NAME `IdealResidueFieldIsomorphism`

PURPOSE Calculates the isomorphism between the residue fields of two prime ideals.

SYNTAX `alpha := IdealResidueFieldIsomorphism(a,b);`

algebraic element	alpha
ideal	a
ideal	b

DESCRIPTION This function calculates an algebraic element α which corresponds to the isomorphism between the residue fields of the prime ideals.

SEE ALSO IdealResidueField,

EXAMPLE

```
kash> O:=OrderMaximal(Order(Poly(Zx,[1,0,-4,0,1])));
Generating polynomial: x^4 - 4*x^2 + 1
Discriminant: 2304

kash> L:=Factor(5*O);
[ [ <5, [1, 1, 1, 0]>, 1 ], [ <5, [1, 4, 1, 0]>, 1 ] ]
kash> IdealResidueFieldIsomorphism(L[1][1],L[2][1]);
[0, 4, 0, 0]
```

NAME `IdealRingOfMultiplicators`

PURPOSE Computes the ring of multiplicators of an ideal.

SYNTAX `o := IdealRingOfMultiplicators (A);`

order	o
ideal	A

DESCRIPTION

EXAMPLE

```
kash> o := OrderMaximal(Z,2,-5);
Generating polynomial: x^2 + 5
Discriminant: -20

kash> A := 6*o;
<6>
kash> IdealDivisors(A);
[ <1>, <2, [1, 1]>, <4, [0, 2]>, <3, [1, 1]>, <
    [6 1]
    [0 1]
    >
    , <6, [2, 2]>, <3, [2, 1]>, <
    [6 5]
    [0 1]
    >
    , <6, [4, 2]>, <3>, <
    [6 3]
    [0 3]
    >
    , <6> ]
```

Name `IdealUpperHNFTrans`

Purpose the transformation matrix from the original basis to the basis in upper HNF

Syntax `M:=IdealUpperHNFTrans(I);`

matrix	M
ideal	I

Description This returns the transformation matrix when transforming the original basis (which can bee seen with IdealBasis) to the basis in upper HNF (which can bee seen with IdealBasisUpperHNF). Once computed the transformation matrix is stored in the ideal data structure so it has not to be computed again.

See Also IdealBasisUpperHNF,

Example

```
kash> O:=OrderMaximal(Order(Poly(Zx,[1,4,1,-4,-3,7])));
Generating polynomial: x^5 + 4*x^4 + x^3 - 4*x^2 - 3*x + 7
Discriminant: 28442269

kash> I:=Ideal(O,Mat(Z,[[1,2,1,4,6],[3,8,4,6,2],[6,4,3,3,8],
> [3,2,7,9,5],[0,2,8,8,4]]),1);
<
[1 2 1 4 6]
[3 8 4 6 2]
[6 4 3 3 8]
[3 2 7 9 5]
[0 2 8 8 4]
>

kash> IdealBasis(I)[2]*IdealUpperHNFTrans(I);
[839 755 349 421 449]
[  0   3   0   1   0]
[  0   0   1   0   0]
[  0   0   0   1   0]
[  0   0   0   0   2]
kash> IdealBasisUpperHNF(I);
[ 1, [839 755 349 421 449]
    [  0   3   0   1   0]
    [  0   0   1   0   0]
    [  0   0   0   1   0]
    [  0   0   0   0   2] ]
```

NAME `IdealValuation`

PURPOSE Computes the valuation of an ideal at a prime ideal.

SYNTAX `val := IdealValuation(p, I);`

`ideal`	`P`	must be prime
`ideal`	`I`	
`integer`	`val`	

DESCRIPTION Consider the factorization of `I` in prime ideals. This function returns the exponent of `P` in the factorization (possibly 0). The prime ideal must not be an index divisor.
The ideal may be fractional, see `IdealFactor` for interpretation.
This function is only implemented for ideals over absolute orders.

SEE ALSO `IdealFactor`,

EXAMPLE

```
kash> O := OrderMaximal (Order (Poly (Zx,[1,6,6,6])));
Generating polynomial: x^3 + 6*x^2 + 6*x + 6
Discriminant: -1836

kash> P := Factor(5*O)[1][1];
<5, [1, 1, 0]>
kash> IdealValuation (P, Elt (O, [5,5,5])*O);
1
kash> P := Factor(2*O)[1][1];
<2, [0, 1, 0]>
kash> IdealValuation (P, Elt (O, [8,8,8])*O);
9
kash> IdealValuation (P, 2*O);
3
```

NAME `IdealWithNorm`

PURPOSE Computed all ideals of `o` of a given norm `n`.

SYNTAX `L := IdealWithNorm(n,o);`

`List`	`L`	
`integer`	`n`	> 1
`MaximalOrder`	`o`	

EXAMPLE We have for example:

```
kash> o := OrderMaximal(Z, 2, 3);
Generating polynomial: x^2 - 3
Discriminant: 12

kash> IdealWithNorm(2,o);
[ <2, [1, 1]> ]
kash> IdealWithNorm(11*13,o);
[ <
    [143  82]
    [  0   1]
    >
    , <
    [143 126]
    [  0   1]
    >
    , <
    [143  17]
    [  0   1]
    >
    , <
    [143  61]
    [  0   1]
    >
     ]
```

NAME `IdemLift`

PURPOSE Lifts an idempotent.

SYNTAX `alpha:=IdemLift(a, p, k);`

algebraic element	alpha
algebraic element	a
integer	p
integer	k

DESCRIPTION `a` is an idempotent for a prime ideal $\mathfrak{p}$ over `p`. This function calculates the idempotent α corresponding to the ideal $\mathfrak{p}^{2^k}$.

EXAMPLE

```
kash> O:=OrderMaximal(Order(Poly(Zx,[1,0,-4,0,1])));
Generating polynomial: x^4 - 4*x^2 + 1
Discriminant: 2304

kash> L:=Factor(5*O);
[ [ <5, [1, 1, 1, 0]>, 1 ], [ <5, [1, 4, 1, 0]>, 1 ] ]
kash> L1:=IdealIdempotents([L[1][1],L[2][1]]);
[ [-2, 0, 0, 2], [3, 0, 0, -2] ]
kash> IdemLift(L1[1],5,3);
[-195312, 112715, 0, -22543]
```

NAME Im

PURPOSE Returns the imaginary part of a complex number.

SYNTAX `a := Im(z);`

`real`	`a`
`complex`	`z`

SEE ALSO `Re`,

EXAMPLE Compute the imaginary part of $1 + 2i$:

```
kash> z := Comp(1, 2);
1 + 2*i
kash> Im(z);
2
```

Name `ImQuadFormCreate`

Purpose Generates a quadratic form.

Syntax
```
g:=ImQuadFormCreate(D,p);
g:=ImQuadFormCreate(D,[a,b,c]);
```

Description Let `D` be a fundamental discriminant and `p` be a prime or `[a,b,c]` a list of integers. This function generates the quadratic form with `a=p` and discriminant `D` if there exists one or proves if the discriminant of the suggested form `[a,b,c]` is `D`. The group operations is written additively.

Example

Example

NAME `ImQuadHilbert`

PURPOSE Determines the Hilbert class field of an imaginary quadratic field.

SYNTAX `O := ImQuadHilbert(o [,repr] [,"roots"] );`

`order`	`o`	imaginary quadratic field
`list`	`repr`	representants for the ideal classes of o
`order`	`O`	Hilbert class field

DESCRIPTION Let K be an imaginary quadratic field and $K|(1)$ its Hilbert class field. Possible generators for $K|(1)$ over K are for example quotients of the Dedekind η–function of the form

$$\left(\frac{\eta(a)\,\eta(b)}{\eta(a\,b)\,\eta(\mathcal{O}|K)}\right)^n, \quad n \in \mathbb{N},$$

where the integral $\mathcal{O}|K$–ideals a and b have to be suitably normalized (this is implicitly contained in [Sch90b]) A defining equation for $\mathcal{O}|K|(1)$ is calculated using complex approximations one gets by evaluating the defining infinite series of the Dedekind η–function.

If `"roots"` is specified, also the roots of the minimal polynomial in the ordering given by the ordering of the used ideal class representants (that correspond to the Galois automorphisms by use of the Artin map) are returned.

SEE ALSO `OrderHilbert`, `ImQuadRayField`, `RayClassField`,

EXAMPLE

```
kash> o := OrderMaximal(Z, 2, -47);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^2 + 47
Discriminant: -47

kash> oK1 := ImQuadHilbert(o);
used real precision is 52 places
      F[1]
```

```
        /
       /
   E1[1]
    |
   E1[2]
  /
 /
Q
F  [ 1]     x^5 + 3*x^2 + 2*x - 1
E 1[ 1]     Given by transformation matrix
E 1[ 2]     x^2 + 47
```

Test for correctness (in other examples the coefficients of the generating polynomial one gets by using the routine `ImQuadHilbert` are usually quite small):

```
kash> O := OrderMaximal(OrderAbs(oK1));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^10 + 235*x^8 + 6*x^7 + 22094*x^6 - 284*x^5 + 1037299*x^4 - 65318\
*x^3 + 24355069*x^2 - 1890344*x + 229780651
Discriminant: -229345007

kash> Factor(Abs(OrderDisc(O)));
[ [ 47, 5 ] ]
```

NAME `ImQuadRayField`

PURPOSE Determines the ray class field modulo ideal f over an imaginary quadratic field.

SYNTAX `O := ImQuadRayField( f [,"char" | "field" | "maxord"] );`

`ideal`	`f`	integral ideal in imaginary quadratic field
`order`	`O`	ray class field modulo `f`

DESCRIPTION Let K be a imaginary quadratic field with maximal order $\mathcal{O}|K$, f an integral $\mathcal{O}|K$–ideal and $K|(f)$ the ray class field over K corresponding to f. Integral generators for $K|(f)$ over the Hilbert class field $K|(1)$ are for example elements of the form

$$\Theta = \alpha \Big(P(1 \mid f) - D \Big),$$

where $P(1 \mid f)$ is a normalized singular value of the Weierstrass $\wp$–function and α and D are suitable choosen elements of $K|(1)$ (see [Sch90b, Sch90a]). Furthermore we have an integral basis for $K|(f)/K|(1)$ given by

$$\begin{aligned} \mathcal{O}|K|(f) = \mathcal{O}|K|(1) + \Theta\, \mathcal{O}|K|(1) \quad &+ \quad \alpha^{-1}\, \Theta^2\, \mathcal{O}|K|(1) + \\ &+ \quad \ldots + \alpha^{-n+2}\, \Theta^{n-1}\, \mathcal{O}|K|(1) \end{aligned}$$

The minimal polynomial for Θ is calculated using complex approximations one gets by evaluating the defining infinite series of the Dedekind η–function and the Weierstrass $\wp$–function.
If `"char"` is specified only the characteristic polynomial for Θ over K is computed (which might be reducible). In case of `"field"` also the Hilbert class field (and the representation of the ray class field over the Hilbert class field) is determined, and if `"maxord"` or nothing is specified (but that might cause a long output) one gets the maximal order of the ray class field as an extension of the maximal order of the Hilbert class field.
Note: For numerical reasons don't use prime ideal powers as a module whenever possible. So for example if 2 is not inert in K, $p|2$ is a prime ideal in K above 2 and f is prime to $p|2$, then we have $K|(f) = K|(p|2f)$ and one should use `ImQuadRayField`$(p|2f)$ instead of `ImQuadRayField`(f).

SEE ALSO `RayClassField`, `ImQuadHilbert`, `OrderHilbert`,

EXAMPLE

```
kash> o := OrderMaximal(Z, 2, -7);
   F[1]
    |
```

```
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^2 + 7
Discriminant: -7

kash> f := Factor(2*o)[1][1] * Factor(7*o)[1][1];
<
[14  3]
[ 0  1]
>

kash> O := ImQuadRayField(f, "maxord");
resulting degrees:
  [RayField:Hilbert]=3  ,  [Hilbert:Imaginary]=1

calculation of Hilbert class field ...
used real precision is 52 places

calculation of ray class field ...
used real precision is 52 places

calculation of representation with integral relative basis...
      F[1]
        /
       /
   E1[1]
    |
   E1[2]
  /
 /
Q
F  [ 1]     x^3 + [0, 1]*x^2 + [-1, 1]*x - 1
E 1[ 1]     Given by transformation matrix
E 1[ 2]     x^2 + 7
```

Here Hilbert class field and imaginary field are the same (E1). F[1] is the maximal order of the ray class field, compare also the corresponding discriminants:

```
kash> Factor(OrderDisc(OrderCoefOrder(O)));
[ [ -1, 1 ], [ 7, 1 ] ]
kash> Factor(OrderDisc(O));
[ [ <7, [-1, 2]>, 2 ] ]
```

Name	`Index`
Purpose	Computes the index of an algebraic element, an order or an alff order
Syntax	`I := Index (a);` `I := Index (o);`

coefficient ring element	I
algebraic element	a
order alff order	o

Description: For an algebraic element a in an arbitrary order o over $\mathbb{Z}$ this function returns the modul index $(o : \mathbb{Z}[\alpha]$ Given an order this function returns the index of the suborder o_1 of o in o. Given an alff order this function returns the index of the equation order in the given order.

See Also: `EltIndexOrderIndexAlffOrderIndex`,

Example

```
kash> o := Order(Z,5,3);
Generating polynomial: x^5 - 3

kash> a := Elt(o, [0,0,1,1,1]);
[0, 0, 1, 1, 1]
kash> I := Index(a);
3501
```

NAME InftyGcd

PURPOSE Computes a greatest common divisor with respect to the degree valuation.

SYNTAX `c := InftyGcd(a, b);`

`quotient field elements a,b,c`

DESCRIPTION Let $a, b \in \mathbb{F}_q(x)$. This function returns $x^{-\min(\nu_\infty(a), \nu_\infty(b))}$.

SEE ALSO InftyVal, InftyQuotRem, InftyLcm,

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(5^2)

kash> x := Poly(kx, [1,0]);
x
kash> InftyGcd(x^2+1, x^3+1);
x^3
```

NAME `InftyLcm`

PURPOSE Computes a least common multiple with respect to the degree valuation.

SYNTAX `c := InftyLcm(a, b);`

`quotient field elements a,b,c`

DESCRIPTION Let $a, b \in \mathbb{F}_q(x)$. This function returns $x^{-\max(\nu_\infty(a),\nu_\infty(b))}$.

SEE ALSO `InftyVal`, `InftyQuotRem`, `InftyGcd`,

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(5^2)

kash> x := Poly(kx, [1,0]);
x
kash> InftyLcm(x^2+1, x^3+1);
x^2
```

NAME `InftyQuotRem`

PURPOSE Computes quotient and remainder with respect to the degree valuation.

SYNTAX `L := InftyQuotRem(a, b);`

`list`	`L`	of q and r
`quotient field elements`	`a,b`	

DESCRIPTION Let $a, b \in \mathbb{F}_q(x)$. The function computes $q, r \in \mathbb{F}_q(x)$ such that $a = qb + r$ and $\nu_\infty(q) \geq 0$. The rest r is chosen such that $r = \sum_{i=\nu_\infty(a)}^{\nu_\infty(b)-1} c_i x^{-i}$ and $c_i \in \mathbb{F}_q$. By these conditions q, r are uniquely determined.

SEE ALSO InftyVal, InftyGcd, InftyLcm,

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(5^2)

kash> x := Poly(kx, [1,0]);
x
kash> a := x + 2 + 3/x + 4/x^2;
(x^3 + 2*x^2 + 3*x + 4)/x^2
kash> InftyQuotRem(a, x);
[ (x^3 + 2*x^2 + 3*x + 4)/x^3, 0 ]
kash> InftyQuotRem(a, x/x);
[ (2*x^2 + 3*x + 4)/x^2, x ]
kash> InftyQuotRem(a, 1/x);
[ (3*x + 4)/x, x + 2 ]
kash> InftyQuotRem(a, 1/x^2);
[ 4, (x^2 + 2*x + 3)/x ]
kash> InftyQuotRem(a, 1/x^3);
[ 0, (x^3 + 2*x^2 + 3*x + 4)/x^2 ]
```

NAME InftyVal

PURPOSE Returns the degree valuation of a rational function or a Puiseux series defined over a finite field.

SYNTAX `n := InftyVal(a);`

integer	n	
quotient field element	a	of $\mathbb{F}_q(x)$

DESCRIPTION This function computes the degree valuation $\nu_\infty(a)$ of an element $a \in \mathbb{F}_q(x)^\times$ or $a \in \mathbb{F}_q((x^{-1/e}))$ (the negative degree).

SEE ALSO InftyQuotRem, InftyGcd, InftyLcm, AlffRoots,

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(5^2)

kash> x := Poly(kx, [1,0]);
x
kash> InftyVal(x);
-1
kash> InftyVal((x^2+x)^-1);
2
```

NAME	`Insert`
PURPOSE	Insert an element into a list at a given position.
SYNTAX	`Insert(L, a, pos);`

list	L
arbitrary object	a
integer	pos

EXAMPLE

```
kash> L := [1, 2, 3];
[ 1, 2, 3 ]
kash> Insert(L, 7, 1);
kash> L;
[ 7, 1, 2, 3 ]
kash> Insert(L, 8, 5);
kash> L;
[ 7, 1, 2, 3, 8 ]
```

Name `IntDivisors`

Purpose Returns a sorted list of all non–negative divisors of a rational integer.

Syntax `L := IntDivisors(d);`

`list`	`L`
`integer`	`d`

Description

See Also `IntFactor`,

Example Divisors of 30030:

```
kash> IntDivisors(30030);
[ 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 21, 22, 26, 30, 33, 35, 39, 42, 55,
  65, 66, 70, 77, 78, 91, 105, 110, 130, 143, 154, 165, 182, 195, 210, 231,
  273, 286, 330, 385, 390, 429, 455, 462, 546, 715, 770, 858, 910, 1001,
  1155, 1365, 1430, 2002, 2145, 2310, 2730, 3003, 4290, 5005, 6006, 10010,
  15015, 30030 ]
```

NAME `IntEulerPhi`

PURPOSE Computes the Euler-ϕ function of n, i.e. the number of coprime integers less than n.

SYNTAX `a := IntEulerPhi(n);`

```
integer  a
integer  n
```

DESCRIPTION

EXAMPLE

EXAMPLE

```
kash> IntEulerPhi(3);
2
kash> IntEulerPhi(81);
54
kash> IntEulerPhi(134);
66
```

NAME IntFactor

PURPOSE

SYNTAX `F := IntFactor(d);`

list	F
integer	d

DESCRIPTION Returns the factorization of the given positive integer $d \in \mathbb{N}$.

SEE ALSO Factor,

EXAMPLE Factorization of 8274626472648264826427648723648276:

```
kash> Factor(8274626472648264826427648723648276);
[ [ 2, 2 ], [ 11, 1 ], [ 41, 1 ], [ 86423, 1 ],
  [ 53074086409412759917474753, 1 ] ]
```

NAME IntGcd

PURPOSE Returns the non-negative greatest common divisor of two integers.

SYNTAX `gcd := IntGcd (a1,a2);`

```
integer  gcd
integer  a1
integer  a2
```

DESCRIPTION IntGcd$(a, 0)$ or IntGcd$(0, a)$ is the absolute value of a. IntGcd$(0, 0)$ is 0.

SEE ALSO IntXGcd, IntLcm,

EXAMPLE

```
kash> IntGcd(9, 0);
9
kash> IntGcd(-9, 0);
9
kash> IntGcd(0, 7);
7
kash> IntGcd(0, -7);
7
kash> IntGcd(2345, -505);
5
kash> IntGcd(2345, 505);
5
```

NAME `IntIsPrime`

PURPOSE Returns `true` iff the argument is a rational prime.

SYNTAX `b := IntIsPrime(n)`

`boolean`	`b`
`integer`	`n`

DESCRIPTION

SEE ALSO `NextPrime`,

EXAMPLE

```
kash> IntIsPrime(1);
false
kash> IntIsPrime(2);
true
```

NAME `IntIsSquare`

PURPOSE Returns `true` iff the argument is a square.

SYNTAX `b := IntIsSquare(n)`

`boolean`	`b`
`integer`	`n`

DESCRIPTION

EXAMPLE

```
kash> IntIsSquare(1);
true
kash> IntIsSquare(2);
false
```

NAME IntLcm

PURPOSE Returns the least positve common multiplier of two integers and zero if one of both is zero.

SYNTAX `c := IntLcm(a, b);`

integer	c
integer	a
integer	b

DESCRIPTION

SEE ALSO IntGcd, IntGcdEx,

EXAMPLE

```
kash> IntLcm(345, 6540);
150420
kash> IntLcm(-345, 6540);
150420
kash> IntLcm(-345, -6540);
150420
kash> IntLcm(345, -6540);
150420
kash> IntLcm(0, 1);
0
kash> IntLcm(1, 0);
0
kash> IntLcm(0, 0);
0
```

NAME `IntMoebiusMy`

PURPOSE Computes the Moebius μ-function.

SYNTAX `a := IntMoebiusMy(n);`

`integers a, n`

SEE ALSO `IntDivisors`,

EXAMPLE

```
kash> IntMoebiusMy(1);
1
kash> IntMoebiusMy(2*3);
1
kash> IntMoebiusMy(2*3*5);
-1
kash> IntMoebiusMy(2^2*3*5);
0
```

NAME IntMoebiusMy

PURPOSE Computes the Moebius μ-function.

SYNTAX `a := IntMoebiusMy(n);`

`integers a, n`

SEE ALSO IntDivisors,

EXAMPLE

```
kash> IntMoebiusMy(1);
1
kash> IntMoebiusMy(2*3);
1
kash> IntMoebiusMy(2*3*5);
-1
kash> IntMoebiusMy(2^2*3*5);
0
```

NAME IntPowerMod

PURPOSE Returns the power of a given integer modulo an integer.

SYNTAX `y := IntPowerMod(x, n, m);`

integer	y
integer	x
integer	n
integer	m

DESCRIPTION This function computes $y = x^n \bmod m$ where n is non-negative and m greater than one.

EXAMPLE

```
kash> IntPowerMod(3, 1000, 17);
16
```

NAME `IntPrimeDivisors`

PURPOSE Returns a sorted list of all prime divisors of a rational integer.

SYNTAX `L := IntPrimeDivisors(d);`

list	L
integer	d

DESCRIPTION

SEE ALSO IntFactor,

EXAMPLE Divisors of 30030:

```
kash> IntPrimeDivisors(30030);
[ 2, 3, 5, 7, 11, 13 ]
```

NAME `IntQuo`

PURPOSE Returns the integer quotient of two integers.

SYNTAX `q := IntQuo(a, b);`

```
integer  q
integer  a
integer  b
```

DESCRIPTION Returns q of $a = q * b + r$ subject to $|r| < |b|$. The integer b must not be zero.

EXAMPLE

```
kash> IntQuo(20, 3);
6
kash> IntQuo(-20, 3);
-7
kash> IntQuo(20, -3);
-7
kash> IntQuo(-20, -3);
6
```

NAME IntRandomBits

PURPOSE Returns a random integerin the range 0 to $2^n - 1$. n must be ≥ 0.

SYNTAX `y := IntRandomBits(n);`

integer	y
small integer	n
	H

DESCRIPTION

EXAMPLE

```
kash> y:=IntRandomBits(100);
648024517884907836259026803\64
```

NAME IntRoot

PURPOSE Returns $\rfloor\sqrt[r]{n}\lfloor$.

SYNTAX `m := IntRoot(r, n)`

`boolean`	`m`
`integer`	`n`
`integer`	`r`

DESCRIPTION

EXAMPLE

```
kash> IntRoot(8, 3);
2
kash> IntRoot(82, 4);
3
```

NAME IntToChar

PURPOSE Returns the integer with (ASCII) code i, where $0 \leq i < 256$

SYNTAX `c := IntToChar(i);`

```
character  c
integer    i
```

SEE ALSO CharToInt,

EXAMPLE

```
kash> IntToChar(123);
'{'
```

NAME `IntValuation`

PURPOSE Returns the p-adic valuation of an integer

SYNTAX `v := IntValuation(p, n);`

```
integer  v
integer  p
integer  n
```

DESCRIPTION Returns the p-adic valuation v of an integer $n \neq 0$, i.e. for a rational prime p it returns a non-negative, maximal v subject to $p^v|n$.

SEE ALSO `IdealValuation`, `EltValuation`,

EXAMPLE

```
kash> IntValuation(1208724496185624589, 1180992730069637804054573);
1
```

NAME IntXGcd

PURPOSE Extended Euclidean algorithm.

SYNTAX

```
G := IntXGcd (a1,a2);
G := IntXGcd (L);
```

list	G
integer	a1
integer	a2
list	L

DESCRIPTION The IntXGcd function computes the greatest common divisor g of integers $a_1, \ldots, a_n$. g is the first element of the list G which will be returned by IntXGcd. The second element of G is another list which contains the representation of g by $a_1, \ldots, a_n$.

IntXGcd (a1,a2);
computes the greatest common divisor of a_1, a_2 and its representation.

IntXGcd (L);
Let L be a list of rational integers $a_1, \ldots, a_n$. The IntXGcd computes the greatest common divisor of $a_1, \ldots, a_n$ and its representation.

SEE ALSO IntGcd,

EXAMPLE

```
kash> G := IntXGcd(2345,-505);
[ 5, [ 14, 65 ] ]
kash> G[2][1]*2345+G[2][2]*(-505);
5
kash> G := IntXGcd([12,18,21]);
[ 3, [ 3, -3, 1 ] ]
kash> G[2][1]*12+G[2][2]*18+G[2][3]*21;
3
```

NAME IntZeta

PURPOSE Returns the value of the Riemann ζ-function

SYNTAX `y := IntZeta(n);`

real	y
integer distinct from 1	n

DESCRIPTION

EXAMPLE

```
kash> IntZeta(2);
1.644934066848226436472415166646025189218949901207
kash> IntZeta(-2);
0
```

NAME `IntegralPoints`

PURPOSE Computes all integral points on an elliptic curve in normal form.

SYNTAX

```
L := IntegralPoints(k);
L := IntegralPoints(a,b);
```

list	L
integer	k
integer	a
integer	b

DESCRIPTION The `IntegralPoints` function computes all integral points on an elliptic curve in normal form over the rational integers. The result is returned as a list of all x-y pairs solving the equation.

`IntegralPoints(k)`
computes all integral points of Mordell's equation $y^2 = x^3 + k$, where k is a rational integer. In this case the `IntegralPoints` function uses the algorithm described in [Wil].

`IntegralPoints(a,b)`
computes all integral points of an elliptic curve in short Weierstraß form $y^2 = x^3 + a\,x + b$ with rational integers a, b. The computation of all integral points is reduced to another problem in which finitely many associated quartic Thue are solved [Str84, TdW89].

EXAMPLE Compute all integral points of $y^2 = x^3 + 17$.

```
kash> IntegralPoints(17);
[ [ -2, -3 ], [ -2, 3 ], [ -1, -4 ], [ -1, 4 ], [ 2, -5 ], [ 2, 5 ],
  [ 4, -9 ], [ 4, 9 ], [ 8, -23 ], [ 8, 23 ], [ 43, -282 ], [ 43, 282 ],
  [ 52, -375 ], [ 52, 375 ], [ 5234, -378661 ], [ 5234, 378661 ] ]
```

NAME `IsAlff`

PURPOSE Returns whether an object is an algebraic function field.

SYNTAX `b := IsAlff(F);`

```
boolean  b
object   F
```

DESCRIPTION Given an object this function returns whether this object is an algebraic function field.

SEE ALSO `Alff`,

EXAMPLE

```
kash> AlffInit(FF(7,2));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff((T^2+1)*y^3+y+T^4+1);
Algebraic function field defined by
$.1^3*$.2^2 + $.1^3 + $.1 + $.2^4 + 1
over
Univariate rational function field over GF(7^2)
Variables: T

kash> IsAlff(F);
true
kash> IsAlff(1);
false
```

Name	`IsAlffDiff`
Purpose	Returns whether argument is an alff differential.
Syntax	`b := IsAlffDiff(a);`

boolean	b
alff differential	a

See Also: AlffDiff,

Example

```
kash> AlffInit(FF(5,1));;
kash> AlffOrders(y^2 + T^3 + 1);
"Defining global variables: F, o, oi, one"
kash> dT := AlffDiff(AlffEltGenT(F));
Alff Differential
[ 1, 0 ] d[ T, 0 ]
kash> IsAlffDiff(dT);
true
```

NAME `IsAlffDivisor`

PURPOSE Returns true if and only if the parameter is an algebraic function field divisor.

SYNTAX `b := IsAlffDivisor(D);`

boolean	b
arbitrary object	D

SEE ALSO IsAlffPlace, IsBound, Unbind,

EXAMPLE

```
kash> AlffInit(FF(2,4));
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> F := Alff(y^3+T^3*y+T);
Algebraic function field defined by
$.1^3 + $.1*$.2^3 + $.2
over
Univariate rational function field over GF(2^4)
Variables: T

kash> P := AlffPlaceSplit(F, T+1)[1];
Alff place < [ T + 1, 0, 0 ] >
kash> IsAlffDivisor(P);
false
kash> D := AlffDivisor(P);
Alff divisor
[ [ Alff place < [ T + 1, 0, 0 ] >, 1 ] ]

kash> IsAlffDivisor(D);
true
```

NAME IsAlffElt

PURPOSE Returns true iff the argument is of type ”KANT function field order elt”.

SYNTAX b := IsAlffElt(T);

boolean	b
function field order element	T

EXAMPLE

```
kash> AlffInit(Q);
"Defining global variables: k, w, kT, kTf, kTy, T, y, AlffGlobals"
kash> AlffOrders(y^3+T^4+1);
"Defining global variables: F, o, oi, one"
kash> a:=AlffElt(o,[0,1,0]);
[ 0, 1, 0 ]
kash> IsAlffElt(a);
true
```

NAME `IsAlffPlace`

PURPOSE Returns whether an object is an algebraic function field place.

SYNTAX `b := IsAlffPlace(P);`

boolean	b
arbitrary object	P

SEE ALSO IsBound, Unbind,

EXAMPLE

```
kash> H := AlffHermitianFunField(2,3);
Algebraic function field defined by
$.1^8 + $.1 + $.2^9
over
Univariate rational function field over GF(2^6)
Variables: T

kash> P := AlffPlacesDegOne(H)[1];
Alff place < [ 1/T, 0, 0, 0, 0, 0, 0, 0 ], [ w^60/T, (w^7*T + w^44)/T, (w^14*T\
 + w^26)/T, (w^37*T + w^19)/T, (w^8*T + w^12)/T, (w^33*T + w^21)/T, (w^12*T + \
w^30)/T, (w^21*T + w^57)/T ] >
kash> IsAlffPlace(P);
true
```

NAME `IsChar`

PURPOSE Returns true if the argument is a character, false otherwise.

SYNTAX `b := IsChar(c);`

character	c
boolean	b

SEE ALSO IsString,

EXAMPLE

```
kash> IsChar('a');
true
kash> IsChar("a");
false
kash> IsChar(1);
false
```

NAME IsEcc

PURPOSE Returns true iff the argument is an elliptic curve.

SYNTAX b := IsEcc(E);

```
boolean  b
         E
```

DESCRIPTION

EXAMPLE

EXAMPLE

```
kash> p:=65112*2^144-1;
1452046121366725933991673688168680114377396846591
kash> IsPrime(p);
true
kash> E := EccInit(p,-3,49);
[ 1452046121366725933991673688168680114377396846588, 49 ]
kash> IsEcc(E);
true
```

NAME `IsElt`

PURPOSE Returns true iff the argument is of type ”KANT algebraic element”.

SYNTAX `b := IsElt(x);`

boolean	b
	x

EXAMPLE

```
kash> o := Order(Z, 2, 5);;
kash> IsElt(1);
false
kash> IsElt(Elt(o, 1));
true
```

NAME `IsFF`

PURPOSE Returns whether object is a finite field or not.

SYNTAX `b := IsFF(k);`

```
boolean  b
object   k
```

DESCRIPTION

SEE ALSO FF,

EXAMPLE

```
IsFF(2);
IsFF(FF(2));
```

NAME `IsFFElt`

PURPOSE Returns true iff the argument is of type ”KANT finite field elt”.

SYNTAX `b := IsFFElt(a);`

boolean	b
	a

EXAMPLE

```
kash> k := FF(2,4);
Finite field of size 2^4
kash> a := FFElt(k,9);
1
kash> IsFFElt(a);
true
```

Name `IsIdeal`

Purpose Returns true iff the argument is an ideal.

Syntax `b := IsIdeal(x);`

```
boolean  b
         x
```

Example We have for example:

```
kash> o := Order(Z, 2, 3);
Generating polynomial: x^2 - 3

kash> IsIdeal(o);
false
kash> IsIdeal(1*o);
true
```

NAME `IsInt`

PURPOSE Returns `true` iff the argument is an integer.

SYNTAX `b := IsInt(x);`

```
boolean  b
         x
```

DESCRIPTION

SEE ALSO `TYPE`, `IsIdeal`, `IsPoly`, `IsOrder`, `IsElt`, ,

EXAMPLE

```
kash> x := Poly(Zx, [1, 0]);
x
kash> IsInt(x);
false
kash> IsInt(2*3);
true
```

NAME IsLat

PURPOSE Returns true iff the argument is of type ”KANT lattice”.

SYNTAX b := IsLat(a);

boolean	b
	a

EXAMPLE

```
kash> lambda := Lat(MatId(Z,8));
Basis:
[1 0 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0]
[0 0 0 1 0 0 0 0]
[0 0 0 0 1 0 0 0]
[0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 1]

kash> IsLat(lambda);
true
```

NAME IsMat

PURPOSE Test if an object is or is convertable into a matrix.

SYNTAX `bool := IsMat(M);`

a matrix-like object	M
a boolean if conversion is possible	bool

DESCRIPTION The object M can be one of different things. Two general cases can be distinguished: in the former the matrix will have a retangular shape, in the latter it will be a one dimensional matrix. KASH does not differentiate vectors and matrices. Vectors are just matrices with only one row.

In the rectangular case M might be a "KANT matrix" or a list. If it is a list it must start from index 1 a need not contain any holes. The list may contain objects of the same dimension n but not necessarily of the same type. The entries might be matrices with one row and n columns, objects of the type "KANT matrow" with dimension n, or lists of length n.

In the vector case M might be a one dimensional "KANT matrix", a "KANT matrow", or a list starting from index 1 and without any holes. The list might contain entries of the "ground–level" entries or matrices with one column. In the latter case the columns are stacked onto each other.

In any case all entries on the "ground–level" must be convertible into objects of exactly one ring. This ring is automatically determined if the conversions are available.

EXAMPLE

```
kash> M := MatSym(Z, [[1,2,3,4], [5,6,7], [8,9], [10]] );
[ 1  2  3  4]
[ 2  5  6  7]
[ 3  6  8  9]
[ 4  7  9 10]
kash> IsMat(M);
true
kash> M := [ [1,2], [3,4] ];
[ [ 1, 2 ], [ 3, 4 ] ]
kash> IsMat(M);
true
kash> M;
[1 2]
[3 4]
```

```
kash> M[1];
[1 2]
kash> M[1][1];
1
kash> IsMat(false=IsMat(IsMat(IsMat(IsMat))));
false
```

NAME IsMat

PURPOSE Returns true iff the argument is of type ”KANT matrix”.

SYNTAX `b := IsMat(x);`

boolean	b
	x

EXAMPLE

```
kash> M := MatSym(Z, [[1,2,3,4], [5,6,7], [8,9], [10]] );
[ 1  2  3  4]
[ 2  5  6  7]
[ 3  6  8  9]
[ 4  7  9 10]
kash> IsMat(M);
true
kash> M := [ [1,2], [3,4] ];
[ [ 1, 2 ], [ 3, 4 ] ]
kash> IsMat(M);
true
kash> M;
[1 2]
[3 4]
kash> M[1];
[1 2]
kash> M[1][1];
1
kash> IsMat(false=IsMat(IsMat(IsMat(IsMat))));
false
```

NAME `IsModule`

PURPOSE Returns true iff the argument is a module.

SYNTAX `b := IsModule(m);`

```
boolean  b
object   m
```

EXAMPLE

```
kash> o := OrderMaximal(Z, 3, 3);;
kash> IsModule(o);
false
kash> IsModule(ModuleId(o, 4));
true
```

NAME IsOrder

PURPOSE Returns true iff the argument is an order.

SYNTAX `b := IsOrder(x);`

boolean	b
	x

EXAMPLE

```
kash> IsOrder(Z);
false
kash> IsOrder(Order(Z, 2, 3));
true
```

NAME `IsPoly`

PURPOSE Returns true iff the argument is a polynomial.

SYNTAX `b := IsPoly(f);`

`boolean`	`b`
	`f`

EXAMPLE

```
kash> f := Poly(Zx, [1, 2, 1, 2]);
x^3 + 2*x^2 + x + 2
kash> IsPoly(f);
true
kash> IsPoly(3);
false
```

NAME IsPrime

PURPOSE Returns `true` iff the argument is either a prime number or a prime ideal.

SYNTAX `b := IsPrime(x);`

```
boolean  b
         x
```

EXAMPLE First some rational integers

```
kash> IsPrime(1997);
true
kash> IsPrime(4);
false
```

Now some ideals in $\mathbb{Q}(\sqrt{3})$

```
kash> o := Order(Z, 2, 3);;
kash> OrderMaximal(o);;
kash> IsPrime(5*o);
true
kash> IsPrime(6*o);
false
kash> IsPrime(Factor(6*o)[1][1]);
true
```

NAME `IsQf`

PURPOSE Returns whether object is a rational function field or not.

SYNTAX `b := IsQf(k);`

boolean	b
object	k

SEE ALSO QuotientField, QfeQf,

EXAMPLE

```
kash> IsQf(1);
false
kash> IsQf(Zx);
false
kash> IsQf(QuotientField(Zx));
true
```

NAME `IsQp`

PURPOSE Checks wether k is an element of a p-adic field.

SYNTAX `b := IsQp(k);`

p-adic element	k
boolean	b

SEE ALSO Qp, QpElt, QpEltToQ, QpEltQp, QpPrec, QpExp, QpLog, QpPrime, QpSqrt, QpValuation,

EXAMPLE

```
kash> F := Qp(2);
2-adic Field mod 2^20
kash> k := QpElt(F, 23);
1 + 2 + 2^2 + 2^4
kash> b := IsQp(k);
true
```

NAME `IsQpElt`

PURPOSE Returns true if an element is a p-adic field element.

SYNTAX `b := IsQpELt(a);`

p-adic element	a
boolean	b

SEE ALSO IsQp, Qp, QpElt, QpEltToQ, QpEltQp, QpPrec, QpExp, QpLog, QpPrime, QpValuation, NAME `IsRecType`

PURPOSE Returns `true` iff the argument is an record with an **Type** enry containing the correct string.

SYNTAX `flag := IsRecType(a, type);`

`boolean`	`flag`
`anything`	`a`
`string`	`type`

DESCRIPTION Useful in User-supplied functions, see the Abelian group package as an example.

EXAMPLE

```
kash> a := rec(Type := "Test", data := 1);
rec(
  Type := "Test",
  data := 1 )
kash> IsRecType(a, "Test");
true
kash> IsRecType(a, "Paul");
false
kash> IsRecType(2, "two");
false
```

NAME `IsThue`

PURPOSE Returns `true` iff the argument is a `Thue` object.

SYNTAX `b := IsThue(x);`

`boolean`	`b`
	`x`

EXAMPLE

```
kash> t := Thue([1,2,2,2,2]);
X^4 + 2 X^3 Y + 2 X^2 Y^2 + 2 X Y^3 + 2 Y^4
kash> IsThue(t);
true
```

NAME JBessel

PURPOSE Returns the value of the J-Bessel function.

SYNTAX `z := JBessel(nu,x);`

integer	nu
real	x
real	z

DESCRIPTION Given a real `x` and an integer `nu` the function returns $K_n u($`x`$)$.

EXAMPLE

```
kash> z := JBessel(2,1.5);
0.2320876721442147272377765399247074797469350619464579
```

NAME `JacobiSymbol`

PURPOSE Returns the Jacobi symbol of two integers.

SYNTAX `y := JacobiSymbol(n, m);`

```
integer  n
integer  m
```

DESCRIPTION

EXAMPLE

```
kash> JacobiSymbol(1000, 17);
-1
```

Name `KASHLEVEL`

Purpose Reads or sets different defaults for KASH.

Syntax
```
x := KASHLEVEL(s);
x := KASHLEVEL(s,level);
```

integer	x
string	s
integer	level

Description The Order command creates a new order and checks if the polynomial which creates the order is irreducible. Furthermore it calculates the real basis of the new order. It is possible to change these defaults. The real basis is only calculated if KASH_REAL_BASIS_CALC is not equal 0. If KASH_ORDER_POLY_FACTOR is 0 the test of irreducibility is omitted. In the case KASH_ORDER_POLY_FACTOR equals 1 kash tests the polynomial only in the case that the polynomial is given over Zx.

Example

```
kash> KASHLEVEL("KASH_ORDER_POLY_FACTOR",2);
2
kash> O:=Order(x^2-1);
Error, polynomial is not irreducible
kash> KASHLEVEL("KASH_ORDER_POLY_FACTOR",0);
0
kash> O:=Order(x^2-1);
Generating polynomial: x^2 - 1
```

Name `KBessel`

Purpose Returns the value of the K-Bessel function.

Syntax `z := KBessel(s,x);`

complex	s
real	x
complex	z

Description Given a complex `s` and a real `x` the function returns $K_s(\texttt{x})$.

Example

```
kash> s := Comp(0.5, 1);
0.5 + 1*i
kash> z := KBessel(s,1);
0.29882498908739134807782547428718795801761085731444223 + 0.1189446943013590937\
37231871224012813256270260507153*i
```

LOFILES

NAME	`LOFILES`
PURPOSE	Lists all open files, for debugging purposes only.
SYNTAX	`LOFILES();`
DESCRIPTION	

NAME Lat

PURPOSE Creates a (relative) lattice.

SYNTAX

```
Lambda := Lat(M,["basis"|"gram"]);
Lambda := Lat(o [,"mink"|"unit"]);
Lambda := Lat(a);
Lambda := Lat(Delta,L);
Lambda := Lat(Delta,M [,"trans"|"basis"|"gram"]);
Lambda := Lat(Module);
```

lattice	Lambda
matrix	M
order	o
ideal	a
list	L
lattice	Delta

DESCRIPTION At the moment there are nine different ways to define a lattice in KASH.

`Lat(M)` (or `Lat(M,"basis")`)
Let $M \in \mathbb{R}^{n\times k}$ be a matrix of rank k. Denote the columns of M by $b_1,\dots,b_k$. The `Lat` function returns the lattice $\Lambda = \mathbb{Z}\,b_1 + \cdots + \mathbb{Z}\,b_k$..

If M is a matrix over an absolute extension o of $\mathbb{Q}$, a relative lattice will be created. This will be an o-module $\Lambda \subset (\mathbb{R}\otimes o)^n$ together with the canonical quadratic form on $(\mathbb{R}\otimes o)^n$.

`Lat(M,"gram")`
Let $M \in \mathbb{R}^{k\times k}$ be a positive definite matrix. The `Lat` function returns the lattice Λ corresponding to the positive definite quadratic form

$$(x_k,\dots,x_k)\cdot M \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}.$$

`Lat(o)` (or `Lat(o,"mink")`)
Let $\omega_1,\dots,\omega_n$ be the $\mathbb{Z}$–basis of the order $\mathfrak{o}$. We denote by $\psi : \mathfrak{o} \mapsto \mathbb{R}^n$ the Minkowski map (see `EltMinkowski`). The `Lat` function returns the lattice

$$\Lambda = \mathbb{Z}\,\psi(\omega_1) + \cdots + \mathbb{Z}\,\psi(\omega_w).$$

Λ has discriminant $\sqrt{|d|}$ where d is the discriminant of the order $\mathfrak{o}$.

For a relative extension $O/o/\mathbb{Q}$ given via a pseudo-basis (a module, cf. `Module`) $O = \sum_{i=1}^{n} \alpha_i \mathfrak{a}_i$ consider the canonical embedding

$$\iota : O \to (\mathbb{R} \otimes O) \to (\mathbb{R} \otimes o)^n,$$

where the first part of the embedding corresponds to the canonical embedding into $\mathbb{C}$. The relative lattice Λ created by this function is the image of O under this embedding together with the canonical quadratic form on $(\mathbb{R} \otimes o)^n$.

`Lat(o,"unit")`
Let $\varepsilon_1, \ldots, \varepsilon_r$ be the current system of independent units of the order $\mathfrak{o}$. The `Lat` function returns the lattice

$$\Lambda = \mathbb{Z} \begin{pmatrix} \log|\varepsilon_1^{(1)}| \\ \vdots \\ \log|\varepsilon_1^{(r+1)}| \end{pmatrix} + \cdots + \mathbb{Z} \begin{pmatrix} \log|\varepsilon_r^{(1)}| \\ \vdots \\ \log|\varepsilon_r^{(r+1)}| \end{pmatrix}.$$

`Lat(a)`
Let $\omega_1, \ldots, \omega_n$ be the $\mathbb{Z}$–basis of the ideal $\mathfrak{a}$. We denote by $\psi : \mathfrak{o} \mapsto \mathbb{R}^n$ the Minkowski map (see `EltMinkowski`). The `Lat` function returns the lattice

$$\Lambda = \mathbb{Z}\,\psi(\omega_1) + \cdots + \mathbb{Z}\,\psi(\omega_w).$$

For an ideal in a relative extension, the relative lattice is similar to the lattice corresponding to the order, i.e. the image under the canonical embedding from O to $(\mathbb{R} \otimes o)^n$.

`Lat(Delta,L)`
Let Δ be a lattice with basis $b_1, \ldots, b_k \in \mathbb{R}^n$ and let `L` be a list of k positive real numbers $\lambda_1, \ldots, \lambda_k$. The `Lat` function returns the lattice

$$\Lambda = \mathbb{Z} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \cdot b_1 + \cdots + \mathbb{Z} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \cdot b_k.$$

Notice that elements can be moved between the lattices Δ and Λ.

`Lat(Delta,M)` (or `Lat(Delta,M,"trans")`)
Let Δ be a lattice with basis $b_1, \ldots, b_k \in \mathbb{R}^n$ and let $M \in \mathbb{Z}^{k \times k}$ a unimodular matrix. The `Lat` function returns the lattice Λ with basis $(b_1, \ldots, b_k) \cdot M$. Notice that elements can be moved between the lattices Δ and Λ.

`Lat(Delta,M,"basis")`
This calling seqeunce creates the same lattice as the calling sequence `Lat(M)`. However, this call enables moving elements between the lattices Δ and Λ.

`Lat(Delta,M,"gram")`
This calling seqeunce creates the same lattice as the calling sequence `Lat(M,"gram")`. However, this call enables moving elements between the lattices Δ and Λ.

`Lat(module)`
Basically the same as `Lat(mat)` eccept, that special care is taken for the coeff. ideals of the module.

EXAMPLE Creating the lattice

$$\Lambda = \mathbb{Z}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 2 \\ 1 \end{pmatrix} :$$

```
kash> B := Mat(Z,[[1,2],[2,1]]);
[1 2]
[2 1]
kash> Lambda := Lat(B);
Basis:
[1 2]
[2 1]
```

NAME `LatBasis`

PURPOSE Return the basis of a lattice.

SYNTAX `M := LatBasis(Lambda);`

matrix	M
lattice	Lambda

DESCRIPTION If the basis is known it is returned as a matrix. Each column of this matrix represents a basis vector. In case of a rel. lattice, a module is returned.

EXAMPLE

```
kash> o := Order(Z, 2, 3);;
kash> Lambda := Lat(o);;
kash> M := LatBasis(Lambda);
[1 1.73205080756887729352744634150587236694280525381]
[1 -1.73205080756887729352744634150587236694280525381]
kash> L := OrderBasis(o);;
kash> List(L, x->EltToList(EltMinkowski(x)));
[ [ 59/3277, 296/16385 ], [ 297/16385, 298/16385 ] ]
```

NAME LatCholesky

PURPOSE Returns a Cholesky type decompostion of the gram matrix of a lattice.

SYNTAX `q := LatCholesky(Lambda);`

matrix	q
lattice	Lambda

DESCRIPTION Let Λ be a lattice of rank n and let G denote its Gram matrix. Then $Q(x) = x^t \cdot G \cdot x$ is a positive definite quadratic form.

The matrix $q = (q_{ij}) \in \mathbb{R}^{n \times n}$ which is returned by the `LatCholesky` function satisfies

$$Q(x) = \sum_{i=1}^{n} q_{ii} \left(x_i + \sum_{j=i+1}^{n} q_{ij} x_j \right)^2 .$$

Additionally $q_{ij} = 0$ for $1 < i \leq n, 1 \leq j < i$. This matrix is used in the Fincke-Pohst enumeration algorithm.

EXAMPLE

```
kash> o := Order(Poly(Zx, [1,1,1,-1]));;
kash> Lambda := Lat(o);
Basis:
[1 0.5436890126920763615708559718017479865252032976509158 0.295597742522084770\
98099659285153861389897544844653 42]
[1.414213562373095048801688724209698078569671875377 -1.091552968917733623999 05\
052688615228256976998159 7 -0.9161259494273487353043121929335785651570335323191\
771]
[0 1.5770496588488276002409312104705702884778662197 78 -2.4344742308347225254 86\
9103325241477096701925799 61]

kash> LatCholesky(Lambda);
[3 -0.3333333333333333333333333333333333333333333333336667 -0.3333333333333333\
3333333333333333333333333333333328219]
[0 3.640837919617073702751368388824778481413999607308 -0.82725650255764032138 9\
61264638048091776315306075213 63]
[0 0 4.028376706263605945611231105883947510766559979681]
```

NAME LatDisc

PURPOSE Computes the discriminant of a lattice.

SYNTAX `disc := LatDisc(Lambda);`

real	disc
lattice	Lambda

EXAMPLE

```
kash> Lambda := Lat(5*MatId(Z, 5));
Basis:
[5 0 0 0 0]
[0 5 0 0 0]
[0 0 5 0 0]
[0 0 0 5 0]
[0 0 0 0 5]

kash> LatDisc(Lambda);
3125
```

Name `LatElt`

Purpose Creates a lattice element.

Syntax

```
a := LatElt(Lambda, L);
a := LatElt(Lambda, v);
```

`lattice elt`	`a`
`lattice`	`Lambda`
`list of reals`	`L`
`vector over reals`	`v`

Description Let Λ be a lattice with basis $b_1, \ldots, b_k \in \mathbb{R}^n$.

`LatElt(Lambda, L)` Let L be a list containing the entries $a_1, \ldots, a_k \in \mathbb{R}$. The `LatElt` function returns the lattice element $a_1 b_1 + \cdots + a_k b_k$.

`LatElt(Lambda,v)` Let $v \in \mathbb{R}^n \cap [\Lambda]$. After computing $v_1, \ldots, v_k \in \mathbb{R}$ with $v = v_1 b_1 + \cdots + v_k b_k$ the `LatElt` function returns the lattice element $v_1 b_1 + \cdots + v_k b_k$.

Example

```
kash> o := Order(Poly(Zx, [1, 13, 13, 13, 169]));;
kash> Lambda1 := Lat(o);;
kash> Lambda2 := LatLLL(Lambda1);
Basis:
[1 -8.898972742509405137989481976338606158885263693444 10.69685215937833774951\
8381293192142719370918398672 -8.3929181756421331436558159231442716924315445312\
98]
[1 0.134261593289898000224869399556756476905813297935 -23.31066585811776801515\
6201504078952000030073676501 79.4175118311570330970156479070963376256057136484\
54]
[1.414213562373095048801688724209698078569671875377 5.490479907567903997165793\
011467947079779898762665 8.919313203002418011662432704182923884009727755229 -5\
0.221971804833625921569542397881159401643756122457]
[0 2.890521001958810373589806560427662461469013267521 36.896665660756767606327\
137523226704052168032822311 64.55872298086831805701859966901857310820073028551\
9]

kash> a := LatElt(Lambda1, [-13,-3,-12, -1]);
[ -13 -3 -12 -1 ]

kash> EltMove(a, Lambda2);
```

```
[ 0 0 0 -1 ]

kash> b := LatElt(Lambda2, Mat(R, [[-8.3929181756421277, 79.4175118311570333, -50.2219718048336188, 64.558722980868315]]));
[3.8982415889784072959401304801339719125234441844412e-15 -1.250940443446586219\
619498224288088927319302801639e-16 5.27881561428295035666070314675828437059321\
869618135e-18 0.9999999999999999552314203468322215678323061495939787]

kash> EltMove(b, Lambda1);
[13.000000000000002940967920453250310436100992792267 2.999999999999999979866718\
8452950496664814830634314 11.9999999999999999468055859776269609170648376941886 \
0.9999999999999999552314203468322215678323061495939787]
```

NAME `LatEltLength`

PURPOSE Computes the length of a lattice element.

SYNTAX `r := LatEltLength(a);`

real	r
lattice elt	a

EXAMPLE

```
kash> o := Order(Poly(Zx, [1, 13, 13, 13, 169]));;
kash> Lambda := Lat(o);;
kash> a := LatElt(Lambda, [1, 2, 3, 4]);
[ 1 2 3 4 ]

kash> LatEltLength(a);
40167001.09020786180114727023964280465933321626335782
kash> elt := Elt(o, [1,2,3,4]);;
kash> EltT2(elt);
40167001.09020786180114727023964280465933321626335817
```

NAME LatEltToList

PURPOSE missing shortdoc

SYNTAX `L := LatEltToList(a);`

list	L
lattice element	a

DESCRIPTION Let $b_1, \dots, b_k$ be the basis of a certain lattice. Given a lattice element $a_1 b_1 + \dots + a_k b_k$ the `LatEltToList` function returns a list containing $a_1, \dots, a_k$.

EXAMPLE

```
kash> Lambda := Lat(MatId(Z, 4));;
kash> a := LatElt(Lambda, [1, 2, 3, 4]);
[ 1 2 3 4 ]

kash> LatEltToList(a);
[ 1, 2, 3, 4 ]
```

NAME `LatEltVec`

PURPOSE Computes the corresponding point of the $\mathbb{R}^n$.

SYNTAX `v := LatEltVec(le);`

`vector`	`v`
`lattice element`	`le`

DESCRIPTION

EXAMPLE

```
kash> o := Order(Poly(Zx, [1, 13, 13, 13, 169]));;
kash> l := Lat(o);;
kash> a := LatElt(l, [1, 2, 3, 4]);
[ 1 2 3 4 ]

kash> LatEltVec(a);
[-6336.931799991655223227849180410447496200770397385772]
[  -74.233116314471776820846681903034918196904380465164]
[   -69.179851117891019081021249377607185019687717 90037]
[   -0.21237686055823886559060806171109610228712111 4234]
```

NAME LatEnum

PURPOSE Enumeration of lattice points.

SYNTAX `ok := LatEnum(Lambda);`

boolean	ok
lattice	Lambda

DESCRIPTION Please refer to "KASH – A User's Guide" for a detailed description of the `LatEnum` function.

EXAMPLE Let ρ be a zero of the polynomial $x^3 + x^2 + x - 1 \in \mathbb{Z}[x]$. We will enumerate several elements $\alpha \in \mathbb{Z}[\rho]$ with small T_2 norms. Those elements are likely to have a small norm. By assigning `4.2` to the lower bound we avoid enumeration of lattice elements which correspond to torsion units.

```
kash> o := Order(Poly(Zx, [1,1,1,-1]));;
kash> Lambda := Lat(o);;
kash> LatEnumLowerBound(Lambda, 4.2);
4.2
kash> LatEnumUpperBound(Lambda, 2*EltT2(OrderBasis(o)[3]));
13.706659122393255422774258033027130293898341365588
kash> while LatEnum(Lambda) do
> elt := EltMove(LatEnumElt(Lambda), o);
> Print(elt, "\t", EltT2(elt), "\t", Norm(elt), "\n");
> od;
[-2, -2, -1] 12.035720531284966795311115387919719602545072476558 -1
[-1, -2, -1] 9.035720531284966795311115387919719602545072476558 -4
[0, -2, -1] 12.035720531284966795311115387919719602545072476584 -7
[-2, -1, -1] 9.470353793290390217264420480058530559999788639047 -2
[-1, -1, -1] 4.470353793290390217264420480058530559999788639048 -1
[0, -1, -1] 5.470353793290390217264420480058530559999788639051 -2
[1, -1, -1] 12.47035379329039021726442048005853055999978863905 1
[-1, 0, -1] 7.853329561196627711387129016513565146949170682804 -4
[0, 0, -1] 6.853329561196627711387129016513565146949170682794 -1
[1, 0, -1] 11.853329561196627711387129016513565146949170682793 4
[-2, -1, 0] 11.974171252950407036084701722158111814747332940641 -7
[-1, -1, 0] 4.974171252950407036084701722158111814747332940639 -2
[1, -1, 0] 8.974171252950407036084701722158111814747332940643 2
-2 12 -8
```

NAME `LatEnumElt`

PURPOSE Returns the current element of an enumeration process.

SYNTAX `a := LatEnumElt(Lambda);`

```
lattice elt  a
lattice      Lambda
```

EXAMPLE See `LatEnum` for an example.

NAME `LatEnumLowerBound`

PURPOSE Reads or sets the lower bound for enumeration.

SYNTAX `lbound := LatEnumLowerBound(Lambda [,lbound]);`

`real`	`lbound`
`lattice`	`Lambda`

EXAMPLE See `LatEnum` for an example.

NAME `LatEnumPrec`

PURPOSE Sets the precision used by the enumeration function `LatEnum`.

SYNTAX

```
s := LatEnumPrec(Lambda);
s := LatEnumPrec(Lambda,"short");
s := LatEnumPrec(Lambda,"long");
```

list	s
lattice	Lambda

DESCRIPTION If precision problems trouble the validity of the results of `LatEnum` a higher precision is necessary. Only 2 levels of precision are supported: 'short' and 'long'.

NAME LatEnumRefVec

PURPOSE Reads or sets a reference vector for enumeration.

SYNTAX `v := LatEnumRefVec(Lambda [,v]);`

vector	v
lattice	Lambda

DESCRIPTION A reference vector of a lattice $l \subset \mathbb{R}^n$ is a vector $v \in \mathbb{R} \otimes_{\mathbb{Z}} l$. If the reference vector is set, we will enumerate points in $v - l$. Typically it is used to find points with minimal distance to the reference vector.

EXAMPLE

```
kash> Lambda := Lat(MatId(Z, 2));;
kash> LatEnumRefVec(Lambda, LatElt(Lambda, [1.7, 3.4]));
[1.7 3.4]

kash> LatShortestElt(Lambda);
[ [ 2 3 ]
    ]
```

NAME `LatEnumReset`

PURPOSE Resets the enumeration environment of a lattice.

SYNTAX `LatEnumReset(Lambda);`

```
lattice  Lambda
```

SEE ALSO `LatEnum`,

Name LatEnumUpperBound

Purpose Reads or sets the upper bound for enumeration.

Syntax `ubound := LatEnumUpperBound(Lambda [,ubound]);`

real	ubound
lattice	Lambda

Example See `LatEnum` for an example.

NAME `LatFinckeReduce`

PURPOSE Performs a reduction on a lattice specially suited for enumeration.

SYNTAX `Lambda2 := LatFinckeReduce(Lambda1);`

```
lattice  Lambda2
lattice  Lambda1
```

EXAMPLE

```
kash> o := Order(Poly(Zx, [1, 13, 13, 13, 169]));;
kash> Lambda1 := Lat(o);
Basis:
[1 -11.898972742509405137989481976338606158885263693444 141.585552326981794267\
402683032916810467108819026556 -1684.7226278718954489385195663891301788210815 8\
1769638]
[1 -2.8657384067101019997751306004432435230941867020 65 8.212456615693353982370\
235100796726754005980046214 -23.534752337032908692101781501134652853183486 7999\
19]
[1.4142135623730950488016887242096980785696718753 77 1.247839220448618850760726\
838838852844070883136534 -4.8069182219323893467055625230444574007699867466 45 -\
14.667247113841045947806926052590304146022298952 22]
[0 2.8905210019588103735898065604276624614690132675 21 5.100934639209853496839 2\
653585224169760088868 7958 -5.3240556955263550258220043145334179883129520720 04]

kash> Lambda2 := LatFinckeReduce(Lambda1);
Basis:
[1 -8.898972742509405137989481976338606158885263693444 10.696852159378337749 51\
838129319214271937091839 8672 -8.3929181756421331436558159231442716924315445312\
98]
[1 0.1342615932898980002248693995567564769058132979 35 -23.310665858117768015 15\
620150407895200003007367 6501 79.4175118311570330970156479070963376256057136484\
54]
[1.4142135623730950488016887242096980785696718753 77 5.4904799075679039971657 93\
011467947079779898762665 8.9193132030024180116624327041829238840097277552 29 -5\
0.22197180483362592156954239788115940164375612 2457]
[0 2.8905210019588103735898065604276624614690132675 21 36.8966656607567676063 27\
137523226704052168032822311 64.558722980868318057018599669018573108200730285 51\
9]
```

NAME `LatGram`

PURPOSE Computes the Gram matrix of a lattice.

SYNTAX `M := LatGram(Lambda);`

matrix	M
lattice	Lambda

EXAMPLE

```
kash> o := Order(Z, 2, 3);;
kash> Lambda := Lat(o);;
kash> LatGram(Lambda);
[2 0]
[0 6]
```

NAME `LatLLL`

PURPOSE Performs a LLL-reduction on a lattice.

SYNTAX `Lambda2 := LatLLL(Lambda1);`

```
lattice  Lambda2
lattice  Lambda1
```

EXAMPLE

```
kash> o := Order(Poly(Zx, [1, 13, 13, 13, 169]));;
kash> Lambda1 := Lat(o);
Basis:
[1 -11.898972742509405137989481976338606158885263693444 141.585552326981794267\
402683032916810467108819026556 -1684.7226278718954489385195663891301788210815\8
1769638]
[1 -2.865738406710101999775130600443243523094186702065 8.2124566156933539823370\
235100796726754005980046214 -23.534752337032908692101781501134652853183486799\9
19]
[1.414213562373095048801688724209698078569671875377 1.24783922044861885076072\6
8388388528440708831365 34 -4.806918221932389346705562523044457400769986746645 -\
14.667247113841045947806926052590304146022298952 22]
[0 2.890521001958810373589806560427662461469013267521 5.10093463920985349683\92
6535852241697600888687958 -5.324055695526355025822004314533417988312952072004]

kash> Lambda2 := LatLLL(Lambda1);
Basis:
[1 -8.898972742509405137989481976338606158885263693444 10.696852159378337749\51
838129319214271937091839867 2 -8.392918175642133143655815923144271692431544531\2
98]
[1 0.134261593289898000224869399556756476905813297935 -23.310665858117768015\15
620150407895200003007367650 1 79.417511831157033097015647907096337625605713648\4
54]
[1.414213562373095048801688724209698078569671875377 5.49047990756790399716579\3
011467947079779898762665 8.919313203002418011662432704182923884009727755229 -5\
0.221971804833625921569542397881159401643756122457]
[0 2.890521001958810373589806560427662461469013267521 36.89666566075676760632\7
137523226704052168032822311 64.558722980868318057018599669018573108200730285\51
9]
```

NAME `LatShortestElt`

PURPOSE Finds shortest elements in a lattice.

SYNTAX
```
L := LatShortestElt(Lambda);
L := LatShortestElt(Lambda,m);
L := LatShortestElt(Lambda,"all");
```

list of lattice elements	L
lattice	Lambda
integer	m

DESCRIPTION Please refer to "KASH – A User's Guide" for a general description of the `LatShortestElt` function.

If `m` is specified, the `LatShortestElt` function tries to find `m` shortest elements. If the second argument is `"all"`, the `LatShortestElt` function returns all shortest vectors.

EXAMPLE See `Lat` for an example.

NAME LatSuccMins

PURPOSE Computes the successive minima of a lattice.

SYNTAX `L := LatSuccMins(Lambda);`

list	L
lattice	Lambda

DESCRIPTION The `LatSuccMins` function computes the successive minima of a lattice and a set of corresponding lattice elements.

EXAMPLE

```
kash> o := Order(Poly(Zx, [1, 13, 13, 13, 169]));;
kash> Lambda := Lat(o);;
kash> LatSuccMins(Lambda);
[ [ -1 0 0 0 ]
    , 4, [ -3 -1 0 0 ]
    , 117.710223325529930087190601154781784396363864536319, [ 0 -11 -1 0 ]
    , 2098.727873763335276613028969537316945979798627539881, [ -13 -3 -12 -1 ]
    , 13067.65742584099018040789043545504811128068694483 ]
```

Name `Lcm`

Purpose Returns least common multiple of list of arguments.

Syntax

```
lcm := Lcm(L);
lcm := Lcm(a, b);
```

`integer or polynomial or ideal`	`lcm`
`list of integers of polynomials or ideals`	`L`
`integer or polynomial or ideal`	`a`
`integer or polynomial or ideal`	`b`

Description Returns the least common multiple of list of arguments or simply the least common multiple of two arguments. Supported are integers, ideals and polynomials over S where S is a field or $\mathbb{Z}$.

See Also `IntLcm`, `IdealLcm`,

Example Lcm of integers:

```
kash> Lcm([18, 12 ,4]);
36
kash> Lcm(18, 12);
36
kash> Lcm([18, 12]);
36
```

NAME Li

PURPOSE Returns the value of the logarithmic integral.

SYNTAX `a := Li(b);`

real	a
real or integer	b

DESCRIPTION This function computes an approximation of the logarithmic integral. It gives a good idea of the number of prime ideals of norm less than or equal to the argument.

EXAMPLE

```
kash> Li(2);
1.045163780117492784844588889194613136522615578151
kash> Li(3);
2.163588594667191972876922367347721366542116212451
kash> Li(7.5);
5.009500143078375327886075408809948575232609368221
kash> Li(10000);
1246.137215899388459692771107529059792486534653513971
kash> l := []; p := 2;
[  ]
2
kash> while p <= 10000 do Add(l, p); p := NextPrime(p); od;
kash> Length(l);
1229
kash> o := OrderMaximal(x^2+164);;
kash> l := []; p := 2;
[  ]
2
kash> while p <= 10000 do
> pl := Filtered(Factor(p*o), x->IdealNorm(x[1]) <= 10000);
> Append(l, pl);
> p := NextPrime(p);
> od;
kash> Length(l);
1209
```

NAME `ListApplyListAdd`

PURPOSE Computes a linear combination.

SYNTAX `a := ListApplyListAdd(L, S);`

`element`	`a`	
`lists`	`L, S`	of elements

DESCRIPTION Given lists $L := [a_1, \ldots, a_n]$ and $S := [\lambda_1, \ldots, \lambda_n]$ this function returns $a := \sum_{i=1}^{n} \lambda_i a_i$.

SEE ALSO ListApplyMatAdd,

EXAMPLE

```
kash> L := [1, 2, 3];
[ 1, 2, 3 ]
kash> S := [1, 3, 4];
[ 1, 3, 4 ]
kash> ListApplyListAdd(L, S);
19
```

NAME ListApplyMatAdd

PURPOSE Computes linear combinations.

SYNTAX `S := ListApplyMatAdd(L, M);`

lists	S, L	of elements
matrix	M	for linear combinations

DESCRIPTION Given a list $L := [a_1, \ldots, a_n]$ and an $n \times m$ matrix $M = (\lambda_{i,j})_{i,j}$ this function returns a list $S := \left[\sum_{i=1}^{n} \lambda_{i,j} a_i\right]_j$.

SEE ALSO ListApplyListAdd,

EXAMPLE

```
kash> M := MatTrans(Z, [ [ 1, 0 ], [1, 1], [0, 2] ]);
[1 1 0]
[0 1 2]
kash> L := [ 1, 2 ];
[ 1, 2 ]
kash> ListApplyMatAdd(L, M);
[ 1, 3, 4 ]
```

NAME ListSplit

PURPOSE Splits a list into equal parts. All parts but the last have equal length as specified by the parameter $n \geq 1$.

SYNTAX `S := ListSplit(L, n);`

list	S	of lists of length $\leq n$
list	L	to be split
integer	n	block length

SEE ALSO IntDivisors,

EXAMPLE

```
kash> L := [1, 2, 3, 4, 5];
[ 1, 2, 3, 4, 5 ]
kash> ListSplit(L, 1);
[ [ 1 ], [ 2 ], [ 3 ], [ 4 ], [ 5 ] ]
kash> ListSplit(L, 2);
[ [ 1, 2 ], [ 3, 4 ], [ 5 ] ]
kash> ListSplit(L, 101);
[ [ 1, 2, 3, 4, 5 ] ]
```

NAME Log

PURPOSE Returns the principal value of the natural logarithm.

SYNTAX

```
y := Log(x);
y := Log(b,x);
```

complex or real	y
complex or real or rational or integer	x
complex or real or rational or integer	b

DESCRIPTION Given $\mathtt{x} > 0$ in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ the function returns the natural logarithm of x. For $\mathtt{x} = |\mathtt{x}|e^{i\phi} \in \mathbb{C} \setminus \mathbb{R}^{\geq 0}$, $\phi \in (-\pi, \pi]$, the function returns $\mathtt{y} = \log|\mathtt{x}| + i \arg \mathtt{x}$. The computation is done in the current precision of the real (complex) field. If two arguments are given the first is used as the base of the logarithm.

SEE ALSO Exp,

EXAMPLE

```
kash> Log(2);
0.6931471805599453094172321214581765680755001343602553
kash> Log(-2);
0.6931471805599453094172321214581765680755001343605 + 3.1415926535897932384626\
43383279502884197169399375*i
kash> i := Comp(0, 1);
1*i
kash> Log(i);
1.570796326794896619231321691639751442098584699688*i
```

NAME Mat

PURPOSE Creates a matrix.

SYNTAX `M := Mat([S,] L);`

matrix	M
ring	S
list	L

DESCRIPTION Creates the matrix `M` over the coefficient ring `S`. `L` is a list of row vectors. If only a list `L` is given the function tries to extract a common coeffient ring from the entries.

SEE ALSO IsMat,

EXAMPLE Create the integer matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

```
kash> Mat(Z,[[1,2],[3,4]]);
[1 2]
[3 4]
```

NAME MatCharPoly

PURPOSE Returns the characteristic polynomial of a matrix.

SYNTAX `f := MatCharPoly(M);`

matrix	M
Polynomial	f

SEE ALSO MatMinPoly,

EXAMPLE Compute the characteristic polynomial of

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{Z}^{4\times 4}.$$

```
kash> M := Mat(Z,[[0,1,0,-1],[1,0,1,0],[0,1,0,1],[-1,0,1,0]]);
[ 0  1  0 -1]
[ 1  0  1  0]
[ 0  1  0  1]
[-1  0  1  0]
kash> MatCharPoly(M);
x^4 - 4*x^2 + 4
```

NAME `MatCoef`

PURPOSE Returns the coefficient ring of a matrix.

SYNTAX
```
S := MatCoef(M);
```

```
ring    S
matrix  M
```

SEE ALSO Mat,

EXAMPLE

NAME MatCols

PURPOSE Returns the number of columns of a matrix.

SYNTAX `n := MatCols(M);`

```
integer  n
matrix   M
```

SEE ALSO MatRows,

EXAMPLE

```
kash> M := Mat(Z,[[67,4,98,1],[-1,5,3,4],[6,-1,7,8]]);
[67  4 98  1]
[-1  5  3  4]
[ 6 -1  7  8]
kash> MatCols(M);
4
```

NAME MatDet

PURPOSE Computes the determinant of a square matrix.

SYNTAX

```
d := MatDet(M);
d := MatDet(M, bound);
```

element of the ring the matrix entries come from	d
matrix	M

DESCRIPTION If `bound` is given and M is over an absolute numberfield we use an modular algorithm. If `bound`$<=0$ we will compute a suitable `bound`.

EXAMPLE Compute the determinant of the integer matrix $\begin{pmatrix} -1 & 5 \\ 6 & -1 \end{pmatrix}$.

```
kash> M := Mat(Z,[[-1,5],[6,-1]]);
[-1  5]
[ 6 -1]
kash> MatDet(M);
-29
```

NAME `MatDiag`

PURPOSE missing shortdoc

SYNTAX `M := MatDiag(S, L);`

`matrix`	`M`	
`ring`	`S`	
`list`	`L`	diagonal entries

DESCRIPTION Creates the diagonal matrix `M` over the coefficient ring `S`. `L` is a list containing the diagonal entries.

EXAMPLE Create the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \in \mathbb{Z}^{5\times 5}.$$

```
kash> MatDiag(Z, [1, 2, 3, 4, 5]);
[1 0 0 0 0]
[0 2 0 0 0]
[0 0 3 0 0]
[0 0 0 4 0]
[0 0 0 0 5]
```

NAME `MatEchelon`

PURPOSE Returns the rank and an upper row echelon form of a matrix together with a transformation matrix. If the echelon form cannot be computed over the ring, it is computed over its field of fractions.

SYNTAX `L := MatEchelon(M);`

`list`	`L`
`matrix`	`M`

NAME MatElt

PURPOSE Returns or modifies a matrix entry or row.

SYNTAX
```
a := MatElt(M, row, col);
a := M[row][col];
M[row][col] := a;
r := M[row];
M[row] := L;
```

element of the ring the matrix entries come from	a
matrix	M
small integer	row
small integer	col
an object of the type "KANT matrow"	r
either a list, a "KANT matrix", or a "KANT matrow"	L

DESCRIPTION The first two syntax variants are equivalent and return the (`row`, `col`)–th entry of the matrix `M`.

The third variant sets a single entry of the matrix. The object must be convertible to an element of the ring over which the matrix is defined.
The fourth syntax retrieves a matrix row. This is an object of the type "KANT matrow". This object can be converted to a proper matrix (with only one row) with a call of `IsMat`. It can also be used as a row for another matrix.
The fifth syntax sets a row of the matrix. For this purpose the matrix is converted into a list of rows (of the type "KANT matrix"). Then the `row`-th entry of this list is either replaced by of set to the right hand side of the assignment. Now `M` is still a list a should be converted to a matrix with a call of `IsMat`. `IsMat` also determines a ring all entries of the matrix belong to. So by setting a new row or modifying an existing one the matrix might change its ring.

EXAMPLE Get certain entries of the matrix

$$\begin{pmatrix} 3 & 7 & 10 & 2 \\ 90 & 101 & 87 & 2 \\ 12 & 34 & 2 & 0 \\ 20 & 25 & 40 & 1 \end{pmatrix}$$

and modify them.

```
kash> M := Mat(Z,[[3,7,10,2],[90,101,87,2],[12,34,2,0],[20,25,40,1]]);
[  3   7  10   2]
[ 90 101  87   2]
```

```
[ 12  34   2   0]
[ 20  25  40   1]
kash> MatElt(M,1,1);
3
kash> M[2][2];
101
kash> r:=M[4];
[20 25 40  1]
kash> TYPE(r);
"KANT matrow"
kash> IsMat(r);
true
kash> r;
[20 25 40  1]
kash> TYPE(r);
"KANT matrix"
kash> M[4][4]:=22;;M;
[  3   7  10   2]
[ 90 101  87   2]
[ 12  34   2   0]
[ 20  25  40  22]
kash> M[3]:=[-1,2,-1,4];;M;
[ [ 3  7 10  2], [ 90 101  87   2], [ -1, 2, -1, 4 ], [20 25 40 22] ]
kash> IsMat(M);
true
kash> M;
[  3   7  10   2]
[ 90 101  87   2]
[ -1   2  -1   4]
[ 20  25  40  22]
```

NAME MatHermiteColLower

PURPOSE missing shortdoc

SYNTAX `H := MatHermiteColLower(M);`

```
matrix  H
matrix  M
```

DESCRIPTION Computes the lower column Hermite normal form of the integer matrix `M`.

EXAMPLE Compute the lower column Hermite normal form of

$$\begin{pmatrix} 4 & 6 & 2 \\ 3 & 9 & 12 \end{pmatrix}.$$

```
kash> M := Mat(Z,[[4,6,2],[3,9,12]]);
[ 4  6  2]
[ 3  9 12]
kash> MatHermiteColLower(M);
[2 0 0]
[0 3 0]
```

NAME `MatHermiteColLowerTrans`

PURPOSE Computes the lower column Hermite normal form together with a transformation matrix.

SYNTAX `L := MatHermiteColLowerTrans(M);`

`list`	`L`
`matrix`	`M`

DESCRIPTION The `MatHermiteColLowerTrans` routine computes the lower column Hermite normal form `H` of the integer matrix `M`. Additionally it computes the rank of `M` and a transformation matrix `T` such that `H` = `M` ∗ `T`. The result is a list `L` which contains the rank, the Hermite normal form `H` and the transformation matrix `T`.

EXAMPLE Compute the lower column Hermite normal form of

$$\begin{pmatrix} 3 & 7 & 10 \\ 20 & 25 & 40 \end{pmatrix}.$$

```
kash> M := Mat(Z,[[3,7,10],[20,25,40]]);
[ 3  7 10]
[20 25 40]
kash> L := MatHermiteColLowerTrans(M);;
kash> L[1];
2
kash> L[2];
[1 0 0]
[0 5 0]
kash> L[3];
[  1   1   6]
[  4   1  16]
[ -3  -1 -13]
kash> M*L[3];
[1 0 0]
[0 5 0]
```

NAME MatHermiteColModUpper

PURPOSE Computes the modular upper column Hermite normal form of an integer matrix.

SYNTAX H := MatHermiteColModUpper(k, M);

matrix	H
integer	k
matrix	M

DESCRIPTION Let M $\in \mathbb{Z}^{m\times n}$. The MatHermiteColModUpper routine computes the upper column Hermite normal form of the matrix

$$(\mathtt{M} \mid \mathtt{k} \cdot I|m) \in \mathbb{Z}^{m\times(n+m)}$$

and returns the rightmost n columns of the Hermite normal form.

EXAMPLE Compute the upper modular column Hermite normal form of

$$\begin{pmatrix} 3 & 7 & 10 \\ 20 & 25 & 40 \\ 47 & 61 & 90 \end{pmatrix}$$

with respect to the modulus 5.

```
kash> M := Mat(Z,[[3,7,10],[20,25,40],[47,61,90]]);
[ 3  7 10]
[20 25 40]
[47 61 90]
kash> MatHermiteColModUpper(5,M);
[1 0 0]
[0 5 0]
[0 0 1]
```

NAME `MatHermiteColUpper`

PURPOSE missing shortdoc

SYNTAX `H := MatHermiteColUpper(M);`

```
matrix  H
matrix  M
```

DESCRIPTION Computes the upper column Hermite normal form of the integer matrix `M`.

EXAMPLE Compute the upper column Hermite normal form of

$$\begin{pmatrix} 4 & 6 & 2 \\ 3 & 9 & 12 \end{pmatrix}.$$

```
kash> M := Mat(Z,[[4,6,2],[3,9,12]]);
[ 4  6  2]
[ 3  9 12]
kash> MatHermiteColUpper(M);
[0 2 0]
[0 0 3]
```

NAME `MatHermiteColUpperTrans`

PURPOSE Computes the upper column Hermite normal form together with a transformation matrix.

SYNTAX `L := MatHermiteColUpperTrans(M [,"int"]);`

`list`	`L`
`matrix`	`M`

DESCRIPTION The `MatHermiteColUpperTrans` routine computes the upper column Hermite normal form `H` of the integer matrix `M`. Additionally it computes the rank of `M` and a transformation matrix `T` such that $\mathtt{H} = \mathtt{M} * \mathtt{T}$. The result is a list `L` which contains the rank, the Hermite normal form `H` and the transformation matrix `T`.
If the matrix is defined over a local field and has integral coefficients, the Hermite Normal Form over the respective valuation ring is returned.

EXAMPLE Compute the upper column Hermite normal form of

$$\begin{pmatrix} 3 & 7 & 10 \\ 20 & 25 & 40 \end{pmatrix}.$$

```
kash> M := Mat(Z,[[3,7,10],[20,25,40]]);
[ 3  7 10]
[20 25 40]
kash> L := MatHermiteColUpperTrans(M);;
kash> L[1];
2
kash> L[2];
[0 1 0]
[0 0 5]
kash> L[3];
[ -6   1  -5]
[-16   4 -15]
[ 13  -3  12]
kash> M*L[3];
[0 1 0]
[0 0 5]
```

NAME `MatHermiteRowMod`

PURPOSE Computes the modular row Hermite normal form of an integer matrix.

SYNTAX `H := MatHermiteRowMod(k, M);`

`matrix`	`H`
`integer`	`k`
`matrix`	`M`

DESCRIPTION Let $\mathtt{M} \in \mathbb{Z}^{m\times n}$. The `MatHermiteRowMod` routine computes the row Hermite normal form of the matrix

$$\begin{pmatrix} \mathtt{M} \\ \mathtt{k}\cdot I_n \end{pmatrix} \in \mathbb{Z}^{(m+n)\times n}$$

and returns the upper m rows of the Hermite Normal form.

EXAMPLE Compute the row Hermite normal form of

$$\begin{pmatrix} 3 & 7 & 10 \\ 20 & 25 & 40 \end{pmatrix}$$

with respect to the modulus 5.

```
kash> M := Mat(Z,[[3,7,10],[20,25,40]]);
[ 3  7 10]
[20 25 40]
kash> MatHermiteRowMod(5,M);
[1 4 0]
[0 5 0]
[0 0 5]
```

NAME	`MatId`
PURPOSE	Returns the identity matrix of given dimension.
SYNTAX	`I := MatId(S, n);`

matrix	I
ring	S
small integer	n

DESCRIPTION Returns the identity matrix $I|n$ of dimension `n` over the (unitary) ring `S`.

EXAMPLE Create $I|3$ over $\mathbb{Z}[\sqrt[4]{-1}]$.

```
kash> O := Order(Z,4,-1);
Generating polynomial: x^4 + 1

kash> MatId(O,3);
[1 0 0]
[0 1 0]
[0 0 1]
```

NAME `MatIndex`

PURPOSE Computes the determinant of a row normal form of an integral matrix.

SYNTAX `n := MatIndex(m);`

`integer`	`n`	index
`matrix`	`m`	over $\mathbb{Z}$

DESCRIPTION Suppose we have $\mathbb{Z}$ modules $M|1 \subset M|2$ given via a basis of $M|2$ and a set of elements generating $M|1$. Furthermore let m be the matrix which rows contain the aforementioned generators. `MatIndex`(m) will compute the index of $M|1$ in $M|2$. If the index is not finite, a 0 will be returned.

EXAMPLE

```
kash> m := RandomMatrix(Z, 3);
[ 19  11   5]
[ -6 -18  -6]
[ 17  18   5]
kash> MatIndex(m);
540
kash> m := m{[1..2]};IsMat(m);
[ 19  11   5]
[ -6 -18  -6]
true
kash> MatIndex(m);
0
kash> m := Concatenation(m, RandomMatrix(Z, 3));IsMat(m);
[ [19 11  5], [ -6 -18  -6], [-16  -1 -16], [-4 -3 13], [17 11 -9] ]
true
kash> MatIndex(m);
1
```

NAME `MatInv`

PURPOSE Returns the inverse matrix (if possible).

SYNTAX `A := MatInv(M);`

```
matrix  A
matrix  M
```

DESCRIPTION Returns the inverse matrix `A` of `M`. Returns an error if `M` is not invertible.

EXAMPLE Compute the inverse of the integer matrix $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

```
kash> M := Mat(Z,[[1,2],[1,1]]);
[1 2]
[1 1]
kash> MatInv(M);
[-1  2]
[ 1 -1]
```

NAME MatKernel

PURPOSE Returns a basis for the kernel of the given matrix.

SYNTAX `K := MatKernel(M); K := MatKernel(M, d);`

matrix	K
matrix	M
integer	d

DESCRIPTION Let M be a $m \times n$–matrix. The MatKernel routine computes a basis K for the linear subspace

$$\{x = (x|1, \dots, x|m) : x \cdot \mathtt{M} = (0, \dots, 0)\}.$$

.

If a second parameter d is used, the kernel over $\mathbb{Z}/d\mathbb{Z}$ will be computed. The used algorithm is based on [BN96].

EXAMPLE Compute the kernel of the integer matrix $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{pmatrix}$.

```
kash> M := Mat(Z,[[2,3,4],[3,4,5],[4,5,6],[6,7,8]]);
[2 3 4]
[3 4 5]
[4 5 6]
[6 7 8]
kash> MatKernel(M);
[ 1  0 -2  1]
[ 0  2 -3  1]
```

Compute the kernel modulo 3

```
kash> MatKernel(M, 3);
[0 2 0 1]
[1 1 1 0]
```

Compute kernel

```
kash> o := Order(Z, 2,3);;
kash> a := Elt(o, [1,2/19]);;
kash> one := Elt(o, [1,0]);;
kash> m := Mat(o, [[a,2*a,3*a^2],[a^2,2*a^2,3*a^3],[one,2*one,3*a]]);
[[19, 2] / 19 [38, 4] / 19 [1119, 228] / 361]
[[373, 76] / 361 [746, 152] / 361 [22629, 6570] / 6859]
[1 2 [57, 6] / 19]
kash> k := MatKernel(m);
[1 0 [-19, -2] / 19]
[0 1 [-373, -76] / 361]
kash> k*m;
[0 0 0]
[0 0 0]
```

NAME `MatLLL`

PURPOSE Performs an LLL reduction on the columns of an integer or real matrix.

SYNTAX

```
L := MatLLL(M [,r]);
L := MatLLL(M,"short"|"long" [,r]);
```

list of A, T	L
matrix	M
LLL constant (default 0.75)	r

DESCRIPTION Performs an LLL reduction on the columns of an integer or real matrix `M`. The result is a list containing the reduced matrix `A` and the unimodular transformation matrix `T` such that `M` = `A` ∗ `T`. The `MatLLL` function requires that the columns of `M` are linearly independent.

If `M` is an integer matrix the `MatLLL` function normally omits any floating–point arithmetic. When the `"SHORT"` option is specified the `MatLLL` function uses floating–point arithmetic basing on machine doubles.

If `M` is a real matrix the `MatLLL` function normally makes use of the long floating–point arithmetic of KASH. When the `"SHORT"` option is specified the `MatLLL` function uses floating–point arithmetic basing on machine doubles

EXAMPLE

```
kash> M := Mat(Z, [[234, 469], [-6546, -13126],
> [-6312, -12657]]);
[   234    469]
[ -6546 -13126]
[ -6312 -12657]
kash> L := MatLLL(M);
[ [   1  42]
    [-34 -18]
    [-33  24], [  -2  385]
    [   1 -192] ]
kash> L[1];
[  1  42]
[-34 -18]
[-33  24]
kash> M*L[2];
[  1  42]
[-34 -18]
[-33  24]
```

NAME MatMLLL

PURPOSE Performs an MLLL reduction on the columns of an integer or real matrix.

SYNTAX `L := MatLLL(M);`

```
list    L
matrix  M
```

DESCRIPTION Performs a MLLL reduction on the columns of an integer or real matrix `M`. Let `M` $\in \mathbb{R}^{m\times(n+1)}$ and let $a|1, \ldots, a|n+1 \in \mathbb{R}^m$ be the columns of `M`. The `MatMLLL` function expects that $a|1, \ldots, a|n$ are linearly independent and that $a|n+1$ has a non-zero entry. It computes columns $b|1, \ldots, b|n+1 \in \mathbb{R}^m$ of `B` such that

$$\mathbb{Z} \cdot a|1 + \cdots + \mathbb{Z} \cdot a|n+1 = \mathbb{Z} \cdot b|1 + \cdots + \mathbb{Z} \cdot b|n+1$$

and additionally a transformation matrix `T` $\in \mathbb{R}^{(n+1)\times(n+1)}$ with `B` $=$ `M` $\cdot$ `T`. Furthermore a flag is returned. If it is true, $b|1, \ldots, b|n+1$ are linearly independent. Otherwise all entries of $b|n+1$ are zero. These results are stored in a list containing the reduced matrix `B`, the unimodular transformation matrix `T` and the flag.

EXAMPLE MLLL-Reduction of a matrix:

```
kash> M := Mat(Z, [ [ 234, 469, -54411 ],
> [ -6546, -13126, 1531763 ], [ -6312, -12657, 1477352 ] ]);
[    234     469  -54411]
[  -6546  -13126 1531763]
[  -6312  -12657 1477352]
kash> L := MatMLLL(M);
[ [ 1  1  0]
    [-1  0  0]
    [ 0  1  0], [ -794943 -1327888  4198061]
    [  427600   714271 -2258136]
    [     267      446    -1410], true ]
kash> L[1];
[ 1  1  0]
[-1  0  0]
[ 0  1  0]
kash> M*L[2];
[ 1  1  0]
[-1  0  0]
[ 0  1  0]
```

NAME `MatMinPoly`

PURPOSE Returns the minimal polynomial of a matrix.

SYNTAX `f := MatMinPoly(M);`

`matrix`	`M`
`Polynomial`	`f`

SEE ALSO `EltMinPoly`, `MatCharPoly`,

EXAMPLE Compute the minimal polynomial of

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{Z}^{4\times 4}.$$

```
kash> M := Mat(Z,[[0,1,0,-1],[1,0,1,0],[0,1,0,1],[-1,0,1,0]]);
[ 0  1  0 -1]
[ 1  0  1  0]
[ 0  1  0  1]
[-1  0  1  0]
kash> MatMinPoly(M);
x^2 - 2
kash> M^2;
[2 0 0 0]
[0 2 0 0]
[0 0 2 0]
[0 0 0 2]
```

NAME MatMove

PURPOSE missing shortdoc

SYNTAX `V := MatMove (M, r);`

```
matrix  V
matrix  M
ring    r
```

DESCRIPTION The `MatMove` function casts the matrix `M` to a matrix over the ring `R`.

SEE ALSO `PolyMove`, `EltMove`,

EXAMPLE Compute the representation of an interger matrix in $\mathbb{Z}/23\mathbb{Z}$.

```
kash> M := Mat(Z,[[0,23,0,45456],[35,33,85,14],[86,79,80,-3],[-1,0,1,0]]);
[    0    23     0 45456]
[   35    33    85    14]
[   86    79    80    -3]
[   -1     0     1     0]
kash> F23 := FF (23);
Finite field of size 23
kash> MatMove (M, F23);
[ 0  0  0  8]
[12 10 16 14]
[17 10 11 20]
[22  0  1  0]
```

NAME MatRows

PURPOSE Returns the number of rows of a matrix.

SYNTAX `n := MatRows(M);`

integer	n
matrix	M

SEE ALSO MatCols,

EXAMPLE

```
kash> M := Mat(Z,[[67,4,98,1],[-1,5,3,4],[6,-1,7,8]]);
[67  4 98  1]
[-1  5  3  4]
[ 6 -1  7  8]
kash> MatRows(M);
3
```

NAME MatSmith

PURPOSE missing shortdoc

SYNTAX `L := MatSmith(M);`

list	L
matrix	M

DESCRIPTION Computes the Smith normal form and the rank of the integer matrix `M`. The result is a list `L` which contains the rank and the Smith normal form `S` of `M`.

EXAMPLE Compute the Smith normal form of

$$\begin{pmatrix} 4 & 6 & 2 \\ 3 & 9 & 12 \end{pmatrix}.$$

```
kash> M := Mat(Z,[[4,6,2],[3,9,12]]);
[ 4  6  2]
[ 3  9 12]
kash> L := MatSmith(M);;
kash> rank := L[1];
2
kash> S := L[2];
[1 0 0]
[0 6 0]
```

NAME `MatSmithTrans`

PURPOSE Computes the Smith normal form together with transformation matrices.

SYNTAX `L := MatSmithTrans(M);`

```
list    L
matrix  M
```

DESCRIPTION The `MatSmithTrans` routine computes the Smith normal form `S` of the integer matrix `M`. Additionally it computes the rank of `M` and transformation matrices `A` and `B` such that $\mathtt{S} = \mathtt{A} * \mathtt{M} * \mathtt{B}$. The result is a list `L` which contains the rank, the Smith normal form `S` and the transformation matrices `A` and `B`.

EXAMPLE Compute the upper column Smith normal form of

$$\begin{pmatrix} 3 & 7 & 10 \\ 20 & 25 & 40 \\ 17 & 6 & 9 \end{pmatrix}.$$

```
kash> M := Mat(Z,[[3,7,10],[20,25,40],[17,6,9]]);
[ 3  7 10]
[20 25 40]
[17  6  9]
kash> L := MatSmithTrans(M);;
kash> rank := L[1];
3
kash> S := L[2];
[   1   0   0]
[   0   1   0]
[   0   0 405]
kash> A := L[3];
[   1   0   0]
[-56 -11   4]
[-70 -14   5]
kash> B := L[4];
[    1    1 -318]
[    4    1 -308]
[   -3   -1  311]
kash> A*M*B;
[   1   0   0]
[   0   1   0]
[   0   0 405]
```

NAME `MatSolve`

PURPOSE Solves a linear equation.

SYNTAX

```
L := MatSolve (A, b);
L := MatSolve(A, b, N);
```

`list`	`L`	
`matrix`	`A`	
`matrix`	`b`	
`integer`	`N`	a module

DESCRIPTION Given matrix A and vector b, the `MatSolve` function finds the vector x which satisfies the matrix equation $A \cdot x = b$. The `MatSolve` function returns a list whose first entry is x. The second of L is the null space of A. If $A \cdot x = b$ has no solution, then `FALSE` is returned. If the module N is given, solutions modulo N are returned.

EXAMPLE

```
kash> M := Mat(Z,[[1,0],[0,1]]);
[1 0]
[0 1]
kash> b := Mat(Z,[[1],[2]]);
[1]
[2]
kash> MatSolve(M, b);
[ [1]
    [2], Matrix with 2 rows and 0 columns ]
kash> M := Mat(Z,[[1,0,1],[0,1,1]]);
[1 0 1]
[0 1 1]
kash> MatSolve(M, b);
[ [1]
    [2]
    [0], [ 1]
    [ 1]
    [-1] ]
```

NAME	`MatSym`
PURPOSE	missing shortdoc
SYNTAX	`M := MatSym(S, L);`

```
matrix  M
ring    S
list    L
```

DESCRIPTION Creates the symmetric matrix `M` over the coefficient ring `S`.

EXAMPLE Create the symmetric matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix} \in \mathbb{Z}^{4\times 4}.$$

```
kash> MatSym(Z, [[1,2,3,4], [5,6,7], [8,9], [10]]);
[ 1  2  3  4]
[ 2  5  6  7]
[ 3  6  8  9]
[ 4  7  9 10]
```

NAME MatSymDiag

PURPOSE missing shortdoc

SYNTAX `L := MatSymDiag (A);`

list	L
matrix	A

DESCRIPTION Let A be a symmetric matrix over $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. The `MatSymDiag` function computes Δ, T and λ such that $\Delta = \lambda T^t A T$ is a diagonal matrix. The diagonal entries of Δ are squarefree integers if the coefficient ring of A is $\mathbb{Z}$ or $\mathbb{Q}$. When the coefficient ring of A equals $\mathbb{R}$ each diagonal entry of Δ is ± 1 or 0.

EXAMPLE

```
kash> A := Mat(Z,[[2,3,4],[3,1,4],[4,4,0]]);
[2 3 4]
[3 1 4]
[4 4 0]
kash> L := MatSymDiag(A);;
kash> Delta := L[1];
[ 7  0  0]
[ 0 -1  0]
[ 0  0 -6]
kash> lambda := L[2];
14/1
kash> T := L[3];
[  1/2 -3/14  -2/7]
[    0   1/7  -1/7]
[    0     0   1/4]
kash> Delta;
[ 7  0  0]
[ 0 -1  0]
[ 0  0 -6]
kash> lambda * MatTrans(T) * A * T;
[ 7  0  0]
[ 0 -1  0]
[ 0  0 -6]
```

NAME MatToColList

PURPOSE Returns a list of the columns.

SYNTAX `L := MatToColList(M);`

list	L
matrix	M

EXAMPLE

```
kash> M := Mat(Z, [[1,2,3], [4,5,6], [7,8,9], [10,11,12]]);
[ 1  2  3]
[ 4  5  6]
[ 7  8  9]
[10 11 12]
kash> L := MatToColList(M);
[ [ 1, 4, 7, 10 ], [ 2, 5, 8, 11 ], [ 3, 6, 9, 12 ] ]
```

NAME `MatToRowList`

PURPOSE Returns a list of the rows.

SYNTAX `L := MatToRowList(M);`

```
list    L
matrix  M
```

EXAMPLE

```
kash> M := Mat(Z, [[1,2,3], [4,5,6], [7,8,9], [10,11,12]]);
[ 1  2  3]
[ 4  5  6]
[ 7  8  9]
[10 11 12]
kash> L := MatToRowList(M);
[ [ 1, 2, 3 ], [ 4, 5, 6 ], [ 7, 8, 9 ], [ 10, 11, 12 ] ]
```

The rows can also be accessed through:

```
kash> M[1];
[1 2 3]
kash> IsList(M);
true
```

NAME MatTrace

PURPOSE Computes the trace of a matrix.

SYNTAX `t := MatTrace(M);`

element of the ring the matrix elements come from	t
matrix	M

EXAMPLE Compute the trace of the integer matrix $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

```
kash> M := Mat(Z,[[1,1],[2,2]]);
[1 1]
[2 2]
kash> MatTrace(M);
3
```

NAME MatTrans

PURPOSE Returns the tranpose of a matrix.

SYNTAX `A := MatTrans(M);`

```
matrix  A
matrix  M
```

DESCRIPTION Returns the transposed matrix `A` of `M`.

EXAMPLE Compute the transpose of the integer matrix $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$.

```
kash> M := Mat(Z,[[1,2,4],[1,3,9]]);
[1 2 4]
[1 3 9]
kash> MatTrans(M);
[1 1]
[2 3]
[4 9]
```

NAME Min

PURPOSE Computes the minimum of an ideal.

SYNTAX `n := Min(a);`

int | rational | ideal	n
ideal	a

DESCRIPTION

EXAMPLE All examples take place in $\mathbb{Q}(\sqrt{5})$:

```
kash> O := OrderMaximal(Order(x^3 - 10*x^2 - 3*x - 2));;
kash> o := Order(O, 3, 3);;
kash> IdealMin(Ideal(5, Elt(O, [1, 1, 0])) / 4);
5/4
kash> IdealMin(Ideal(Elt(O, [3, 1, 0]), Elt(O, [1, 1, 0])));
2
kash> IdealMin(5*o + Elt(o, [3, 1, 0])*o);
<5>
```

NAME MinPoly

PURPOSE Computes the minimal polynomial of an algebraic element, a matrix, or an alff element

SYNTAX
```
p := MinPoly (a [,PA]);
p := MinPoly (a [,O]);
p := MinPoly( M );
p := MinPoly(a);
```

polynomial	p
polynomial algebra	PA
suborder	O
algebraic element alff order element	a
matrix	M

DESCRIPTION For an algebraic element a in an Order O and a polynomial algebra PA over another order o or this order, this function computes the minimal polynomial of a over o i.e. the returned polynomial is contained in PA.
Given a matrix this function returns its minimal polynomial.
Given a alff element this function computes the minimal polynomial

SEE ALSO EltMinPolyMatMinPolyAlffEltMinPoly,

EXAMPLE

```
kash> o := Order(Z,2,2);
Generating polynomial: x^2 - 2

kash> a := Elt(o,[0,1]);
[0, 1]
kash> p := MinPoly(a);
x^2 - 2
kash> M := Mat(o,[[1,a],[1,-a]]);
[1 [0, 1]]
[1 [0, -1]]
kash> p := MinPoly(M);
x^2 + [-1, 1]*x + [0, -2]
```

NAME MinVec

PURPOSE Returns a list of minimal vectors of a real Humbert form in two variables and his minimum

SYNTAX `M := MinVec(HF,o,n);`

list	M
Humbert form with $S_i \in \overline{Q}^{2\times 2}$	HF=[S1,S2]
maximal order of an quadratic field	o
integer	n

DESCRIPTION Given a real Humbert form in two variables it returns a list of minimal vectors and the minimum of the real Humbert form. If `n` $=0$, then MinVec uses the default value which is always computed to get minimal vectors. In the other case, `n` would be

- the value for computing minimal vectors or
- the value to receive an upper bound for the minimum

The minimal vectors are conjugated in the same order KANT conjugates an element of the given order `o`.

SEE ALSO EutacticCoef,

EXAMPLE

```
kash> o := OrderMaximal(x^2-3);
Generating polynomial: x^2 - 3
Discriminant: 12

kash> ur := Re(EltCon(Elt(o, [2, 1]))[1][1]);
3.732050807568877293527446341505872366942805253381
kash> urs := Re(EltCon(Elt(o, [2, 1]))[1][2]);
0.267949192431122706472553658494127633057194746619

kash> S1 := Mat(R, [ [ 1, ur/2 ], [ur/2, ur ] ]);
[1 1.866025403784438646763723170752936183471402626905]
[1.866025403784438646763723170752936183471402626905 3.732050807568877293527446\
341505872366942805253381]
kash> S2 := Mat(R, [ [ 1, urs/2], [urs/2, urs] ]);
[1 0.133974596215561353236276829247063816528597373095]
[0.133974596215561353236276829247063816528597373095 0.267949192431122706472553\
```

```
65849412763305719474619]
kash> HF := [S1,S2];
[ [1 1.866025403784438646763723170752936183471402626905]
    [1.866025403784438646763723170752936183471402626905 3.73205080756887729352\
744634150587236694280525381],
  [1 0.133974596215561353236276829247063816528597373095]
    [0.133974596215561353236276829247063816528597373095 0.26794919243112270647\
255365849412763305719474619] ]
kash> L := MinVec(HF,o,0);
[ [ [ [0, 1], [1, -1] ], [ [0, 1], -1 ], [ [1, 1], [0, -1] ], [ [1, 1], -1 ],
      [ [2, 1], [0, -1] ], [ [2, 1], -2 ], [ 0, 1 ], [ 1, [1, -1] ],
      [ 1, [-2, 1] ], [ 1, -1 ], [ 1, 0 ], [ 2, -1 ] ],
  0.99999999999999999999999999999999999999999999999156162 ]
```

Name `Module`

Purpose Creates a module over an order.

Syntax
```
M1 := Module([IL,] M);
M1 := Module(EL);
```

`module`	`M1`	
`list`	`IL`	of ideals over a maximal order $\mathcal{O}$
`matrix`	`M`	of algebraic elements over $\mathcal{O}$
`list`	`EL`	of relative algebraic elements of $\mathfrak{o}$

Description Let $\mathfrak{a}|1, \ldots, \mathfrak{a}|m$ be fractional $\mathcal{O}$–ideals (the list `IL`, they are called the coefficient ideals) and $A|1, \ldots, A|m$ be the columns of the matrix M of dimension n over the quotient field of $\mathcal{O}$. This function creates a pseudomatrix which is a representation of the $\mathcal{O}$–module

$$M|1 = A|1\mathfrak{a}|1 + \cdots + A|m\mathfrak{a}|m.$$

If the list of coefficient ideals (`IL`) is omitted, the module with trivial coefficient ideals ($1\mathcal{O}$) will be created.
If `Module` is invoked with a list of relative algebraic elements, the representations of those in a relative order $\mathfrak{o}$ as column vectors over $\mathcal{O}$, the coefficient order of $\mathfrak{o}$. Again the ideals are assumed to be trivial.

See Also ModuleIdeals, ModuleMatrix, ModuleOrder,

Example

```
kash> O:=OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> IL:=List(Factor(2*O),f->f[1]);;
kash> List(IL,IdealGenerators);
[ [ 2, [0, 2, 0, 3, 0, 2] ], [ 2, [0, 0, 3, 0, 1, 2] ],
  [ 2, [3, 3, 1, 2, 0, 0] ] ]
kash> EL1:=List(Factor(5*O),f->IdealGen(f[1],2));
[ [12, 17, 15, 7, 9, 5], [19, 18, 15, 1, 10, 24], [4, 22, 4, 10, 23, 4] ]
kash> EL2:=List(Factor(13*O),f->IdealGen(f[1],2));
[ [10, 8, 0, 1, 0, 0], [1, 12, 5, 1, 0, 0] ]
kash> EL2[3]:=IdealGen(Factor(7*O)[1][1],2);
[1, 1, 0, 0, 0, 0]
kash> EL3:=List(Factor(3*O),f->IdealGen(f[1],2));
[ [1, 1, 0, 0, 0, 0], [1, 0, 1, 0, 0, 0] ]
kash> EL3[3]:=IdealGen(Factor(7*O)[2][1],2);
```

```
[3, 1, 0, 0, 0, 0]
kash> M:=Module(IL,Mat(O,[EL1,EL2,EL3]));
{<2, [0, 2, 0, 3, 0, 2]><2, [0, 0, 3, 0, 1, 2]><2, [3, 3, 1, 2, 0, 0]>
[[12, 17, 15, 7, 9, 5] [19, 18, 15, 1, 10, 24] [4, 22, 4, 10, 23, 4]]
[[10, 8, 0, 1, 0, 0] [1, 12, 5, 1, 0, 0] [1, 1, 0, 0, 0, 0]]
[[1, 1, 0, 0, 0, 0] [1, 0, 1, 0, 0, 0] [3, 1, 0, 0, 0, 0]]
}
```

NAME ModuleDen

PURPOSE Returns the denominator of the given module.

SYNTAX `den := ModuleDen(M);`

integer	den
module	M

DESCRIPTION The denominator of a module M is defined as the smallest natural number d that the module dM is an integral module. An integral module over the Dedekind ring R is a subset of R^n where n is the degree of the module.

EXAMPLE

```
kash>  O := OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> Ids:=[ Ideal(5, Elt(O,[-27, 1, -2, 1, -3, 15])),
> Ideal(1, Elt(O,[-18, 0, -3, 3, -1, 0]/5)) ];
[ <5, [-27, 1, -2, 1, -3, 15]>, <1, [-18, 0, -3, 3, -1, 0] / 5> ]
kash> M := Module(Ids,Mat(O,[[1,2],[2,1]])/15);
{<5, [-27, 1, -2, 1, -3, 15]><1, [-18, 0, -3, 3, -1, 0] / 5>
[1 / 15 2 / 15]
[2 / 15 1 / 15]
}

kash> ModuleDen(M);
75
```

Name `ModuleDet`

Purpose Returns the determinant of the given module.

Syntax `det := ModuleDet(M)`

module	M
ideal	det

Description If the module is represented by a square pseudomatrix, the product of the coefficient ideals multiplied by the determinant of the matrix is computed.
Otherwise there are more columns than rows in the representation. In this case the gcd of the determinants of all n–subpseudomatrices of the pseudomatrix is computed. For further information have a look at [Hop98].
If there are many more columns than rows it would be impractiacal to compute all determinants (because there are exponentially many of them) so this function considers just a few (set `PRINTLEVEL(ORDER_MODULE_DET,1)` to see how many) determinants and gets a reasonable multiple of the degree minor gcd.

See Also `ModuleModul`,

Example

```
kash> O := OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> Ids:=[ Ideal(5, Elt(O,[-27, 1, -2, 1, -3, 15])),
> Ideal(1, Elt(O,[-18, 0, -3, 3, -1, 0]/5)) ];
[ <5, [-27, 1, -2, 1, -3, 15]>, <1, [-18, 0, -3, 3, -1, 0] / 5> ]
kash> M := Module(Ids,Mat(O,[[1,2],[2,1]])/15);
{<5, [-27, 1, -2, 1, -3, 15]><1, [-18, 0, -3, 3, -1, 0] / 5>
[1 / 15 2 / 15]
[2 / 15 1 / 15]
}

kash> ModuleDet(M);
<
[-5  0  0 -1 -4  0]
[ 0 -5  0 -1 -4  0]
[ 0  0 -5 -2 -1  0]
[ 0  0  0 -1  0  0]
[ 0  0  0  0 -1  0]
[ 0  0  0  0  0 -5]
/375>
```

Name `ModuleDual`

Purpose Computes the dual module of the given module.

Syntax `M2 := ModuleDual( M1 );`

modules	M1, M2

Description The dual module is defined as $M|2 = \{x \in Q(\mathcal{O})M|1; Tr(xM|1) \subseteq \mathcal{O}\}$. The module $M|1$ must be of full rank. For details see [Hop98].

See Also ModuleIntersection,

Example

```
kash> O:=OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> IL:=List(Factor(2*O),f->f[1]);;
kash> List(IL,IdealGenerators);;
kash> EL1:=List(Factor(5*O),f->IdealGen(f[1],2));;
kash> EL2:=List(Factor(13*O),f->IdealGen(f[1],2));;
kash> EL2[3]:=IdealGen(Factor(7*O)[1][1],2);;
kash> EL3:=List(Factor(3*O),f->IdealGen(f[1],2));;
kash> EL3[3]:=IdealGen(Factor(7*O)[2][1],2);;
kash> M:=Module(IL,Mat(O,[EL1,EL2,EL3]));
{<2, [0, 2, 0, 3, 0, 2]><2, [0, 0, 3, 0, 1, 2]><2, [3, 3, 1, 2, 0, 0]>
[[12, 17, 15, 7, 9, 5] [19, 18, 15, 1, 10, 24] [4, 22, 4, 10, 23, 4]]
[[10, 8, 0, 1, 0, 0] [1, 12, 5, 1, 0, 0] [1, 1, 0, 0, 0, 0]]
[[1, 1, 0, 0, 0, 0] [1, 0, 1, 0, 0, 0] [3, 1, 0, 0, 0, 0]]
}

kash> ModuleDual(M);
{<1, [6883335932560408235907630751944302230589929986403502304 25, 4157727723404\
9412961809251677, 688333593256040823590763075213190812482229202013839062953, 4\
00521075905880685987392871 8, 880056335481757290935697071 43, 3] / 2623611238838\
6371287536615054><1 / 2><1, [0, 3, 3, 3, 0, 2] / 2>
[1 0 0]
[[1021824636960371063894162871 0, 0, -1, 0, 0, -1] / 2 1 0]
[[-3553866725934899057416486544 3, 1021824636960371063894161612 1, 1021824636960\
371063894164055 1, -1021824636960371063894162503 9, -204364927392074212778832286\
83, -66089] / 4 [-2, 1, 1, -1, -2, 0] / 2 1]
}
```

NAME `ModuleId`

PURPOSE Returns the trivial module over an order with a certain degree.

SYNTAX `m := ModuleId(o, n);`

`Module`	`m`	
`order`	`o`	
`positive integer`	`n`	the degree

DESCRIPTION Creates the trivial module $\mathfrak{o}^n$.

EXAMPLE

```
kash> o := OrderMaximal(Poly(Zx, [1, 1, 10]));;
kash> ModuleId(o, 4);
{<1><1><1><1>
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
}
```

NAME	`ModuleIdeals`
PURPOSE	Retrieves the coefficient ideals of the module representation.
SYNTAX	`L := ModuleIdeals(M);`

`list`	`L`	of ideals
`module`	`M`	

DESCRIPTION

The module is represented by a pseudomatrix over a Dedekind ring and consists of a matrix and a list of ideals (one for each column). See description of `Module` for details.
If the module is a zero module false is returned.

SEE ALSO

`Module`, `ModuleMatrix`,

EXAMPLE

```
kash> O:=OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> IL:=List(Factor(2*O),f->f[1]);;
kash> List(IL,IdealGenerators);;
kash> EL1:=List(Factor(5*O),f->IdealGen(f[1],2));;
kash> EL2:=List(Factor(13*O),f->IdealGen(f[1],2));;
kash> EL2[3]:=IdealGen(Factor(7*O)[1][1],2);;
kash> EL3:=List(Factor(3*O),f->IdealGen(f[1],2));;
kash> EL3[3]:=IdealGen(Factor(7*O)[2][1],2);;
kash> M:=Module(IL,Mat(O,[EL1,EL2,EL3]));
{<2, [0, 2, 0, 3, 0, 2]><2, [0, 0, 3, 0, 1, 2]><2, [3, 3, 1, 2, 0, 0]>
[[12, 17, 15, 7, 9, 5] [19, 18, 15, 1, 10, 24] [4, 22, 4, 10, 23, 4]]
[[10, 8, 0, 1, 0, 0] [1, 12, 5, 1, 0, 0] [1, 1, 0, 0, 0, 0]]
[[1, 1, 0, 0, 0, 0] [1, 0, 1, 0, 0, 0] [3, 1, 0, 0, 0, 0]]
}

kash> ModuleIdeals(M);
[ <2, [0, 2, 0, 3, 0, 2]>, <2, [0, 0, 3, 0, 1, 2]>, <2, [3, 3, 1, 2, 0, 0]> ]
```

NAME ModuleIntersection

PURPOSE Computes the intersection of two modules.

SYNTAX `M :=ModuleIntersection(M1, M2 [,d | I] [,"PBNF"] [,"lower"] );`

modules	M, M1, M2	
integer	d	used for reduction
ideal	I	used for reduction

DESCRIPTION This function computes the intersetion $M_1 \cap M_2$ of the given modules. Both modules must be of full rank.
The additional parameters control the normal form algorithm application. For their description see `ModuleNF`.
The algorithm computes the dual module of the sum of the dual moduls of the input, it is described in [Hop98].

SEE ALSO `ModuleConcat`, `ModuleNF`,

EXAMPLE

```
kash>  O:=OrderMaximal(Poly(Zx, [1,-10,-3,-2]));;
kash> o:=OrderMaximal(O,3,3);;
kash> Oa:=OrderMaximal(OrderAbs(o));;
kash> L:=List(Factor(10*Oa),i->i[1]);;
kash> M1:=IdealBasis(IdealMove(L[2]*L[6], o));
{<10, [26, 1, 0]><2, [0, 0, 1]><1>
[1 -2 -1]
[0 1 1]
[0 0 1]
}

kash> M2:=IdealBasis(IdealMove(L[1]*L[6], o));
{<10, [-12714, -20615, -67306]><1><1>
[1 3 1]
[0 1 0]
[0 0 1]
}

kash> ModuleIntersection(M1,M2);
{<10, [88, 2, 99]><2, [0, 2, 1]><2, [2, 2, 1]>
[1 -2 -1]
```

```
[0 1 1]
[0 0 1]
}
```

NAME `ModuleIntersectionVS`

PURPOSE Computes the intersection of a module and a VS.

SYNTAX `M :=ModuleIntersectionVS(M1, mat );`

`modules`	`M, M1, M2`	
`integer`	`d`	used for reduction
`ideal`	`I`	used for reduction

DESCRIPTION

SEE ALSO `ModuleConcat`, `ModuleNF`,

EXAMPLE

```
kash>  O:=OrderMaximal(Poly(Zx, [1,-10,-3,-2]));;
kash> o:=OrderMaximal(O,3,3);;
kash> Oa:=OrderMaximal(OrderAbs(o));;
kash> L:=List(Factor(10*Oa),i->i[1]);;
kash> M1:=IdealBasis(IdealMove(L[2]*L[6], o));
{<10, [26, 1, 0]><2, [0, 0, 1]><1>
[1 -2 -1]
[0 1 1]
[0 0 1]
}

kash> M2:=IdealBasis(IdealMove(L[1]*L[6], o));
{<10, [-12714, -20615, -67306]><1><1>
[1 3 1]
[0 1 0]
[0 0 1]
}

kash> ModuleIntersection(M1,M2);
{<10, [88, 2, 99]><2, [0, 2, 1]><2, [2, 2, 1]>
[1 -2 -1]
[0 1 1]
[0 0 1]
}
```

Name `ModuleMap`

Purpose Computes the image and kernel under the multiplication by map

Syntax `L :=ModuleMap(M1, mat [ , "nf" | "kernel" ] );`

`modules`	`M, M1, M2`	
`integer`	`d`	used for reduction
`ideal`	`I`	used for reduction

Description

See Also `ModuleConcat`, `ModuleNF`,

Example

```
kash>  O:=OrderMaximal(Poly(Zx, [1,-10,-3,-2]));;
kash> o:=OrderMaximal(O,3,3);;
kash> Oa:=OrderMaximal(OrderAbs(o));;
kash> L:=List(Factor(10*Oa),i->i[1]);;
kash> M1:=IdealBasis(IdealMove(L[2]*L[6], o));
{<10, [26, 1, 0]><2, [0, 0, 1]><1>
[1 -2 -1]
[0 1 1]
[0 0 1]
}

kash> M2:=IdealBasis(IdealMove(L[1]*L[6], o));
{<10, [-12714, -20615, -67306]><1><1>
[1 3 1]
[0 1 0]
[0 0 1]
}

kash> ModuleIntersection(M1,M2);
{<10, [88, 2, 99]><2, [0, 2, 1]><2, [2, 2, 1]>
[1 -2 -1]
[0 1 1]
[0 0 1]
}
```

NAME `ModuleMatrix`

PURPOSE Retrieves the matrix of the module representation.

SYNTAX `m := ModuleMatrix(M);`

matrix	m
module	M

DESCRIPTION The module is represented by a pseudomatrix over a Dedekind ring and consists of a matrix and a list of ideals (one for each column). See description of `Module` for details.
If the module is a zero module false is returned.

SEE ALSO `Module`,

EXAMPLE

```
kash> O:=OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> IL:=List(Factor(2*O),f->f[1]);;
kash> List(IL,IdealGenerators);;
kash> EL1:=List(Factor(5*O),f->IdealGen(f[1],2));;
kash> EL2:=List(Factor(13*O),f->IdealGen(f[1],2));;
kash> EL2[3]:=IdealGen(Factor(7*O)[1][1],2);;
kash> EL3:=List(Factor(3*O),f->IdealGen(f[1],2));;
kash> EL3[3]:=IdealGen(Factor(7*O)[2][1],2);;
kash> M:=Module(IL,Mat(O,[EL1,EL2,EL3]));
{<2, [0, 2, 0, 3, 0, 2]><2, [0, 0, 3, 0, 1, 2]><2, [3, 3, 1, 2, 0, 0]>
[[12, 17, 15, 7, 9, 5] [19, 18, 15, 1, 10, 24] [4, 22, 4, 10, 23, 4]]
[[10, 8, 0, 1, 0, 0] [1, 12, 5, 1, 0, 0] [1, 1, 0, 0, 0, 0]]
[[1, 1, 0, 0, 0, 0] [1, 0, 1, 0, 0, 0] [3, 1, 0, 0, 0, 0]]
}

kash> ModuleMatrix(M);
[[12, 17, 15, 7, 9, 5] [19, 18, 15, 1, 10, 24] [4, 22, 4, 10, 23, 4]]
[[10, 8, 0, 1, 0, 0] [1, 12, 5, 1, 0, 0] [1, 1, 0, 0, 0, 0]]
[[1, 1, 0, 0, 0, 0] [1, 0, 1, 0, 0, 0] [3, 1, 0, 0, 0, 0]]
```

NAME ModuleMember

PURPOSE Tests if a column vector is element of the given module.

SYNTAX `L:=ModuleMember(M,V|EL);`

list	L	see below
module	M	
vector	V	
list	EL	of n algebraic elements of $\mathcal{O}$

DESCRIPTION The column vector v is a member of the module iff it is a linear combination of the columns of the matrix of M with coefficients in the corresponding coefficient ideals.
This function returns a list of a boolean value (if the element is indeed a member of the module) and a representation column vector x where $Mx = v$. x is returned even if its entries are not members of the corresponding ideals of the module.

EXAMPLE

```
kash> O:=OrderMaximal(Poly(Zx,[1,-10,-3,-2]));;
kash> M:=Module([2*O,2*O,1*O],Mat(O,[[1,0,1],[0,1,1],[0,0,1]]));
{<2><2><1>
[1 0 1]
[0 1 1]
[0 0 1]
}

kash> V:=MatTrans(O,[[1,3,1]]);
[1]
[3]
[1]
kash> L:=ModuleMember(M,V);
[ true, [0]
    [2]
    [1] ]
kash> ModuleMatrix(M)*L[2];
[1]
[3]
[1]
```

NAME ModuleModul

PURPOSE Returns an ideal which can be used for reduction.

SYNTAX `a := ModuleModul(M)`

ideal	a
module	M

DESCRIPTION Let R be a Dedekind ring, $n \in \mathbb{N}$, $M \subset \mathrm{Q}(R)^n$ be a module of degree n over the ring R. The ideal $\mathfrak{a}$ guarantees $\mathfrak{a}R^n \subset M$ where R^n denotes the 1–module of degree n over R.
It is computed as follows: Let d be the denominator of M, $\mathfrak{a} := \frac{1}{d}\det(dM)$.

SEE ALSO ModuleDet, ModuleDen,

EXAMPLE

```
kash>  O:=OrderMaximal(Poly(Zx, [1,-10,-3,-2]));;
kash> o:=OrderMaximal(O,3,3);;
kash> Oa:=OrderMaximal(OrderAbs(o));;
kash> L:=List(Factor(10*Oa),i->i[1]);;
kash> M:=IdealBasis(IdealMove(L[2]*L[6]/2, o));
{<5, [4, 1, 2] / 2><1, [2, 3, 2] / 2><1 / 2>
[1 -2 -1]
[0 1 1]
[0 0 1]
}

kash> I := ModuleModul(M);
<
[20  6  4]
[ 0  1  0]
[ 0  0  1]
/16>

kash> I*ModuleId(O,3) < M;
false
```

Name `ModuleMove`

Purpose Changes the coefficient order of the module.

Syntax `M2 := ModuleMove(M1, o);`

`same type as M1`	`M2`
`module	list of modules`	`M1`
`order`	`o`

Description Suppose `M1` is an $\mathfrak{o}_1$-module and there is an order $\mathfrak{o}_2$ such that there is an homomorphism from $\phi : \mathfrak{o}_1 \to \mathfrak{o}_2$ this function will compute $\phi($`M1`$)$. This is done simply by applying ϕ to all the matrix elements and to the coefficient ideals of `M1`. ϕ must be known to the system before using this function.
If `M1` is a list of modules all modules contained are moved.

See Also `EltMove`, `IdealMove`,

Example

```
kash> O:=Order(Poly(Zx,[1,1,-4,-14,3,1]));;
kash> o:=OrderMaximal(O);;
kash> M:=Module([Ideal(2,Elt(o,[0, 0, 0, 1, 1])),1*o],
> Mat(o,[[1,Elt(o, [2, 2, 0, 2, 0])],[0,1]]));
{<2, [0, 0, 0, 1, 1]><1>
[1 [2, 2, 0, 2, 0]]
[0 1]
}

kash> M1:=ModuleMove(M, O);
{<2, [2, 1, 1, 1, 1] / 2><1>
[1 [3, 3, 1, 1, 0]]
[0 1]
}

kash> M2:=ModuleMove(M1, o);
{<2, [0, 0, 0, 1, 1]><1>
[1 [2, 2, 0, 2, 0]]
[0 1]
}

kash> M=M2;
true
```

Name `ModuleNF`

Purpose Computes a normal form of the given module.

Syntax `M2 := ModuleNF(M1 [,d|I] [,"PBNF"] [,"lower"] );`

`modules`	`M1, M2`	
`integer`	`d`	used for reduction
`ideal`	`I`	used for reduction

Description The module will be represented by a pseudomatrix whose matrix is in triangular form with ones on the diagonal.
Two different algorithms are implemented. One is according to [Coh96] (the default), the other one according to [BP91] (which is invoked by the string parameter `PBNF`).
The pseudomatrix will be either in upper triangular form which is the default or in lower triangular form if requested by the string parameter `lower`.
Modular reduction is implemented using either an integer or an ideal.

See Also `Module`,

Example

```
kash> O:=OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> IL:=List(Factor(2*O),f->f[1]);;
kash> List(IL,IdealGenerators);;
kash> EL1:=List(Factor(5*O),f->IdealGen(f[1],2));;
kash> EL2:=List(Factor(13*O),f->IdealGen(f[1],2));;
kash> EL2[3]:=IdealGen(Factor(7*O)[1][1],2);;
kash> EL3:=List(Factor(3*O),f->IdealGen(f[1],2));;
kash> EL3[3]:=IdealGen(Factor(7*O)[2][1],2);;
kash> M:=Module(IL,Mat(O,[EL1,EL2,EL3]));
{<2, [0, 2, 0, 3, 0, 2]><2, [0, 0, 3, 0, 1, 2]><2, [3, 3, 1, 2, 0, 0]>
[[12, 17, 15, 7, 9, 5] [19, 18, 15, 1, 10, 24] [4, 22, 4, 10, 23, 4]]
[[10, 8, 0, 1, 0, 0] [1, 12, 5, 1, 0, 0] [1, 1, 0, 0, 0, 0]]
[[1, 1, 0, 0, 0, 0] [1, 0, 1, 0, 0, 0] [3, 1, 0, 0, 0, 0]]
}

kash> ModuleNF(M);
{<26236112388386371287536615054, [294144153883328, 688333593256040823590763075\
215506079571362956748104814491, 106413296427452, 30897359779322, 2714188044413\
95, 688333593256040823590763075215506079571362445884846952856]><2><2, [3, 2, 1\
```

```
, 2, 2, 0]>
[1 [-10218246369603710638941628710, 0, 1, 0, 0, 1] / 2 [7551087260070784648140\
818381, -1, 1, 1, 1, 1] / 2]
[0 1 [2, -1, -1, 1, 2, 0] / 2]
[0 0 1]
}
```

NAME `ModuleOrder`

PURPOSE Retrieves the order over which the module is defined.

SYNTAX `o := ModuleOrder(M);`

order	o
module	M

SEE ALSO Module,

EXAMPLE

```
kash> O:=OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> IL:=List(Factor(2*O),f->f[1]);;
kash> List(IL,IdealGenerators);;
kash> EL1:=List(Factor(5*O),f->IdealGen(f[1],2));;
kash> EL2:=List(Factor(13*O),f->IdealGen(f[1],2));;
kash> EL2[3]:=IdealGen(Factor(7*O)[1][1],2);;
kash> EL3:=List(Factor(3*O),f->IdealGen(f[1],2));;
kash> EL3[3]:=IdealGen(Factor(7*O)[2][1],2);;
kash> M:=Module(IL,Mat(O,[EL1,EL2,EL3]));
{<2, [0, 2, 0, 3, 0, 2]><2, [0, 0, 3, 0, 1, 2]><2, [3, 3, 1, 2, 0, 0]>
[[12, 17, 15, 7, 9, 5] [19, 18, 15, 1, 10, 24] [4, 22, 4, 10, 23, 4]]
[[10, 8, 0, 1, 0, 0] [1, 12, 5, 1, 0, 0] [1, 1, 0, 0, 0, 0]]
[[1, 1, 0, 0, 0, 0] [1, 0, 1, 0, 0, 0] [3, 1, 0, 0, 0, 0]]
}

kash> ModuleOrder(M);
F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^6 + 5*x^5 - 6*x^4 - 53*x^3 + 3*x^2 + 206*x + 244
Discriminant: -182099043
```

NAME `ModuleSmith`

PURPOSE Computes the structure of the quotient module.

SYNTAX `L :=ModuleSmith(M1, M2, ["U" | "V" | "UV" );`

`modules`	`M1, M2`	
`integer`	`d`	used for reduction
`ideal`	`I`	used for reduction

DESCRIPTION

SEE ALSO `ModuleConcat`, `ModuleNF`,

EXAMPLE

```
kash>  O:=OrderMaximal(Poly(Zx, [1,-10,-3,-2]));;
kash> o:=OrderMaximal(O,3,3);;
kash> Oa:=OrderMaximal(OrderAbs(o));;
kash> L:=List(Factor(10*Oa),i->i[1]);;
kash> M1:=IdealBasis(IdealMove(L[2]*L[6], o));
{<10, [26, 1, 0]><2, [0, 0, 1]><1>
[1 -2 -1]
[0 1 1]
[0 0 1]
}

kash> M2:=IdealBasis(IdealMove(L[1]*L[6], o));
{<10, [-12714, -20615, -67306]><1><1>
[1 3 1]
[0 1 0]
[0 0 1]
}

kash> ModuleIntersection(M1,M2);
{<10, [88, 2, 99]><2, [0, 2, 1]><2, [2, 2, 1]>
[1 -2 -1]
[0 1 1]
[0 0 1]
}
```

NAME `ModuleSteinitz`

PURPOSE Computes a Steinitz form of the given module.

SYNTAX `M2 := ModuleSteinitz(M1);`

```
modules   M1, M2
```

DESCRIPTION The representation of a module is called to be in Steinitz form if all but possibly the last coefficient ideal are trivial. The ideal class of this non-trivial ideal is called the Steinitz class of the module, and is uniquely determined by the module.

The given module must be represented by a square pseudomatrix, `ModuleNF` could be applied to recieve a square pseudomatrix.

Even if the given matrix was in triangular form the result is usually not in triangular form. A description of the algorithm can be found in [Coh96, Hop98].

SEE ALSO `Module`,

EXAMPLE

```
kash> O:=OrderMaximal(Poly(Zx,[1, 5, -6, -53, 3, 206, 244]));;
kash> IL:=List(Factor(2*O),f->f[1]);;
kash> List(IL,IdealGenerators);;
kash> EL1:=List(Factor(5*O),f->IdealGen(f[1],2));;
kash> EL2:=List(Factor(13*O),f->IdealGen(f[1],2));;
kash> EL2[3]:=IdealGen(Factor(7*O)[1][1],2);;
kash> EL3:=List(Factor(3*O),f->IdealGen(f[1],2));;
kash> EL3[3]:=IdealGen(Factor(7*O)[2][1],2);;
kash> M:=Module(IL,Mat(O,[EL1,EL2,EL3]));
{<2, [0, 2, 0, 3, 0, 2]><2, [0, 0, 3, 0, 1, 2]><2, [3, 3, 1, 2, 0, 0]>
[[12, 17, 15, 7, 9, 5] [19, 18, 15, 1, 10, 24] [4, 22, 4, 10, 23, 4]]
[[10, 8, 0, 1, 0, 0] [1, 12, 5, 1, 0, 0] [1, 1, 0, 0, 0, 0]]
[[1, 1, 0, 0, 0, 0] [1, 0, 1, 0, 0, 0] [3, 1, 0, 0, 0, 0]]
}

kash> ModuleSteinitz(M);
{<1><1><1>
[[4394728, -1966075, 1823048, 571045, 4429120, -10180704] [-886494713833578911\
17763, 4930959390474156685802, -5207051362719771705169 2, -1650569801283810864\
2379, -1056656188351775787245 52, 2502541358003563030733 04] [149910328613511789\
69776672140483616, -229672252831431500221652858070939 73, -346734888860880361 07\
```

```
962483287508403, -44372242268972323659181951240442 50, -74602028023951564691914\
894717311488, 197126665681095115272266885027081077]]
[[-329720, 131751, -131932, -39035, -319020, 734938] [5821306724808390554304 8,\
 -270057592804693528529 93, 253865003016274388789 53, 794121678637459308240 8, 59\
825169409273038698515, -13824539799817077548419 3] [-11853849383701969927973650\
52296325118, 544037966810989998767976423878766838, -48534301452621879401121006\
6434824596, -146424099146002933785473535422180681, -11324954441447246556135245\
21832008039, 261424261841649931043198623495343243 8]]
[[-7556, 2694, -2857, -791, -7018, 16204] [325864287111255254043 0, -1479582178\
071356246319, 134613725986826261265 0, 426410933164074407916, 32313579395021170\
17165, -744914667525599089919 6] [-170193889031977664076567799469094156, 748152\
087474392119099747789520220 06, -626681870185774646681970756890184 08, -19947714\
48152074519237061657813826 5, -151478708262378372278211079885467801, 3490309531\
12850724274499079221773238]]
}
```

NAME `ModuleUnion`

PURPOSE Computes the union of two modules.

SYNTAX `M :=ModuleUnion(M1, M2 [,d|I] [,"PBNF"] [,"lower"] );`

`modules`	`M, M1, M2`	
`integer`	`d`	used for reduction
`ideal`	`I`	used for reduction

DESCRIPTION The result of this function is the union or sum of the two modules in normal form.
For a detailed description of the parameters for the modular normal form computation see `ModuleNF`. This function is basically a `ModuleConcat` of the modules and a `ModuleNF` on the result.
The operator '+' on modules is identical to `ModuleUnion` except the additionsl parameters for the normal form computation.

SEE ALSO `ModuleConcat`, `ModuleNF`,

EXAMPLE

```
kash>  O:=OrderMaximal(Poly(Zx, [1,-10,-3,-2]));;
kash> o:=OrderMaximal(Order(O,3,3));;
kash> Oa:=OrderMaximal(OrderAbs(o));;
kash> L:=List(Factor(10*Oa),i->i[1]);;
kash> M1:=IdealBasis(IdealMove(L[2]*L[6], o));
{<10, [26, 1, 0]><2, [0, 0, 1]><1>
[1 -2 -1]
[0 1 1]
[0 0 1]
}

kash> M2:=IdealBasis(IdealMove(L[1]*L[6], o));
{<10, [-12714, -20615, -67306]><1><1>
[1 3 1]
[0 1 0]
[0 0 1]
}

kash> ModuleUnion(M1,M2);
{<5, [0, 2, 2]><1><1>
```

```
[1 -2 1]
[0 1 0]
[0 0 1]
}
```

NAME `Move`

PURPOSE Moves an object to another structure (order, ring)

SYNTAX
```
a := Move(L, O);
N := Move(M, r);
a := Move(f, r)
```

`integer or polynomial or ideal`	`a`
`list of integers of polynomials or ideals`	`L`
`Order`	`O`
`matrix`	`N, M`
`polynomial`	`f`
`ring`	`r`

DESCRIPTION Moves algebraic numbers (Elt), ideals (Ideal), pseudomatrices (Module), or lists of those to the order `O`.
Moves a matrix (Mat) to a matrix over another ring R.
Tries to compute a representation of a polynomial in a different polynomial algebra.

SEE ALSO `EltMove`, `IdealMove`, `ModuleMove`, `MatMove`, `PolyMove`, `FFEltMove`,

EXAMPLE This is the same example as IdealMove apart from using the function Move.

```
kash> O := OrderMaximal(Order(x^6 - 9*x^4 - 4*x^3 + 27*x^2 - 36*x - 23));;
kash> o := OrderShort(O);;
kash> Move(2*o, O);
<2>
kash> Move(Factor(2*o), O);
[ [ <2, [1, -2, 2, 1, 4, -4]>, 6 ] ]
```

NAME MultiCounterInc

PURPOSE Increments a multi counter subject to a bound.

SYNTAX `b := MultiCounterInc(L, n);`

boolean	b
list	L
integer | list	n

DESCRIPTION Increments a multi counter `L`. Incrementation is done if `L[i] < n` for some `i`, if `n` is an integer or `L[i] < n[i]` for some i, if `n` is a list. In case the counter was incremented, "true" is returned. Otherwise "false" is returned.

SEE ALSO MultiCounterInit,

EXAMPLE A counter with 2 items:

```
kash> L := MultiCounterInit(2);;
kash> while MultiCounterInc(L, 2) do
> Print(L, " ");
> od; Print("\n");[ 0, 0 ] [ 1, 0 ] [ 2, 0 ] [ 0, 1 ] [ 1, 1 ] [ 2, 1 ] [ 0, 2 ] [ 1, 2 ]
[ 2, 2 ] >
```

NAME MultiCounterInit

PURPOSE Initializes a multi counter.

SYNTAX `L := MultiCounterInit(n);`

list	L
integer	n

DESCRIPTION Initializes a list `L` of length `n` to be used by MultiCounterInc.

SEE ALSO `MultiCounterInc`,

EXAMPLE A counter with 2 items:

```
kash> L := MultiCounterInit(2);;
kash> while MultiCounterInc(L, 2) do
> Print (L, " ");
> od; Print("\n");[ 0, 0 ] [ 1, 0 ] [ 2, 0 ] [ 0, 1 ] [ 1, 1 ] [ 2, 1 ] [ 0, 2 ] [ 1, 2 ]
[ 2, 2 ] >
```

NextPrime

Name	`NextPrime`
Purpose	Returns the smallest rational prime greater a given rational integer.
Syntax	`p := NextPrime(n);`

```
integer  p
integer  n
```

Description

Example

```
kash> NextPrime(-1000);
2
kash> NextPrime(2);
3
kash> NextPrime(NextPrime(NextPrime(NextPrime(0))));
7
```

NAME Nice

PURPOSE If called with no parameter, this function returns the current priority (nice) value. Otherwise the priority (nice) value is set to the parameter.

SYNTAX
```
Nice(v);
v := Nice();
```

```
small integer  v
```

EXAMPLE

```
kash> Nice();
0
```

Norm

NAME Norm

PURPOSE Computes the norm of an algebraic number, an ideal, a polynomial or an alff element or divisor.

SYNTAX
```
n := Norm(a);
n := Norm(b, o);
```

int | rational | algebraic number | polynomial	n
algebraic element | ideal | polynomial | alff element | ff element	a
algebraic element | ff element	b
order	o

DESCRIPTION Depending on the type of `a` or `b` the norm of an algebraic element, the norm of an ideal, polynomial or an alff element or divisor is computed.

EXAMPLE This examples take place in $\mathbb{Q}(\sqrt{5})$:

```
kash> o := Order(Z, 2, 5);;
kash> Norm(Elt(o, [1, 2]));
-19
kash> u := OrderUnitsFund(o);
[ [2, 1] ]
kash> Norm(u[1]);
-1
kash> Norm(2*o);
4
kash> Norm(Ideal(2, Elt(o,[2,3 ])));
1
```

NAME Num

PURPOSE Returns the numerator of an object.

SYNTAX

```
d := Num( q );
d := Num( a );
d := Num(I);
```

integer	d
quotient field element or polynomial	d
rational	q
algebraic function field order element	a
algebraic element	a
quotient field element or polynomial	q
ideal	I

DESCRIPTION For a rational number the function returns the numerator of the rational argument.
For an algebraic number a or for a fractional ideal $\mathfrak{a}$ the function returns $d \cdot a$ or $d \cdot \mathfrak{a}$. d is the smallest integer that $d \cdot a$ is an algebraic integral or an integral ideal
For an algebraic function field order element see `AlffEltDen`.

EXAMPLE An absolute example for an algebraic element:

```
kash> o := Order (x^3 - 23*x^2 + 146*x - 244);;
kash> a := Elt (o, [1/2,3/5,45/5657567]);
[28287835, 33945402, 450] / 56575670
kash> Num ( a );
[28287835, 33945402, 450]
```

Open

NAME Open

PURPOSE (Re)opens a file.

SYNTAX

```
f := Open(name, mode);
ok := Open(f);
```

File	f
string	name
string	mode
boolean	ok

DESCRIPTION The `Open` function opens the file specified in the argument `name`. The type of operations you intend to perform on the file must be given in the argument `mode`. The following table explains the values that the `mode` string can take.

Access mode	Interpretation
"r"	Opens file for read operations.
"w"	Opens a new file for writing. If the file exists, its content is destroyed.
"a"	Opens file for appending. A new file is created if the file does not exist.
"R"	Opens file for *unbuffered* read operations. This mode should be used in connection with the `FLDin` function.

If the file is successfully opened, `Open` returns a `File` object `f`, an object to be used in subsequent I/O operations on the file. The `File` object is a structure which contains information about the name of the file, the access mode and the status indicating whether the file is opened or closed. In case of an error, `Open` returns `false`.

SEE ALSO BagRead, BagWrite, Close, ECHOon, ECHOoff, FLDin, FLDout, LOFILES,

EXAMPLE We will open, close and reopen a file named ”dummy”.

```
kash> f := Open("dummy","w");
Filename: dummy / Mode: w / Open (fid): 5
kash> Close(f);
true
kash> f;
Filename: dummy / Mode: w / Closed
```

```
kash> Open(f);
true
kash> f;
Filename: dummy / Mode: w / Open (fid): 5
```

NAME `Order`

PURPOSE Returns the order defined by the given arguments.

SYNTAX
```
o1 := Order (f);
o1 := Order (o,d,alpha);
o1 := Order (o,T,d);
o1 := Order (o,T,L);
o1 := Order (o,L);
```

order	o1
order	o
polynomial	f
integer	d
matrix	T
algebraic element	alpha
list	L

DESCRIPTION In the following $\mathfrak{o}$ may be an arbitrary order or the ring of rational integers $\mathbb{Z}$. At the moment there are no checks to ensure that the created module is a ring. So it is possible that a module is returned which is not a ring. In this case the representation of the module can be used as a representation of the defined number field. Basic arithmetic with the elements of this number field is supported. But be careful when using this module as an argument in functions which require an order. Usually these functions do not check this, so it may lead to an error or even to wrong results (for example calling `OrderMaximal`).

At the moment there are five different ways of defining an order:

`Order (f)`
creates an equation order $\mathfrak{o}|1 = \mathfrak{o}[\beta]$ with $f \in \mathfrak{o}[t]$ and $f(\beta) = 0$. The polynomial f must have degree greater than 1 and must be irreducible. If f is not monic, $\mathfrak{o}|1$ may not be a ring (see above).

`Order (o,d,alpha)`
returns the order $\mathfrak{o}|1 = \mathfrak{o}[\sqrt[d]{\alpha}]$, where $d > 1$ is a rational integer and α is an element of the number field defined by $\mathfrak{o}$. If α is not integral, this leads to the difficulties mentioned above.

`Order (o,T,d)`
T must be an $n \times n$-matrix over the coefficient ring of $\mathfrak{o}$ and should be non-singular, $d \neq 0$ must be a rational integer.

In the absolute case (coefficient ring is $\mathbb{Z}$) an order $\mathfrak{o}|1$ with basis

$$(\omega|1, \ldots, \omega|n) = \frac{1}{d}(a|1, \ldots, a|n)T$$

is returned, where $a|1, \ldots, a|n$ is the basis of $\mathfrak{o}$. It is possible that $\omega|1, \ldots, \omega|n$ is not a basis for an order (see above).

In the relative case an order $\mathfrak{o}|1$ with pseudo basis $\mathfrak{a}|1 \cdot \omega|1, \ldots, \mathfrak{a}|n \cdot \omega|n$ is returned, where

$$(\omega|1, \ldots, \omega|n) = \frac{1}{d}(a|1, \ldots, a|n)T,$$

and $\mathfrak{a}|1 \cdot a|1, \ldots, \mathfrak{a}|n \cdot a|n$ is the pseudo basis of $\mathfrak{o}$. For a detailed description of relative orders and pseudo bases we refer to [Fri97]. What happens if $\mathfrak{a}|1\omega|1, \ldots, \mathfrak{a}|n\omega|n$ is not a pseudo basis of a relative order is explained above.

`Order (o,T,L)`

This is for the relative case: $\mathfrak{o}$ must be a relative order with coefficient ring $\overline{\mathfrak{o}}$, a maximal order, and relative degree n, T a $n \times n$-Matrix over the number field defined by $\overline{\mathfrak{o}}$ and $L = \{\mathfrak{a}|1, \ldots, \mathfrak{a}|n\}$ a list of (fractional) ideals defined in $\overline{\mathfrak{o}}$. A relative order $\mathfrak{o}|1$ with pseudo basis $\mathfrak{a}|1\omega|1, \ldots, \mathfrak{a}|n\omega|n$ is returned, where

$$(\omega|1, \ldots, \omega|n) = (a|1, \ldots, a|n)T$$

and $\mathfrak{a}|1a|1, \ldots, \mathfrak{a}|na|n$ is the pseudo basis of $\mathfrak{o}$. For a detailed description of relative orders and pseudo bases we refer to [Fri97]. What happens if $\mathfrak{a}|1\omega|1, \ldots, \mathfrak{a}|n\omega|n$ is not a pseudo basis of a relative order is explained above.

`Order (o,L)`

with $L = \{a|1, \ldots, a|m\}$, a list of algebraic elements, the function returns the order $\mathfrak{o}|1 = \mathfrak{o} + a|1\mathfrak{o} + \cdots + a|m\mathfrak{o}$. In the case that $\mathfrak{o} + a|1\mathfrak{o} + \cdots + a|m\mathfrak{o}$ does not define an order, see above.

Example Creation of $\mathbb{Q}(\sqrt{2})(\sqrt[3]{3})$ by polynomials.

```
kash> o := Order (Poly (Zx,[1,0,-2]));
Generating polynomial: x^2 - 2

kash> ox := PolyAlg (o);
Univariate Polynomial Ring in x over Generating polynomial: x^2 - 2

kash> zero := Elt (o,0);
0
kash> o1 := Order (Poly (ox,[Elt(o,1),zero,zero,Elt (o,-3)]));
F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^3 - 3
```

```
E 1[ 1]     x^2 - 2
```

NAME `OrderAbs`

PURPOSE Creates an absolute extension from a simple relative extension or creates a simple relative extension from a double relative extension.

SYNTAX

```
Oa := OrderAbs(O);
Oa := OrderAbs(O,"no hom");
```

order	Oa
order	O

DESCRIPTION Given a simple relative order $\mathcal{O}$ – the coefficient ring (`OrderCoefOrder`) of the coefficient ring of $\mathcal{O}$ is $\mathbb{Z}$ – this function computes an absolute equation order $\mathcal{O}|a$ with quotient field isomorphic to the quotient field of $\mathcal{O}$ over $\mathbb{Q}$.
Given a double relative order $\mathcal{O}$ – the coefficient ring of the coefficient ring of $\mathcal{O}$, denoted by $\mathfrak{o}$, has the coefficient ring $\mathbb{Z}$ – this function computes a relative equation order $\mathcal{O}|a$ with quotient field isomorphic to the quotient field of $\mathcal{O}$ over the quotient field of $\mathfrak{o}$. $\mathfrak{o}$ has to be maximal.
Usually the homomorphisms are computed and installed (cf. `OrderInstallHom`), so that elements can be moved between the orders $\mathcal{O}$ and $\mathcal{O}|a$. But this may take a while. If you want to omit the computation of the homomorphism, use the second optional argument `"no hom"`.

SEE ALSO `OrderInstallHom`,

EXAMPLE The following function merges two absolute orders

```
kash> Merge := function(o1,o2)
> local o1x, f, fl;
> o1x := PolyAlg(o1);
> f := PolyMove(OrderPoly(Zx, o2), o1x);
> fl := Factor (f);
> return OrderAbs(Order(fl[Length(fl)][1]));
> end;
function ( o1, o2 ) ... end
```

We create $\mathbb{Q}(\sqrt{5}, \sqrt{3})$:

```
kash> o := Order(Z, 2, 3);
Generating polynomial: x^2 - 3

kash> O := Order(o, 2, 5);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 5
E 1[ 1]     x^2 - 3

kash> Oa := OrderAbs(O);
Generating polynomial: x^4 - 16*x^2 + 4
```

The element $\sqrt{3}$ of $\mathfrak{o}$ is represented in $\mathcal{O}|a$:

```
kash> a:=EltMove(Elt(o, [0,1]), Oa);
[0, -14, 0, 1] / 4
kash> EltMinPoly(a, Zx);
x^2 - 3
```

A double relative extension:

```
kash> o := OrderMaximal(Z,8,2);
Generating polynomial: x^8 - 2
Discriminant: -2147483648

kash> oo := Order(o,2,5);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
```

```
F  [ 1]     x^2 - 5
E 1[ 1]     x^8 - 2

kash> O := Order(oo,2,7);
         F[1]
           /
          /
      E2[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 7
E 2[ 1]     x^2 - 5
E 1[ 1]     x^8 - 2

kash> Oa := OrderAbs(O);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^4 - 24*x^2 + 4
E 1[ 1]     x^8 - 2

kash> Oaa:= OrderAbs(Oa);
Generating polynomial: x^32 - 192*x^30 + 16160*x^28 - 779520*x^26 + 23611832*x\
^24 - 461456256*x^22 + 5726131264*x^20 - 42165517824*x^18 + 155056995992*x^16 \
- 167363594496*x^14 + 79016151424*x^12 + 17149424640*x^10 + 206341230816*x^8 -\
 1685897697792*x^6 - 781759831296*x^4 - 194736998400*x^2 + 415926965776
```

NAME OrderAutomorphisms

PURPOSE Computes or stores automorphisms of the given extension.

SYNTAX

```
aut := OrderAutomorphisms(o);
aut := OrderAutomorphisms(o, L);
aut := OrderAutomorphisms(o, "normal");
aut := OrderAutomorphisms(o, "abel");
```

list	aut	list of automorphisms
order	o	the given order
list	L	list of some known automorphisms

DESCRIPTION This function computes or stores the automorphisms of the given extension. The automorphisms are represented by algebraic numbers which are zeros of the generating polynomials of the given extension. They can be applied to algebraic numbers with the function `EltAutomorphism`.
In the case that some automorphisms are known in the list `L`, they can be stored with `OrderAutomorphisms(o, L);`.
In the other cases the automorphisms will be computed. The option `"normal"` can be omitted. With the option `"abel"` you can turn on the more efficient algorithm for abelian extensions. The computation of automorphisms is only possible for absolute normal extensions. In the case that an extension is not normal the function will return `false`. The algorithms are described in [Klü97, AK99].

SEE ALSO `EltAutomorphism`, `OrderAutomorphismsAbel`, `OrderAutomorphismsNormal`,

EXAMPLE Store an automorphism:

```
kash> o := Order(x^4-4*x^2+1);;
kash> a := Elt(o, [0,-1,0,0]);;
kash> OrderAutomorphisms(o, [a]);
[ [0, 1, 0, 0], [0, -1, 0, 0] ]
```

Compute the automorphisms:

```
kash> o := Order(x^4-4*x^2+1);;
kash> OrderAutomorphisms(o);
[ [0, 1, 0, 0], [0, -4, 0, 1], [0, 4, 0, -1], [0, -1, 0, 0] ]
```

Compute the automorphisms in the abelian case (faster):

```
kash> o := Order(x^4-4*x^2+1);;
kash> OrderAutomorphisms(o, "abel");
[ [0, 1, 0, 0], [0, -4, 0, 1], [0, 4, 0, -1], [0, -1, 0, 0] ]
```

NAME `OrderAutomorphismsAbel`

PURPOSE Computes of the given Abelian extension.

SYNTAX `aut := OrderAutomorphismsAbel(o);`

`logical`	`aut`	IsAbelian
`order`	`o`	the given order

DESCRIPTION This function computes the automorphisms of the given Abelian extension. The automorphisms are represented by algebraic numbers which are zeros of the generating polynomials of the given extension. They can be applied to algebraic numbers with the function `EltAutomorphism`.
The computation of automorphisms is only possible for absolute Abelian extensions. In the case that an extension is Abelian the function will return `true` otherwise `false`. Using the function `OrderAutomorphisms` one gets the explicit automorphisms. The algorithms are described in [Klü97, AK99].

SEE ALSO `EltAutomorphism`, `OrderAutomorphisms`, `OrderAutomorphismsNormal`,

EXAMPLE Compute the automorphisms:

```
kash> o := Order(x^4-4*x^2+1);;
kash> OrderAutomorphismsAbel(o);
true
```

NAME `OrderAutomorphismsNormal`

PURPOSE Computes automorphisms of the given normal extension.

SYNTAX `aut := OrderAutomorphismsNormal(o);`

`list`	`aut`	list of automorphisms
`order`	`o`	the given order

DESCRIPTION This function computes the automorphisms of the given normal extension. The automorphisms are represented by algebraic numbers which are zeros of the generating polynomials of the given extension. They can be applied to algebraic numbers with the function `EltAutomorphism`.
The computation of automorphisms is only possible for absolute normal extensions. In the case that an extension is normal the function will return `true` otherwise `false`. Using the function `OrderAutomorphisms` one gets the explicit automorphisms. The algorithms are described in [Klü97, AK99]. If it is known that the given extension is Abelian it is strongly recommended to use the function `OrderAutomorphismsAbel` which is much faster.

SEE ALSO `EltAutomorphism`, `OrderAutomorphisms`, `OrderAutomorphismsAbel`,

EXAMPLE Compute the automorphisms:

```
kash> o := Order(x^4-4*x^2+1);;
kash> OrderAutomorphismsNormal(o);
true
```

NAME `OrderBach`

PURPOSE Returns the value of the floor function of the Bach bound of the algebraic number field which is generated by the given order.

SYNTAX `b := OrderBach(O);`

`integer`	`b`
`order`	`O`

DESCRIPTION The given order `o` must be maximal, otherwise an error is returned.
A theorem of Bach asserts (under the assumption of GRH) that all prime ideals with norm below this bound generate the class group.

SEE ALSO `OrderMinkowski`,

EXAMPLE Compute the floor of the Bach bound of the algebraic number field $\mathbb{Q}(\sqrt[17]{2})$.

```
kash> o := Order(Z,17,2);
Generating polynomial: x^17 - 2

kash> O := OrderMaximal(o);
Generating polynomial: x^17 - 2
Discriminant: 54214017802982966177103872

kash> OrderBach(O);
42133
```

NAME `OrderBasis`

PURPOSE Returns a basis whose elements are elements of the given order with a denominator.

SYNTAX
```
L := OrderBasis(o);
L := OrderBasis(o, O);
```

list of algebraic elements	L
order	o
order	O

DESCRIPTION If the second order is omitted, the basis returned consists of elements of the equation order. Otherwise the elements are moved into the second order. In the relative case the basis elements of the pseudo basis are returned.

SEE ALSO OrderCoefIdeals,

EXAMPLE A basis of the maximal order of $x^4 + 73x^2 - 280x - 2399$ represented in the equation order and in the maximal order itself.

```
kash> O := OrderMaximal (Poly (Zx,[1,0, 73, -280, -2399]));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975

kash> OrderBasis (O);
[ 1, [0, 1, 0, 0], [1, 1, 1, 0] / 2, [414, 857, 795, 1] / 1646 ]
kash> OrderBasis (O, O);
[ 1, [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] ]
```

NAME `OrderBasisIsPower`

PURPOSE Returns `true` iff the given order is an equation order.

SYNTAX `B := OrderBasisIsPower(o);`

boolean	B
order	o

DESCRIPTION `OrderBasisIsPower` checks whether in the actual KANT-representation a given order `o` is an equation order or not. It does not check whether a given order could be represented as an equation order.

SEE ALSO `OrderBasisIsRel`, `OrderEquationOrder`,

EXAMPLE Check, whether or not a certain order is an equation order.

```
kash> o := Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> O := OrderMaximal (o);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176
Discriminant: 60050488100625

kash> OrderBasisIsPower (O);
false
kash> OrderBasisIsPower (OrderEquationOrder (O));
true
```

NAME `OrderBasisIsRel`

PURPOSE Returns true iff the basis of an order is given by a transformation matrix.

SYNTAX `b := OrderBasisIsRel(o);`

boolean	b
order	o

SEE ALSO `OrderBasisIsPower`, `OrderEquationOrder`,

EXAMPLE Checks, whether a certain order is an equation order or given by a transformation matrix.

```
kash> O:= Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> OrderBasisIsRel(OrderEquationOrder(O));
false
kash> OrderBasisIsRel(O);
false
```

NAME OrderClassGroup

PURPOSE Computes the class group.

SYNTAX `L := OrderClassGroup( O [,b] [,"fast"] [,"Euler"] );`

list	L
order	O
integer	b

DESCRIPTION The `OrderClassGroup` function computes the class group of the algebraic number field which is generated by the order $\mathcal{O}$. It returns a list `L` whose first entry is the class number. The second entry is a list which contains the orders of the cyclic subgroups.

Notice that the class group is stored in the order $\mathcal{O}$. To get more information on the class group print out the order.

Several (optional) arguments can be passed to this function: The first argument has always to be the order. As a second argument a bound `b` for prime ideals to be considered may be given. If it is omitted, `OrderClassGroup` uses the Minkowski bound for the computation of the class group. The Minkowski bound always guarantees correct results. However, when the field discriminant is *large*, the Minkowski bound requires very time consuming computations.

Additionally, the following options can be passed to speed up the computation: `"fast"` indicates a fast computation without checking the results `"Euler"` uses the Euler product for computation. The check omitted when using `"fast"` may be done later by using `OrderClassGroupCheck`.

SEE ALSO `OrderClassGroupCheck`, `OrderClassGroupCyclicFactors`, `IdealClassRep`, `IdealIsPrincipal`,

EXAMPLE We are going to compute the class group of $\mathcal{F} = \mathbb{Q}(\sqrt[4]{-65})$.

```
kash> o := OrderMaximal(Z,4,-65);;
kash> Time(true);
true
Time: 0 ms
kash> OrderClassGroup(o);
[ 128, [ 2, 4, 4, 4 ] ]
Time: 440 ms
```

The class group is of order 128 and is isomorphic to $C_2 \times C_4 \times C_4 \times C_4$.

In the example above, the time display was activated. Compare the runtime difference when computing the class group of $\mathcal{F}$ by taking 30 as a bound for the prime ideals. Notice, that the Minkowski bound is 1274.

```
kash> o := OrderMaximal(Z,4,-65);;
kash> OrderMinkowski(o);
1274
Time: 0 ms
kash> OrderClassGroup(o,30);
[ 128, [ 2, 4, 4, 4 ] ]
Time: 180 ms
```

NAME `OrderClassGroupCheck`

PURPOSE Checks the results of class group computations done by using the Euler product.

SYNTAX `OrderClassGroupCheck(o, [[lb,] ub ] | [p, "pmax"]);`

`order`	`o`	
`integer`	`ub`	upper resp. lower bound
`integer`	`p`	prime number

DESCRIPTION This function checks the computed class group to be the ideal class group generated by all prime ideals with norm below the used ideal bound. The units are proved to be fundamental. You may specify by `lb` and `ub` a lower and upper bound for primes p for which the units are checked to be p-maximal. In this way you may compute a class group using the option "fast" first and then perform this test later on. This function returns after quite a while if the results are wrong. It is therefore sometimes convenient to switch on output using the `PRINTLEVEL` function.
The last calling sequence is used to ensure that subsequently computed S-units are p-maximal (special case of the previous calling sequences).

SEE ALSO `OrderClassGroup`,

EXAMPLE Of $\mathbb{Q}(\sqrt[6]{-13})$ we check the results:

```
kash> o := OrderMaximal(x^6+13);;
kash> OrderClassGroup(o, 500, fast);
[ 6, [ 6 ] ]
kash> OrderClassGroupCheck(o);
true
```

NAME `OrderClassGroupCyclicFactors`

PURPOSE Returns a list with the generators of the cyclic factors.

SYNTAX `L := OrderClassGroupCyclicFactors(O);`

`list`	`L`
`order`	`O`

DESCRIPTION Returns a list with the generators of the cyclic factors of the class group together with their orders in the class group. The list contains sublists and each sublist contains two items. The first is an ideal, and the second item is the order of this ideal in the class group.
This function does not compute the class group so that `OrderClassGroup` has to be called in advance.

SEE ALSO `OrderClassGroup`, `OrderClassGroupCyclicFactorsPrincipal`,

EXAMPLE Compute the class group structure of $\mathbb{Q}(\sqrt[4]{-65})$.

```
kash> O := OrderMaximal(Z,4,-65);;
kash> OrderClassGroup (O, 100);
[ 128, [ 2, 4, 4, 4 ] ]
kash> OrderClassGroupCyclicFactors (O);
[ [ <30, [885, 896, 0, 899]>, 2 ], [ <3, [2, 1, 0, 0]>, 4 ],
  [ <3, [1, 0, 1, 0]>, 4 ], [ <2, [1, 1, 0, 0]>, 4 ] ]
```

NAME `OrderClassGroupCyclicFactorsPrincipal`

PURPOSE Returns a list of the generators of the cyclic factors of the class group to the power of their orders.

SYNTAX `L := OrderClassGroupCyclicFactorsPrincipal(O, ["raw"]);`

`list`	`L`
`order`	`O`

DESCRIPTION This function computes the generating elements of the powers of the cyclic factors of the class group to their orders (which are principal integral ideals). Given a second string parameter `"raw"` a power product representation is returned.
This function does not compute the class group so that `OrderClassGroup` has to be called in advance.

SEE ALSO `OrderClassGroup`, `OrderClassGroupCyclicFactors`,

EXAMPLE Compute the class group structure of $\mathbb{Q}(\sqrt[4]{-65})$.

```
kash> O := OrderMaximal(Z,4,-65);;
kash> OrderClassGroup (O, 100);
[ 128, [ 2, 4, 4, 4 ] ]
kash> L1 := OrderClassGroupCyclicFactors(O);
[ [ <30, [885, 896, 0, 899]>, 2 ], [ <3, [2, 1, 0, 0]>, 4 ],
  [ <3, [1, 0, 1, 0]>, 4 ], [ <2, [1, 1, 0, 0]>, 4 ] ]
kash> L2 := OrderClassGroupCyclicFactorsPrincipal(O);
[ [5, 0, -1, 0], [-2, -1, 0, 0], [4, 0, 1, 0], 2 ]
kash> L1[2][1]^4/L2[2] = 1*O;
true
```

NAME `OrderClassGroupFactorBasisProve`

PURPOSE Checks whether a small factorbasis generates the same subgroup of the class-group as a large factorbasis.

SYNTAX `r := OrderClassGroupFactorBasisProve(o, lb, ub);`

`boolean`	`r`	
`order`	`o`	
`integer`	`lb, ub`	lower resp. upper bound

DESCRIPTION This function is used to check whether a given small factorbasis generates the same subgroup of the classgroup as a large one. It is recommended first to compute the class group using a small bound, i. e. a small factorbasis, and then afterwards to check if the class group was computed itself.
Actually this function is not really needed since one has a good criterion by the euler product but which is not fully proven. This function is only able to detect the equality of the factor bases. It should be used only for small fields and may take a very long time.

SEE ALSO `OrderClassGroup`,

EXAMPLE Compute the class group structure of $\mathbb{Q}(\sqrt[4]{-65})$.

```
kash> O := OrderMaximal(Z,4,-65);;
kash> OrderClassGroup (O, 100, "Euler");
[ 128, [ 2, 4, 4, 4 ] ]
kash> OrderMinkowski(O);
1274
kash> OrderClassGroupFactorBasisProve(O, 100, 1274);
true
```

NAME OrderCoefIdeals

PURPOSE Returns the list of coefficient ideals of a relative order.

SYNTAX `L := OrderCoefIdeals(o);`

```
list of ideals  L
relative order  o
```

DESCRIPTION Let $\mathfrak{o}$ be a relative order over a maximal order $\mathcal{O}$. Then $\mathfrak{o}$ can be represented via a pseudo basis:

$$\mathfrak{o} = \mathfrak{a}|1\xi|1 + \ldots + \mathfrak{a}|n\xi|n$$

where the $\mathfrak{a}|i$ are $\mathcal{O}$–ideals and the $\xi|i$ (possibly fractional) elements of $\mathcal{O}$. The $\mathfrak{a}|i$ are the coefficient ideals (returned by this function), the $\xi|i$ are the basis elements (returned by `OrderBasis`).
Returns an error if the order is not relative.

SEE ALSO `OrderBasis`,

EXAMPLE The coefficient ideals of a relative maximal order.

```
kash> O := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> o := OrderMaximal(O, 2, 5);
      F[1]
       |
      F[2]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^2 - 5
E 1[ 1]     x^2 - 10
Discriminant: <1>
Coef. Ideals are: <1>, <1, [5, 1] / 10>

kash> OrderCoefIdeals(o);
[ <1>, <1, [5, 1] / 10> ]
```

NAME	`OrderCoefOrder`
PURPOSE	Returns the coefficient ring of a given order.
SYNTAX	`c := OrderCoefOrder(o);`

order, ring of integers	c
order	o

DESCRIPTION	Returns the coefficient ring c of the given order o.
SEE ALSO	`OrderEquationOrder`, `OrderSubOrder`,

EXAMPLE A simple relative extension

```
kash> o1:=Order(Z,2,2);
Generating polynomial: x^2 - 2

kash> o2:=Order(o1,2,3);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 3
E 1[ 1]     x^2 - 2

kash> OrderCoefOrder(o2);
Generating polynomial: x^2 - 2
```

NAME `OrderCyclotomic`

PURPOSE Computes the n'th cyclotomic field.

SYNTAX `o := OrderCyclotomic(n);`

```
order    o
integer  n
```

DESCRIPTION

EXAMPLE

```
kash> o := OrderCyclotomic(15);
Generating polynomial: x^8 - x^7 + x^5 - x^4 + x^3 - x + 1

kash> OrderIsMaximal(o);
true
kash> OrderDisc(o);
1265625
kash> OrderAutomorphisms(o, []);
[ [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0],
  [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1],
  [-1, 1, 0, -1, 1, -1, 0, 1], [0, -1, 0, 0, 0, 0, -1, 0],
  [1, -1, 0, 0, -1, 1, 0, -1], [1, 0, -1, 1, -1, 0, 1, -1] ]
```

NAME `OrderCyclotomicRealSubfield`

PURPOSE Computes the maximal real subfield of the n'th cyclotomic field.

SYNTAX `o := OrderCyclotomicRealSubfield(n);`

```
order    o
integer  n
```

DESCRIPTION

EXAMPLE

```
kash> o := OrderCyclotomicRealSubfield(11);
Generating polynomial: x^5 + x^4 - 4*x^3 - 3*x^2 + 3*x + 1
```

NAME `OrderDeg`

PURPOSE Returns the relative degree of the given order.

SYNTAX `d := OrderDeg(o);`

```
integer  d
order    o
```

DESCRIPTION Returns the degree $[\mathbb{Q}(\mathfrak{o}) : \mathbb{Q}(C(\mathfrak{o}))]$ of $\mathbb{Q}(\mathfrak{o})$ as vector space over $\mathbb{Q}(C(\mathfrak{o}))$, where $C(\mathfrak{o})$ is the coefficient ring of the given order.

SEE ALSO `OrderDegAbs`,

EXAMPLE Some relative degrees of extensions.

```
kash> F := Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> E := Order (F, 2, 7);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 7
E 1[ 1]     x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> OrderDeg (E);
2
kash> OrderDeg (F);
8
```

NAME OrderDegAbs

PURPOSE Returns the absolute degree of the given order.

SYNTAX `d := OrderDegAbs(o);`

```
integer  d
order    o
```

DESCRIPTION Returns $[\mathbb{Q}(\mathfrak{o}) : \mathbb{Q}]$, the absolute degree of the given order $\mathfrak{o}$.

SEE ALSO OrderDeg,

EXAMPLE Absolute degrees of extensions.

```
kash> F := Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> E := Order (F, 2, 7);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 7
E 1[ 1]     x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> OrderDegAbs (E);
16
kash> OrderDegAbs (F);
8
```

NAME `OrderDisc`

PURPOSE Returns the discriminant of the given order.

SYNTAX `D := OrderDisc(o);`

`integer or ideal`	`D`	
`order`	`o`	over Z or over a maximal order

DESCRIPTION Returns the discriminant of the order `o` given over $\mathbb{Z}$ or over a maximal order. The discriminant of an absolute order is a rational integer, of a relative order an ideal.

SEE ALSO `OrderDisc`, `PolyDisc`,

EXAMPLE Compute the discriminant of an order over $\mathbb{Z}$.

```
kash> o := Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> OrderDisc (o);
822201012939308277619413178871929896960000
kash> o := OrderMaximal (o);;
kash> OrderDisc (o);
60050488100625
```

NAME `OrderEquationOrder`

PURPOSE Returns the equation suborder of the given order.

SYNTAX `oe := OrderEquationOrder(o);`

```
order  oe
order  o
```

DESCRIPTION `OrderEquationOrder` returns the equation suborder $\mathfrak{o}|E$ of a given order $\mathfrak{o}$.

SEE ALSO `OrderBasisIsPower`, `OrderBasisIsRel`,

EXAMPLE Extracting the equation order of an arbitrary order.

```
kash> o := Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> O := OrderMaximal (o);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176
Discriminant: 60050488100625

kash> oe := OrderEquationOrder (O);
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176
Discriminant: 822201012939308277619413178871929896960000
```

NAME `OrderExcepSequence`

PURPOSE Computes a sequence of exceptional units of maximal length in a given order.

SYNTAX `L := OrderExcepSequence (o);`

`list`	`L`	exceptional sequence
`order`	`o`	

DESCRIPTION Let $\mathfrak{o}$ be an order over $\mathbb{Z}$ and let $\omega_1, \ldots, \omega_m$ $(m \geq 2)$ be a sequence of elements of $\mathfrak{o}$. We say that this sequence is an exceptional sequence if $\omega_1 = 0, \omega_2 = 1$ and all the mutual differences $\omega_i - \omega_j$ $(1 \leq i < j \leq m)$ are units. Note that for such an exceptional sequence the elements $\omega_3, \ldots, \omega_m$ are exceptional units in the terminology of T. Nagell [Nag69].

The `OrderExcepSequence` function returns an exceptional sequence of maximal length.

SEE ALSO `OrderUnitsExcep`,

EXAMPLE Compute an exceptional sequence of maximal length in $\mathbb{Z}[\zeta_7]$:

```
kash> o := Order(Poly(Zx,[1,1,1,1,1,1,1]));
Generating polynomial: x^6 + x^5 + x^4 + x^3 + x^2 + x + 1

kash> OrderExcepSequence(o);
[ 0, 1, [3, 0, 2, 1, 1, 2], [0, -1, 0, -1, 0, -1], [0, 0, -1, 0, 0, -1],
  [1, 0, 1, 0, 0, 1], [-2, 0, -2, -1, -1, -2] ]
```

NAME OrderFincke

PURPOSE Computes or sets the Fincke constants.

SYNTAX

```
OrderFincke(o);
OrderFincke(o, x);
OrderFincke(o, x, y);
```

```
order  o
real   x
real   y
```

DESCRIPTION Computes the optimal λ and $\gamma|1$ as described in [PZ89, p. 341]. λ is defined as the unique zero of a function.
If the second optional argument x is specified, the constant λ is set to be the value of x and $\gamma|1$ is computed in an iteration beginning with the third optional argument y or 0 otherwise.

SEE ALSO OrderNormEquation,

EXAMPLE

```
kash> Time(true);
true
Time: 0 ms
kash> o:=Order(Poly(Zx,[1,0,-4,0,1]));
Generating polynomial: x^4 - 4*x^2 + 1

Time: 10 ms
kash> OrderUnitsFund(o);;
kash> OrderNormEquation(o, 3, 1, "abs");
[  ]
Time: 220 ms
kash> OrderNormEquation(o, 3, 1, "abs");
[  ]
Time: 0 ms
kash> OrderFincke(o, 10.0);
kash> OrderNormEquation(o, 3, 1, "abs");
[  ]
Time: 0 ms
kash> Time(false);
false
```

NAME `OrderGalois`

PURPOSE Computation of Galois groups.

SYNTAX

DESCRIPTION This function computes the Galois group of an irreducible polynomial with coefficients in $\mathbb{Q}$ and with degree up to 15. For description see `Galois()`.

SEE ALSO `Galois`, `GaloisT`, `GaloisGlobals`, `GaloisGroupsPossible`, `GaloisModulo`, `GaloisTree`, `GaloisRoots`, `GaloisNumberToName`, `GaloisBlocks`, NAME `OrderIndex`

PURPOSE Returns the index of the given order relative to its suborder.

SYNTAX `I := OrderIndex(o);`

```
integer  I
order    o
```

DESCRIPTION This function returns the index of the suborder $\mathfrak{o}|1$ of $\mathfrak{o}$ in $\mathfrak{o}$. The index equals $\sqrt{\mathrm{disc}(\mathfrak{o}|1)/\mathrm{disc}(\mathfrak{o})}$. In the absolute case the index is a rational integer, otherwise it is an ideal. If $\mathfrak{o}$ has no suborder 1 resp. the ideal generated by the coefficient order is returned.

SEE ALSO `OrderDisc`,

EXAMPLE Computing and checking the index of a maximal order.

```
kash> o := Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> OrderIndex (o);
1
kash> O := OrderMaximal (o);
   F[1]
     |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176
Discriminant: 60050488100625

kash> OrderIndex (O);
117012089012224
kash> OrderDisc (o) / OrderDisc (O) - OrderIndex (O)^2;
0
```

NAME `OrderIndexFormEquation`

PURPOSE Solves an index form equation.

SYNTAX `L := OrderIndexFormEquation(o,index);`

`list`	`L`
`order`	`o`
`integer`	`index`

DESCRIPTION Let $\mathfrak{o}$ be an order over $\mathbb{Z}$. We say that algebraic integers $a, b \in \mathfrak{o}$ are $\mathbb{Z}$-equivalent iff $a-b$ or $a+b$ is a rational integer. The `OrderIndexFormEquation` function computes a full set of pairwise inequivalent $\alpha \in \mathfrak{o}$ such that the module index $(\mathfrak{o} : \mathbb{Z}[\alpha])$ equals `index`. Note that this set is always finite due to a result of K. Győry.

At present, the `OrderIndexFormEquation` function can only handle cubic and quartic fields. Algorithms are taken from [GPP93, GPP96, GS89].

SEE ALSO `EltIndex`,

EXAMPLE Compute - up to $\mathbb{Z}$-equivalence - all generators of power integral bases of $\mathbb{Z}[\sqrt[3]{2}]$
:

```
kash> o := Order(Z,3,2);
Generating polynomial: x^3 - 2

kash> OrderIndexFormEquation(o,1);
[ [0, 1, 0], [0, 1, 1] ]
```

NAME `OrderInstallHom`

PURPOSE Installs an embedding from the first given order into the second given order.

SYNTAX
```
OrderInstallHom(o1,o2,alpha);
OrderInstallHom(o1,o2,b);
```

order	o1
order	o2
algebraic element	alpha
boolean	b

DESCRIPTION Installs the embedding given via the image of the primitive element of $\mathfrak{o}|1$ in $\mathfrak{o}|2$ which is defined to be α in case $\mathfrak{o}|1$ is an equation order. If the third argument is a boolean instead of an algebraic element, the corresponding number fields are assumed to be isomorphic, iff the boolean is 'true' and this will be used in the sequel.

EXAMPLE Moving of an element from an absolute extension into a relative one.

```
kash> o1 := Order (Z,2,3);;
kash> o2 := Order (o1,2,2);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 2
E 1[ 1]     x^2 - 3

kash> gen1 := OrderBasis (OrderEquationOrder (o1))[2];
[0, 1]
kash> gen2 := OrderBasis (OrderEquationOrder (o2))[2];
[0, 1]
kash> gen := EltMove (gen1, o2) + gen2;
[[0, 1], 1]
kash> o3 := Order (MatMinPoly (EltRepMat(gen, Z)[2]));
Generating polynomial: x^4 - 10*x^2 + 1

kash> OrderInstallHom (o3, o2, gen);
kash> EltMove (Elt (o3, [1,2,3,4]), o2);
```

```
[[16, 38], [46, 6]]
kash> EltMove (Elt (o3, [1,2,3,4]/2), o2);
[[8, 19], [23, 3]]
```

NAME OrderIsMaximal

PURPOSE Tests whether the given order is known to be maximal or not.

SYNTAX `B := OrderIsMaximal(o);`

```
boolean  B
order    o
```

DESCRIPTION This function checks if the given order `o` is known to be maximal according to the OrderIsMaximal flag.

SEE ALSO `OrderSetMaximal`,

EXAMPLE The order $\mathbb{Z}[\alpha]$ with $a^4 + 73\alpha^2 - 280\alpha - 2399 = 0$ is not known to be maximal.

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));;
kash> OrderIsMaximal (o);
false
kash> O := OrderMaximal (o);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975

kash> OrderIsMaximal (O);
true
```

NAME `OrderIsRelative`

PURPOSE Tests whether the given order is given as a releative extension.

SYNTAX `B := OrderIsMaximal(o);`

```
boolean  B
order    o
```

SEE ALSO `OrderSetMaximal`,

NAME `OrderIsSubfield`

PURPOSE Tests Q(o|1) ⊆ Q(o|2).

SYNTAX `OrderIsSubfield(o1, o2);`

```
order  o1
order  o2
```

DESCRIPTION This function checks whether Q(o|1) ⊆ Q(o|2) and returns true or false. If true is returned, a homomorphism between o|1 and o|2 is saved (globally) for further usage in, say, moving elements from o|1 and o|2.

EXAMPLE Check a certain subfield.

```
kash> o1 := Order (Z,2,3);
Generating polynomial: x^2 - 3

kash> o2 := Order (o1, 2, 2);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 2
E 1[ 1]     x^2 - 3

kash> M := EltRepMat (Elt (o2,[[0,1], 1]),Z)[2];
[0 3 2 0]
[1 0 0 2]
[1 0 0 3]
[0 1 1 0]
kash> o3 := Order (MatMinPoly (M));
Generating polynomial: x^4 - 10*x^2 + 1

kash> EltMove (Elt (o1,[0,1]), o3);
[0, -11, 0, 1] / 2
kash> OrderIsSubfield (o1, o3);
true
```

NAME OrderKextGenAbs

PURPOSE Computes a set of absolute generators for the ring of integers.

SYNTAX `Gen := OrderKextGenAbs(F);`

list	Gen	of algebraic elements
order	F	

DESCRIPTION Computes absolute generators for the ring of integers of a Kummer extension of prime degree [Dab95].

EXAMPLE Absolute generators in $\mathbb{Q}(\sqrt{10}, \sqrt{5})$ over $\mathbb{Q}(\sqrt{10})$

```
kash> E := OrderMaximal (Z,2,10);;
kash> F := Order (E,2,5);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> OrderKextGenRel(F);
[ 1, [1, 1] / 2, [[0, 5], [0, 9]] / 10 ]
```

NAME `OrderKextGenRel`

PURPOSE Computes relative generators for the ring of integers.

SYNTAX `Gen := OrderKextGenRel(F);`

`list`	`Gen`	of algebraic elements
`order`	`F`	

DESCRIPTION Computes a set of relative generators for the ring of integers of a Kummer externsion of prime degree [Dab95].

SEE ALSO `OrderKextGenRel`, `OrderKextDisc`,

EXAMPLE Relative generators in $\mathbb{Q}(\sqrt{10}, \sqrt{5})$ over $\mathbb{Q}(\sqrt{10})$

```
kash> E := OrderMaximal (Z,2,10);;
kash> F := Order (E,2,5);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> OrderKextGenAbs(F);
[ 0, [[0, 2], 0], [1, 1] / 2, [[0, 5], [0, 1]] / 10 ]
```

NAME `OrderKextModularPower`

PURPOSE Computes a maximal exponent.

SYNTAX `L := OrderKextModularPower(pI, mu);`

`list`	`L`	containing alpha and l
`ideal`	`pI`	a prime ideal
`algebraic element`	`mu`	

DESCRIPTION Given an prime ideal $\mathfrak{p} \subset \mathfrak{o}$ and an integral element $\mu \in \mathfrak{o} \setminus \mathfrak{p}$, such that $\zeta | p \in \mathfrak{o}$ for the prime number p dividing $\mathfrak{p}$. This function will compute the maximal exponent $l \leq p\frac{v|\mathfrak{p}\,(p)}{p-1}$ such that there is an α subject to $\alpha^p \equiv \mu \pmod{\mathfrak{p}^l}$. α and l will be returned.
This function is useful mainly in the context of Kummer extensions, where conditions like the above are used to compute the discriminant and the splitting of $\mathfrak{p}$. This is based on [Dab95, p 50].

EXAMPLE

```
kash> o := OrderCyclotomic(7);;
kash> pI := Factor(7*o)[1][1];
<7, [6, 1, 0, 0, 0, 0]>
kash> mu := Elt(o, [1,2,3,4,5,8]);
[1, 2, 3, 4, 5, 8]
kash> L := OrderKextModularPower(pI, mu);
[ 2, 1 ]
kash> EltIsInIdeal(mu - L[1]^7, pI^L[2]);
true
```

NAME `OrderLLL`

PURPOSE Creates an order with LLL-reduced basis.

SYNTAX `o1 := OrderLLL(O);`

```
order  o1
order  o
```

DESCRIPTION Creates an overorder o|1 of the given order o via transformation, so that the basis of the new order o|1 is LLL-reduced.

EXAMPLE

```
kash> o := Order (Poly(Zx,[1,0,73,-280,-2399]));;
kash> O1 := OrderMaximal (o);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975

kash> O2 := OrderSimplify (OrderLLL (O1));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975

kash> OrderBasis (O1);
[ 1, [0, 1, 0, 0], [1, 1, 1, 0] / 2, [414, 857, 795, 1] / 1646 ]
kash> OrderBasis (O2);
[ 1, [409, -34, 28, -1] / 1646, [-4045, 1775, 39, 28] / 1646,
  [2055, -1680, 28, -1] / 1646 ]
```

NAME `OrderMaximal`

PURPOSE Returns the maximal overorder of an order.

SYNTAX `O := OrderMaximal(`*def*`);`
`O := OrderMaximal(`*def*`,` *str*`);`

`order`	`O`
`see below`	*def*
`up to 4 optional strings`	*str*

DESCRIPTION The `OrderMaximal` function returns the maximal overorder of the order `o` by using the algorithm and techniques which are specified in the optional strings. The argument *def* may be the order `o` or one of the argument lists of the function `Order` which defines the order `o`.
The optional argument list *str* (1–4 strings) is used to control the algorithm: You can choose the `"Round2" XOR "Round4"` algorithm for the computation with additional techniques `"Split"` to use the algebra splitting due to the `Round4`-Algorithm or `"NoSplit"` to avoid this. The reduced discriminant can be useful to factorize the discriminant of the order `o`, which is necessary if you want to compute the maximal order. `"RD"` uses the reduced discriminant and `"NoRD"` avoids the reduced discriminant. The Dedekind-test allows a short-cut if `o` is an equation order, a simple test if `o` is already p-maximal. Using the `Round2`-Algorithm, we implemented the extended Dedekind-test which computes an overorder of the equation order if it is not p-maximal. If you want the Dedekind-test use `"Dedekind"`, otherwise type `"NoDedekind"`. A special order of these strings is not necessary, but you can use just one of the above mentioned devices in each string. If you leave one of them unspecified, the algorithm tries to find the best solution (with respect to the running time).
At present we use `"Round4"` and `"Split"` only for absolute equation orders and `RD` only for equation orders, so that these devices are changed to `"Round2"`, `"NoSplit"` and `"NoRD"` in the other cases.
If you want further information about the `Round2`-Algorithm and the Dedekind-test, see [Fri97], about the `Round4`-Algorithm, algebra-splitting and the reduced-discriminant see [Bai96].

SEE ALSO `OrderPMaximal`, `Order`,

EXAMPLE Compute the maximal order of $x^4 + 73x^2 - 280x - 2399$ with the `Round4`-Algorithm

```
kash> f := Poly (Zx,[1,0, 73, -280, -2399]);
x^4 + 73*x^2 - 280*x - 2399
```

```
kash> O := OrderMaximal (f, "Round4");
F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975
```

NAME `OrderMerge`

PURPOSE Computes extension fields which contain the given ones.

SYNTAX `L := OrderMerge(o1,o2);`

`list`	`L`	return value
`order`	`o1`	
`order`	`o2`	

DESCRIPTION This function computes fields which contain the given ones. It is possible to get more than one field (see example). The output is a list of generating polynomials for these fields.
Let $\alpha|i$, $\beta|j$ be the zeroes of the defining polynomials of $\mathfrak{o}|1$ and $\mathfrak{o}|2$. After a substitution of the form $\beta|i \mapsto \beta|i + s$, $s \in \mathbb{N}$ appropriate, all $\alpha|i + \beta|j$ are pairwise distinct. The polynomial $f := \prod|i,j(x - \alpha|i - \beta|j)$ is computed, and the irreducible factors are returned.

EXAMPLE Compute the following splitting field.

```
kash> o:=Order(Z,3,2);
Generating polynomial: x^3 - 2

kash> L:=OrderMerge(o,o);
[ Generating polynomial: x^3 - 54
    , Generating polynomial: x^6 + 108
     ]
```

NAME OrderMinIdeal

PURPOSE Computes a non-zero integral ideal of smallest norm.

SYNTAX `a := OrderMinIdeal (o);`

ideal	a
order	o

DESCRIPTION The `OrderMinIdeal` function computes an ideal $\mathfrak{a} \neq \{0\}$ of $\mathfrak{o}$ with smallest norm. Note that $\mathfrak{a}$ is always a prime ideal.

EXAMPLE

```
kash> o := Order(Poly(Zx,[1,-1,-6,4,10,-4,-4,1]));
Generating polynomial: x^7 - x^6 - 6*x^5 + 4*x^4 + 10*x^3 - 4*x^2 - 4*x + 1

kash> a := OrderMinIdeal(o);
<7, [5, 1, 0, 0, 0, 0, 0]>
kash> Norm(a);
7
```

NAME `OrderMinkowski`

PURPOSE Returns the floor of the Minkowski bound of the algebraic number field which is generated by the given order.

SYNTAX `m := OrderMinkowski(O);`

```
integer  m
order    O
```

DESCRIPTION Returns the floor of the Minkowski bound of the algebraic number field which is generated by the maximal order $\mathcal{O}$. Returns an error if $\mathcal{O}$ is not known to be maximal.

SEE ALSO `OrderBach`,

EXAMPLE Compute the floor of the Minkowski bound of the algebraic number field $\mathbb{Q}(\sqrt[4]{-27})$.

```
kash> o := Order(Z,4,-27);
Generating polynomial: x^4 + 27

kash> O := OrderMaximal(o);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 27
Discriminant: 432

kash> OrderMinkowski(O);
3
```

NAME `OrderNormEquation`

PURPOSE Solves a (relative) norm equation.

SYNTAX `L := OrderNormEquation(o, a [,n|"all" [,"exact"|"abs"|"ineq"]]);`

`list`	`L`
`int` | `element of the coefficient ring`	`a`
`int`	`n`

DESCRIPTION Searches for elements $\alpha \in \mathfrak{o}$ with norm a. The third argument specifies the number of different non associated solutions. `OrderNormEquation` tries to find n, all or, if unspecified, 1 solution.
The default for the fourth argument is `"exact"`: the function solves the above mentioned equation. If `"abs"` is given, the norm equation $\mathrm{Norm}(\alpha) = \epsilon \mathrm{a}$ is solved, where ϵ is a torsion unit of the coefficient ring of the given order. The `"ineq"`-Option is supported only in the absolute case, here the norm equation $0 \leq \mathrm{Norm}(\alpha) \leq \mathrm{a}$ is solved.
Remember that solving relative norm equations may take quite a long time. Due to the used algorithm it is faster to call the `OrderNormEquation` function using the `"abs"`-Option than using the `"exact"`-Option or the default value.
For further information about the algorithms we refer the reader to [Fin84] for the absolute case and to [Fie97] otherwise.

SEE ALSO `Solve`,

EXAMPLE First we solve an absolute norm equation in $\mathbb{Z}[\sqrt[3]{5}]$.

```
kash> o := Order(Z, 3, 5);
Generating polynomial: x^3 - 5

kash> L := OrderNormEquation(o, 4);
[ [29, 17, 10] ]
kash> L := OrderNormEquation(o, 4, "all");
[ [29, 17, 10], [-1, -1, 1] ]
kash> EltNorm(L[1]);
4
kash> EltNorm(L[2]);
4
```

Now a relative norm equation is solved in the maximal order of $\mathbb{Q}(\sqrt[3]{5}, \sqrt{2})$.

```
kash> O := Order(o, 2, 2);;
kash> EltNorm(Elt(O, [[1, 2, 3], [3, 2, 2]]));
[-37, -15, -22]
kash> a:=OrderNormEquation(O, Elt(o, [-37, -15, -22]), 1, "abs");
[ [[-5, -2, -1], [2, 0, -1]] ]
kash> EltNorm(a[1]);
[37, 15, 22]
```

NAME `OrderPMaximal`

PURPOSE Computes the p-maximal overorder of an order.

SYNTAX `op := OrderPMaximal(o,p,`*str*`);`
`op := OrderPMaximal(o,p,b,`*str*`);`

order	op
order	o
rational prime or prime ideal	p
integer	b
up to 3 optional strings	*str*

DESCRIPTION Let $\mathfrak{o}$ be an arbitrary order and let p be a prime (in the absolute case: a rational prime number, in the relative case a prime ideal). `OrderPMaximal` computes the p–maximal overorder $\mathfrak{o}_p$ of $\mathfrak{o}$ subject to the constraint $\nu_p(\mathfrak{o}_p : \mathfrak{o}) \leq b$. If the third argument b is omitted, its default value is ∞.
You can control the algorithm and additional techniques using the optional argument list *str* (1–3 strings) with the values `"Round2" XOR "Round4", "Split" XOR "NoSplit", "Dedekind" XOR "NoDedekind".` Their behavior is explained in the description of `OrderMaximal.`
A special order of these strings is not necessary, but you can use just one of these devices in each string. If you leave one of them unspecified, the algorithm tries to find the best solution (with respect to the running time).
If you want further information about the `Round2`-Algorithm and the Dedekind-test, see [Fri97], about the `Round4`-Algorithm and algebra-splitting see [Bai96].

SEE ALSO `OrderMaximal`, `Order`,

EXAMPLE Compute a maximal order by computing p-maximal orders:

```
kash> o := Order (Poly (Zx,[1,0, 73, -280, -2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> Factor (Abs (Disc (o)));
[ [ 2, 4 ], [ 5, 2 ], [ 11, 1 ], [ 19, 1 ], [ 191, 1 ], [ 823, 2 ] ]
kash> O := OrderPMaximal (o, 2, 4,"Round4");;
kash> O := OrderPMaximal (O, 5, 2,"Round2", "NoSplit");;
kash> O := OrderPMaximal (O, 823);;
kash> Disc (O);
-997975
```

We test our results:

```
kash> O := OrderMaximal (O);;
kash> Disc (O);
-997975
```

NAME `OrderPoly`

PURPOSE Returns the defining polynomial of the associated equation order

SYNTAX
```
f := OrderPoly(P,o);
f := OrderPoly(o);
```

`polynomial`	`f`
`polynomial algebra`	`P`
`order` | `algebraic function field order`	`o`

DESCRIPTION Returns the polynomial defining the equation order of `o` represented in the polynomial algebra P.
The polynomial algebra P has to be defined over the coefficient ring of `o`. If the polynomial algebra P is omitted, the polynomial f is represented over the coefficient ring of the order `o`.
For an order of an algebraic function field, the polynomial f is an element of $k[T, y]$ or $\mathcal{O}_\infty[y]$.
IDEAL order; equation; defining polynomial equation; order; defining polynomial polynomial; defining; of an equation order

SEE ALSO `OrderEquationOrder`, `AlffOrderDeg`, `Alff`,

EXAMPLE

NAME `OrderPolyHenselLift`

PURPOSE Calculates a p-adic factorization of the defining polynomial of an order.

SYNTAX `L := OrderPolyHenselLift(o, d, p, k);`

list	L
order	o
integer	d
prime	p
integer	k

DESCRIPTION Let o be an order defined by a polynomial $f \in \mathbb{Z}[x]$ and let $p \in \mathbb{Z}$ be a prime which does not divide the discriminant of f. This function computes the factorization of f over the unramified extension of $\mathbb{Q}|p$ of degree d. It returns a list containing the factors modulo p^{2^k}.

EXAMPLE

```
kash> o :=Order(Poly(Zx,[1,0,-4,0,1]));
Generating polynomial: x^4 - 4*x^2 + 1

kash> P :=OrderPolyHenselLift(o,2,5,2);
[ x + [258, 312], x + [-258, 312], x + [-258, -312], x + [258, -312] ]
kash> P[1]*P[2]*P[3]*P[4];
x^4 - 522504*x^2 + 16415759376
kash> P :=OrderPolyHenselLift(o,3,7,2);
[ x^2 - 235*x - 1, x^2 + 235*x - 1 ]
kash> P[1]*P[2];
x^4 - 55227*x^2 + 1
```

NAME `OrderPrec`

PURPOSE Sets or returns the internal precision for calculations in orders.

SYNTAX

```
P := OrderPrec(p);
P := OrderPrec();
L := OrderPrec(o,p);
L := OrderPrec(o,p,u);
L := OrderPrec(o);
```

`small integer`	`P`	
`small integer`	`p`	
`order`	`o`	
`small integer`	`u`	
`list`	`L`	precisions of both real rings belonging to `o`

DESCRIPTION To every order in KASH internally there belong two ”real rings”. One of these rings is used for computations with (real) approximations to the elements of `o`, the other ring is used for computations in which units and their logarithms are involved. When a *new* order is created, both rings obtain the same precision which is taken from an internal KANT constant named `Real_Std_Prec`.

Note however, that all precisions are automatically enlarged to a multiple of 4.

`OrderPrec(p)`
Sets the internal KANT constant `Real_Std_Prec` and returns the new value of this constant. Already existing orders are not affected.

`OrderPrec()`
Returns the current value of Real_Std_Prec.

`OrderPrec(o,p)`
The precisions of both ”real rings” in the order `o` are set to p. A list L containing the precisions of both ”real rings” of `o` is returned.

`OrderPrec(o,p,u)`
The precision for the first ”real ring” of the order `o` is set to p and the precision of the second is set to u. A list L containing the precisions of both ”real rings” of `o` is returned.

`OrderPrec(o)`
Returns a list L containing the precisions of both ”real rings” of `o`.

SEE ALSO `Prec`,

EXAMPLE We compute the regulator of the maximal order $\mathcal{O}$ in $\mathbb{Q}(\sqrt[4]{2})$. First, we use the default precision of 50 digits. Then we increase the precision of the order and print the regulator again. Remark that we must use the Prec function to increase the precision in order to print all 60 digits.

```
kash> OrderPrec();
50
kash> O := OrderMaximal(Order(Z,4,2));
Generating polynomial: x^4 - 2
Discriminant: -2048

kash> OrderPrec(O);
[ 52, 52 ]
kash> OrderUnitsFund(O);
[ [1, 0, 1, 0], [1, -1, 0, 0] ]
kash> OrderReg(O);
2.15800131645680564826065544584339217422724449652
kash> OrderPrec(O,60);
[ 60, 60 ]
kash> OrderReg(O);
2.158001316456805648260655445843392174227244496522
kash> Prec(60);
60
kash> OrderReg(O);
2.15800131645680564826065544584339217422724449652236075411
```

NAME `OrderPrintFlags`

PURPOSE Set or get Flags to reduce or extend the output level of print_order.

SYNTAX
```
OrderPrintFlags(a);
a := OrderPrintFlags();

record  a
```

DESCRIPTION OrderPrint can be used to set or get Flags, which are saved in a record. The record entrys can be the following

name	type	description
disc	int	information about discriminant
reg	int	information about regulator
fac_disc	int	information about factorizing
unit	int	information about units
index	int	information about index
class_group	bool	information about class group
class	bool	information about class number
trafo	bool	information about transform. Matrix
trafo_inv	bool	information about inverse Trafomatrix
trafo_den	bool	information about Matrix denominator
galois	bool	information about galois groups
sign	bool	information about sign
autom	bool	information about automorphisms

where

number = -1: maximal output
number = 0: no output
number = $\langle$ any number $\rangle$: prints if length of output string is less or equal then num-
ber, otherwise prints "Known"
and

bool = true: prints if known
bool = false: prints nothing

To get the settings call OrderPrintFlags();

EXAMPLE Set output level of unit and signature to 0 (prints nothing) even if unit or signature are known.

```
kash> OrderPrintFlags(rec(unit:=0,sign:=false));

kash> a:=OrderPrintFlags();
rec(
  disc := -1,
  reg := -1,
  fac_disc := 0,
  unit := 0,
  index := 0,
  class_group := true,
  class := false,
  trafo_inv := false,
  trafo_den := false,
  trafo := false,
  galois := false,
  sign := false,
  autom := false,
  coef_ideals := true )
kash> a.unit:=0;
0
kash> a.sign:=false;
false
kash> OrderPrintFlags(a);

kash> q:=Order(Z, 2, 2^15);; OrderUnitsFund(q);;
kash> o:= OrderMaximal(Z, 4, 2);; O := Order(o, 5, 3);;
kash> OrderSig(O);;
kash> q;
Generating polynomial: x^2 - 32768
Discriminant: 131072
Regulator: 112.815819138501507229773993597413415555604522017489

kash> O;
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^5 - 3
E 1[ 1]     x^4 - 2
```

```
kash> OrderPrintFlags(rec(unit:=-1,sign:=true));
kash> q;
Generating polynomial: x^2 - 32768
Discriminant: 131072
Regulator: 112.815819138501507229773993597413415555604522017489
Fundamental unit:
[4946041176255201878775086487573351061418968498177, -2732327543560891540641290\
5094301272808109095411]
Signature: [2, 0]

kash> O;
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^5 - 3
E 1[ 1]     x^4 - 2
Signature: [ [1, 2]  [1, 2]  [5, 0] ]

kash> OrderPrintFlags(rec(unit:=3));
kash> q;
Generating polynomial: x^2 - 32768
Discriminant: 131072
Regulator: 112.815819138501507229773993597413415555604522017489
Fundamental unit: Known
Signature: [2, 0]
```

NAME OrderReg

PURPOSE Returns the regulator of the *current* maximal system of independent units.

SYNTAX

```
x := OrderReg(o);
x := OrderReg(o,"classgroup");
x := OrderReg(o, reg);
```

```
real   x
order  o
real   reg
```

DESCRIPTION Let $\mathfrak{o}$ be an absolute order with unit rank $r \geq 1$.

`OrderReg(o)`
Let $\mathcal{U} = \{\varepsilon_1, \ldots, \varepsilon_k\} \quad (0 \leq k \leq r)$ be the current system of independent units which is stored in $\mathfrak{o}$. If k equals r we say that the system $\mathcal{U}$ is *maximal*. In this case the regulator of the system $\mathcal{U}$ can be computed by the `OrderReg` function.

The regulator $\mathrm{Reg}_{\mathfrak{o}}$ of the order $\mathfrak{o}$ is defined as the regulator of a system of fundamental units of the order $\mathfrak{o}$. To obtain $\mathrm{Reg}_{\mathfrak{o}}$, a system of fundamental units for $\mathfrak{o}$ must be computed first, which can be done by the `OrderUnitsFund` function.

`OrderReg(o,"classgroup")`
The `OrderReg` function returns the regulator of the units found by class group computations using the `"euler"` option.

`OrderReg(o, reg)`
Sets the regulator of the order $\mathfrak{o}$ to the value of *reg* and returns it.

SEE ALSO `OrderRegLowBound`, `OrderUnitsFund`, `OrderUnitsIndep`,

EXAMPLE Let $\mathcal{O}$ denote the maximal order of $\mathbb{Q}(\rho)$, where $\rho^4 + 73\rho^2 - 280\rho - 2399 = 0$. First, we compute the regulator of a maximal system of independent units in $\mathcal{O}$.

```
kash> O := OrderMaximal (Poly(Zx,[1,0,73,-280,-2399]));
   F[1]
    |
   F[2]
  /
 /
Q
```

```
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975

kash> OrderUnitsIndep (O);
[ [0, 0, 1, -1], [109589, -13889, 743946, -766419] ]
kash> OrderReg (O);
12.399824579954190488548880525089080810892337788058
```

Next we compute the regulator of a fundamental system of units in $\mathcal{O}$.

```
kash> OrderUnitsFund (O);
[ [-1, 0, -1, 1], [-253011, -13889, 519847, -542320] ]
kash> OrderReg (O);
12.39982457995419048854888052508908081089233778807
```

From this result we find that the system of independent units which we had computed in the first step was fundamental.

Finally, we set the regulator of the maximal order to 12.4:

```
kash> OrderReg (O, 12.4);
12.4
kash> OrderReg (O);
12.4
```

NAME `OrderRegLowBound`

PURPOSE Returns a lower bound for the regulator of the given order.

SYNTAX `x := OrderRegLowBound(o);`
`x := OrderRegLowBound(o, reg);`

real	x
order	o
real	reg

DESCRIPTION Let `o` be an absolute order.
`OrderRegLowBound(o)` computes a lower bound for the regulator of the order `o`.

`OrderRegLowBound(o, reg)` sets the regulator of the order `o` to the value of reg and returns it.

SEE ALSO `OrderReg`,

EXAMPLE Compute a lower bound for the regulator of $\mathbb{Q}(\rho)$ where $\rho^4 + 73\rho^2 - 280\rho - 2399 = 0$:

```
kash> O := OrderMaximal (Order (Poly(Zx,[1,0,73,-280,-2399])));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975

kash> OrderRegLowBound (O);
2.590616643223093069474635679775415624303102251639
```

Set the lower bound to 2.0:

```
kash> OrderRegLowBound (O, 2.0);
2
kash> OrderRegLowBound (O);
2
```

NAME OrderRelNormEq

PURPOSE Calculates an element of given norm.

SYNTAX `elt := OrderRelNormEq(O,n [, "true"]);`

order	O	
list	elt	list of elements in O
element	n	element in coefficient ring of O

DESCRIPTION Let o be the coefficient order of O and O a simple relative extension. This means that o is an absolute extension. (Thus not Z!) Let f and F be the quotient fields of o and O. This function computes for a given $n \in o$ an element $elt \in O$ with $N^{F/f}(elt) = n$.
If no such element exists an empty list is returned. If "true" is set we try to find an integral element. Sometimes two solutions are returned. That is if the reduced version is different from the non-reduced version.

EXAMPLE

```
kash> o:= Order(Z, 2, 2);
Generating polynomial: x^2 - 2

kash> O:= Order(o, 2, 3);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 3
E 1[ 1]     x^2 - 2

kash> n := Elt(o, [-3856938, 1576756]);
[-3856938, 1576756]
kash> elt := OrderRelNormEq(O,n);
[ [[-3961, -4022], [3679, 1372]], [[-449, 140], [361, -786]] ]
```

NAME `OrderRelUnits`

PURPOSE Computes the relative units.

SYNTAX `L := OrderRelUnits(o [,S | I] [,"raw"]);`

`list`	`L`	of L1, T, hs or L2, hs
`list`	`L1`	of algebraic integers
`list`	`L2`	of S-units
`matrix`	`T`	transformation matrix
`integer`	`hs`	S-class number
`order`	`o`	
`list`	`S`	of pairwise distinct prime ideals.
`ideal`	`I`	

DESCRIPTION The `OrderRelUnits` function returns a basis of the S-units modulo split into torsion units and other S-units. You may specify S either as a list of prime ideals or as an ideal $\mathfrak{I}$. In this case S consists of all primes dividing $\mathfrak{I}$. Omitting the argument S means to choose for S the ideal $1{*}o$.
If the last argument is `"raw"`, this function will return some algebraic elements $\alpha|1, \ldots, \alpha|s$ and two transformation matrices T and U such that

$$(\alpha|1, \ldots, \alpha|s)\, T$$

interpreted as power products gives a basis of the relative torsion units and

$$(\alpha|1, \ldots, \alpha|s)\, U$$

interpreted as power products gives a basis of the relative S-units.
In both cases it will also return the cardinality of the subgroup of the class group which is generated by the prime ideals of S.

SEE ALSO `OrderSUnits`,

EXAMPLE We have for example:

```
kash> o:= Order(Poly(Zx,[1,1,1,1,1]));
Generating polynomial: x^4 + x^3 + x^2 + x + 1

kash> zeta:= Elt(o,[0,1,0,0]);
[0, 1, 0, 0]
kash> ox:=PolyAlg(o);
Univariate Polynomial Ring in x over Generating polynomial: x^4 + x^3 + x^2 + \
x + 1
```

```
kash> O:= Order(Poly(ox,[1,0,-zeta,0,zeta^2]));
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^4 + [0, -1, 0, 0]*x^2 + [0, 0, 1, 0]
E 1[ 1]     x^4 + x^3 + x^2 + x + 1

kash> I:=31*o;
<31>
kash> m:=OrderRelUnits(O,I);
[ [ [0, [-2, -2, -1, -1], 0, [0, 1, 0, -1]],
      [0, [-1, -1, -1, 0], 0, [-1, -1, -2, -1]],
      [[0, -1, 0, 0], [0, 0, -1, -1], [1, 1, 1, 0], [0, 1, 1, 1]],
      [[0, -1, 0, 0], [-1, 0, -1, -1], [1, 0, 0, 1], [0, 0, 0, -1]],
      [[-1, -1, -1, -1], [0, 1, 0, 0], [0, 0, 0, -1], -1],
      [[0, 1, 0, 0], [1, 1, 0, 1], 0, [0, 0, 0, 1]],
      [[-10, -19, -24, -44], 0, [-14, -8, 16, -12], 0] / 31,
      [[17, 10, 22, -3], 0, [-4, -37, -7, -23], 0] / 31,
      [[-66, -67, -137, -77], 0, [6, -39, -58, -117], 0] / 31,
      [[-19, -26, 84, 56], 0, [27, -43, -59, -91], 0] / 31 ],
  [ [[0, 0, 1, 1], 0, [-1, -1, -1, 0], 0],
      [0, [0, -1, -1, 1], 0, [2, 1, 1, 1]],
      [[-49, -38, -43, -52], 0, [27, -5, 18, -15], 0] / 31,
      [[-2, -18, 71, 8], 0, [34, -4, 33, 81], 0] / 31,
      [[-13, 69, -19, -103], 0, [66, 67, 137, -16], 0] / 31 ] ]
```

NAME `OrderRelativeOrder`

PURPOSE Computes a relative representation of the given orders.

SYNTAX `Or := OrderRelativeOrder(O, o);`

order	Or
order	O
order	o

DESCRIPTION This function computes a relative representation for `O` and `o` where `o` is a suborder of `O`. The homomorphisms are installed.

SEE ALSO `OrderAbs`,

EXAMPLE A simple example

```
kash> O:=Order(x^4-4*x^2+1);
Generating polynomial: x^4 - 4*x^2 + 1

kash> o:=OrderSubfield(O)[2];
Generating polynomial: x^2 + 4*x + 1

kash> Or:=OrderRelativeOrder(O, o);
F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 + [0, 1]
E 1[ 1]     x^2 + 4*x + 1
```

NAME `OrderSUnits`

PURPOSE Computes the S-unit group.

SYNTAX `L := OrderSUnits(o [,S | I] [,"raw"]);`

`list`	`L`	of L1, T, hs or L2, hs
`list`	`L1`	of algebraic integers
`list`	`L2`	of S-units
`matrix`	`T`	transformation matrix
`integer`	`hs`	S-class number
`order`	`o`	
`list`	`S`	of pairwise distinct prime ideals.
`ideal`	`I`	

DESCRIPTION The `OrderSUnits` function returns a basis of the S-units modulo torsion units. You may specify S either as a list of prime ideals or as an ideal $\mathfrak{I}$. In this case S consists of all primes dividing $\mathfrak{I}$. Omitting the argument S means to choose for S the factor basis used during class group computation. The first elements of this basis are fundamental units.

If the last argument is `"raw"`, this function will return some algebraic integers $\alpha|1, \ldots, \alpha|s$ and a transformation matrix T such that

$$(\alpha|1, \ldots, \alpha|s)\, T$$

interpreted as power products gives a basis of the S-unit group modulo torsion units.

In both cases it will also return the cardinality of the subgroup of the class group which is generated by the prime ideals of S.

The first basis elements are not fully proven to be fundamental units. This may be verified using `OrderClassGroupCheck`.

SEE ALSO `OrderClassGroup`, `OrderClassGroupCheck`, `OrderSUnitsPositive`,

EXAMPLE For $\mathbb{Q}(\sqrt[3]{333})$ we have for example:

```
kash> o := OrderMaximal(Z, 3, 333);;
kash> OrderClassGroup(o, 500, euler, fast);
[ 3, [ 3 ] ]
kash> OrderSUnits(o, 7*11*13*o);
[ [ [-7309383190, 288532187, 331545960], -7, [256, 37, 16], [-13, -12, 6],
      [3422, -161, -144], [4852, 700, 303], [16, 0, -1] ], 3 ]
```

NAME `OrderSUnitsPositive`

PURPOSE Returns a list of positive S-units

SYNTAX `P := OrderSUnitsPositive(L);`

```
list  P
list  L
```

DESCRIPTION Given a list of independent S-units this function returns a list consisting of a list of positive S-units and the index in the original S-units.

SEE ALSO `OrderSUnits`, `OrderUnitsFund`, `EltMinkowski`,

EXAMPLE

```
kash> o := OrderMaximal(x^3 + x^2 - 6*x - 1);
Generating polynomial: x^3 + x^2 - 6*x - 1
Discriminant: 985

kash> OrderSig(o);
[ 3, 0 ]
kash> L := OrderUnitsFund(o);
[ [0, 1, 0], [2, -1, 0] ]
kash> List(L, EltMinkowski);
[ [ -0.162961856777529942222421038865924402542853454927 4, -2.93080160017275807\
0023094586767733368753328418998, 2.09376345695028801224551562563365777129618 18\
73926],
  [ 2.162961856777529942222421038865924402542853454927, 4.930801600172758070 02\
3094586767733368753328418998, -0.09376345695028801224551562563365777129618 1873\
926] ]
kash> P := OrderSUnitsPositive(L);
[ [ [0, 1, 0], [2, -1, 0] ], 1 ]
kash> List(P[1], EltMinkowski);
[ [ -0.162961856777529942222421038865924402542853454927 4, -2.93080160017275807\
0023094586767733368753328418998, 2.09376345695028801224551562563365777129618 18\
73926],
  [ 2.162961856777529942222421038865924402542853454927, 4.930801600172758070 02\
3094586767733368753328418998, -0.09376345695028801224551562563365777129618 1873\
926] ]
```

NAME `OrderSetMaximal`

PURPOSE Sets the `OrderIsMaximal` flag in the given order.

SYNTAX `OrderSetMaximal(o);`
`OrderSetMaximal(o,flag);`

`order`	`o`
`optional boolean`	`flag`

DESCRIPTION Sets the `OrderIsMaximal` flag in the order `o` to flag. If flag is not given, the order is assumed to be maximal.

SEE ALSO `OrderIsMaximal`,

EXAMPLE Define the order $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ as maximal.

```
kash> o := Order (Poly (Zx,[1,0,-5]));
Generating polynomial: x^2 - 5

kash> o1 := Order (o,[Elt(o,1),Elt (o,[1,1]/2)]);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^2 - 5

kash> OrderIsMaximal(o1);
false
kash> OrderSetMaximal (o1);
kash> OrderIsMaximal(o1);
true
```

NAME OrderSetTorsionUnit

PURPOSE Sets a generator of the torsion unit group in the given order.

SYNTAX `OrderSetTorsionUnit(o,alpha,r);`

order	o	
algebraic element	alpha	
integer	r	rank

DESCRIPTION Sets the generator for the torsion units in the order o to be α of exponent r, no check is done, so be very careful.

SEE ALSO OrderTorsionUnit, OrderTorsionUnitRank,

EXAMPLE

```
kash> o := Order(Poly(Zx,[1,1,1,1,1]));
Generating polynomial: x^4 + x^3 + x^2 + x + 1

kash> zeta := Elt(o,[0,-1,0,0]);
[0, -1, 0, 0]
kash> OrderSetTorsionUnit(o,zeta,10);
```

NAME `OrderShort`

PURPOSE Tries to find a ”better” primitive polynomial for the given order.

SYNTAX

```
o1 := OrderShort(o);
o1 := OrderShort(o, modus);
o1 := OrderShort(o, modus, iterations);
```

`order`	`o1`
`order`	`o`
`integer`	`modus`
`integer`	`iterations`

DESCRIPTION Tries to find a polynomial for which the corresponding equation order is of smaller index. If modus is given (and not zero), the search is for a polynomial for which modus does not divide the index of the equation order. When using on large fields, this routine may take a long time ($O(2^{deg})$). You can specify an upper bound for the number of iterations with the optional third argument. If an invalid or no argument is given, this bound is set to 15. Note that this function only tries to find a better polynomial, it may fail, even if there is a better one.

SEE ALSO `Order`,

EXAMPLE This function is useful, if you merge two fields. The final equation order is a maximal order.

```
kash> o1 := Order(Z,2,2);
Generating polynomial: x^2 - 2

kash> o2:= Order(o1,2,3);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 3
E 1[ 1]     x^2 - 2

kash> o3:=OrderAbs(o2);
Generating polynomial: x^4 - 10*x^2 + 1
```

```
kash> o4:=OrderShort(o3);
Generating polynomial: x^4 - 4*x^2 + 1
Discriminant: 2304
```

NAME `OrderShortAbs`

PURPOSE Creates an absolute extension from a simple relative extension and tries to find a "good" primitive polynomial for the given order.

SYNTAX `Oa := OrderShortAbs(O);`

order	Oa
order	O

DESCRIPTION Given a simple relative order $\mathcal{O}$ – the coefficient ring (`OrderCoefOrder`) of the coefficient ring of $\mathcal{O}$ is $\mathbb{Z}$ – this function computes an absolute equation order $\mathcal{O}|a$ with quotient field isomorphic to the quotient field of $\mathcal{O}$ over $\mathbb{Q}$.
At the same time this algorithm tries to find a polynomial for which the corresponding equation order is of small index.

SEE ALSO `OrderShort`, `OrderAbs`,

EXAMPLE The following example creates the absolute extension of a relative extension:

```
kash> o:=OrderMaximal(x^2 + 19*x + 11);
Generating polynomial: x^2 + 19*x + 11
Discriminant: 317

kash> ox:=PolyAlg(o);
Univariate Polynomial Ring in x over Generating polynomial: x^2 + 19*x + 11
Discriminant: 317

kash> O:=Order(Poly(ox,[1,-18,-6]));
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 18*x - 6
E 1[ 1]     x^2 + 19*x + 11

kash> Oa:=OrderShortAbs(O);
F[1]
    |
```

```
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 - 2*x^3 - 331*x^2 + 332*x - 23
```

NAME `OrderSig`

PURPOSE Returns the signature of the number field defined by the given order.

SYNTAX `L := OrderSig(o);`

list	L
order	o

DESCRIPTION Computes the signature of the polynomial which defines an equation suborder of the given order `o`. The two entries of are the number of real roots and one half of the number of complex roots of the corresponding generating polynomial. For further information see the description of `PolySig`.

SEE ALSO `PolySig`,

EXAMPLE Compute the signature of $x^4 + 73x^2 - 280x - 2399$.

```
kash> o := Order (Poly(Zx,[1,0, 73, -280, -2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> L := OrderSig (o);
[ 2, 1 ]
```

NAME OrderSimplify

PURPOSE Returns the given order in a simplified representation.

SYNTAX `o1 := OrderSimplify(o);`

```
order  o1
order  o
```

DESCRIPTION The order is simplified in the sense that the number of suborders might be reduced.

SEE ALSO OrderTransformationMatrix, Order,

EXAMPLE Simplification of an order.

```
kash> o := Order (Poly(Zx,[1,0,73,-280, -2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> M1 := Mat (Z, [[2,0,-1,-2],[0,2,-1,0],[0,0,1,0],[0,0,0,1]]);
[ 2  0 -1 -2]
[ 0  2 -1  0]
[ 0  0  1  0]
[ 0  0  0  1]
kash> M2 := Mat (Z, [[823,0,0,-409],[0,823,0,-789],[0,0,823,-28],[0,0,0,1]]);
[ 823    0    0 -409]
[   0  823    0 -789]
[   0    0  823  -28]
[   0    0    0    1]
kash> o := Order (o, M1, 2);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399

kash> o := Order (o, M2, 823);
   F[1]
    |
```

```
   F[2]
    |
   F[3]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     Given by transformation matrix
F  [ 3]     x^4 + 73*x^2 - 280*x - 2399

kash> O := OrderSimplify (o);
F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
```

NAME `OrderSplittingField`

PURPOSE Computes the normal closure of an algebraic number field.

SYNTAX `O := OrderSplittingField (o);`

order	O
order	o

DESCRIPTION Let `o` be an absolute order. The `OrderSplittingField` function computes an equation order of the normal closure of the quotient field of `o`.

EXAMPLE Computation of an equation order $\mathcal{O}$ of the normal closure of the quotient field of the absolut order `o` and the factorization of the generating polynomial f for `o` in $\mathcal{O}$

```
kash> o := Order(Z,3,2);
Generating polynomial: x^3 - 2

kash> O := OrderSplittingField(o);
Generating polynomial: x^6 - 3*x^5 + 5*x^3 - 3*x + 1
Discriminant: -34992

kash> f := PolyMove(x^3-2, O );
x^3 - 2
kash> Factor(f);
[ [ x + [-1, -1, 1, 0, 0, 0], 1 ], [ x + [5, -5, -10, 3, 5, -2], 1 ],
  [ x + [-4, 6, 9, -3, -5, 2], 1 ] ]
```

NAME `OrderSubOrder`

PURPOSE Returns a suborder of the given order.

SYNTAX `os := OrderSubOrder(o);`

order or boolean	os
order	o

DESCRIPTION If the given order $\mathfrak{o}$ is an overorder of $\mathfrak{o}|s$, for example $\mathfrak{o}$ is defined via transformation (matrix and denominator or matrix and list of ideals; see `Order`) over $\mathfrak{o}|s$, this order will be returned. This function will return `false` if $\mathfrak{o}$ is an equation order.

SEE ALSO `OrderEquationOrder`, `OrderCoefOrder`,

EXAMPLE Compute suborder $\mathfrak{o}|s$ of an equation order and of a maximal overorder.

```
kash> o:=Order(Z,2,5);
Generating polynomial: x^2 - 5

kash> OrderSubOrder(o);
false
kash> O:=OrderMaximal(o);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^2 - 5
Discriminant: 5

kash> OrderSubOrder(O);
Generating polynomial: x^2 - 5
Discriminant: 20
```

NAME `OrderSubfield`

PURPOSE Returns all non-trivial subfields of given degree `m`. If no `m` is specified, all subfields are calculated.

SYNTAX
```
L := OrderSubfield(o);
L := OrderSubfield(o, m);
```

`list`	`L`	list of suborders
`order`	`O`	the given order
`small integer`	`m`	the prescribed degree of subfields

DESCRIPTION This function calculates all non-trivial subfields of given degree `m`. If no `m` is specified, all non-trivial subfields are calculated. The result is a list `L`, which contains the calculated subfields as orders. The function uses the algorithms described in [Klü95, KP97, Klü97]
It is possible that this function returns the same order for two subfields. In this case, these subfields are isomorphic, but not identical. In the example below, the three subfields are isomorphic If the running time for this function seems too long, the user may want to try OrderSubfieldSub.

SEE ALSO `OrderSubfieldSub`,

EXAMPLE Computation of subfields of given degree.

```
kash> O:=Order(Poly(Zx,[1,-3,5,-5,5,-3,1]));
Generating polynomial: x^6 - 3*x^5 + 5*x^4 - 5*x^3 + 5*x^2 - 3*x + 1

kash> L:=OrderSubfield(O,3);
[ Generating polynomial: x^3 - 3*x^2 + 2*x - 1
    , Generating polynomial: x^3 - 2*x^2 + 3*x - 1
    , Generating polynomial: x^3 - 3*x^2 + 2*x - 1
     ]
kash> elt:=Elt(L[2],[1,1,1]);
[1, 1, 1]
kash> EltMove(elt,O);
[1, 1, 0, -2, 1, 0]
```

NAME `OrderSubfieldSub`

PURPOSE Calculates subfields with specified block systems.

SYNTAX
```
L := OrderSubfieldSub(o, p);
L := OrderSubfieldSub(o, p, d);
L := OrderSubfieldSub(o, p, d, L1);
```

list	L
order	o
integer	p
integer	d
list	L1

DESCRIPTION This function calculates non-trivial subfields of Q(`o`) using a special prime number `p`, which may not divide the discriminant of the equation order of `o`. If `d` is specified, only subfields of relative degree `d` are calculated. `L1` is a potential block system of the form $[block|1, \ldots, block|r]$.
The computation time is dependent on the cycle decomposition corresponding to the prime `p`. It is very difficult to choose a "good" prime. In some cases it is better to use this function instead of `OrderSubfield`.

SEE ALSO `OrderSubfield`,

EXAMPLE

```
kash> o:=Order(Poly(Zx,[1,0,-4,0,1]));
Generating polynomial: x^4 - 4*x^2 + 1

kash> OrderSubfieldSub(o,5);
[ Generating polynomial: x^2 - 4*x - 2
    , Generating polynomial: x^2 - 2
    , Generating polynomial: x^2 + 4*x + 1
     ]
kash> o:=Order(Poly(Zx,[1,0,-4,0,1]));
Generating polynomial: x^4 - 4*x^2 + 1

kash> OrderSubfieldSub(o,5);
[ Generating polynomial: x^2 - 4*x - 2
    , Generating polynomial: x^2 - 2
    , Generating polynomial: x^2 + 4*x + 1
     ]
```

NAME	`OrderTorsionUnit`
PURPOSE	Returns a generator for the group of torsion units of the given order.
SYNTAX	`u := OrderTorsionUnit(o);`

algebraic element	u
order	o

DESCRIPTION `o` must be an absolute order. `OrderTorsionUnit` returns an algebraic element u, which is a generator for the group of roots of unity in the order `o`.

SEE ALSO `OrderSetTorsionUnit`, `OrderTorsionUnitRank`,

EXAMPLE Computing generators for the groups of roots of unity in an equation order and the corresponding maximal order.

```
kash> o := Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0, 4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> OrderTorsionUnit (o);
-1
kash> O := OrderMaximal (o);;
kash> OrderTorsionUnit (O);
[15, -80, 10, -9, 10, -20, -51, 102]
```

NAME OrderTorsionUnitRank

PURPOSE Returns the number of roots of unity in the given order.

SYNTAX `r := OrderTorsionUnitRank(o);`

integer	r	number of roots of unity
order	o	

DESCRIPTION o must be an absolute order. OrderTorsionUnitRank returns the number of roots of unity in the order o, the rank of the stored generator of the roots of unity in the order o.

SEE ALSO OrderTorsionUnit, OrderSetTorsionUnit,

EXAMPLE Compute the number of roots of unity in $\mathbb{Z}[\sqrt{-3}]$ and in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$

```
kash> o := Order(Z,2,-3);
Generating polynomial: x^2 + 3

kash> O := OrderMaximal(o);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^2 + 3
Discriminant: -3

kash> OrderTorsionUnitRank(o);
2
kash> OrderTorsionUnitRank(O);
6
```

NAME `OrderTraceMat`

PURPOSE Returns the trace matrix of the order.

SYNTAX `M := OrderTraceMat(o);`

```
Matrix  M
order   o
```

DESCRIPTION `OrderTraceMat` returns the the trace matrix of the given order `o`. Let o be given by a (pseudo) basis $\omega|1, \ldots, \omega|n$ (we ignore the coefficient ideals here). The `OrderTraceMat` $M = (\mathrm{Tr}(\omega|i\omega|j))_{i,j}$.

EXAMPLE Extracting the equation order of an arbitrary order.

```
kash> O := OrderMaximal(Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176
Discriminant: 60050488100625

kash> M := OrderTraceMat(O);
[      8       0     186       4    1120     657     774     387]
[      0     372     186    2426     186   24657       0    5304]
[    186     186    4294    1306   24471   26711   16664   10984]
[      4    2426    1306   15364    1773  147638     387   32097]
[   1120     186   24471    1773  133394   91728   89382   47343]
[    657   24657   26711  147638   91728 1282603   53410  298835]
[    774       0   16664     387   89382   53410   59566   29783]
[    387    5304   10984   32097   47343  298835   29783   74701]
```

NAME `OrderTransformationMatrix`

PURPOSE Returns a list containing information about the transformation from a suborder of the given order to the given order.

SYNTAX `L := OrderTransformationMatrix(O);`

`list`	`L`
`order`	`O`

DESCRIPTION In the absolute case the list L contains the denominator d and the $n \times n$-matrix T with

$$(\omega|1, \dots, \omega|n) = \frac{1}{d}(a|1, \dots, a|n)T,$$

where $\omega|1, \dots, \omega|n$ is a basis of the given order $\mathcal{O}$ and $a|1, \dots, a|n$ is a basis of a suborder of the given order. If the order $\mathcal{O}$ is the equation order, simply $d = 1$ and $T = I|n$ are returned.
In the relative case the list L contains a list of ideals (1 or n, where n is the relative degree of $\mathcal{O}$), called the coefficient ideals, and an $n \times n$-matrix T.
If we get only one coefficient ideal $\mathfrak{a}$, the pseudo basis of the order $\mathcal{O}$ is $\mathfrak{o}\omega|1, \dots, \mathfrak{o}\omega|n-1, \mathfrak{a}\omega|n$, where

$$(\omega|1, \dots, \omega|n) = (a|1, \dots, a|n)T,$$

$a|1, \dots, a|n$ are the basis elements of the pseudo basis of the suborder of $\mathcal{O}$ and $\mathfrak{o}$ is the coefficient order of $\mathcal{O}$. In this case the relative order is called "free" ($\mathfrak{a} = \mathfrak{o}$) or "almost free" ($\mathfrak{a} \neq \mathfrak{o}$).
If we get n coefficient ideals $\mathfrak{a}|1, \dots, \mathfrak{a}|n$, the pseudo basis of the order $\mathcal{O}$ is $\mathfrak{a}|1\omega|1, \dots, \mathfrak{a}|n\omega|n$, with $\omega|1, \dots, \omega|n$ as above.
A relative equation order $\mathcal{O}$ is "free", hence `OrderTransformationMatrix(O)` returns the ideal $\mathfrak{o}$ (the coefficient order) and the $n \times n$ identity matrix.

SEE ALSO `OrderBasis`, `OrderCoefIdeals`, `Order`,

EXAMPLE An absolute extension:

```
kash> o := Order(Z,2,165);
Generating polynomial: x^2 - 165

kash> O := OrderMaximal(o);
   F[1]
    |
   F[2]
  /
```

```
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^2 - 165
Discriminant: 165

kash> OrderTransformationMatrix(O);
[ 2, [2 1]
    [0 1] ]
```

NAME `OrderUnitsAreFund`

PURPOSE Tests whether the current unit system of the given order is known to be fundamental or not.

SYNTAX `b := OrderUnitsAreFund(o);`
`b := OrderUnitsAreFund(o,c);`

`boolean`	`b, c`
`order`	`o`

DESCRIPTION The order `o` must be an absolute order. The `OrderUnitsAreFund` function can be used to check whether the current unit system of the order `o` is known to be fundamental.
Additionally, the `OrderUnitsAreFund` function makes it possible to set an internal flag in the order `o` which indicates that the current unit system is known to be fundamental. This can be achieved by invoking the `OrderUnitsAreFund` function with second optional argument c.

SEE ALSO `OrderUnitsFund`,

EXAMPLE In a CM field it is quite easy to compute fundamental units.

```
kash> O := OrderMaximal (Order (x^3 + x^2 - 16*x - 8));;
kash> units := OrderUnitsFund (O);
[ [1, 2, -1], [1, 2, 1] ]
kash> Ox := PolyAlg (O);;
kash> f := Poly (Ox, [1,1,1]);
x^2 + x + 1
kash> O2 := OrderMaximal (OrderAbs (Order (f)));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^6 + 5*x^5 - 20*x^4 - 95*x^3 + 136*x^2 + 465*x + 475
Discriminant: -502020288
```

The fundamental units of $\mathcal{O}$ are moved to $\mathcal{O}|2$, the index of the lifted group in $U(O|2)$ is shown not to be divisible by 2.

```
kash> Apply (units, u -> EltMove (u, O2));
kash> for u in units do OrderUnitsMerge (O2, u); od;
kash> OrderReg (O2);
28.915447106812858929100272734448621763738691622816
kash> OrderUnitsPFund (O2, 2);
[ [6, -1, -1, 1, 1, -1], [-8, 1, 1, -1, -1, 1] ]
kash> OrderReg (O2);
28.915447106812858929100272734448621763738691622816
kash> OrderUnitsAreFund (O2);
false
kash> \# Mark the units as fundamental
kash> OrderUnitsAreFund (O2,true);
true
```

NAME OrderUnitsEquation

PURPOSE Solves a unit equation.

SYNTAX

```
L := OrderUnitsEquation (alpha,beta,gamma);
L := OrderUnitsEquation (alpha,beta);
```

list	L
algebraic elements	alpha, beta, gamma

DESCRIPTION Let α, β, γ be non-zero elements of a certain order $\mathfrak{o}$. We denote by U the unit group of $\mathfrak{o}$. The `OrderUnitsEquation` function computes all $(\varepsilon_1, \varepsilon_2) \in \mathrm{U} \times \mathrm{U}$ satisfying

$$\alpha \cdot \varepsilon_1 + \beta \cdot \varepsilon_2 = \gamma.$$

If the third argument is omitted, $\gamma = 1$ is used by default.

SEE ALSO OrderUnitsExcep,

EXAMPLE Compute all units $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}[\sqrt[4]{5}]$ with $1 \cdot \varepsilon_1 + 2 \cdot \varepsilon_2 = 3$.

```
kash> o := Order(Z,4,5);
Generating polynomial: x^4 - 5

kash> one := Elt(o,1);
1
kash> OrderUnitsEquation(one,2*one,3*one);
[ [ 1, 1 ] ]
```

NAME OrderUnitsExcep

PURPOSE Computes all exceptional units of an order.

SYNTAX
```
L := OrderUnitsExcep(o);
L := OrderUnitsExcep(o,"orbits"|"list");
n := OrderUnitsExcep(o,"number");
```

list	L
integer	n
order	o

DESCRIPTION Let $\mathfrak{o}$ be an arbitrary order. In the terminology of T. Nagell [Nag69] a unit $\varepsilon \in \mathfrak{o}$ is exceptional if $1-\varepsilon$ is also a unit.

For an exceptional unit $\varepsilon \in \mathfrak{o}$ all the units

$$\varepsilon_1 = \varepsilon, \quad \varepsilon_2 = \frac{1}{\varepsilon}, \quad \varepsilon_3 = 1-\varepsilon, \quad \varepsilon_4 = \frac{1}{1-\varepsilon}, \quad \varepsilon_5 = \frac{\varepsilon-1}{\varepsilon}, \quad \varepsilon_6 = \frac{\varepsilon}{\varepsilon-1}$$

are exceptional. The set formed by $\varepsilon_1, \ldots, \varepsilon_6$ is called the orbit of ε.

`OrderUnitsExcep(o)` or `OrderUnitsExcep(o,"orbits")`
returns a list whose entries represent all orbits.

`OrderUnitsExcep(o,"number")`
returns the number of exceptional units in $\mathfrak{o}$.

`OrderUnitsExcep(o,"list")`
returns the list `L` of all exceptional units in $\mathfrak{o}$.

SEE ALSO EltExcepUnitOrbit, OrderExcepSequence, OrderUnitsEquation,

EXAMPLE Compute exceptional units in $\mathbb{Z}[\zeta_7]$.

```
kash> O := Order(Poly(Zx,[1,1,1,1,1,1,1]));
Generating polynomial: x^6 + x^5 + x^4 + x^3 + x^2 + x + 1

kash> OrderUnitsExcep(O);
[ [0, -1, 0, 0, 0, 0], [0, -1, -1, 0, 0, 0], [0, 0, -1, 0, 0, 0],
  [0, -1, -1, -1, 0, 0], [0, -1, -1, -1, -1, 0], [-1, 0, 0, -1, -1, 0],
  [0, 0, 0, -1, -1, 0], [1, 0, 0, -1, -1, 0],
  [-684, 0, -549, -244, -244, -549], [-7, 0, -6, -3, -3, -6],
  [-3, 0, -3, -1, -1, -3], [-2, 0, -2, -1, -1, -2] ]
kash> OrderUnitsExcep(O,"number");
```

```
72
kash> OrderUnitsExcep(O,"list");
[ [0, -1, 0, 0, 0, 0], [1, 1, 0, 0, 0, 0], [0, -1, -1, -1, -1, -1],
  [0, -1, 0, -1, 0, -1], [1, 1, 0, 1, 0, 1], [1, 1, 1, 1, 1, 1],
  [0, -1, -1, 0, 0, 0], [1, 1, 1, 0, 0, 0], [0, 0, -1, 0, -1, 0],
  [0, -1, 0, 0, -1, 0], [1, 1, 0, 0, 1, 0], [1, 0, 1, 0, 1, 0],
  [0, 0, -1, 0, 0, 0], [1, 0, 1, 0, 0, 0], [0, -1, 0, 0, -1, -1],
  [0, 0, 0, 0, 0, -1], [1, 0, 0, 0, 0, 1], [1, 1, 0, 0, 1, 1],
  [0, -1, -1, -1, 0, 0], [0, 0, 0, -1, 0, 0], [1, 0, 0, 1, 0, 0],
  [1, 1, 1, 1, 0, 0], [0, 0, 0, 0, -1, 0], [1, 0, 0, 0, 1, 0],
  [0, -1, -1, -1, -1, 0], [1, 1, 1, 1, 1, 0], [0, -1, -1, 0, -1, -1],
  [0, 0, 0, -1, 0, -1], [1, 0, 0, 1, 0, 1], [1, 1, 1, 0, 1, 1],
  [-1, 0, 0, -1, -1, 0], [2, 0, 0, 1, 1, 0], [1, 0, -1, -2, -2, -1],
  [-1, 0, -1, -1, -1, -1], [2, 0, 1, 1, 1, 1], [0, 0, 1, 2, 2, 1],
  [0, 0, 0, -1, -1, 0], [1, 0, 0, 1, 1, 0], [0, 0, -1, -1, -1, -1],
  [0, 0, -1, 0, 0, -1], [1, 0, 1, 0, 0, 1], [1, 0, 1, 1, 1, 1],
  [1, 0, 0, -1, -1, 0], [0, 0, 0, 1, 1, 0], [-1, 0, -1, 0, 0, -1],
  [2, 0, -1, 1, 1, -1], [-1, 0, 1, -1, -1, 1], [2, 0, 1, 0, 0, 1],
  [-684, 0, -549, -244, -244, -549], [136, 0, -305, -549, -549, -305],
  [441, 0, -244, 305, 305, -244], [-440, 0, 244, -305, -305, 244],
  [-135, 0, 305, 549, 549, 305], [685, 0, 549, 244, 244, 549],
  [-7, 0, -6, -3, -3, -6], [2, 0, -3, -6, -6, -3], [5, 0, -3, 3, 3, -3],
  [-4, 0, 3, -3, -3, 3], [-1, 0, 3, 6, 6, 3], [8, 0, 6, 3, 3, 6],
  [-3, 0, -3, -1, -1, -3], [1, 0, -2, -3, -3, -2], [3, 0, -1, 2, 2, -1],
  [-2, 0, 1, -2, -2, 1], [0, 0, 2, 3, 3, 2], [4, 0, 3, 1, 1, 3],
  [-2, 0, -2, -1, -1, -2], [1, 0, -1, -1, -1, -1], [1, 0, -1, 0, 0, -1],
  [0, 0, 1, 0, 0, 1], [0, 0, 1, 1, 1, 1], [3, 0, 2, 1, 1, 2] ]
```

NAME `OrderUnitsFund`

PURPOSE Returns a list whose entries are a maximal set of fundamental units (algebraic numbers) of the given order.

SYNTAX `L := OrderUnitsFund(o);`

```
list   L
order  o
```

DESCRIPTION The order `o` must be an absolute order. For further information about units, the underlying theory and the implemented algorithms, we refer the reader to [Wil93]

SEE ALSO `OrderSUnits`, `OrderSUnitsPositive`,

EXAMPLE A set of fundamental units in the maximal order of $x^4 + 73x^2 - 280x - 2399$.

```
kash> O := OrderMaximal (Order (Poly(Zx,[1,0,73,-280,-2399])));
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975

kash> OrderUnitsFund (O);
[ [-1, 0, -1, 1], [109589, -13889, 743946, -766419] ]
```

NAME `OrderUnitsIndep`

PURPOSE Computes a maximal system of independent units.

SYNTAX
```
L := OrderUnitsIndep(o);
L := OrderUnitsIndep(o,"classgroup");
L := OrderUnitsIndep(o,"classgroup","list");
```

list	L
order	o

DESCRIPTION Let $\mathfrak{o}$ be an absolute order with unit rank $r \geq 1$.

`OrderUnitsIndep(o)`
The `OrderUnitsIndep` function computes a maximal system $\varepsilon_1, \ldots, \varepsilon_r$ of independent units of $\mathfrak{o}$ which are returned in a list.

`OrderUnitsIndep(o,"classgroup")`
The `OrderUnitsIndep` function returns a list containing a maximal system $\varepsilon_1, \ldots, \varepsilon_r$ of independent units of $\mathfrak{o}$ found during class group computations done by using the `"euler"` option. Note: It is not necessary to call the function `OrderClassGroup` first. It is done automatically.

When the class group computation is done without the `"fast"` option, the units are proven to be p-maximal for primes p dividing the class number of $\mathfrak{o}$.

An empty list is returned if a system of fundamental units has been computed before the computation of the class group.

If a system of units has been computed before the computation of the class group, `OrderUnitsIndep(o,"classgroup")` merges the two systems of units. This resulting system of units is stored in the order $\mathfrak{o}$ and returned. No merging occurs when giving `"list"` as additional parameter.

For further information about units, the underlying theory and the implemented algorithms, we refer the reader to [Wil93]

SEE ALSO `OrderUnitsFund`, `OrderClassGroup`,

EXAMPLE A set of independent units in the maximal order of $x^4 + 73x^2 - 280x - 2399$.

```
kash> O := OrderMaximal (Order (Poly(Zx,[1,0,73,-280,-2399])));
   F[1]
    |
   F[2]
  /
 /
```

```
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 + 73*x^2 - 280*x - 2399
Discriminant: -997975

kash> OrderUnitsIndep (O);
[ [0, 0, 1, -1], [109589, -13889, 743946, -766419] ]
```

NAME `OrderUnitsLLL`

PURPOSE Computes a LLL reduced system of independent units.

SYNTAX `OrderUnitsLLL (O);`

`order O`

DESCRIPTION The given absolute order $\mathcal{O}$ has to be maximal. Computes a LLL reduced representation of the units given in $\mathcal{O}$. If no units have been computed so far, an arbitrary system of independent units is computed in advance.

SEE ALSO OrderLLL, OrderUnitsPFund,

EXAMPLE

```
kash>  O := OrderMaximal(x^3+x^2-2*x-1);
Generating polynomial: x^3 + x^2 - 2*x - 1
Discriminant: 49

kash> OrderUnitsLLL(O);
[ [-1, 1, 0], [5, -1, -2] ]
```

NAME OrderUnitsMerge

PURPOSE Extends a system of units of the given order.

SYNTAX `B := OrderUnitsMerge(o,eta); B := OrderUnitsMerge(o,eta,"append");`

boolean	B	
order	o	
algebraic element	eta	a unit in o

DESCRIPTION The OrderUnitsMerge function performs a MLLL reduction to expand the current unit group of the given absolute order $\mathfrak{o}$ by merging η in it. If the rank of the unit group increases `OrderUnitsMerge` will return TRUE. Otherwise it will return FALSE.
`OrderUnitsMerge(o,eta,"append")` just appends the unit η to the system of units of the order $\mathfrak{o}$, MLLL is not performed.

EXAMPLE Extending a system of units of an order.

```
kash> o := Order (Poly (Zx,[1,0,-186,0,13097,0,-412704,0,4946176]));
Generating polynomial: x^8 - 186*x^6 + 13097*x^4 - 412704*x^2 + 4946176

kash> O := OrderMaximal (o);;
kash> OrderUnitsMerge (O,Elt (O,[-14, 5, 1, 0, -1, 1, 3, -5]));
true
kash> OrderUnitsMerge (O,Elt (O,[177, -301, 26, -31, 39, -74, -190, 377]));
true
kash> OrderReg (O);
Error, no maximal set of independent units known
kash> OrderUnitsMerge (O,Elt (O,[227, -555, 57, -63, 74, -140, -361, 714]));
true
kash> a := Elt (O,[-14, 5, 1, 0, -1, 1, 3, -5]);;
kash> b := Elt (O,[227, -555, 57, -63, 74, -140, -361, 714]);;
kash> OrderUnitsMerge (O,a*b);
false
kash> OrderReg (O);
121.297795062765195428716897829517493027623458635
```

NAME `OrderUnitsPFund`

PURPOSE Computes the p maximal overgroup of the units in the given order.

SYNTAX `OrderUnitsPFund (O,p);`

order	O
rational prime	p

DESCRIPTION The given absolute order $\mathcal{O}$ must be maximal. This function computes the p maximal overgroup of the units given in $\mathcal{O}$. If no units have been computed so far, an arbitrary system of independent units is computed in advance. For further information we refer the reader to [Wil93].

SEE ALSO `OrderUnitsFund`,

EXAMPLE Let ρ be a root of $t^3 + 15t^2 + 15t + 15$. We consider the order $\mathbb{Z}[\rho]$, which is already maximal with signature $[1, 1]$.
$2729 + 573\rho + 27\rho^2$ is a unit in this order.

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,15,15,15])));
Generating polynomial: x^3 + 15*x^2 + 15*x + 15
Discriminant: -110700

kash> a := Elt(O,[2729, 573, 27]);
[2729, 573, 27]
kash> OrderUnitsMerge(O,a);
true
kash> OrderReg(O);
15.630874731280170083241764717443071385772563284107
kash> OrderUnitsPFund(O,2);
[ [2729, 573, 27] ]
kash> OrderUnitsPFund(O,3);
[ [-14, -1, 0] ]
kash> OrderReg(O);
5.210291577093390027747254905814357128590623830734
```

NAME PRINTLEVEL

PURPOSE Reads or sets the printlevel for a certain function.

SYNTAX
```
x := PRINTLEVEL(s);
x := PRINTLEVEL(s,level);
x := PRINTLEVEL("all",level);
```

small integer	x
string	s
small integer	level

DESCRIPTION The `PRINTLEVEL` function reads or sets the printlevel for a certain function, i.e. the level of how much information of a computation is displayed for the KANT function specified by the debug string `s`. For valid debug strings refer to the DEBUG entries in the reference manual. To read the printlevel corresponding to `s` enter `PRINTLEVEL(s)`. A printlevel can be set by calling the `PRINTLEVEL` function with two arguments the second one being the new printlevel. In both cases the current printlevel will be returned.

The command `PRINTLEVEL("all",level)` sets the printlevel for all KANT functions to `level`. Please note, that this function should be used carefully, e.g. for a first try it is useful to start with `PRINTLEVEL("all", 0)`.

EXAMPLE Compute a maximal order with printlevel 2:

```
kash> f := Poly(Zx,[1,0,-2750]);
x^2 - 2750
kash> O := Order(f);
Generating polynomial: x^2 - 2750

kash> PRINTLEVEL("ROUND2",2);
2
kash> OrderMaximal(O,"round2");

Factorization of discriminant: 2^3 * 5^3 * 11^1

Factors of discriminant with (possibly) R(p) != ZZ[a] : 5^3
Calculation of p-maximal overorder...
Prime: 5 maximal exponent: 3
relative degree found: 5
Index exponent found until now: 1
relative degree found: 1
```

```
Transformation for 5...
Index of order: 5
(FLD=) 2, -2, 0, 0, 0, 0, 0, 0, 0, 0, 2, *
0 -2750
5 0 5
0 1 5
F[1]
|
F[2]
/
/
Q
F [ 1] Given by transformation matrix
F [ 2] x^2 - 2750
Generating polynomial: x^2 - 2750
Discriminant: 440
```

NAME `Poly`

PURPOSE Creates a polynomial.

SYNTAX `f := Poly(A, L);`

`polynomial`	`f`
`polynomial algebra`	`A`
`list`	`L`

DESCRIPTION Given a polynomial algebra A and a list of elements (coefficients) $L := [c_n, c_{n-1}, \ldots, c_0]$ the function returns the polynomial $f(x) := \sum_{i=0}^{n} c_i x^i$. The c_i must be elements of the coefficient ring of A.

SEE ALSO `PolyAlg`,

EXAMPLE Creation of the polynomial $x^5 + 2 * x^4 + 3 * x^3 + 4 * x^2 + 5 * x + 6 \in \mathbb{Z}[x]$:

```
kash> f := Poly(Zx, [1, 2, 3, 4, 5, 6]);
x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6
```

NAME `PolyAlg`

PURPOSE Creates a univariate polynomial algebra over a ring or returns the polynomial algebra of a polynomial.

SYNTAX `Sx := PolyAlg(S [, name]);`

polynomial algebra	Sx
ring or polynomial	S
string	name

DESCRIPTION Creates a univariate polynomial algebra $S[x]$ over a ring S. `Zx` is a predefined constant for the polynomial algebra over the integers. A name for the variable can be given as a second parameter.
If S is a polynomial the polynomial algebra of S is returned.

SEE ALSO `Poly`, `PolyAlgCoef`, `Zx`,

EXAMPLE Creating a polynomial algebra over an order and computing a minimal polynomial:

```
kash> o := Order(Poly(Zx, [1 ,0 , 73, -280, -2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> ox := PolyAlg(o);
Univariate Polynomial Ring in x over Generating polynomial: x^4 + 73*x^2 - 280\
*x - 2399

kash> M := Mat (o,[[1,2],[2,3]]);
[1 2]
[2 3]
kash> MatMinPoly (M);
x^2 - 4*x - 1
```

NAME `PolyAlgCoef`

PURPOSE Returns the coefficient ring of a polynomial algebra.

SYNTAX `S := PolyAlgCoef(Sx);`

ring	S
polynomial algebra	Sx

DESCRIPTION Returns the coefficient ring S of a polynomial algebra $S[x]$.

SEE ALSO `PolyAlg`,

EXAMPLE

```
kash> o := Order(Poly(Zx, [1 , 0, 73, -280, -2399]));
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399

kash> ox := PolyAlg(o);
Univariate Polynomial Ring in x over Generating polynomial: x^4 + 73*x^2 - 280\
*x - 2399

kash> PolyAlgCoef(ox);
Generating polynomial: x^4 + 73*x^2 - 280*x - 2399
```

NAME `PolyDeg`

PURPOSE Returns the degree of a given polynomial.

SYNTAX `n := PolyDeg(f);`

integer	n
polynomial	f

DESCRIPTION Returns the degree n of a polynomial f. The degree of the zero-polynomial is -1.

EXAMPLE

```
kash> f := Poly(Zx, [1, 2, 3, 4, 5, 6]);
x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6
kash> PolyDeg(f);
5
kash> f := Poly(Zx, [0]);
0
kash> PolyDeg(f);
-1
```

NAME `PolyDeriv`

PURPOSE Returns the derivative of a given polynomial.

SYNTAX `h := PolyDeriv(f);`

```
polynomial  h
polynomial  f
```

DESCRIPTION Returns the derivative $h \in S[x]$ of a polynomial $f \in S[x]$.

EXAMPLE The derivative of $x^4 + 73 * x^2 - 280 * x - 2399$:

```
kash> f := Poly(Zx, [1, 0, 73, -280, -2399]);
x^4 + 73*x^2 - 280*x - 2399
kash> PolyDeriv(f);
4*x^3 + 146*x - 280
```

NAME `PolyDisc`

PURPOSE Returns the discriminant of a polynomial.

SYNTAX `d := PolyDisc(f);`

`discriminant (integer or finite field element)`	`d`
`polynomial`	`f`

DESCRIPTION Returns the discriminant $d \in \mathbb{Z}$ of a polynomial $f \in \mathbb{Z}[x], \mathbb{F}_q[x]$.

SEE ALSO `PolyRedDisc`,

EXAMPLE The discriminant of $x^4 + 73 * x^2 - 280 * x - 2399$:

```
kash> f := Poly(Zx, [1, 0, 73, -280, -2399]);
x^4 + 73*x^2 - 280*x - 2399
kash> PolyDisc(f);
-10815318540400
```

NAME PolyFactor

PURPOSE Returns the factorization of the given polynomial.

SYNTAX

```
F := PolyFactor(f);
F := PolyFactor(f, p);
F := PolyFactor(f, p, m);
```

list	F
polynomial	f
prime number	p
integer	m

DESCRIPTION This function returns the factorization of the given polynomial over its coefficient ring. Supported ring types are: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}|p$, $\mathcal{O}$, and $\mathbb{Q}_p$.
If the polynomial has coefficients in $\mathbb{Z}$, a second argument p may be specified to obtain the factorization modulo (the prime) p.
If the polynomial has coefficients in $\mathbb{Z}$ or $\mathbb{Q}$ and it is monic an approximation to a p-adic factorization can be obtained by giving a prime number p and a precision m.

SEE ALSO Factor,

EXAMPLE Factorization of the polynomial $f(x) = 4x^7+6x^6+12x^5+14x^4+27x^3+24x^2+30x+9 \in \mathbb{Z}[x]$:

```
kash> f := Poly(Zx, [4,6,12,14,27,24,30,9]);
4*x^7 + 6*x^6 + 12*x^5 + 14*x^4 + 27*x^3 + 24*x^2 + 30*x + 9
kash> Factor(f);
[ [ 2*x^2 + x + 3, 1 ], [ 2*x^5 + 2*x^4 + 2*x^3 + 3*x^2 + 9*x + 3, 1 ] ]
```

NAME `PolyGcd`

PURPOSE Returns the gcd of two polynomials.

SYNTAX `g := PolyGcd(f, h);`

```
polynomial  g
polynomial  f
polynomial  h
```

DESCRIPTION Given two polynomials $f, h \in S[x]$ where S is a field or $\mathbb{Z}$ the function returns the gcd g of f and h.

SEE ALSO `PolyXGcd`,

EXAMPLE

```
kash> f := Poly(Zx, [1, 2, 3, 4, 5, 6]);
x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6
kash> h := Poly(Zx, [1, 4, 10, 16, 22, 28, 27, 18]);
x^7 + 4*x^6 + 10*x^5 + 16*x^4 + 22*x^3 + 28*x^2 + 27*x + 18
kash> PolyGcd(f, h);
x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6
```

NAME `PolyHenselLift`

PURPOSE Lifts a factorization of a polynomial modulo a prime ideal to a factorization modulo the prime ideal to a given exponent.

SYNTAX `PolyHenselLift(f, A, n);`

`polynomial over an order`	`f`
`ideal of the same order`	`A`
`integer`	`n`

DESCRIPTION

NAME `PolyIsIrreducible`

PURPOSE missing shortdoc

SYNTAX `b := PolyIsIrreducible(f);`

boolean	b
polynomial	f

DESCRIPTION Checks if a given polynomial is irreducible.

EXAMPLE

```
kash> f := Poly(Zx, [1, 0, -4, 0, 1]);
x^4 - 4*x^2 + 1
kash> PolyIsIrreducible(f);
true
```

NAME `PolyIsSquarefree`

PURPOSE missing shortdoc

SYNTAX `b := PolyIsSquarefree(f);`

boolean	b
polynomial	f

DESCRIPTION Checks if a given polynomial is square free

EXAMPLE

```
kash> f := Poly(Zx, [1, 0, -4, 0, 1]);
x^4 - 4*x^2 + 1
kash> PolyIsSquarefree(f);
true
```

NAME PolyIsZero

PURPOSE Returns true iff the argument is a polynomial and equal to the zero polynomial

SYNTAX b := PolyIsZero(f);

boolean	b
	f

EXAMPLE

```
kash> f := Poly(Zx, [1, 2, 1, 2]);
x^3 + 2*x^2 + x + 2
kash> PolyIsZero(f);
false
kash> f := Poly(Zx, [0, 0, 0]);
0
kash> PolyIsZero(f);
true
```

NAME `PolyMakeMonicInOrder`

PURPOSE Make $f(x)$ monic by rational transformation, so that the resulting monic polynomial $g(x)$ is defined over o and creates the same algebra as $f(x)$. $g(x)$ may be the same as $f(x)$.

SYNTAX `g:= PolyMakeMonicInOrder(f, o);`

order	o
polynomial	f
polynomial	g

DESCRIPTION

EXAMPLE

```
kash> f:= -1/3*x^3+7;
-1/3*x^3 + 7
kash> g:= PolyMakeMonicInOrder(f, Z);
x^3 - 21
```

NAME `PolyMakeMonicInZ`

PURPOSE Given $f(x)$ in $Q[x]$, the function returns $g(x)$ monic in $Z[x]$, such that $g(x)$ generates the same algebra as $f(x)$.

SYNTAX `g:= PolyMakeMonicInZ(f);`

`polynomial`	`f`
`polynomial`	`g`

DESCRIPTION

EXAMPLE

```
kash> f:= -1/3*x^3+7;
-1/3*x^3 + 7
kash> g:= PolyMakeMonicInZ(f);
x^3 - 21
```

NAME `PolyMove`

PURPOSE Tries to compute a representation of a polynomial in a different polynomial algebra.

SYNTAX `g := PolyMove (f, S);`

polynomial	g
polynomial	f
ring	S

DESCRIPTION Given a polynomial and a polynomial algebra or a ring, this function computes a representation of the polynomial either in the given polynomial algebra or in the polynomial algebra associated to the ring. In case only a ring is given, the function automatically generates the polynomial algebra, which is then accessible by the function call PolyAlg (g).

SEE ALSO `PolyAlg`,

EXAMPLE We will factor a polynomial over its equation order.

```
kash> o := Order(Z, 4, 5);;
kash> ox := PolyAlg(o);;
kash> f := PolyMove(OrderPoly(Zx, o), ox);
x^4 - 5
kash> Factor(f);
[ [ x + [0, -1, 0, 0], 1 ], [ x + [0, 1, 0, 0], 1 ],
  [ x^2 + [0, 0, 1, 0], 1 ] ]
```

NAME PolyMoveIntegral

PURPOSE Returns the polynomial moved to integral coefficients.

SYNTAX `g := PolyMoveIntegral(f);`

```
polynomial  f
polynomial  g
```

DESCRIPTION Given a polynomial defined over the quotient field of R such that the denominators of the coefficients are all one, the function returns this polynomial defined over R.

SEE ALSO PolyMove,

EXAMPLE

```
kash> a := 2*(x/2);
x
kash> PolyAlg(a);
Univariate Polynomial Ring in x over Rational Field

kash> b := PolyMoveIntegral(a);
x
kash> PolyAlg(b);
Univariate Polynomial Ring in x over Integer Ring
```

NAME PolyNewtonLift

PURPOSE Lifts an algebraic element with the Newton lifting method.

SYNTAX

```
beta := PolyNewtonLift(f, alpha, k);
beta := PolyNewtonLift(f, alpha, k, a);
```

algebraic element	alpha
integer (0 if omitted)	a
polynomial	f
integer	k
polynomial	beta

DESCRIPTION Given an algebraic element alpha with $\mathtt{f}(\mathtt{alpha}) \equiv 0 \bmod (\mathtt{t} - \mathtt{a})$, this function calculates an algebraic element $\beta \in o[t]$ with $\mathtt{f}(\beta) \equiv 0 \bmod (\mathtt{t} - \mathtt{a})^{\mathtt{k}}$.

EXAMPLE

```
kash> AlffInit(Q,"t","X");
"Defining global variables: k, w, kt, ktf, ktX, t, X, AlffGlobals"
kash> f:=X^4-2+t;
X^4 + t - 2
kash> o:=Order(Z,4,2);
Generating polynomial: x^4 - 2

kash> alpha:=OrderBasis(o)[2];
[0, 1, 0, 0]
kash> beta:=PolyNewtonLift(f,alpha,5);
[0, -77, 0, 0] / 32768*t^4 + [0, -7, 0, 0] / 1024*t^3 + [0, -3, 0, 0] / 128*t^\
2 + [0, -1, 0, 0] / 8*t + [0, 1, 0, 0]
```

NAME `PolyNorm`

PURPOSE Returns the norm of a polynomial.

SYNTAX `n := PolyNorm(f);`

norm (polynomial)	n
polynomial	f

DESCRIPTION Returns the norm $n \in \mathfrak{o}[x]$ of a polynomial $f \in \mathcal{O}[x]$. $\mathfrak{o}$ is the coefficient ring of the order $\mathcal{O}$.

EXAMPLE

```
kash> O := Order(Z,2,3);
Generating polynomial: x^2 - 3

kash> Ox := PolyAlg(O);
Univariate Polynomial Ring in x over Generating polynomial: x^2 - 3

kash> f := Poly(Ox,[1,0,Elt(O,[1,4]),Elt(O,[3,4]),9]);
x^4 + [1, 4]*x^2 + [3, 4]*x + 9
kash> PolyNorm(f);
x^8 + 2*x^6 + 6*x^5 - 29*x^4 - 90*x^3 - 21*x^2 + 54*x + 81
```

NAME	`PolyPowerMod`
PURPOSE	missing shortdoc
SYNTAX	`h := PolyPowerMod(f, n, g);`

```
polynomial        g
polynomial        f
polynomial        h
positive integer  n
```

DESCRIPTION Returns the remainder of a power product of a polynomial modulo another polynomial.

NAME	`PolyPrimeList`
PURPOSE	Returns a list containing all monic prime polynomials of degree d in $k[x]$ for a finite field k.
SYNTAX	`L := PolyPrimeList(kx, d);`

`list`	`L`	
`polynomial algebra`	`kx`	over finite field k
`integer`	`d`	

SEE ALSO	`PolyPrimeNum`, `PolyPrimeRandom`,
EXAMPLE	

NAME PolyPrimeNum

PURPOSE Return the number of monic prime polynomials over a finite field.

SYNTAX `f := PolyPrimeNum(kx, d);`

polynomial	f
polynomial algebra over k	kx
integer	d

DESCRIPTION For given degree d and finite field k this function returns the number of monic prime polynomials of degree d in $k[x]$.

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k, "x");
Univariate Polynomial Ring in x over GF(5^2)

kash> PolyPrimeNum(kx, 11);
216744162819600
```

NAME `PolyPrimeRandom`

PURPOSE Return a random prime polynomial over a finite field.

SYNTAX `f := PolyPrimeRandom(kx, d);`

polynomial	f
polynomial algebra over k	kx
integer	d

DESCRIPTION For given degree d and finite field k this function returns a random prime polynomial of degree d in $k[x]$.

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k, "x");
Univariate Polynomial Ring in x over GF(5^2)

kash> PolyPrimeRandom(kx, 11);
x^11 + 4*x^10 + w^14*x^9 + x^8 + 3*x^7 + 2*x^6 + w^22*x^5 + w^19*x^4 + w^15*x^\
3 + w^16*x^2 + w^23*x + 4
```

NAME PolyQuotRem

PURPOSE missing shortdoc

SYNTAX `L := PolyGcd(f, g);`

```
polynomial  g
polynomial  f
list        L
```

DESCRIPTION Returns the quotient and remainder of two polynomials.

EXAMPLE

```
kash> f := Poly(Zx, [1, 2, 3, 4, 5, 6]);
x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6
kash> h := Poly(Zx, [1, 4, 10, 16, 22, 28, 27, 18]);
x^7 + 4*x^6 + 10*x^5 + 16*x^4 + 22*x^3 + 28*x^2 + 27*x + 18
kash> PolyQuotRem(f, h);
[ 0, x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6 ]
```

NAME PolyRedDisc

PURPOSE Returns the reduced discriminant of a separable polynomial.

SYNTAX `d := PolyRedDisc(f);`

integer	d
polynomial	f

DESCRIPTION Returns the reduced discriminant $d \in \mathbb{Z}$ of a separable polynomial $f \in \mathbb{Z}[x], \mathbb{F}_q[x]$.

SEE ALSO PolyDisc,

EXAMPLE The discriminant of $x^4 + 73 * x^2 - 280 * x - 2399$:

```
kash> f := Poly(Zx, [1, 0, 73, -280, -2399]);
x^4 + 73*x^2 - 280*x - 2399
kash> PolyRedDisc(f);
270382963510
```

NAME `PolyResultant`

PURPOSE Computes the resultant of the two given polynomials.

SYNTAX `r := PolyResultant (f, g);`

```
polynomial  r
polynomial  f
polynomial  g
```

DESCRIPTION PolyResultant(f,g) = 0 if and only if f and g have a common factor which has positive degree in x

SEE ALSO `PolyGcd`,

EXAMPLE

```
kash> Zxy := PolyAlg(Zx);
Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Inte\
ger Ring

kash> f := Poly(Zxy,[1,2*x+4,x^2 +x,6*x]);
y^3 + (2*x + 4)*y^2 + (x^2 + x)*y + 6*x
kash> f_ := PolyDeriv(f);
3*y^2 + (4*x + 8)*y + x^2 + x
kash> PolyResultant(f,f_);
-12*x^5 - 64*x^4 + 460*x^3 + 2828*x^2 + 1536*x
```

NAME	`PolyRoundFour`
PURPOSE	Returns the factorization of the given polynomial over the p-adic integers and certificates for the irreducibility of the factors.

SYNTAX

```
L := PolyRoundFour(f);
L := PolyRoundFour(f, p);
L := PolyRoundFour(f, p, m);
```

`list`	`L`
`polynomial`	`f`
`prime number`	`p`
`integer`	`m`

DESCRIPTION

This function returns the factorization of the given polynomial over the p-adic inetegers. Supported ring types are: $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q}_p$. If the polynomial is not given over $\mathbb{Q}_p$ a prime number p as a second parameter is needed. The desired precision can be given as a third parameter.

SEE ALSO

`Factor`, `PolyFactor`, `OrderMaximal`,

NAME `PolySig`

PURPOSE Computes the signature of a monic squarefree polynomial.

SYNTAX `s := PolySig(f);`

`polynomial`	`f`
`list`	`s`

DESCRIPTION Computes the signature of a monic squarefree polynomial $f(x)$ of degree n over $\mathbb{Z}$ or over an absolut order o.
In the first case the `PolySig` function returns a list containing the number r_1 of real roots and the number r_2 of pairs of complex roots of the given polynomial.
In the second case the `PolySig` function returns a list s containing lists of the type above.
Let $[r_1, r_2]$ be the signature of the generating polynomial of o and for $1 \le i \le r_1$ let $f^{(i)}(x) \in \mathbb{R}[x]$ be the polynomial received by applying the i–the real embedding of $Q(o)$ in $\mathbb{C}$ to the coefficients of f. The i–th entry of s is the signature of the polynomial $f^{(i)}(x) \in \mathbb{R}[x]$ $(1 \le i \le r_1)$. For $r_1+1 \le i \le r_1+r_2$ the i–th entry of s is $[n, 0]$.
The algorithm is based on [Coh95, Algorithm 4.1.11].

EXAMPLE

```
kash> f:= Poly(Zx,[1,7,4,9,6,1]);
x^5 + 7*x^4 + 4*x^3 + 9*x^2 + 6*x + 1
kash> s:= PolySig(f);
[ 3, 1 ]
kash> p:= Poly(Zx,[1,1,1,0,0,-2]);
x^5 + x^4 + x^3 - 2
kash> o:= Order(p);
Generating polynomial: x^5 + x^4 + x^3 - 2

kash> ox:= PolyAlg(o);
Univariate Polynomial Ring in x over Generating polynomial: x^5 + x^4 + x^3 - \
2

kash> f:= Poly(ox,[Elt(o,[1,0,0,0,0]),Elt(o,[-12,13,3,20,-19]),
> Elt(o,[-1,12,-5,7,-12])]);
x^2 + [-12, 13, 3, 20, -19]*x + [-1, 12, -5, 7, -12]
kash> PolySig(f);
[ [ 2, 0 ], [ 2, 0 ], [ 2, 0 ] ]
```

NAME PolySwapVars

PURPOSE Changes the variables of the given bivariate polynomial.

SYNTAX

```
g := PolySwapVars (f);
```

```
polynomial  g
polynomial  f
```

DESCRIPTION

EXAMPLE

```
kash> Zxy := PolyAlg(Zx);
Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Inte\
ger Ring

kash> f := Poly(Zxy,[1,2*x+4,x^2 +x,6*x]);
y^3 + (2*x + 4)*y^2 + (x^2 + x)*y + 6*x
kash> PolySwapVars(f);
x*y^2 + (2*x^2 + x + 6)*y + x^3 + 4*x^2
```

NAME `PolyToList`

PURPOSE Returns the coefficients of a polynomial in a list.

SYNTAX `L := PolyToList(f);`

`list`	`L`
`polynomial`	`f`

DESCRIPTION Given a polynomial $f(x) := \sum_{i=0}^{n} c_i x^i$ the function returns $L := [c_n, c_{n-1}, \ldots, c_0]$.

SEE ALSO `Poly`,

EXAMPLE List of coefficients of the polynomial $x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6 \in \mathbb{Z}[x]$:

```
kash> f := Poly(Zx, [1, 2, 3, 4, 5, 6]);
x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6
kash> PolyToList(f);
[ 1, 2, 3, 4, 5, 6 ]
```

NAME PolyXGcd

PURPOSE Returns the extended gcd of two polynomials.

SYNTAX `l := PolyXGcd(f, h);`

list of polynomials	l
polynomial	f
polynomial	h

DESCRIPTION Given two polynomials $f, h \in S[x]$ where S is a field the function returns the gcd g of f and h together with polynomials a b such that $g = af + bh$.

SEE ALSO PolyGcd,

EXAMPLE

```
kash> Qx := PolyAlg(Q);
Univariate Polynomial Ring in x over Rational Field

kash> f := Poly(Qx, [1, 2, 3, 4, 5, 6]);
x^5 + 2*x^4 + 3*x^3 + 4*x^2 + 5*x + 6
kash> h := Poly(Qx, [1, 4, 10, 16, 22, 28, 27, 18]);
x^7 + 4*x^6 + 10*x^5 + 16*x^4 + 22*x^3 + 28*x^2 + 27*x + 18
kash> PolyXGcd(f, h);
Error, Panic: can't eval bag of type 116
```

NAME `PolyZeros`

PURPOSE Computes roots of a polynomial.

SYNTAX `L := PolyZeros (f [[, "complex"|"int"] [, p [, m ]]]);`

`list`	`L`
`polynomial`	`f`

DESCRIPTION The `PolyZeros` function returns a list containing roots of a polynomial.

`PolyZeros(f)` (or `PolyZeros(f,"complex")`)
Let f be a polynomial over $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. The `PolyZeros` function returns a list containing all complex roots of f.

`PolyZeros(f,"int")`
Let f be a polynomial with integral coefficients, The `PolyZeros` function returns a list containing all integral roots of f.

`PolyZeros(f,p)`
Let f be a polynomial with integral coefficients, the functions returns p-adic approximations of the roots.

`PolyZeros(f,p,m)`
Let f be a polynomial with integral coefficients, then functions returns p-adic approximations of the roots to a p-adic precision m.

EXAMPLE Compute all complex roots of $x^3 + 2$:

```
kash> f := Poly(Zx,[1,0,0,2]);
x^3 + 2
kash> PolyZeros(f);
[ -1.2599210498948731647672106072782283505702514647 01,
  0.62996052494743658238360530363911417528512573235 07919 + 1.09112363597172140\
35600726141898088813258733387 4*i,
  0.62996052494743658238360530363911417528512573235 07919 - 1.09112363597172140\
35600726141898088813258733387 4*i ]
```

NAME Prec

PURPOSE Sets and gets the precision for real and complex computations in the shell.

SYNTAX
```
Prec(n);
Prec();

 integer  n
```

DESCRIPTION All real and complex computations in the shell are performed in a global precision. The `Prec` function sets and gets this global precision. Remark that the global precision has **no** influence at all on computations done in internal KANT functions. The only way to set this is using `OrderPrec`.

`Prec(n)` Sets the global precision to `n` and returns the new global precision. This is the smallest positive integer which is divisible by 4 and greater than or equal to $max(\mathtt{n}, 12)$.

`Prec()` Returns the current global precision.

SEE ALSO `OrderPrec`,

EXAMPLE Compute Euler's constant with different precisions:

```
kash> e;
2.7182818284590452353602874713526624977572470936999
kash> Prec(100);
100
kash> e;
2.718281828459045235360287471352662497757247093699959574966967627724076630353\
47594571382178525166
```

NAME `PvmClear`

PURPOSE Remove all broadcast-jobs, all failed-jobs and all slave-jobs.

SYNTAX `PvmClear();`

EXAMPLE

```
PvmInit();
PvmStartSlave(["nicky","julia","markov"]);
PvmSendAll("h:=GetEnvironment(HOSTNAME);;\n");
PvmShowBroadcastJobs();
PvmClear();
PvmShowBroadcastJobs();
PvmExit();
Exec("echo halt | pvm"); \# kill pvmd on all hosts
```

NAME `PvmExit`

PURPOSE Stops KASH-Pvm.

SYNTAX `PvmExit`

EXAMPLE see function PvmStartSlave

Name `PvmGet`

Purpose Receives data from PVM.

Syntax `L := PvmGet();`

```
List  L
```

Description If there is (valid) data `PvmGet` will receive one (complete) item, otherwise the return value is false. If it successfully receives data it will return a list containing a (small) integer as first entry as a flag. If flag equals 1 the other list entries are (exactly) as sent e.g. by PvmSend.
If flag equals 2 this is send by Error(fmt) on a slave. The second entry will contain the hostname, the third the output of Error(fmt).
If flag equals 4 the data comes from an ordenary Print() on the slave. The second parameter will be the hostname, the third the data string.
Attention, it is possible to send arbitrary data with flag of 2 and 4 using `PvmSlavePrint` or `PvmSlaveError`. Data sent this way may not contain the hostname.
In the examples given below we will NOT describe how to set up the pvm-network. Lines beginning with `slave>` indicate commands executed on the slave. They will not work an the master.

Example

```
PvmInit();
PvmStartSlave("nicky");
Sleep(5);;
PvmGet();
PvmSendNext("PvmSlaveSend(1,2,3,4,5,\"Hallo\");\n");
\# Sending this, the Slave will call PvmSlaveSend(...);
\# don't forget the \n
PvmGet();
PvmGet();
```

NAME `PvmGetAnswer`

PURPOSE Waits for a valid answer to arrive from the slave.

SYNTAX `L := PvmGetAnswer(true | false);`

`list` `L` first entry is in $\{1, 2, 4\}$, all others arbitrary

DESCRIPTION This function performs a while-loop until a valid answer arrives (i. e. a data list starting with 1).
Master only: if the parameter is `true` all incoming data is echoed to the screen. The function will return after the first answer arrived.

SEE ALSO `PvmGet`,

EXAMPLE See `PvmClassgroup` (in `PvmClassgroup.kash`) for an example.

NAME `PvmGetB`

PURPOSE Waits for any data to arrive.

SYNTAX `L := PvmGetB();`

`list L`

DESCRIPTION Calls alternately `PvmGet` and `Sleep(1)` until valid data arrives.

SEE ALSO `PvmGet`,

EXAMPLE See `PvmGet` or `PvmClassgroup` (in `PvmClassgroup.kash`) for an example.

NAME `PvmGetEval`

PURPOSE

SYNTAX *not intended to be called by a user*

DESCRIPTION This function serves both as an example for a slave program and to provide a minimal slave. This function performs a typical cycle for the slave: It waits for any data to arrive, evaluates it and returns an answer. You may stop this loop by sending a simple `true`.

EXAMPLE See `PvmClassgroup` in `PvmClassgroup.kash` for an example.

Name `PvmInit`

Purpose Starts KASH-Pvm on the master.

Syntax `PvmInit();`

Example see function PvmStartSlave

NAME `PvmKashIsSlave`

PURPOSE true iff we are running as slave, false otherwise.

SYNTAX `s := PvmKashIsSlave();`

```
boolean  s
```

EXAMPLE

```
PvmKashIsSlave;
PvmInit();
PvmStartSlave("nicky");
Sleep(15);
PvmGet();
PvmGet();
PvmGet();
PvmSendNext("PvmKashIsSlave;\n");
Sleep(5);
PvmGet();
PvmExit();
Exec("echo halt | pvm"); \# kill pvmd on all hosts
```

Name `PvmLengthOfQueue`

Purpose If called with no parameter, this function returns the maximal number of jobs that can be stored in queue. Otherwise the number is set to the parameter.

Syntax

```
PvmLengthOfQueue(n1);
n2 := PvmLengthOfQueue();

 integer  n1, n2
```

NAME PvmMaxRestartSlave

PURPOSE If called with no parameter, this function returns the maximal number of times a slave may be restarted by security system. Otherwise the number is set to the parameter.

SYNTAX

```
PvmMaxRestartSlave(n1);
n2 := PvmMaxRestartSlave();

 integer  n1, n2
```

NAME `PvmMaxRetransmitJob`

PURPOSE If called with no parameter, this function returns the maximal number of times a job may be retransmitted by security system. Otherwise the number is set to the parameter.

SYNTAX

```
PvmMaxRetransmitJob(n1);
n2 := PvmMaxRetransmitJob();

 integer  n1, n2
```

NAME `PvmPread`

PURPOSE Reads and prints all arrived data in plain style, i.e. as provided by KASH.

SYNTAX `PvmPread();`

DESCRIPTION

SEE ALSO `PvmGet`,

EXAMPLE

NAME PvmRead

PURPOSE Reads (and prints) all data already received from the slave(s).

SYNTAX `PvmRead();`

DESCRIPTION The incoming data is “beautyfied” before printing.

SEE ALSO PvmGet,

EXAMPLE

NAME `PvmSecurity`

PURPOSE If called with no parameter, this function returns the current status of the security system (activated or not). Otherwise it is set to the parameter.

SYNTAX

```
PvmSecurity(true|false);
b := PvmSecurity();
```

```
boolean  b
```

Name `PvmSendAll`

Purpose Sends data to all slaves.

Syntax `PvmSendAll(data, data, ...);`

Description Sends data to all slaves, corresponds to a broadcast. Only data is beeing send, no jobs, so that the slaves are not allowed to send an answer.
Master only: Note that no evaluation is done on the master, so if you send 1+2 the addition is performed on all slaves! The job count on the slaves will be unaffected.

Example see function PvmSendNext

NAME `PvmSendLast`

PURPOSE Sends all parameters to the next free slave.

SYNTAX
```
ok := PvmSendLast(data, data, ...);
```

DESCRIPTION Master only: Note that no evaluation is done on the master, so if you send 1+2 the addition is performed on the slave!
Data sent this way is not regarded as a new job for the security system, the slave is not allowed to send an answer to this. If there is no last slave it will return false, true otherwise.

SEE ALSO `PvmSendNext`,

EXAMPLE see function PvmSendNext

NAME `PvmSendNext`

PURPOSE Sends all parameters to the next free slave.

SYNTAX `PvmSendNext(data, data, ...);`

DESCRIPTION Master only: Note that no evaluation is done on the master, so if you send 1+2 the addition is performed on the slave!
Data sent this way will cause a new job to be started. The job count will be incremented by 1.
For each job received from a `PvmSendNext` exacly one answer has to be send back.

EXAMPLE

```
kash> PvmInit();
Error, KANT failure
kash> PvmStartSlave(["marlin","julia"]);
```

NAME `PvmSetPrintLevel`

PURPOSE Set the `pvm_master->print_level` to lev. Only for debuging.

SYNTAX `PvmSetPrintLevel(lev);`

```
small integer  lev
```

EXAMPLE

```
PvmInit();
PvmSetPrintLevel(1);
PvmExit();
```

NAME `PvmShowBroadcastJobs`

PURPOSE Returns the formatstrings of the broadcast—job as a list. Intendend maily for debugging.

SYNTAX `PvmShowBroadcastJobs();`

EXAMPLE see function PvmSendNext and PvmClear

NAME `PvmSlaveError`

PURPOSE Slave only: Sends all data given as parameter to the master.

SYNTAX `KashPvmError(data, data, ...);`

DESCRIPTION Data send this way will not be regarded as an answer, no updates of the watch and the security system take place! Flag will be set to 2 on the receiving system.

SEE ALSO `PvmGet`,

EXAMPLE see function PvmGet

NAME `PvmSlaveInfo`

PURPOSE Returns information of all Slaves in a list of records.

SYNTAX `PvmSlaveInfo();`

EXAMPLE see function PvmStartSlave and PvmStopSlave

NAME `PvmSlavePrint`

PURPOSE Slave only: Sends all data given as parameter to the master.

SYNTAX `PvmSlavePrint(data, data, ...);`

DESCRIPTION Data send this way will not be regarded as an answer, no updates of the watch and the security system take place! Flag will be set to 4 on the receiving system.

SEE ALSO `PvmGet`,

EXAMPLE see function PvmGet

NAME `PvmSlaveSend`

PURPOSE Slave only: Sends all data given as parameter to the master.

SYNTAX `PvmSlaveSend(data, data, ...);`

DESCRIPTION The data send this way is regarded as an answer, so the job count is decremented by one, and the watch (and the security system) is updated. Flag will be set to 1 on the receiving system.

SEE ALSO `PvmGet`,

EXAMPLE see function PvmGet

NAME PvmStartSlave

PURPOSE Starts the slave(s).

SYNTAX

```
noSlaves := PvmStartSlave(num);
noSlaves := PvmStartSlave(host);
```

integer	noSlaves	the number of started slaves
integer	num	the number of slaves to start. Use 0 to start as many as possible.
string	host	start one slave at host.
list	L	List of hostnames to start one slave at.

EXAMPLE

```
PvmInit();
PvmStartSlave(["nicky","marlin"]);
PvmSlaveInfo();
PvmStartSlave("julia");
List(PvmSlaveInfo(),slave->[slave.host,slave.arch,slave.tid]);
PvmExit();
Exec("echo halt | pvm"); \# kill pvmd on all hosts
```

Name `PvmStopSlave`

Purpose Kills the slaves. If no argument is given, all slaves will be killed.

Syntax

```
noSlaves := PvmStopSlave();
noSlaves := PvmStopSlave(stid);
noSlaves := PvmStopSlave(L);
```

`integer`	`noSlaves`	the number of killed slaves
`integer`	`stid`	slave-tid to kill
`list`	`L`	List of slave-tids to kill

Example

```
PvmInit();
PvmStartSlave(["markov","markov","julia","daisy","goofy"]);
a:=List(PvmSlaveInfo(),slave->[slave.host,slave.tid]);
a1:=a[1][2];
PvmStopSlave(a1);
a:=List(PvmSlaveInfo(),slave->[slave.host,slave.tid]);
PvmStopSlave([a[1][2],a[4][2]]);
a:=List(PvmSlaveInfo(),slave->[slave.host,slave.tid]);
PvmStopSlave();
a:=List(PvmSlaveInfo(),slave->[slave.host,slave.tid]);
PvmExit(); \# kill all kash-slaves
Exec("echo halt | pvm"); \# kill pvmd on all hosts
```

Name	`PvmStoreOrders`
Purpose	Toggels/ Queries some internal flags.
Syntax	`PvmStoreOrders`

NAME PvmUse

PURPOSE Enables or disables the use of KANT-PVM. Initial state is false. If no argument is given, this function gets the status of `PvmUse`.

SYNTAX
```
PvmUse(true|false);
s := PvmUse();

 boolean  s
```

EXAMPLE Enable KANT-PVM.

```
PvmUse(true);
PvmUse();
```

NAME PvmUseMastersHost

PURPOSE Enables or disables the use of the master's host for `PvmStartSlave`. Initial state is false. If no argument is given, it gets the status of `PvmUseMastersHost`.

SYNTAX
```
PvmUseMastersHost(true|false);
s := PvmUseMastersHost();

 boolean  s
```

EXAMPLE Enable `PvmUseMastersHost`.

```
PvmUseMastersHost(true);
PvmUseMastersHost();
```

NAME `PvmUseMsg`

PURPOSE Enables or disables output for debugging.

SYNTAX

```
PvmUseMsg(true|false);
s := PvmUseMsg();

 boolean  s
```

DESCRIPTION Initial state is false. If no argument is given, it gets the status of `PvmUseMsg`. The output is written to files of the name "pvm_master_*uid*.old" or "pvm_slave_*uid*.old" in the directory /tmp. *uid* is the user-id of the particular PVM-process.

EXAMPLE Enable PvmUseMsg.

```
PvmUseMsg(true);
PvmUseMsg();
```

NAME `PvmUseWatch`

PURPOSE Enables or disables the PVM-Watch for debugging. If h is given, the PVM-Watch starts on this host. Initial state is false. If no argument is given, it gets the status of `PvmUseWatch`.

SYNTAX

```
PvmUseWatch(x, h);
s := PvmUseWatch(x);
s := PvmUseWatch();
```

`boolean`	`x`	
`string`	`h`	hostname
`boolean`	`s`	

EXAMPLE Enable and disable the PVM-Watch.

```
PvmUseWatch(true);
PvmUseWatch();
PvmUseWatch(false);
```

Q

Name Q

Purpose Predefined constant: Rational field $\mathbb{Q}$.

Syntax Q;

```
ring  Q
```

Description This is a predefined constant for the rational field $\mathbb{Q}$ and is referenced by the variable Q. It is of arbitrary precision.

See Also Z, R, C,

Example

```
kash> Q;
Rational Field
```

NAME QfName

PURPOSE missing shortdoc

SYNTAX `s := QfName(S);`

string	s
polynomial algebra or quotient field	S

DESCRIPTION Given a polynomial algebra or its quotient field this function returns the name of the (outmost) variable.

EXAMPLE

```
kash> Zx;
Univariate Polynomial Ring in x over Integer Ring

kash> QfName(Zx);
"x"
```

NAME `QfRank`

PURPOSE missing shortdoc

SYNTAX `n := QfRank(S);`

`integer`	`n`
`polynomial algebra or quotient field`	`S`

DESCRIPTION Given a polynomial algebra or its quotient field this function returns the number of variables.

EXAMPLE

```
kash> Zxy := PolyAlg(Zx);
Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Inte\
ger Ring

kash> QfRank(Zxy);
2
```

NAME `QfScalarRing`

PURPOSE missing shortdoc

SYNTAX `A := QfScalarRing(S);`

`ring`	`A`
`polynomial algebra or quotient field`	`S`

DESCRIPTION Given a polynomial algebra or its quotient field this function returns its ring of scalars.

SEE ALSO `PolyAlgCoef`,

EXAMPLE

```
kash> Zxy := PolyAlg(Zx);
Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over Inte\
ger Ring

kash> QfScalarRing(Zxy);
Integer Ring

kash> QfScalarRing(QuotientField(Zxy));
Rational Field
```

NAME QfeDen

PURPOSE This function returns the denominator of a rational function (quotient field element). Works also for polynomials.

SYNTAX `d := QfeDen(f);`

`quotient field elements d,f`

SEE ALSO QfeNum, Num, Den,

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(5^2)

kash> x := Poly(kx, [1,0]);
x
kash> f := x;
x
kash> g := x+1;
x + 1
kash> h := f/g;
x/(x + 1)
kash> QfeDen(h);
x + 1
```

Name	`QfeDeriv`
Purpose	Computes the derivation of an rational function.
Syntax	`dhdT := QfeDeriv(h);`
	`qf elements  dhdT, h`
Description	This function computes dh/dT for $h \in k(T)$.
See Also	`AlffDiff`, `PolyDeriv`,
Example	

```
kash> AlffInit(FF(5,1));;
kash> h := 1/T;
1/T
kash> QfeDeriv(h);
4/T^2
```

NAME QfeNum

PURPOSE This function returns the numerator of a rational function (quotient field element). Works also for polynomials.

SYNTAX `d := QfeNum(f);`

`quotient field elements d,f`

SEE ALSO QfeDen, Num, Den,

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(5^2)

kash> x := Poly(kx, [1,0]);
x
kash> f := x;
x
kash> g := x+1;
x + 1
kash> h := f/g;
x/(x + 1)
kash> QfeNum(h);
x
```

NAME `QfePthRoot`

PURPOSE Returns the p-th root of a rational function with scalars in a finite field of characteristic p, if existent.

SYNTAX `r := QfePthRoot(a);`

`qf element or bool`	`r`	r p-th root of a or `false`
`qf element`	`a`	

SEE ALSO `AlffEltPthRoot`,

EXAMPLE

```
kash> k := FF(3);
Finite field of size 3
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(3)

kash> kxy := PolyAlg(kx);
Univariate Polynomial Ring in y over Univariate Polynomial Ring in x over GF(3\
)

kash> X := Poly(kx, [1,0]);
x
kash> y := Poly(kxy, [1,0]);
y
kash> QfePthRoot(1/X^3);
1/x
kash> QfePthRoot(y);
false
kash> QfePthRoot(1/X + y^3);
false
kash> QfePthRoot(1/X^3 + y^3);
y + 1/x
kash> QfePthRoot(1/y^3);
1/y
```

NAME `QfeQf`

PURPOSE Returns the quotient field in which a rational function (quotient field element) is defined.

SYNTAX `F := QfeQf(a);`

quotient field	F
quotient field element	a

SEE ALSO `QuotientField`,

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(5^2)

kash> x := Poly(kx, [1,0]);
x
kash> QfeQf(1/x);
Univariate rational function field over GF(5^2)
Variables: x
```

NAME `QfeVal`

PURPOSE Computing valuations in a rational function field.

SYNTAX `v := QfeVal(p, a);`

`integer`	`v`
`qf elements`	`p, a`

DESCRIPTION Given the rational function field $k(x)$ and a prime element $p \in k(x)$, that is a prime polynomial or $1/x$, compute the valuation $v_p(a)$ of a at the place given by p.

SEE ALSO `InftyVal`, `PolyFactor`,

EXAMPLE

```
kash> k := FF(5, 2);
Finite field of size 5^2
kash> kx := PolyAlg(k);
Univariate Polynomial Ring in x over GF(5^2)

kash> x := Poly(kx, [1,0]);
x
kash> QfeVal(x, x^3);
3
kash> QfeVal(1/x, x^3);
-3
kash> QfeVal(x, (x+1)^3);
0
kash> QfeVal(1/x, (x+1)^3);
-3
```

Name	`Qp`
Purpose	Creates the p-adic field $\mathbb{Q}_p$
Syntax	`F := Qp(p[,n]);`

p-adic field	F
prime number	p
integer	n

Description: This function creates the p-adic Field $\mathbb{Q}_p$ with given precision n. If no precision is given it is set to 20.

See Also: IsQp, QpElt, QpEltToQ, QpEltQp, QpPrec, QpExp, QpLog, QpPrime, QpSqrt, QpValuation,

Example

```
kash> F := Qp(2);
2-adic Field mod 2^20
```

NAME QpElt

PURPOSE Returns an element of $\mathbb{Q}_p$.

SYNTAX `k := QpElt(F, n);`

p-adic field	F
integer or rational	n
p-adic element	k

DESCRIPTION This function returns the series of an integral or rational number, e.g. an element k of the p-adic field.

SEE ALSO IsQp, Qp, QpEltToQ, QpEltQp, QpExp, QpLog, QpPrec, QpPrime, QpSqrt, QpValuation,

EXAMPLE

```
kash> F := Qp(3);
3-adic Field mod 3^20
kash> q := 1/2;
1/2
kash> k := QpElt(F, q);
2 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 3^7 + 3^8 + 3^9 + 3^10 + 3^11 + 3^12 + 3\
^13 + 3^14 + 3^15 + 3^16 + 3^17 + 3^18 + 3^19 + O(3^20)
```

Name QpEltQp

Purpose This function creates the p-adic field of the element k.

Syntax `F := QpEltQp(k);`

p-adic field element	k
p-adic field	F

See Also IsQp, Qp, QpElt, QpEltToQ, QpPrec, QpExp, QpLog, QpPrime, QpSqrt, QpValuation,

Example

```
kash> F := Qp(3);
3-adic Field mod 3^20
kash> k := QpElt(F, 44);
2 + 2*3 + 3^2 + 3^3
kash> F := QpEltQp(k);
3-adic Field mod 3^20
```

NAME `QpEltToQ`

PURPOSE Computes the rational number q of an element in $\mathbb{Q}_p$.

SYNTAX `q := QpEltToQ(k);`

p-adic field	F
p-adic element	k
rational number	q

SEE ALSO IsQp, Qp, QpElt, QpEltQp, QpExp, QpLog, QpPrec, QpPrime, QpSqrt, QpValuation,

EXAMPLE

```
kash> F := Qp(3);
3-adic Field mod 3^20
kash> k := QpElt(F, 27);
3^3
kash> q := QpEltToQ(k);
27
```

Name
: `QpExp`

Purpose
: Computes the exponent function of an element in $\mathbb{Q}_p$.

Syntax
: `exp:= QpExp(k);`

p-adic element `k`
p-adic element `exp`

Description
: The exponent function *exp* of the element k of a p-adic field is returned. If *prec* is the used precision, the result is correct modulo p^{prec}.

See Also
: `IsQp`, `Qp`, `QpElt`, `QpEltToQ`, `QpEltQp`, `QpLog`, `QpPrec`, `QpPrime`, `QpSqrt`, `QpValuation`,

Example

```
kash> F := Qp(3);
3-adic Field mod 3^20
kash> k := QpElt(F, 18);
2*3^2
kash> exp := QpExp(k);
1 + 2*3^2 + 2*3^4 + 3^5 + 3^6 + 2*3^7 + 2*3^9 + 3^10 + 3^11 + 2*3^13 + 3^17 + \
2*3^18 + 2*3^19 + O(3^20)
```

NAME QpLog

PURPOSE Computes the logarithm of an element in $\mathbb{Q}_p$.

SYNTAX `l := QpLog(k);`

p-adic element	k
p-adic element	l

DESCRIPTION This function returns the logarithm l of the element k of a p-adic field. If $prec$ is the used precision, the result is correct modulo p^{prec}.

SEE ALSO IsQp, Qp, QpElt, QpEltToQ, QpEltQp, QpPrec, QpExp, QpPrime, QpSqrt, QpValuation,

EXAMPLE

```
kash> F := Qp(3);
3-adic Field mod 3^20
kash> k := QpElt(F, 18);
2*3^2
kash> l := QpLog(k);
2*3 + 2*3^2 + 3^5 + 3^6 + 2*3^8 + 3^9 + 2*3^10 + 3^12 + 2*3^13 + 3^14 + 2*3^16\
 + 2*3^17 + O(3^21)
```

NAME QpPrec

PURPOSE Prints or sets the precision of a p-adic field.

SYNTAX `k:= QpPrec(F [,n]);`

p-adic field	F
integer	n
integer	k

SEE ALSO IsQp, Qp, QpElt, QpEltToQ, QpEltQp, QpExp, QpLog, QpPrime, QpSqrt, QpValuation,

EXAMPLE

```
kash> F:= Qp(2);
2-adic Field mod 2^20
kash> k:= QpPrec(F, 30);
30
```

NAME QpPrime

PURPOSE This function creates the prime number which defines the p-adic field.

SYNTAX `p := QpPrime(F);`

p-adic field	F
prime number	p

SEE ALSO IsQp, Qp, QpElt, QpEltToQ, QpEltQp, QpPrec, QpExp, QpLog, QpSqrt, QpValuation,

EXAMPLE

```
kash> F := Qp(2);
2-adic Field mod 2^20
kash> p := QpPrime(F);
2
```

NAME QpSqrt

PURPOSE Computes the square root of an element in $\mathbb{Q}_p$.

SYNTAX `s := QpSqrt(k);`

p-adic element `k`
p-adic element `s`

DESCRIPTION This function returns the square root s of the element k of a p-adic field. If $prec$ is the used precision, the result is correct modulo p^{prec}.

SEE ALSO IsQp, Qp, QpElt, QpEltToQ, QpEltQp, QpPrec, QpExp, QpLog, QpPrime, QpValuation,

EXAMPLE

```
kash> F := Qp(3);
3-adic Field mod 3^20
kash> k := QpElt(F, 19);
1 + 2*3^2
kash> s := QpSqrt(k);
1 + 3^2 + 3^4 + 3^5 + 2*3^8 + 3^9 + 3^10 + 2*3^11 + 3^13 + 2*3^15 + 3^16 + 2*3\
^18 + 2*3^19 + O(3^20)
```

NAME QpValuation

PURPOSE Computes the valuation of an element in $\mathbb{Q}_p$.

SYNTAX `v := QpValuation(F, k);`

p-adic field	F
p-adic element	k
integer	v

SEE ALSO IsQp, Qp, QpElt, QpEltToQ, QpEltQp, QpExp, QpLog, QpPrec, QpPrime, QpSqrt,

EXAMPLE

```
kash> F := Qp(3);
3-adic Field mod 3^20
kash> k := QpElt(F, 27);
3^3
kash> v := QpValuation(k);
3
```

QuotientField

NAME `QuotientField`

PURPOSE missing shortdoc

SYNTAX `QF := QuotientField(S);`

quotient field	QF
ring	S

DESCRIPTION Creates the quotient field of a ring, intended for polynomial algebras.

SEE ALSO PolyAlg,

EXAMPLE Creating the quotient field of a polynomial algebra over a finite field:

```
kash> Fp := FF(5);
Finite field of size 5
kash> Fpx := PolyAlg(Fp);
Univariate Polynomial Ring in x over GF(5)

kash> Fpxf := QuotientField(Fpx);
Univariate rational function field over GF(5)
Variables: x
```

NAME R

PURPOSE Predefined constant: Real field $\mathbb{R}$.

SYNTAX `R;`

```
ring  R
```

DESCRIPTION This is a predefined constant for the real field $\mathbb{R}$ and is referenced by the variable `R`. The precision can be set or displayed with `Prec`. Default precision is 50.

SEE ALSO `Z`, `Q`, `C`, `Prec`,

EXAMPLE

```
kash> R;
Real Field of precision 52
```

NAME RandomEcc

PURPOSE Returns a random elliptic curve in Weierstrass form.

SYNTAX `E := RandomEcc(F);`

elliptic curve	E
finite field	F

DESCRIPTION

EXAMPLE

EXAMPLE

```
kash> p:=65112*2^144-1;
1452046121366725933991673688168680114377396846591
kash> IsPrime(p);
true
kash> RandomEcc(FF(p));
[ 1042675512671975645588756792447528436479728692810,
  932028267535118580676757976094182914034159852790 ]
```

NAME `RandomElt`

PURPOSE Computes a “random” element of a given order.

SYNTAX `alpha := RandomElt(R [, l | b, deg | degl]);`

`element`	`alpha`	of `o`
`ring`	`R`	may be an order, an ideal, $\mathbb{Z}$, a module, a polynomial algebra or an function field order.
`list`	`l`	of integers
`integer`	`b`	equivalent to $l := [-b..b]$
`integer`	`deg`	degree of a random element of the
	`polynomial algebra` R	
`list`	`degl`	of positive integers

DESCRIPTION A “random” element *aplpha* with coefficients in l will be returned (by default $l := [-20, 20]$). Of course, it is not possible to do a truly random element from an infinite set.
If R is a polynomial algebra, and you want to create a random element with a certain degree *deg*, then you also have to give l or b as a parameter to `RandomElt`. If *deg* is a list, then the degree of *alpha* will be an element of *deg*. If R is a function field order, then *deg* is the range of the degree of the coefficients of α, and l the range of coefficients of these coefficients (only for absolute extensions in the moment). By default the degree is an integer between 0 and 20.

EXAMPLE

```
kash> o := Order(Z, 5, 3);
Generating polynomial: x^5 - 3

kash> RandomElt(o);
[19, 11, 5, -6, -18]
kash> RandomElt(o, 100);
[-29, 86, 87, 22, -77]
kash> RandomElt(o, [-1,1]);
[-1, -1, -1, -1, 1]
```

NAME `RandomIdeal`

PURPOSE A random ideal over a given order is computed.

SYNTAX `id := RandomIdeal(o);`

order	o
ideal	id

DESCRIPTION

EXAMPLE

```
kash> o := OrderMaximal(Z, 5, 3);
Generating polynomial: x^5 - 3
Discriminant: 253125

kash> id := RandomIdeal(o);
<
[30221858653155826767789393356661370243131978936076711612754461465704355739193\
61247556678677790696389158652435905798869586147566445476339207974268596145615 6\
962323208667984344470385964474097208077148437 5 852047693423259124067478502694 0\
58106827437321604118758741520952262050290153703205001069322518997562101902191 5\
52469101948783314677600625748214471906560165442142975019758636626600319066442 8\
5115888671875 23816376024820232547001188887159039866762011677965683011367506 91\
82576837888462167829016853979600513608116408212409109691761908541022773824730 6\
340608441903976611695930824662086399999781445319262890625 00 205138702125977431\
92034233219582095634716038319005641300183887530496530031891653442758902037410 4\
63523338485622644404687311502271884392566966735343747668601497973078513391242 5\
462572002447925251933593 75 2777697367620085223107590662101719717172625700566 49\
96002379430202512802343016633638899961790568801069452139845974244634791572783 5\
2876277680101245483943381393159362510114103363219919851851805936562500 00]
[0 617906921611055298621727122045948267927573408240201614029487224472609818283\
70285935288805261937879994200301005080207304605912038987292198488888282614251 7\
1266812143181382327148437 5 0 2471627686444221194486908488183793071710293632960\
80645611794889789043927313481143741155221047751519976801204020320829218423648 1\
55949168793955553130457006850672485727255293085937 50 152361451240245104297572 7\
55008067051138312392938802301051664902348823705549990466796214881286910146980 0\
39885060127376717332196555730877135680993196304294807162796063423361640625 00]
[0 0 6179069216110552986217271220459482679275734082402016140294872244726098182\
83702859352888052619378799942003010050802073046059120389872921984888888282614 25\
171266812143181382327148437 5 123581384322211059724345424409189653585514681648 0\
```

```
403228058974448945219636567405718705776105238757599884006020101604146092118240\
7797458439697777656522850342533624286362764654296875 5236738144655066600424429\
584652494977523477732689116091261137434737377598973805051150653892245491459124\
0224151737804141789496164296487346917600159181767975872641726792257248046875]
[0 0 0 12358138432221105972434542440918965358551468164804032280589744489452196\
365674057187057761052387575998840060201016041460921182407797458439697777656522\
850342533624286362764654296875 16425823889565787100790840234242390693672473764\
862421365217535087150211602695303929742010807150070841791067627570060156602027\
46487079386135363732060428168073754840037437089843750]
[0 0 0 0 803628477662010217173357389370142650855419626318440491520629646672245\
144719795607173644668491194018865334378398584170064876074355033808838298070701\
7644456828732421875]
>
```

NAME	`RandomMatrix`
PURPOSE	Using `RandomMatrix` a square matrix with “random” entries is returned.
SYNTAX	`m := RandomMatrix(R, n, l);`

`matrix`	`m`	in $\mathtt{R}^{\mathtt{n}\times\mathtt{n}}$
`ring`	`R`	may be an order, an ideal, $\mathbb{Z}$, a module or an function field order.
`integer`	`n`	
`list`	`l`	of possible (coeffiecents of the) entries.

DESCRIPTION

EXAMPLE

```
kash> m := RandomMatrix(Z, 5);
[ 19  11   5  -6 -18]
[ -6  17  18   5 -16]
[ -1 -16  -4  -3  13]
[ 17  11  -9  11  18]
[ -5   3  -7  12  -3]
kash> m := RandomMatrix(Z, 5);
[  7  17  -6  -9   5]
[ 11   3 -10 -18  10]
[ 17  16 -14   3 -15]
[ -2   0   3  -9  10]
[  1 -11   6   9  -4]
```

NAME `RandomOrder`

PURPOSE Using `RandomOrder` a random order is computed.

SYNTAX `o := RandomOrder(R, n [, l]);`

`order`	`o`	
`ring`	`R`	may be $\mathbb{Z}$, an order or an functions field order.
`integer`	`n`	degree
`list`	`l`	

DESCRIPTION

EXAMPLE

```
kash> o := RandomOrder(Z, 4);
Generating polynomial: x^4 + 19*x^3 + 11*x^2 + 5*x - 6
```

NAME `RandomPoly`

PURPOSE Produces a monic random polynomial of given degree.

SYNTAX `f := RandomPoly(R, d [, l]);`

`polynomial`	`f`	in $\mathtt{R}[x]$
`ring`	`R`	may be an order, an ideal, $\mathbb{Z}$, a module or an function field order.
`integer`	`d`	degree of f
`list`	`l`	of possible (coeffiecents of the) coefficients.

DESCRIPTION

EXAMPLE

```
kash> f := RandomPoly(Z, 3);
x^3 + 19*x^2 + 11*x + 5
kash> o := Order(f);
Generating polynomial: x^3 + 19*x^2 + 11*x + 5

kash> g := RandomPoly(o, 4);
x^4 + [-6, -18, -6]*x^3 + [17, 18, 5]*x^2 + [-16, -1, -16]*x + [-4, -3, 13]
```

NAME RationalReconstruct

PURPOSE Lifts an integer modulo m to a rational.

SYNTAX `q := RationalReconstruct (u,m);`

```
rational  q
integer   u
integer   m
```

DESCRIPTION Given positive integers u, m, the function `RationalReconstruct` computes a rational number $q = \frac{a}{b}$ such that $a \equiv bu \bmod m$ and $0 \leq |a|, b < \sqrt{\frac{m}{2}}, b \neq 0$, if such a pair exists. Otherwise the `0` is returned.

SEE ALSO EltReconstruct,

EXAMPLE

```
kash> RationalReconstruct(17,49);
2/3
```

NAME RayCantoneseRemainder

PURPOSE missing shortdoc

SYNTAX `elt := RayCantoneseRemainder(m0,minf,elt0,sig);`

ideal	m0
list of integers/infinite primes	minf
algebraic number	elt0
list	sig
algebraic number	elt

DESCRIPTION Returns an algebraic number `elt`, which is congruent to `elt0` (mod `m0`) and whose real conjugates have the signature as chosen by `sig` at the places of `minf`. The algorithm is described in [Pau96]. `sig` is a vector/matrix, which consists of 0's and 1's. A 1 in the i'th position means that elt has a negative conjugate at the i'th place of `minf` (which must be sorted!), a 0 means that it has positive conjugate.

SEE ALSO EltApproximation, EltCon, IdealChineseRemainder, OrderSig, RayResidueRing,

EXAMPLE

```
kash> O := OrderMaximal(Z,4,5);
   F[1]
    |
   F[2]
  /
 /
Q
F  [ 1]     Given by transformation matrix
F  [ 2]     x^4 - 5
Discriminant: -2000

kash> m0 := 12*O;; minf := [1,2];;
kash> elt0 := Elt(O,[1,2,3,4]);;sig := [1,0];;IsMat(sig);;
kash> elt := RayCantoneseRemainder(m0,minf,elt0,sig);
[5029, -2590, 3591, -2300]
kash> elt mod m0;
[1, 2, 3, 4]
kash> EltCon(elt);
```

```
[-5910.97931590567721573697080615820292291362269435155 12964.7430748579057138\
01214012490280542867460996 61 640.1181205238857509678783968339611900230808488 71\
131 - 1747.34768815293818873191206684979046442813714147799 1*i 640.118120523885\
7509678783968339611900230808488 71131 + 1747.34768815293818873191206684979046 44\
28137141477991*i]
```

NAME `RayClassFieldAbelianTest`

PURPOSE Tests if a class field is also abelian over $\mathbb{Q}$.

SYNTAX `a:=RayClassFieldIsAbelian(G [,m]);`

`AbelianGroup`	`G`	defining a class group
`boolean`	`a`	
`integer`	`m`	

DESCRIPTION Let $\mathcal{O}$ be the ring of integers of an abelian extention over $\mathbb{Q}$. $\mathfrak{m}$ is a multiple of its conductor $\mathfrak{f} \in \mathbb{Z}$. G is a class group in $\mathcal{O}$, K the abelian extension over $\mathcal{O}$ defined by G. The function determines if K is also abelian over $\mathbb{Q}$. The methode is described in [Has70, II, p.24].

EXAMPLE

```
kash> o:=OrderMaximal(x^2-10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> G:=RayClassGroupToAbelianGroup(9*o);
RayClassGroupToAbelianGroup(<9>, [  ])
Group with relations:
[2 0]
[0 3]

kash> RayClassFieldAbelianTest(G);
true
```

Let's check this. First we'll compute the `RayClassField`

```
kash> L:=RayClassField(G);
[ x^2 - 2, x^3 - 3*x - 1 ]
```

The only critical part is the cubic polynomial, therfore we'll show that this polynomial defines a $C|3$ extension over $\mathbb{Q}$.

```
kash> g:=PolyMove(L[2], Z);
x^3 - 3*x - 1
kash> OrderAutomorphismsAbel(Order(g));
true
kash> o:=OrderMaximal(x^4-2);
Generating polynomial: x^4 - 2
Discriminant: -2048

kash> G:=RayClassGroupToAbelianGroup(9*o);
RayClassGroupToAbelianGroup(<9>, [  ])
Group with relations:
[3 0]
[0 3]
kash> RayClassFieldIsAbelian(G);
false
```

NAME RayClassFieldArtin

PURPOSE Given an ideal, this function computes the corresponding automorphism.

SYNTAX `aut := RayClassFieldArtin(id, O);`

an automorphism	aut	
ideal	id	coprime to the defining module
order	O	output of RayClassFieldAuto

DESCRIPTION This function essentially provides $(\mathfrak{a}, O/o) \in \mathrm{Gal}(O/o)$ for given unramified ideals. Note, that this function won't check if the conductor of the abelian extension is known. Therefore it is only possible to compute automorphisms for ideals coprime to the defining module.
Note, that this function won't work on an arbitraily defined abelian extensions `O` of `o`. It is necessary to compute `O` using `RayClassFieldAuto`.

SEE ALSO RayClassFieldAuto, RayClassField,

EXAMPLE

```
kash> o := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> i := 9*o;
<9>
kash> O := RayClassFieldAuto(RayClassField(i))[1];
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^6 - 12*x^4 - 2*x^3 + 21*x^2 - 6*x - 1
E 1[ 1]     x^2 - 10
Discriminant: <17901347328>

kash> l := RayClassGroupCyclicFactors(i);
[ [ <[-4, -6], [-30, -2]>, 2 ], [ <[-2, -3]>, 3 ] ]
kash> ap := RayClassFieldArtin(l[1][1], O);
```

```
a
kash> ap2 := RayClassFieldArtin(l[1][1]^2, 0);
e
kash> ap^2 = ap2;
true
kash> OrderAutomorphisms(o);
[ [0, 1], [0, -1] ]
kash> RayClassFieldArtin(l[2][1], 0);
b*b
kash> RayClassFieldArtin(IdealAutomorphism(l[2][1], 2), 0);
b*b
```

NAME `RayClassFieldAuto`

PURPOSE Given the output of `RayClassField`, this function computes a primitive element for the Ray Class Field togther with the automorphisms.

SYNTAX `L := RayClassFieldAuto(c);`

`list`	`L`	containing `O` and a list of automorphisms
`list`	`c`	output of `RayClassField`

DESCRIPTION Starting with the data computed by `RayClassField`, this function first computes a primitive element for `O/o`. Next all the automorphisms of `O/o` are computed, together with information allowing one to use the Artin isomorphism.
If automorphisms of the base field (`o`) are known (prior to the call of `RayClassField`), and if they extend to `O/Q` they to are computed.

SEE ALSO `RayClassField`,

EXAMPLE We will consider various extensions of $\mathbb{Q}(\sqrt{10})$ to demonstrate some effects and possibilities:

```
kash> o := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40
```

NAME `RayClassFieldIsAbelian`

PURPOSE Tests if a class field is also abelian over $\mathbb{Q}$.

SYNTAX `a:=RayClassFieldIsAbelian(G [,m]);`

`AbelianGroup`	`G`	defining a class group
`boolean`	`a`	
`integer`	`m`	

DESCRIPTION Let $\mathcal{O}$ be the ring of integers of an abelian extention over $\mathbb{Q}$. $\mathfrak{m}$ is a multiple of its conductor $\mathfrak{f} \in \mathbb{Z}$. G is a class group in $\mathcal{O}$, K the abelian extension over $\mathcal{O}$ defined by G. The function determines if K is also abelian over $\mathbb{Q}$.

EXAMPLE

```
kash> o:=OrderMaximal(x^2-10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> G:=RayClassGroupToAbelianGroup(9*o);
RayClassGroupToAbelianGroup(<9>, [  ])
Group with relations:
[2 0]
[0 3]
kash> m:=Disc(o);
40
kash> RayClassFieldIsAbelian(G,m);
true
```

Let's check this. First we'll compute the `RayClassField`

```
kash> L:=RayClassField(G);
[ x^2 - 2, x^3 - 3*x - 1 ]
```

The only critical part is the cubic polynomial, therfore we'll show that this polynomial defines a $C|3$ extension over $\mathbb{Q}$.

```
kash> g:=PolyMove(L[2], Z);
x^3 - 3*x - 1
kash> OrderAutomorphismsAbel(Order(g));
true
kash> o:=OrderMaximal(x^4-2);
Generating polynomial: x^4 - 2
Discriminant: -2048

kash> G:=RayClassGroupToAbelianGroup(9*o);
RayClassGroupToAbelianGroup(<9>, [  ])
Group with relations:
[3 0]
[0 3]
kash> RayClassFieldIsAbelian(G);
false
```

NAME	`RayClassFieldIsCentral`
PURPOSE	Tests whether the Ray Class Field defined by the given data will be a central extension.
SYNTAX	`flag := RayClassFieldIsCentral(G [, l]);`

`boolean`	`flag`	
`AbelianGroup`	`G`	a quotient from RayClassGroupToAbelianGroup
`list`	`l`	of automorphisms, if not present all known automorphisms are used.

DESCRIPTION

EXAMPLE We'll investigate the ray class field mod $m := (6)\mathfrak{p}|1^\infty\mathfrak{p}|2^\infty$ in $k := \mathbb{Q}(\sqrt{10})$:

```
kash> o := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> OrderAutomorphisms(o);
[ [0, 1], [0, -1] ]
kash> G1 := RayClassGroupToAbelianGroup(6*o, [1, 2]);
RayClassGroupToAbelianGroup(<6>, [ 1, 2 ])
Group with relations:
[2 0 0]
[0 4 0]
[0 0 2]
kash> rcg := FindMaximalCentralField(G1);
[2 0 0]
[0 2 0]
[0 0 2]
[0 0 0]
[0 0 0]
[0 0 0]
kash> G2 := AbelianQuotientGroup(G1, AbelianSubGroup(G1, rcg));
Group with relations:
[2 0 0]
[0 2 0]
[0 0 2]
[0 0 0]
[0 0 0]
[0 0 0]
```

```
[2 0 0]
[0 4 0]
[0 0 2]
kash> RayClassFieldIsCentral(G1);
false
kash> RayClassFieldIsCentral(G2);
true
```

NAME `RayClassFieldIsNormal`

PURPOSE Tests whether the Ray Class Field defined by the given data will be a normal extension.

SYNTAX `flag := RayClassFieldIsNormal(G [, l]);`

`boolean`	`flag`	
`AbelianGroup`	`G`	a quotient from RayClassGroupToAbelianGroup
`list`	`l`	of automorphisms, if not present all known automorphisms are used.

DESCRIPTION

EXAMPLE See `RayClassFieldSplittingField` for an example.

NAME `RayClassFieldSplittingField`

PURPOSE Computes data to get the splitting field of the Ray Class Field.

SYNTAX `sg := RayClassFieldSplittingField(G, l);`

`AbelianGroup`	`sg`	
`AbelianGroup`	`G`	quotient of RayClassGroupToAbelianGroup
`list`	`l`	of automorphisms. Must be complete, not just generators.

DESCRIPTION

EXAMPLE

```
kash> o := OrderMaximal(Z, 2, 10);
Generating polynomial: x^2 - 10
Discriminant: 40

kash> OrderAutomorphisms(o);
[ [0, 1], [0, -1] ]
kash> p3 := Factor(3*o)[2][1];
<3, [2, 1]>
kash> G1 := RayClassGroupToAbelianGroup(p3, [2]);
RayClassGroupToAbelianGroup(<3, [2, 1]>, [ 2 ])
Group with relations:
[2]
kash> G2 := RayClassFieldSplittingField(G1);
Group with relations:
[2 0]
[0 1]
[4 0]
[0 2]
kash> RayClassFieldIsNormal(G1);
true
kash> RayClassFieldIsNormal(G2);
true
kash> O1 := RayClassField(G1);
[ x^2 - 2 ]
kash> Galois(OrderAbs(Order(O1[1])));
"E(4)"
```

NAME `RayClassGroup`

PURPOSE Returns the ray class number and the orders of the generators of the ray class group modulo a congruence module.

SYNTAX
```
L := RayClassGroup(m0 [,minf]);
L := RayClassGroup(o [,minf]);
```

list	L	
ideal	m0	
list	minf	of integers/infinite primes
order	o	

DESCRIPTION Computes the ray class group modulo a congruence module $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_\infty$, where $\mathfrak{m}_0$ is an integral ideal of a maximal order $\mathcal{O}$ and $\mathfrak{m}_\infty$ is a formal product of primes at infinity — here a subset of the real embeddings of $\mathcal{O}$. $\mathfrak{m}_\infty$ is represented by a list of integers, where each entry represents a real embedding (as given by `EltCon`). `RayClassGroup` returns a list consisting of the ray class number and a list of the orders of the cyclic factors of the ray class group.
It might be necessary to use a higher precision (set by `OrderPrec`) in order to get correct results when working with primes at infinity. Computing the class group and the units of the maximal order first can speed up the computation of the ray class group, as it is possible to influence the calculation of the class group, for instance by setting a lower bound for the regulator. For a description of the algorithm see [Pau96, PP98].

SEE ALSO `EltCon`, `IdealRayClassRep`, `OrderClassGroup`, `OrderPrec`, `RayClassGroupCyclicFactors`, `RayConductor`,

EXAMPLE

```
kash> O := OrderMaximal(Order(Z,2,10));;
kash> OrderClassGroup(O,500,"euler","fast");
[ 2, [ 2 ] ]
kash> m0 := 27*O;;
kash> minf := [1,2];;
kash> L := RayClassGroup(m0,minf);
[ 72, [ 36, 2 ] ]
kash> o:=OrderMaximal(Poly(Zx,[1,1,1]));
Generating polynomial: x^2 + x + 1
Discriminant: -3

kash> L:=RayClassGroup(o);
[ 1, [ 1 ] ]
```

compute the ray class group of an ideal in $\mathbb{Z}$

```
kash> a := ZIdealCreate(9);
<9>
kash> RayClassGroup(a);
[ 3, [ 3 ] ]
```

NAME `RayClassGroupCyclicFactors`

PURPOSE Returns the generators and the orders of the generators of the ray class group modulo a congruence module as computed using `RayClassGroup`.

SYNTAX `L := RayClassGroupCyclicFactors(m0 [,minf]);`

`list`	`L`	
`ideal`	`m0`	
`list`	`minf`	of integers/infinite primes

SEE ALSO `EltCon`, `IdealRayClassRep`, `OrderClassGroup`, `RayClassGroup`,

EXAMPLE

```
kash> O := OrderMaximal(Order(Z,2,10));;
kash> OrderClassGroup(O,500,"euler");
[ 2, [ 2 ] ]
kash> m0 := 7*13*11*O;;
kash> L := RayClassGroupCyclicFactors(m0);
[ [ <[184, 728], [3640, 92]>, 12 ], [ <[463, -77]>, 12 ],
  [ <[92, 364]>, 120 ] ]
kash> IdealRayClassRep(L[1][1],m0);
[1 0 0]
kash> IdealRayClassRep(L[2][1],m0);
[0 1 0]
kash> IdealRayClassRep(L[3][1],m0);
[0 0 1]
```

NAME	`RayClassGroupToAbelianGroup`
PURPOSE	Returns the ray class group for a congruence module.
SYNTAX	`g := RayClassGroupToAbelianGroup(m0 [, minf] [, rels | expo]);`

`group`	`g`	
`ideal`	`m0`	
`list of integers`	`minf`	infinite primes
`matrix of integers`	`rels`	gives additional relations for a quotient
`integer`	`expo`	equivalent to `rels` = `expo*MatId`.

DESCRIPTION Returns the ray class group for a congruence module as an abstract group g. For more information see `RayClassGroup`.

SEE ALSO `RayClassGroup`,

EXAMPLE

```
kash> o := OrderMaximal(x^2+6*x+2);
Generating polynomial: x^2 + 6*x + 2
Discriminant: 28

kash> m0 :=Elt(o, [0,3])*o;
<[0, 3]>
kash> g := RayClassGroupToAbelianGroup(m0, [1,2]);
RayClassGroupToAbelianGroup(<[0, 3]>, [ 1, 2 ])
Group with relations:
[2 0]
[0 2]
```

NAME RayConductor

PURPOSE Calculates the conductor of the ray class group modulo a congruence module.

SYNTAX `L := RayConductor(m0 [, minf] [, rels]);`

list	L	
ideal	m0	
list	minf	of integers/infinite primes
matrix	rels	relation matrix over $\mathbb{Z}$

DESCRIPTION Calculates the minimal ideal and the minimal set of infinite primes with the same ray class group as the given congruence module. The conductor is represented by a list of an ideal and a list of primes at infinity — the numbers correspond to the real embeddings of the order.
This is done by a careful analysis of the structure of the ray class group as described in [Pau96, PP98].

SEE ALSO EltCon, EltRayResidueRingRep, RayResidueRing, RayClassGroup,

EXAMPLE

```
kash> O := OrderMaximal(Order(x^3+9*x^2-8*x-9));;
kash> m0 := 94*O;;
kash> minf:= [1,2];;
kash> L := RayConductor(m0,minf);
[ <47>, [  ] ]
kash> OrderClassGroup(O,500,euler,fast);
[ 1, [ 1 ] ]
kash> RayClassGroup(m0,minf);
[ 23, [ 23 ] ]
kash> RayClassGroup(L[1],L[2]);
[ 23, [ 23 ] ]
```

Name	`RayConductorTest`
Purpose	Tests iff the given module is the true conductor.
Syntax	`b := RayConductorTest(m0 [, minf] [, rels]);`

`boolean`	`b`	
`ideal`	`m0`	
`list`	`minf`	of integers/infinite primes
`matrix`	`rels`	relation matrix over $\mathbb{Z}$

See Also: EltCon, EltRayResidueRingRep, RayResidueRing, RayClassGroup,

Example

```
kash> O := OrderMaximal(Order(x^3+9*x^2-8*x-9));;
kash> m0 := 94*O;;
kash> minf := [1,2];;
kash> RayConductorTest(m0,minf);
false
kash> L := RayConductor(m0,minf);
[ <47>, [  ] ]
kash> OrderClassGroup(O,500,euler,fast);
[ 1, [ 1 ] ]
kash> RayClassGroup(m0,minf);
[ 23, [ 23 ] ]
kash> RayClassGroup(L[1],L[2]);
[ 23, [ 23 ] ]
```

NAME RayDiscSig

PURPOSE Returns the relative discriminant and the absolute signature of the ray class field belonging to a congruence module.

SYNTAX `L := RayDiscSig(m0 [,minf] [,rels]);`

`ideal`	`m0`	
`list`	`minf`	of integer/infinite primes
`matrix`	`rels`	relation matrix over $\mathbb{Z}$
`list`	`L`	

DESCRIPTION A list which contains the relative discriminant and the signature in the form [r1,r2] is returned.
If the units of the maximal order or the multiplicative group of the residue ring modulo the ideal or the class group of the order are not known, they will be calculated. The algorithm is based on the formulas for the discriminant and the signature in [CDO96, CDO97].

SEE ALSO `EltCon`, `OrderDisc`, `RayConductor`, `RayClassGroup`, `RayClassField`,

EXAMPLE

```
kash> O := OrderMaximal(Order(x^3+9*x^2-8*x-9));
Generating polynomial: x^3 + 9*x^2 - 8*x - 9
Discriminant: 42953

kash> m0 := 3*O;
<3>
kash> minf:= [1,2];
[ 1, 2 ]
kash> RayDiscSig(m0,minf);
[ <
    [27  0  9]
    [ 0 27  0]
    [ 0  0  9]
    >
    , [ 8, 2 ] ]
```

NAME RayResidueRing

PURPOSE This function computes the multiplicative group of the residue class ring.

SYNTAX `L := RayResidueRing(m0 [,minf]);`

list	L	
ideal	m0	
list	minf	of integers/infinite primes

DESCRIPTION This function computes the multiplicative group of the residue class ring of an ideal or a congruence module $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_\infty$, where $\mathfrak{m}_0$ is an integral ideal of a maximal order $\mathcal{O}$ and $\mathfrak{m}_\infty$ is a formal product of primes at infinity. $\mathfrak{m}_\infty$ is represented by a list of integers, where each entry represents a real embedding (as given by `EltCon`). `RayResidueRing` returns a list of the order of the multiplicative group and a list of the orders of the cyclic factors.
This structure can be used to solve multiplicative congruences which may involve infinite places. This algorithm bases an the presentation of the Einseinheiten in [Has63] and is described in [Pau96].

SEE ALSO EltCon, EltRayResidueRingRep, RayResidueRingRepToElt, RayResidueRingCyclicFactors,

EXAMPLE

```
kash> O := OrderMaximal(Order(Z,2,10));
Generating polynomial: x^2 - 10
Discriminant: 40

kash> P3 := Factor(3*O)[1][1];;
kash> IdealDegree(P3);
1
kash> IdealRamIndex(P3);
1
kash> m0 := P3^2;; L := RayResidueRing(m0);
[ 6, [ 2, 3 ] ]
kash> m0 := P3^4;; L := RayResidueRing(m0);
[ 54, [ 2, 27 ] ]
kash> P5 := Factor(5*O)[1][1];;
kash> IdealDegree(P5);
1
kash> IdealRamIndex(P5);
```

```
2
kash> m0 := P5^2;; L := RayResidueRing(m0);
[ 20, [ 4, 5 ] ]
kash> m0 := P5^4;; L := RayResidueRing(m0);
[ 500, [ 4, 25, 5 ] ]
kash> P7 := 7*O;;
kash> IdealDegree(P7);
2
kash> IdealRamIndex(P7);
1
kash> m0 := P7^2;; RayResidueRing(m0);
[ 2352, [ 48, 7, 7 ] ]
kash> m0 := P7^4;; RayResidueRing(m0);
[ 5647152, [ 48, 343, 343 ] ]
kash> m0 := P5^4*P7^2;; L := RayResidueRing(m0);
[ 1176000, [ 4, 25, 5, 48, 7, 7 ] ]
kash> minf := [2];; L := RayResidueRing(1*O,minf);
[ 2, [ 1, 2 ] ]
kash> L := RayResidueRing(m0,minf);
[ 2352000, [ 4, 25, 5, 48, 7, 7, 2 ] ]
```

compute the ray residue ring of an ideal in $\mathbb{Z}$

```
kash> a := ZIdealCreate(9);
<9>
kash> RayResidueRing(a);
[ 6, [ 6 ] ]
```

Name `RayResidueRingCyclicFactors`

Purpose Returns generators (and their orders) for the multiplicative group of the residue class ring of an ideal or a congruence module, as computed by `RayResidueRing`.

Syntax `L := RayResidueRingCyclicFactors(m0 [,minf]);`

`list`	`L`	
`ideal`	`m0`	
`list`	`minf`	of integers/infinite primes

Description A list containing generators and their orders is computed. This structure can be used to solve multiplicative congruences which may involve infinite places.

See Also `EltCon`, `IdealRayClassRep`, `OrderClassGroup`, `RayClassGroup`,

Example

```
kash> O := OrderMaximal(Order(Z,2,10));;
kash> m0 := 7^3*O;; minf := [1,2];;
kash> L := RayResidueRingCyclicFactors(m0,minf);
[ [ [243657, 43632], 48 ], [ 15, 49 ], [ [48021, 357], 49 ], [ [1, -343], 2 ],
  [ [1, 343], 2 ] ]
kash> EltRayResidueRingRep(L[1][1],m0,minf);
[1 0 0 0 0]
kash> EltCon(L[1][1]);
[381633.49886846672701377572313068837506325162983903 105680.501131533272986224\
27686931162493674837016097]
kash> EltRayResidueRingRep(L[2][1]^2*L[3][1],m0,minf);
[0 2 1 0 0]
kash> EltRayResidueRingRep(L[4][1],m0,minf);
[0 0 0 1 0]
kash> L[4][1] mod m0;
1
kash> EltRayResidueRingRep(L[2][1]^2*L[3][1]*L[4][1],m0,minf);
[0 2 1 1 0]
```

NAME `RayResidueRingRepToElt`

PURPOSE Returns a canonical representative of an element of the multiplicative group of the residue class ring of a congruence module.

SYNTAX `b := RayResidueRingRepToElt( r, m0, minf);`

`algebraic element`	`b`	
`matrix`	`r`	
`ideal`	`m0`	
`list`	`minf`	of integers/infinite primes

DESCRIPTION This is the inverse to `EltRayResidueRingRep`. The function will compute the following product:

$$\prod |i = 1^l a|i^{\mathfrak{r}|i}$$

with $a|i$ the multiplicative basis for $(\mathfrak{o}/\mathfrak{m}_0)^*$ as returned by `RayResidueRingCyclicFactors`.

SEE ALSO RayResidueRing, RayResidueRingCyclicFactors, EltRayResidueRingRep,

EXAMPLE

```
kash> O := OrderMaximal(Order(Poly(Zx,[1,3,-4,7])));;
kash> m0 := 15*O;;
kash> a := Elt(O,[7,1,8]);;
kash> r := EltRayResidueRingRep(a,m0);
[20 65]
kash> b := RayResidueRingRepToElt(r,m0);
[7, 1, -7]
```

NAME RayResidueRingToAbelianGroup

PURPOSE Returns the ray residue ring modulo a congruence module.

SYNTAX `g := RayResidueRingToAbelianGroup(m0 [, minf]);`

group	g	
ideal	m0	
list of integers	minf	infinite primes

DESCRIPTION Returns the multiplicative group g of the ray residue ring modulo a congruence module. For more information see RayResidueRing.

SEE ALSO RayResidueRing,

EXAMPLE

```
kash> o := OrderMaximal(Poly(Zx, [1,6,2]));
Generating polynomial: x^2 + 6*x + 2
Discriminant: 28

kash> m0 := Ideal(Elt(o, [0,3]));
<[0, 3]>
kash> g := RayResidueRingToAbelianGroup(m0, [1,2]);
Group with relations:
[2 0 0 0]
[0 2 0 0]
[0 0 2 0]
[0 0 0 2]
```

NAME Re

PURPOSE Returns the real part of a complex number.

SYNTAX `a := Re(z);`

`real`	`a`
`complex`	`z`

SEE ALSO Im,

EXAMPLE Compute the real part of $1 + 2i$:

```
kash> z := Comp(1, 2);
1 + 2*i
kash> Re(z);
1
```

NAME Read

PURPOSE Read a file containing KASH commands.

SYNTAX `Read(name);`

`string name`

DESCRIPTION The file ”name” must be both existing and readable. KASH looks first in the given path, then in the current directory and finally in `LIBNAME../src`.

EXAMPLE Assume that we have a file named ”in” in our current working directory. First, we take a look at its contents:

```
kash> Exec("cat in");
Time(true);
x := Poly(Zx, [1,0]);
Palg := function(O)
local px;
px := PolyAlg(O);
x := Poly(px, [1,0]);
return px;
end;

kash> Read("in");
kash> x;
x
Time: 0 ms
```

NAME ReadLib

PURPOSE The same as Read, but here the file should be in the KASH lib directory and it must have the extension ".g".

SYNTAX ReadLib(name);

string name

SEE ALSO Read,

EXAMPLE We will read the neq.g library. This will define a function solving norm equations by suitable principal ideal tests.

```
kash> ReadLib("neq");
kash> Neq(o, 7);
Prime 7: 1 [ 2 ]
No solution
false
Time: 40 ms
kash> Neq(o, 11);
Prime 11: 1 [ 1, 1 ]
[ [ [ [ <11, [5, 2]>, 1 ], [ <11, [8, 2]>, 1 ] ], [ 0, 1 ], [ 1, 0 ] ] ]
at most 2 different solutions!
Ideals : [ [ <11, [5, 2]>, 1 ], [ <11, [8, 2]>, 1 ] ]
Ideal Basis of all (possible) solutions: [ [ <11, [8, 2]>, <11, [5, 2]> ] ]
[ [1, 3], [-2, -3] ]
Time: 260 ms
kash> OrderNormEquation(o, 11, -1);
[ [-2, -3], [-1, -3] ]
Time: 180 ms
```

Round

NAME Round

PURPOSE Returns the integer closest to a given number. Note that if the decimal part of the number given is .5, then the number is rounded up. If the number is negative, it is rounded down.

SYNTAX `y := Round(x);`

integer	y
real	x

SEE ALSO Trunc, Floor, Ceil,

EXAMPLE

```
kash> Round(1.4);
1
kash> Round(-1.4);
-1
kash> Round(1.5);
2
kash> Round(-1.5);
-2
```

NAME	`SPrint`
PURPOSE	Creates a string instead of printing on the screen.
SYNTAX	`s := SPrint( obj1, obj2, ... );`

`string`	`s`
`string or kash object`	`obj1, obj2`

DESCRIPTION

SEE ALSO `Print`, `SScan`,

EXAMPLE

EXAMPLE converting a polynomial into a string

```
kash> f := Poly(Zx, [1,0,2]);
x^2 + 2
kash> s := SPrint("f equals ",f);;
kash> Print(s,"\n");
f equals x^2 + 2
```

convert an algebraic element into a string

```
kash> o := OrderMaximal(f);;
kash> alpha := Elt(o, [0,1]);
[0, 1]
kash> s := SPrint("alpha equals ",alpha);;
kash> Print(s,"\n");
alpha equals [0, 1]
```

NAME SScan

PURPOSE Reads kash objects out of a string.

SYNTAX `L := SScan( s, fmt, R1, R2, ...);`

`list`	`L`	containing the kash objects
`string`	`s`	
`string`	`fmt`	format string
`rings`	`R1, R2`	

DESCRIPTION Supported kash objects are: integers (`%Z`), reals (`%R`), logicals (`%B`), polynomials (`%P`) and algebraic elements (`%E`); also the usual standard C-format is supported. If you want to scan an algebraic element resp. a polynomial you have to pass the order resp. the polynomial algebra as parameter to the `SScan` function. For a better understanding see the detailed list of examples below.

SEE ALSO `SPrint`, `SScan`,

EXAMPLE

EXAMPLE reading a list of integers

```
kash> L := SScan("111, 12, 13","\%Z,\%Z,\%Z");
[ 111, 12, 13 ]
```

reading an algebraic element

```
kash> o := OrderMaximal(Z,2,5);;
kash> L := SScan("alpha: [0,27]/12 ","alpha: \%E",o);
[ [0, 9] / 4 ]
```

or

```
kash> L := SScan("alpha: [0,27]/12 ","\%s \%E",o);
[ "alpha:", [0, 9] / 4 ]
```

reading a logical and a polynomial and a real

```
kash> s := "true, 17*x^3 + 14*x +2, 2.1382";;
kash> fmt := "\%B,\%P,\%R";;
kash> L := SScan(s, fmt, Zx);
[ true, 17*x^3 + 14*x + 2, 2.1382 ]
```

and now an interesting example

```
kash> s := "foobar : foobar 17*x^3 + 14*x +2, 2.1382";;
kash> fmt := "\%s : \%s \%P,\%R";;
kash> L := SScan(s, fmt, Zx);
[ "foobar", "foobar", 17*x^3 + 14*x + 2, 2.1382 ]
```

Name	`SimplexElt`
Purpose	Returns the current point.
Syntax	`L := SimplexElt(s);`

`list of coordinates of one integral point`	`L`
`simplex`	`s`

Description	When enumerating all points in the given simplex using `SimplexNext`, this function will return the actual point.
See Also	`SimplexNext`, `SimplexInit`, `SimplexReInit`,

NAME `SimplexInit`

PURPOSE Initializes the enumeration of all integral points contained in a bounded simplex.

SYNTAX `s := SimplexInit(A, b` [`, delta`] `);`

`Simplex`	`s`	
`real matrix`	`A`	$A \in \mathbb{R}^{m \times n}$
`real matrix`	`b`	$b \in \mathbb{R}^{m}$
`real`	`delta`	should be $1 + \epsilon$

DESCRIPTION Initializes the enumeration of all integral points contained in an bounded simplex. This function performs Fourier-Motzkin elimination to produce a triangular system. Be careful: The size of the system may be exponential in the number of initial constraints.
A simplex `s` if defined as `s`$:= \{x \in \mathbb{R}^n \mid \mathtt{A}x \leq \mathtt{b}\}$ where $\leq$ means $\leq$ componentwise. If `delta` is given, it is used as a correction for round-off errors. You may get points outside of `s`.

SEE ALSO `SimplexNext`, `SimplexElt`, `SimplexReInit`,

EXAMPLE Find all x, y in $\mathbb{Z}^2| \geq 0$ subject to $x + y \leq 3$:

```
kash> A := Mat(R, [[1,1],[-1, 0],[0,-1]]);
[ 1  1]
[-1  0]
[ 0 -1]
kash> b := MatTrans(Mat(R, [[3, 0, 0]]));
[3]
[0]
[0]
kash> s := SimplexInit(A, b);;
kash> while SimplexNext(s) do Print(SimplexElt(s), "\n"); od;
[ 0, 0 ]
[ 1, 0 ]
[ 2, 0 ]
[ 3, 0 ]
[ 0, 1 ]
[ 1, 1 ]
[ 2, 1 ]
[ 0, 2 ]
[ 1, 2 ]
[ 0, 3 ]
```

NAME SimplexNext

PURPOSE Tries to find a "next" integral point in the given simplex.

SYNTAX

```
ok := SimplexNext(s);

boolean  ok
simplex  s
```

DESCRIPTION Tries to find a "next" integral point in the given simplex. If there is no next point, it will return false; otherwise true.

SEE ALSO SimplexInit, SimplexElt, SimplexReInit, NAME `SimplexReInit`

PURPOSE Reinitializes a given simplex. May be used to set `delta`.

SYNTAX `SimplexReInit(s [, delta ]);`

```
simplex  s
real     delta
```

DESCRIPTION Restarts the enumeration of the integral points of the given simplex. In addition you may give a (`delta`) that is used as a scaling factor on the bounds of the enumeration environment. It is intended to be used to correct round-off errors for border points.

SEE ALSO `SimplexNext`, `SimplexElt`, `SimplexInit`,

EXAMPLE Find all x, y in $\mathbb{Z}^2| \geq 0$ subject to $x + y \leq 2$:

```
kash> A := Mat(R, [[1,1],[-1, 0],[0,-1]]);;
kash> b := MatTrans(Mat(R, [[2, 0, 0]]));;
kash> s := SimplexInit(A, b);;
kash> while SimplexNext(s) do Print(SimplexElt(s), "\n"); od;
[ 0, 0 ]
[ 1, 0 ]
[ 2, 0 ]
[ 0, 1 ]
[ 1, 1 ]
[ 0, 2 ]
```

NAME Sin

PURPOSE Returns the sine of a number.

SYNTAX `y := Sin(x);`

```
complex  y
complex  x
```

DESCRIPTION Given an `x` (in radians) the function returns the sine of `x`. The computation is done in the current precision of the real (complex) field.

SEE ALSO Cos, Tan,

EXAMPLE

```
kash> Sin(.5);
0.4794255386042030002732879352155713880818033679406006
kash> i := Comp(0, 1);
1*i
kash> Sin(i);
1.175201193643801456882381850595600815155717981334*i
```

NAME Sleep

PURPOSE Lets kash sleep for nSec seconds. Useful for KashPvm

SYNTAX `Sleep(nSec);`

`small integer  nSec`

EXAMPLE

```
kash> Date();
22.2.2004 16:17:06
kash> Sleep(5);
0
kash> Date();
22.2.2004 16:17:11
```

NAME Solve

PURPOSE Solves an equation or computes the roots of a polynomial.

SYNTAX

```
L := Solve(t, A);
L := Solve(o, a);
L := Solve(f);
```

list	L
Thue object	t
int	A
order	o
int | algebraic element	a
polynomial	f

DESCRIPTION At the moment two kinds of equations can be solved using `Solve`. Furthermore it is possible to compute the roots of a polynomial.

`Solve(t,A)` Solves a Thue equation. The command `Solve(t,A)` is tantamount to the calling sequence `ThueSolve(t,A)`.

`Solve(o,a)` Solves a norm equation. The command `Solve(o,a)` is tantamount to the calling sequence `OrderNormEquation(o,a)`.

`Solve(f)` Computes the roots of a polynomial.

SEE ALSO OrderNormEquation, ThueSolve, PolyZeros,

EXAMPLE Compute all $x, y \in \mathbb{Z}$ with $x^3 + x^2y - 6xy^2 + 2y^3 = 2$:

```
kash> t := Thue( [1, 1, -6, 2] );
X^3 + X^2 Y - 6 X Y^2 + 2 Y^3
kash> Solve(t, 2);
[ [ -724, -411 ], [ -4, -11 ], [ -3, 1 ], [ -1, -1 ], [ 0, 1 ], [ 2, 1 ] ]
```

NAME Sqrt

PURPOSE Returns a square root of a number.

SYNTAX `y := Sqrt(x);`

complex or real	y
complex or real or rational or integer	x

DESCRIPTION Given x in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ the function returns $\sqrt{\mathtt{x}}$ if $\mathtt{x} > 0$ or $i\sqrt{|\mathtt{x}|}$, otherwise. For $\mathtt{x} = |\mathtt{x}|e^{i\phi} \in \mathbb{C}$, $\phi \in (-\pi, \pi]$, the function returns $\sqrt{|\mathtt{x}|}\, e^{i\frac{\phi}{2}}$. The computation is done in the current precision of the real (complex) field.

EXAMPLE

```
kash> Sqrt(2);
1.4142135623730950488016887242096980785696718753 77
kash> Sqrt(-2);
1.414213562373095048801688724209698078569671875377*i
kash> i := Comp(0, 1);
1*i
kash> Sqrt(i);
0.7071067811865475244008443621048490392848359376883159 + 0.7071067811865475244\
008443621048490392848359376886321*i
kash> Sqrt(-i);
0.7071067811865475244008443621048490392848359376883159 - 0.7071067811865475244\
008443621048490392848359376886321*i
```

NAME `SubfieldAdd`

PURPOSE Adds a subfield to the given order.

SYNTAX `SubfieldAdd(o, sub, alpha);`

`order`	`o`	
`order`	`sub`	subfield
`algebraic element`	`alpha`	primitive element of sub in o

DESCRIPTION This function adds a subfield to the given order. It checks if this subfield was known. In this case, true is returned. The subfield must be given as an equation order.

SEE ALSO `OrderSubfield`, `OrderSubfieldSub`, `SubfieldGet`,

EXAMPLE

```
kash> o:=Order(Z,2,2);
Generating polynomial: x^2 - 2

kash> O:=Order(o,2,3);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 3
E 1[ 1]     x^2 - 2

kash> Oa:=OrderAbs(O);
Generating polynomial: x^4 - 10*x^2 + 1

kash> SubfieldAdd(Oa,o,EltMove(OrderBasis(o)[2],Oa));
true
```

NAME `SubfieldGet`

PURPOSE Returns all computed subfields.

SYNTAX `L:=SubfieldGet(o);`

list of subfields	L
order	o

DESCRIPTION This function returns all computed subfields. Use `OrderSubfield` if you want to get all subfields.

SEE ALSO `OrderSubfield`, `OrderSubfieldSub`, `SubfieldAdd`,

EXAMPLE

```
kash> o:=OrderMaximal(Order(Poly(Zx,[1,0,-4,0,1])));
Generating polynomial: x^4 - 4*x^2 + 1
Discriminant: 2304

kash> OrderSubfield(o);;
kash> L:=SubfieldGet(o);
[ Generating polynomial: x^2 - 2
    , Generating polynomial: x^2 + 4*x + 1
    , Generating polynomial: x^2 - 4*x - 2
     ]
```

Name	`SubfieldSetDegreeMax`
Purpose	Sets the maximal number of subfield of degree `d`.
Syntax	`SubfieldSetDegreeMax(O, d)`

`order`	`O`
`integer`	`d`

Description	This function sets the maximal number of degree `d` of a given field to the number of known subfields of degree `d`. This function is very useful if one knows from theory that there are only a specified number of subfields which can be computed by a special algorithm. One has to be very carefull in using this function. If the number of computed subfields of degree `d` is smaller than the number of subfields of degree `d`, the Subfield command has no chance to find the other ones.

NAME Tan

PURPOSE Returns the tangent of a number.

SYNTAX `y := Tan(x);`

```
complex  y
complex  x
```

DESCRIPTION Given an $\mathtt{x} \in \mathbb{C} \setminus \{k\pi + \pi/2 : k \in \mathbb{Z}\}$ (in radians) the function returns the tangent of `x`. The computation is done in the current precision of the real (complex) field.

SEE ALSO Sin, Cos,

EXAMPLE

```
kash> Tan(1);
1.557407724654902230506974807458360173087250772381
kash> i := Comp(0, 1);
1*i
kash> Tan(i);
0.761594155955764888119458282604793590412768597257912*i
```

NAME Thue

PURPOSE Creates a thue object for solving Thue equations.

SYNTAX
```
t := Thue(o);
t := Thue(L);
t := Thue(f);
```

Thue object	t
order	o
list	L
polynomial	f

DESCRIPTION One classical object of number theory is the Diophantine equation of Thue

$$f(X,Y) = a,$$

where $f(X,Y) \in \mathbb{Z}[X,Y]$ is a homogeneous polynomial of degree $n \leq 3$ and a is an integer.

Before solving a Thue equation in KASH the form f on the left side has to be created by invoking the `Thue` function.

`Thue(o)`
takes the coefficients of the form from the polynomial defining the order `o`.

`Thue(L)`
takes the coefficients from the list `L`.

`Thue(f)`
takes the coefficients from the polynomial `f`.

SEE ALSO ThueEval, ThueSolve,

EXAMPLE Create the Thue object corresponding to the form

$$f(X,Y) = X^3 + X^2Y - 6XY^2 + 2Y^3.$$

```
kash> o := Order(Poly(Zx,[1,1,-6,2]));
Generating polynomial: x^3 + x^2 - 6*x + 2

kash> t := Thue(o);
X^3 + X^2 Y - 6 X Y^2 + 2 Y^3
kash> L := [1,1,-6,2];
[ 1, 1, -6, 2 ]
kash> t := Thue(L);
```

```
X^3 + X^2 Y - 6 X Y^2 + 2 Y^3
kash> f := Poly(Zx,[1,1,-6,2]);
x^3 + x^2 - 6*x + 2
kash> t := Thue(f);
X^3 + X^2 Y - 6 X Y^2 + 2 Y^3
```

NAME ThueEval

PURPOSE missing shortdoc

SYNTAX `a := ThueEval(t,x,y);`

int	a
Thue object	t
int	x
int	y

DESCRIPTION Let $f(X,Y) \in \mathbb{Z}[X,Y]$ be the homogeneous polynomial of the Thue object t generated by the KASH function Thue. The ThueEval function evaluates f at (x,y).

SEE ALSO Thue, ThueSolve,

EXAMPLE With $f(X,Y) = X^3 + X^2Y - 6XY^2 + 2Y^3$ compute $f(1,1)$, $f(3,1)$ and $f(-724,-411)$.

```
kash> t := Thue([1,1,-6,2]);
X^3 + X^2 Y - 6 X Y^2 + 2 Y^3
kash> ThueEval(t,1,1);
-2
kash> ThueEval(t,3,1);
20
kash> ThueEval(t,-724,-411);
2
```

NAME ThueSolve

PURPOSE Solves a Thue equation.

SYNTAX `L := ThueSolve(t,a [,"exact"|"abs"]);`

list	L
Thue object	t
int	a

DESCRIPTION Let $f(X,Y) \in \mathbb{Z}[X,Y]$ be the homogeneous polynomial of the Thue object t generated by the KASH function Thue. The ThueSolve function determines all $x, y \in \mathbb{Z}$ such that $f(x,y) = a$.

If the third argument equals "abs" the ThueSolve function will compute all $x, y \in \mathbb{Z}$ with $|f(x,y)| = a$. The calling sequence ThueSolve(t,a,"exact") is tantamount to ThueSolve(t,a).

The implementation of the ThueSolve function in KASH bases on the algorithm of Y. Bilu and G. Hanrot [BH96].

SEE ALSO Solve, Thue, ThueEval,

EXAMPLE Compute all $x, y \in \mathbb{Z}$ with $x^3 + x^2y - 6xy^2 + 2y^3 = 2$.

```
kash> t := Thue([1,1,-6,2]);
X^3 + X^2 Y - 6 X Y^2 + 2 Y^3
kash> ThueSolve(t,2);
[ [ -724, -411 ], [ -4, -11 ], [ -3, 1 ], [ -1, -1 ], [ 0, 1 ], [ 2, 1 ] ]
```

Time

Name `Time`

Purpose Toggles the time display.

Syntax `b := Time([true|false]);`

`boolean`	`b`	current status of time display

Description Activates or deactivates the time display. Default is `false`. If no argument is given the mode will be toggled. The CPU-time used is displayed after a result of a computation is printed. If there is no printed result, the time will not be printed even if the time display is on.

Example Toggling of time display:

```
kash> o := Order(Z,2,110);
Generating polynomial: x^2 - 110

kash> Time(true);
true
Time: 0 ms
kash> O := OrderMaximal(o);
Generating polynomial: x^2 - 110
Discriminant: 440

Time: 0 ms
kash> OrderUnitsFund(O);
[ [21, 2] ]
Time: 0 ms
kash> Time();
false
kash> OrderClassGroup(O);
[ 2, [ 2 ] ]
```

NAME `Trace`

PURPOSE Computes the trace of an algebraic number or alff element or a matrix

SYNTAX
```
n := Trace(a);
n := Trace(b, o);
```

`algebraic element` | `alff elt` | `matrix` | `ff element`	`a`
`algebraic element` | `ff element`	`b`
`order`	`o`

DESCRIPTION Depending on the type of `a` the trace of an algebraic element, alff element or matrix is computed.

EXAMPLE All examples take place in $\mathbb{Q}(\sqrt{5})$:

```
kash> o := Order(Z, 2, 5);
Generating polynomial: x^2 - 5

kash> Trace(Elt(o, [1, 2]));
2
```

TrialDivision

NAME TrialDivision

PURPOSE

SYNTAX `F := TrialDivision(d, b);`

```
list     F
integer  d
integer  b
```

DESCRIPTION Finds all factors up to b. Returns the rest.

EXAMPLE

```
kash> TrialDivision(8274626472648264826427648723648276,10^6);
[ [ [ 2, 2 ], [ 11, 1 ], [ 41, 1 ], [ 86423, 1 ],
      [ 53074086409412759917474753, 1 ] ], 1 ]
```

NAME `Trunc`

PURPOSE Returns the integer part of a number.

SYNTAX `y := Trunc(x);`

```
integer  y
real     x
```

DESCRIPTION Given an `x` in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ the function returns

$$\begin{cases} \max\{k \in \mathbb{Z} : k \le \mathtt{x}\} & : \quad \mathtt{x} \ge 0 \\ \min\{k \in \mathbb{Z} : k \ge \mathtt{x}\} & : \quad \mathtt{x} < 0 \end{cases}.$$

The computation is done in the current precision of the real (complex) field.

SEE ALSO `Round`, `Floor`, `Ceil`,

EXAMPLE

```
kash> Trunc(1);
1
kash> Trunc(-1);
-1
kash> Trunc(1.1);
1
kash> Trunc(-1.1);
-1
kash> Trunc(1.6);
1
kash> Trunc(-1.6);
-1
```

NAME	`Valuation`
PURPOSE	missing shortdoc
SYNTAX	`d := Valuation( [P,] a); SHOTDOC Computes the valuation of the argument at a prime (if possible).`

`integer`	`d`	
`ideal` | `integer` | `alff order ideal`	`P`	must be prime.
`ideal` | `algebraic element` | `integer`	`a`	to valuate.
`alff order ideal`	`a`	to valuate.

DESCRIPTION Returns the valuation of a p-adic number, an integer, an algebraic element, an (fractional) ideal at a prime ideal, a rational prime or an alff order ideal at a prime ideal. If the first argument is an integer then the second argument must also be an integer. For alff order ideals they have to be defined in the same order which must be maximal.
This function calls EltValuation, IdealValuation, IntValuation, or AlffIdeal-Valuation.

EXAMPLE Computation of the valuation of an algebraic number.

```
kash> O := OrderMaximal(Order(Z, 2, 3));
Generating polynomial: x^2 - 3
Discriminant: 12

kash> L := Factor(2*O);
[ [ <2, [1, 1]>, 2 ] ]
kash> P := L[1][1];
<2, [1, 1]>
kash> Valuation(P, Elt(O, 2));
2
kash> Valuation(P, Elt(O, [1, 1]));
1
```

NAME Vec

PURPOSE Returns a vector (matrix with one column).

SYNTAX `v := Vec(r,L);`

ring	r
list	L
matrix	M

DESCRIPTION Returns a matrix with one column, used for VecDotProduct for example.

EXAMPLE

```
kash> v := Vec(Z, [1,2,3]);
[1]
[2]
[3]
```

NAME `VecDotProduct`

PURPOSE Computes the dot product of two vectors.

SYNTAX `r := VecDotProduct (u,v);`

```
list    r
matrix  u
matrix  v
```

EXAMPLE

```
kash> u := Mat(Z,[[1],[2],[3]]);
[1]
[2]
[3]
kash> v := Mat(Z,[[4],[5],[6]]);
[4]
[5]
[6]
kash> VecDotProduct(u,v);
32
kash> v := Comp(0,1)*v;
[4*i]
[5*i]
[6*i]
kash> VecDotProduct(u,v);
-32*i
```

NAME `WeierstrassP`

PURPOSE Calculates the value of the Weierstrass $\wp$–function related to the given lattice.

SYNTAX `u := WeierstrassP(z, w1, w2);`

`complex`	`u`	
`complex`	`z`	a non-zero element of the complex torus $\mathbb{C}/\mathbb{Z}\,w_1 \oplus \mathbb{Z}\,w_2$
`complex`	`w1,w2`	complex values with $\mathrm{Im}\,(w_1/w_2) > 0$

DESCRIPTION Let z, w_1 and w_2 be complex numbers with $\mathrm{Im}\,(w_1/w_2) > 0$ and with $z \notin \mathbb{Z}\,w_1 \oplus \mathbb{Z}\,w_2$. The value of the Weierstrass $\wp$–function $\wp(z; \mathbb{Z}\,w_1 \oplus \mathbb{Z}\,w_2)$ is then a well–defined complex number. It is calculated by use of the infinite q-expansion of $\wp$ described in [Lan87, Chapt. 4.2, Prop. 3], where the precision of the calculated value depends on that of the defining complex field.

EXAMPLE Compute $\wp(\sqrt{-11}; (1+\sqrt{-11})\,\mathbb{Z} \oplus 12\,\mathbb{Z})$:

```
kash> z := Sqrt(-11);
3.3166247903553998491149327366706866839270885455589*i
kash> w1 := 1+Sqrt(-11);
1 + 3.3166247903553998491149327366706866839270885455589*i
kash> w2 := Comp(12,0);
12
kash> WeierstrassP(z,w1,w2);
1.0182193657561417476670755549115740763102063217 85 + 0.0362297178481350322691 5\
10244384538520650475633 2*i
```

WeilPairing

NAME WeilPairing

PURPOSE Calculates the value of the Weil-Pairing on given places p,q

SYNTAX `a := WeilPairing(p,q,m);`

place of degree one in a global function field of genus 1	p
place of degree one in a global function field of genus 1	q
integer	m
finite field element or false	a

DESCRIPTION Return em(p,q) if p,q have m-torsion in the classgroup and m is coprime to the characteristic of the ground field OR return false if it is not possible to compute the pairing using the below algorithm (this may happen when the curve has very few elements over the ground field (e.g. 4)). em(p,q) is of type FFElt.
Raise an error if:

- Either p or q have not m torsion
- p,q have no common function field,
- Either p or q are not of degree 1,
- Either p or q are not of type alffPlace
- m is not of type int or negative,
- F is not global

Raise a warning if the infinite place of the underlying rational field splits in F (-> No canonical choice can be made for a neutral element of the point group).

EXAMPLE

```
kash> prime := 5;
5
kash> d := 3;
3
kash> k:=FF(prime,d);
Finite field of size 5^3
kash> w:=FFPrimitiveElt(k);
w
kash> kT := PolyAlg(k, "T");
Univariate Polynomial Ring in T over GF(5^3)
```

```
kash> kTy := PolyAlg(kT, "y");
Univariate Polynomial Ring in y over Univariate Polynomial Ring in T over GF(5\
^3)

kash> L:=[1,2,3,7,4];
[ 1, 2, 3, 7, 4 ]
kash> f:= y^2 + L[1]*T*y + L[3]*y - (T^3 + L[2]*T^2 + L[4]*T + L[5]);
y^2 + (T + 3)*y + 4*T^3 + 3*T^2 + 3*T + 1
kash> AlffPolyIsIrrSep(f);
true
kash> F:=Alff(f);
Algebraic function field defined by
$.1^2 + $.1*$.2 + 3*$.1 + 4*$.2^3 + 3*$.2^2 + 3*$.2 + 1
over
Univariate rational function field over GF(5^3)
Variables: T

kash> AlffGenus(F);
1
kash> P := AlffPlaces(F,1);;
kash> p := P[1];
Alff place < [ 1/T, 0 ], [ 0, 1 ] >
kash> q := P[2];
Alff place < [ T, 0 ], [ 4, 1 ] >
kash> WeilPairing(p,q,18);
1
```

World

Name	`World`
Purpose	Returns some information on objects.
Syntax	`World (arg1, arg2, ...);`

object 1	arg1
⋮	⋮
object n	argn

Description

Example Information on integers:

```
kash> i:=2^28;
268435456
kash> World(i);
Big integer: 268435456
kash> i:=2^28-1;
268435455
kash> World(i);
Small integer:
268435455
```

NAME Z

PURPOSE Predefined constant: Integer ring $\mathbb{Z}$.

SYNTAX `Z;`

`ring Z`

DESCRIPTION This is a predefined constant for the integer ring $\mathbb{Z}$ and is referenced by the variable `Z`. It is of arbitrary precision.

SEE ALSO `Q`, `R`, `C`,

EXAMPLE

```
kash> Z;
Integer Ring
```

NAME ZIdealCreate

PURPOSE Returns the ideal in $\mathbb{Z}$ generated by the given integer.

SYNTAX `I := IdealRayClassRep(a);`

ideal in $\mathbb{Z}$	I
integer	a

SEE ALSO Ideal,

EXAMPLE

```
kash> I := ZIdealCreate(6);
<6>
```

NAME Zx

PURPOSE Predefined constant: polynomial algebra over integer ring.

SYNTAX `Zx;`

SEE ALSO PolyAlg, PolyAlgCoef,

EXAMPLE

```
kash> Zx;
Univariate Polynomial Ring in x over Integer Ring
```

NAME e

PURPOSE Predefined constant: Euler's constant in the current precision of the real field.

SYNTAX e;

```
real   e
```

SEE ALSO pi, Prec, R,

EXAMPLE

```
kash> e;
2.718281828459045235360287471352662497757247093699
kash> Prec(100);
100
kash> e;
2.718281828459045235360287471352662497757247093699959574966967627724076630353\
47594571382178525166
```

NAME `mod`

PURPOSE The remainder modulo an ideal or an element (in case of principal ideal domains)

SYNTAX `a := b mod c;`

`ring element`	`a`
`ring element`	`b`
`ideal or (in the case of principal ideal domains) ring element`	`c`

DESCRIPTION The function computes the relative discriminant of a Kummer extension of prime degree using results of [Dab95]. The function will first check, whether or not F is such an extension. The Discriminant is an ideal in the coefficient ring of F.

SEE ALSO `EltIdealReduce`,

EXAMPLE Discriminant of $\mathbb{Q}(\sqrt{10}, \sqrt{5})$ over $\mathbb{Q}(\sqrt{10})$

```
kash> E := OrderMaximal (Z,2,10);;
kash> F := Order (E,2,5);
      F[1]
        /
       /
   E1[1]
  /
 /
Q
F  [ 1]     x^2 - 5
E 1[ 1]     x^2 - 10

kash> OrderKextDisc (F);
<1, 1>
```

undefined function name

NAME `undefined function name`

PURPOSE missing shortdoc

SYNTAX

NAME	`undefined function name`
PURPOSE	missing shortdoc
SYNTAX	

pi

NAME pi

PURPOSE Predefined constant: π in the current precision of the real field.

SYNTAX pi;

```
real  pi
```

SEE ALSO e, Prec, R,

EXAMPLE

```
kash> pi;
3.1415926535897932384626433832795028841971693993751
kash> Prec(100);
100
kash> pi;
3.14159265358979323846264338327950288419716939937510582097494459230781640628620\
8998628034825342117
```

Chapter 2

The Programming Language

This chapter describes the GAP programming language. It should allow you in principle to predict the result of each and every input. In order to know what we are talking about, we first have to look more closely at the process of interpretation and the various representations of data involved.

First we have the input to GAP, given as a string of characters. How those characters enter GAP is operating system dependent, e.g., they might be entered at a terminal, pasted with a mouse into a window, or read from a file. The mechanism does not matter. This representation of expressions by characters is called the *external representation* of the expression. Every expression has at least one external representation that can be entered to get exactly this expression.

The input, i.e., the external representation, is transformed in a process called *reading* to an internal representation. At this point the input is analyzed and inputs that are not legal external representations, according to the rules given below, are rejected as errors. Those rules are usually called the *syntax* of a programming language.

The internal representation created by reading is called either an *expression* or a *statement*. Later we will distinguish between those two terms, however now we will use them interchangeably. The exact form of the internal representation does not matter. It could be a string of characters equal to the external representation, in which case the reading would only need to check for errors. It could be a series of machine instructions for the processor on which GAP is running, in which case the reading would more appropriately be called compilation. It is in fact a tree–like structure.

After the input has been read it is again transformed in a process called *evaluation* or *execution*. Later we will distinguish between those two terms too, but for the moment we will use them interchangeably. The name hints at the nature of this process, it replaces an expression with the value of the expression. This works recursively, i.e., to evaluate an expression first the subexpressions are evaluated and then the value of the expression is computed according to rules given below from those values. Those rules are usually called the *semantics* of a programming language.

The result of the evaluation is, not surprisingly, called a *value*. The set of values is of course a much smaller set than the set of expressions; for every value there are several expressions that will evaluate to this value. Again the form in which such a value is represented internally does not matter. It is in fact a tree–like structure again.

The last process is called *printing*. It takes the value produced by the evaluation and creates an external representation, i.e., a string of characters again. What you do with this external representation is up to you. You can look at it, paste it with the mouse into another window, or write it to a file.

Lets look at an example to make this more clear. Suppose you type in the following string of 8 characters

```
1 + 2 * 3;
```

GAP takes this external representation and creates a tree like internal representation, which we can picture as follows

```
    +
   / \
  1   *
     / \
    2   3
```

This expression is then evaluated. To do this GAP first evaluates the right subexpression `'2\*3'`. Again to do this GAP first evaluates its subexpressions 2 and 3. However they are so simple that they are their own value, we say that they are self–evaluating. After this has been done, the rule for `'\*'` tells us that the value is the product of the values of the two subexpressions, which in this case is clearly 6. Combining this with the value of the left operand of the `'+'`, which is self–evaluating too gives us the value of the whole expression 7. This is then printed, i.e., converted into the external representation consisting of the single character `'7'`.

In this fashion we can predict the result of every input when we know the syntactic rules that govern the process of reading and the semantic rules that tell us for every expression how its value is computed in terms of the values of the subexpressions. The syntactic rules are given in sections "Lexical Structure", "Symbols", "Whitespaces", "Keywords", "Identifiers", and "The Syntax in BNF", the semantic rules are given in sections "Expressions", "Variables", "Function Calls", "Comparisons", "Operations", "Statements", "Assignments", "Procedure Calls", "If", "While", "Repeat", "For", "Functions", and the chapters describing the individual data types.

2.1 Lexical Structure

The input of GAP consists of sequences of the following characters.

Digits, uppercase and lowercase letters, <space>, <tab>, <newline>, and the special characters

```
"    '    (    )    *    +    ,    _
.    /    :    ;    <    =    >    ~
[    \    ]    ^    _    {    }    &
```

Other characters will be signalled as illegal. Inside strings and comments the full character set supported by the computer is allowed.

2.2 Symbols

The process of reading, i.e., of assembling the input into expressions, has a subprocess, called *scanning*, that assembles the characters into symbols. A *symbol* is a sequence of characters that form a lexical unit. The set of symbols consists of keywords, identifiers, strings, integers, and operator and delimiter symbols.

A keyword is a reserved word consisting entirely of lowercase letters (see "Keywords"). An identifier is a sequence of letters and digits that contains at least one letter and is not a keyword (see "Identifiers"). An integer is a sequence of digits. A string is a sequence of arbitrary characters enclosed in double quotes.

Operator and delimiter symbols are

```
+       -       *       /       ^       ~
=       <>      <       <=      >       >=
:=      .       ..      ->      ,       ;
[       ]       {       }       (       )
```

Note that during the process of scanning also all whitespace is removed (see "Whitespaces").

2.3 Whitespaces

The characters <space>, <tab>, <newline>, and <return> are called *whitespace characters*. Whitespace is used as necessary to separate lexical symbols, such as integers, identifiers, or keywords. For example `'Thorondor'` is a single identifier, while `'Th  or  ondor'` is the keyword `'or'` between the two identifiers `'Th'` and `'ondor'`. Whitespace may occur between any two symbols, but not within a symbol. Two or more adjacent whitespaces are equivalent to a single whitespace. Apart from the role as separator of symbols, whitespaces are otherwise insignificant. Whitespaces may also occur inside a string, where they are significant. Whitespaces should also be used freely for improved readability.

A *comment* starts with the character `'\#'`, which is sometimes called sharp or hatch, and continues to the end of the line on which the comment character appears. The whole comment, including `'\#'` and the <newline> character is treated as a single whitespace. Inside a string, the comment character `'\#'` looses its role and is just an ordinary character.

For example, the following statement

```
if i<0 then a:=-i;else a:=i;fi;
```

is equivalent to

```
if i < 0  then      & if i is negative
    a := -i;        &     take its inverse
else                & otherwise
    a := i;         &     take itself
fi;
```

(which by the way shows that it is possible to write superfluous comments). However the first statement is not equivalent to

```
ifi<0thena:=-i;elsea:=i;fi;
```

since the keyword 'if' must be separated from the identifier 'i' by a whitespace, and similarly 'then' and 'a', and 'else' and 'a' must be separated.

2.4 Keywords

Keywords are reserved words that are used to denote special operations or are part of statements. They must not be used as identifiers. The keywords are

```
and     do       elif    else    end     fi
for     function if      in      local   mod
not     od       or      repeat  return  then
until   while    quit
```

Note that all keywords are written in lowercase. For example only 'else' is a keyword; 'Else', 'eLsE', 'ELSE' and so forth are ordinary identifiers. Keywords must not contain whitespace, for example 'el if' is not the same as 'elif'.

2.5 Identifiers

An identifier is used to refer to a variable (see "Variables"). An identifier consists of letters, digits, and underscores '_', and must contain at least one letter or underscore. An identifier is terminated by the first character not in this class. Examples of valid identifiers are

```
a                   foo                 aLongIdentifier
hello               Hello               HELLO
x100                100x                _100
some_people_prefer_underscores_to_separate_words
WePreferMixedCaseToSeparateWords
```

Note that case is significant, so the three identifiers in the second line are distinguished.

The backslash \ can be used to include other characters in identifiers; a backslash followed by a character is equivalent to the character, except that this escape sequence is considered to be an ordinary letter. For example G\(2\,5\) is an identifier, not a call to a function 'G'.

An identifier that starts with a backslash is never a keyword, so for example * and \mod are identifier.

The length of identifiers is not limited, however only the first 1023 characters are significant. The escape sequence \<newline> is ignored, making it possible to split long identifiers over multiple lines.

2.6 Expressions

An *expression* is a construct that evaluates to a value. Syntactic constructs that are executed to produce a side effect and return no value are called *statements* (see ”Statements”). Expressions appear as right hand sides of assignments (see ”Assignments”), as actual arguments in function calls (see ”Function Calls”), and in statements.

Note that an expression is not the same as a value. For example `'1 + 11'` is an expression, whose value is the integer 12. The external representation of this integer is the character sequence `'12'`, i.e., this sequence is output if the integer is printed. This sequence is another expression whose value is the integer 12. The process of finding the value of an expression is done by the interpreter and is called the *evaluation* of the expression.

Variables, function calls, and integer, permutation, string, function, list, and record literals (see ”Variables”, ”Function Calls”,”Functions”,), are the simplest cases of expressions.

Expressions, for example the simple expressions mentioned above, can be combined with the operators to form more complex expressions. Of course those expressions can then be combined further with the operators to form even more complex expressions. The *operators* fall into three classes. The *comparisons* are `'='`, `'$<>$'`, `'$<=$'`, `'$>$'`, `'$>=$'`, and `'in'` (see ”Comparisons” and ”In”). The *arithmetic operators* are `'+'`, `'-'`, `'\*'`, `'/'`, `'mod'`, and `'\^'` (see ”Operations”). The *logical operators* are `'not'`, `'and'`, and `'or'`.

```
gap> 2 * 2;    # a very simple expression with value
4
gap> 2 * 2 + 9 = Fibonacci(7) and  Fibonacci(13) in Primes;
true            # a more complex expression
```

2.7 Variables

A *variable* is a location in a GAP program that points to a value. We say the variable is *bound* to this value. If a variable is evaluated it evaluates to this value.

Initially an ordinary variable is not bound to any value. The variable can be bound to a value by *assigning* this value to the variable (see ”Assignments”). Because of this we sometimes say that a variable that is not bound to any value has no assigned value. Assignment is in fact the only way by which a variable, which is not an argument of a function, can be bound to a value. After a variable has been bound to a value an assignment can also be used to bind the variable to another value.

A special class of variables are *arguments* of functions. They behave similarly to other variables, except they are bound to the value of the actual arguments upon a function call (see ”Function Calls”).

Each variable has a name that is also called its *identifier*. This is because in a given scope an identifier identifies a unique variable (see ”Identifiers”). A *scope* is a lexical part of a program text. There is the global scope that encloses the entire program text, and there are local scopes that range from the `'function'` keyword, denoting the beginning of a function definition, to the corresponding 'end' keyword. A local scope introduces new variables, whose identifiers are given in the formal argument list and the 'local' declaration of the function (see "Functions"). Usage of an identifier in a program text refers to the variable in the innermost scope that has this identifier as its name. Because this mapping from identifiers to variables is done when the program is read, not when it is executed, GAP is said to have lexical scoping. The following example shows how one identifier refers to different variables at different points in the program text.

```
g := 0;           & global variable g
x := function ( a, b, c )
   local   y;
   g := c;        & c refers to argument c of function x
   y := function ( y )
       local  d, e, f;
       d := y;    & y refers to argument y of function y
       e := b;    & b refers to argument b of function x
       f := g;    & g refers to global variable g
       return d + e + f;
   end;
   return y( a ); & y refers to local y of function x
end;
```

It is important to note that the concept of a variable in GAP is quite different from the concept of a variable in programming languages like PASCAL. In those languages a variable denotes a block of memory. The value of the variable is stored in this block. So in those languages two variables can have the same value, but they can never have identical values, because they denote different blocks of memory. (Note that PASCAL has the concept of a reference argument. It seems as if such an argument and the variable used in the actual function call have the same value, since changing the argumentś value also changes the value of the variable used in the actual function call. But this is not so; the reference argument is actually a pointer to the variable used in the actual function call, and it is the compiler that inserts enough magic to make the pointer invisible.) In order for this to work the compiler needs enough information to compute the amount of memory needed for each variable in a program, which is readily available in the declarations PASCAL requires for every variable.

In GAP on the other hand each variable justs points to a value.

2.8 Function Calls

'<function-var>()'

'<function-var>(<arg-expr> \{',' <arg-expr>\})'

The function call has the effect of calling the function <function-var>. The precise semantics are as follows.

First GAP evaluates the <function-var>. Usually <function-var> is a variable, and GAP does nothing more than taking the value of this variable. It is allowed though that <function-var> is a more complex expression, namely it can for example be a selection of a list element '<list-var>[<int-expr>]', or a selection of a record component '<record-var>.<ident>'. In any case GAP tests whether the value is a function. If it is not, GAP signals an error.

Next GAP checks that the number of actual arguments <arg-expr>s agrees with the number of formal arguments as given in the function definition. If they do not agree GAP signals an error. An exception is the case when there is exactly one formal argument with the name 'arg', in which case any number of actual arguments is allowed.

Now GAP allocates for each formal argument and for each formal local a new variable. Remember that a variable is a location in a GAP program that points to a value. Thus for each formal argument and for each formal local such a location is allocated.

Next the arguments <arg-expr>s are evaluated, and the values are assigned to the newly created variables corresponding to the formal arguments. Of course the first value is assigned to the new variable corresponding to the first formal argument, the second value is assigned to the new variable corresponding to the second formal argument, and so on. However, GAP does not make any guarantee about the order in which the arguments are evaluated. They might be evaluated left to right, right to left, or in any other order, but each argument is evaluated once. An exception again occurs if the function has only one formal argument with the name 'arg'. In this case the values of all the actual arguments are stored in a list and this list is assigned to the new variable corresponding to the formal argument 'arg'.

The new variables corresponding to the formal locals are initially not bound to any value. So trying to evaluate those variables before something has been assigned to them will signal an error.

Now the body of the function, which is a statement, is executed. If the identifier of one of the formal arguments or formal locals appears in the body of the function it refers to the new variable that was allocated for this formal argument or formal local, and evaluates to the value of this variable.

If during the execution of the body of the function a 'return' statement with an expression (see "Return") is executed, execution of the body is terminated and the value of the function call is the value of the expression of the 'return'. If during the execution of the body a 'return' statement without an expression is executed, execution of the body is terminated and the function call does not produce a value, in which case we call this call a procedure call (see "Procedure Calls"). If the execution of the body completes without execution of a 'return' statement, the function call again produces no value, and again we talk about a procedure call.

```
gap> Fibonacci( 11 );
    # a call to the function \verb|'Fibonacci'| with actual argument \verb|'11'|
89

gap> G.operations.RightCosets( G, Intersection( U, V ) );;
    # a call to the function in \verb|'G.operations.RightCosets'|
    # where the second actual argument is another function call
```

2.9 Comparisons

'<left-expr> = <right-expr>'

'<left-expr> <> <right-expr>'

The operator '=' tests for equality of its two operands and evaluates to 'true' if they are equal and to 'false' otherwise. Likewise '<>' tests for inequality of its two operands. Note that any two objects can be compared, i.e., '=' and '<>' will never signal an error. For each type of objects the definition of equality is given in the respective chapter. Objects of different types are never equal, i.e., '=' evaluates in this case to 'false', and '<>' evaluates to 'true'.

'<left-expr> < <right-expr>'

'<left-expr> > <right-expr>'

'<left-expr> <= <right-expr>'

'<left-expr> >= <right-expr>'

'<' denotes less than, '<=' less than or equal, '>' greater than, and '>=' greater than or equal of its two operands. For each type of objects the definition of the ordering is given in the respective chapter. The ordering of objects of different types is as follows. Rationals are smallest, next are cyclotomics, followed by finite field elements, permutations, words, words in solvable groups, boolean values, functions, lists, and records are largest.

Comparison operators, which includes the operator 'in' (see "In") are not associative, i.e., it is not allowed to write '<a> = <b> <> <c> = <d>', you must use '(<a> = <b>) <> (<c> = <d>)' instead. The comparison operators have higher precedence than the logical operators, but lower precedence than the arithmetic operators (see "Operations"). Thus, for example, '<a> <b> = <c> and <d>' is interpreted, '((<a> <b>) = <c>) and <d>)'.

```
gap> 2 * 2 + 9 = Fibonacci(7);    # a comparison where the left
true                              # operand is an expression
```

2.10 Operations

'+ <right-expr>'

'- <right-expr>'

'<left-expr> +   <right-expr>'

'<left-expr> -   <right-expr>'

'<left-expr> *\ <right-expr>'

'<left-expr> /   <right-expr>'

'<left-expr> mod <right-expr>'

'<left-expr> \^\ <right-expr>'

The arithmetic operators are '+', '-', '*', '/', 'mod', and '\^'. The meanings (semantic) of those operators generally depend on the types of the operands involved, and they are defined in the various chapters describing the types. However basically the meanings are as follows.

'+' denotes the addition, and '-' the subtraction of ring and field elements. '*' is the multiplication of group elements, '/' is the multiplication of the left operand with the inverse of the right operand. 'mod' is only defined for integers and rationals and denotes the modulo operation. '+' and '-' can also be used as unary operations. The unary '+' is ignored and unary '-' is equivalent to multiplication by -1. '\^' denotes powering of a group element if the right operand is an integer, and is also used to denote operation if the right operand is a group element.

The *precedence* of those operators is as follows. The powering operator `'\^'` has the highest precedence, followed by the unary operators `'+'` and `'-'`, which are followed by the multiplicative operators `'\*'`, `'/'`, and `'mod'`, and the additive binary operators `'+'` and `'-'` have the lowest precedence. That means that the expression `'-2 \^\ -2 \*\  3 + 1'` is interpreted as `'(-(2  \^\ (-2)) \*\ 3) + 1'`. If in doubt use parentheses to clarify your intention.

The *associativity* of the arithmetic operators is as follows.`'\^'` is not associative, i.e., it is illegal to write `'2\^3\^4'`, use parentheses to clarify whether you mean `'(2\^3)  \^\  4'` or `'2 \^\ (3\^4)'`. The unary operators `'+'` and `'-'` are right associative, because they are written to the left of their operands. `'\*'`, `'/'`, `'mod'`, `'+'`, and `'-'` are all left associative, i.e., `'1-2-3'` is interpreted as `'(1-2)-3'` not as `'1-(2-3)'`. Again, if in doubt use parentheses to clarify your intentions.

The arithmetic operators have higher precedence than the comparison operators (see ”Comparisons” and ”In”) and the logical operators. Thus, for example, '<a> <b> = <c> and <d>' is interpreted, '((<a> <b>) = <c>) and <d>'.

```
gap> 2 * 2 + 9;    & a very simple arithmetic expression
13
```

2.11 Statements

Assignments (see ”Assignments”), Procedure calls (see ”Procedure Calls”), `'if'` statements (see ”If”), `'while'` (see ”While”), `'repeat'` (see ”Repeat”) and `'for'` loops (see ”For”), and the `'return'` statement (see ”Return”) are called statements. They can be entered interactively or be part of a function definition. Every statement must be terminated by a semicolon.

Statements, unlike expressions, have no value. They are executed only to produce an effect. For example an assignment has the effect of assigning a value to a variable, a `'for'` loop has the effect of executing a statement sequence for all elements in a list and so on. We will talk about *evaluation* of expressions but about *execution* of statements to emphasize this difference.

It is possible to use expressions as statements. However this does cause a warning.

```
gap> if i <> 0  then  k = 16/i;  fi;
Syntax error: warning, this statement has no effect
if i <> 0  then  k = 16/i;  fi;
                        ^
```

As you can see from the example this is useful for those users who are used to languages where `'='` instead of `'\:='` denotes assignment.

A sequence of one or more statements is a statement sequence, and may occur everywhere instead of a single statement. There is nothing like PASCALś BEGIN-END, instead each construct is terminated by a keyword. The most simple statement sequence is a single semicolon, which can be used as an empty statement sequence.

2.12 Assignments

'<var> \:= <expr>;'

The *assignment* has the effect of assigning the value of the expressions <expr> to the variable <var>.

The variable <var> may be an ordinary variable (see "Variables"), a list element selection '<list-var>[<int-expr>]' (see "List Assignment") or a record component selection '<record-var>.<ident>' (see "Record Assignment"). Since a list element or a record component may itself be a list or a record the left hand side of an assignment may be arbitrarily complex.

Note that variables do not have a type. Thus any value may be assigned to any variable. For example a variable with an integer value may be assigned a permutation or a list or anything else.

If the expression <expr> is a function call then this function must return a value. If the function does not return a value an error is signalled and you enter a break loop. As usual you can leave the break loop with 'quit;'. If you enter 'return <return-expr>;' the value of the expression <return-expr> is assigned to the variable, and execution continues after the assignment.

```
gap> S6 := rec( size := 720 );; S6;
rec(
  size := 720 )
gap> S6.generators := [ (1,2), (1,2,3,4,5) ];; S6;
rec(
  size := 720,
  generators := [ (1,2), (1,2,3,4,5) ] )
gap> S6.generators[2] := (1,2,3,4,5,6);; S6;
rec(
  size := 720,
  generators := [ (1,2), (1,2,3,4,5,6) ] )
```

2.13 Procedure Calls

'<procedure-var>();'
'<procedure-var>(<arg-expr> \{',' <arg-expr>\});'

The procedure call has the effect of calling the procedure <procedure-var>. A procedure call is done exactly like a function call (see "Function Calls"). The distinction between functions and procedures is only for the sake of the discussion, GAP does not distinguish between them.

A *function* does return a value but does not produce a side effect. As a convention the name of a function is a noun, denoting what the function returns, e.g., 'Length', 'Concatenation' and 'Order'.

A *procedure* is a function that does not return a value but produces some effect. Procedures are called only for this effect. As a convention the name of a procedure is a verb, denoting what the procedure does, e.g., 'Print', 'Append' and 'Sort'.

```
gap> Read( "myfile.g" );     & a call to the procedure Read
gap> l := [ 1, 2 ];;
gap> Append( l, [3,4,5] );    & a call to the procedure Append
```

2.14 If

'if <bool-expr1> then <statements1>
{ elif <bool-expr2> then <statements2> }
[else <statements3>]
fi;'

The `'if'` statement allows one to execute statements depending on the value of some boolean expression. The execution is done as follows.

First the expression <bool-expr1> following the `'if'` is evaluated. If it evaluates to `'true'` the statement sequence <statements1> after the first `'then'` is executed, and the execution of the `'if'` statement is complete.

Otherwise the expressions <bool-expr2> following the `'elif'` are evaluated in turn. There may be any number of `'elif'` parts, possibly none at all. As soon as an expression evaluates to `'true'` the corresponding statement sequence <statements2> is executed and execution of the `'if'` statement is complete.

If the `'if'` expression and all, if any, `'elif'` expressions evaluate to `'false'` and there is an `'else'` part, which is optional, its statement sequence <statements3> is executed and the execution of the `'if'` statement is complete. If there is no `'else'` part the `'if'` statement is complete without executing any statement sequence.

Since the `'if'` statement is terminated by the `'fi'` keyword there is no question where an `'else'` part belongs, i.e., GAP has no dangling else.
In `'if <expr1>  then  if <expr2>  then <stats1>  else <stats2>  fi;  fi;'`
the `'else'` part belongs to the second `'if'` statement, whereas in
`'if <expr1>  then  if  <expr2>  then  <stats1>  fi;  else  <stats2>  fi;'`
the `'else'` part belongs to the first `'if'` statement.

Since an if statement is not an expression it is not possible to write

```
abs := if x  > 0  then  x;  else  -x;  fi;
```

which would, even if legal syntax, be meaningless, since the `'if'` statement does not produce a value that could be assigned to `'abs'`.

If one expression evaluates neither to `'true'` nor to `'false'` an error is signalled and a break loop is entered. As usual you can leave the break loop with `'quit;'`. If you enter `'return true;'`, execution of the `'if'` statement continues as if the expression whose evaluation failed had evaluated to `'true'`. Likewise, if you enter `'return  false;'`, execution of the `'if'` statement continues as if the expression whose evaluation failed had evaluated to `'false'`.

```
gap> i := 10;;
gap> if 0 < i  then
>        s := 1;
>    elif i < 0  then
>        s := -1;
>    else
>        s := 0;
>    fi;
gap> s;
1        # the sign of i
```

2.15 While

`'while <bool-expr>  do <statements>  od;'`

The `'while'` loop executes the statement sequence <statements> while the condition <bool-expr> evaluates to `'true'`.

First <bool-expr> is evaluated. If it evaluates to `'false'` execution of the `'while'` loop terminates and the statement immediately following the `'while'` loop is executed next. Otherwise if it evaluates to `'true'` the <statements> are executed and the whole process begins again.

The difference between the `'while'` loop and the `'repeat  until'` loop (see "Repeat") is that the <statements> in the `'repeat until'` loop are executed at least once, while the <statements> in the `'while'` loop are not executed at all if <bool-expr> is `'false'` at the first iteration.

If <bool-expr> does not evaluate to `'true'` or `'false'` an error is signalled and a break loop is entered. As usual you can leave the break loop with `'quit;'`. If you enter `'return false;'`, execution continues with the next statement immediately following the `'while'` loop. If you enter `'return  true;'`, execution continues at <statements>, after which the next evaluation of <bool-expr> may cause another error.

```
gap> i := 0;;  s := 0;;
gap> while s <= 200  do
>        i := i + 1;  s := s + i^2;
>    od;
gap> s;
204        # first sum of the first i squares larger than 200
```

2.16 Repeat

`'repeat <statements>  until <bool-expr>;'`

The `'repeat'` loop executes the statement sequence <statements> until the condition <bool-expr> evaluates to `'true'`.

First <statements> are executed. Then <bool-expr> is evaluated. If it evaluates to `'true'` the `'repeat'` loop terminates and the statement immediately following the `'repeat'` loop is executed next. Otherwise if it evaluates to `'false'` the whole process begins again with the execution of the <statements>.

The difference between the `'while'` loop (see "While") and the 'repeat until' `loop is that the <statements> in the` 'repeat until' loop are executed at least once, while the <statements> in the `'while'` loop are not executed at all if <bool-expr> is `'false'` at the first iteration.

If <bool-expr> does not evaluate to `'true'` or `'false'` a error is signalled and a break loop is entered. As usual you can leave the break loop with `'quit;'`. If you enter `'return true;'`, execution continues with the next statement immediately following the `'repeat'` loop. If you enter `'return false;'`, execution continues at <statements>, after which the next evaluation of <bool-expr> may cause another error.

```
gap> i := 0;;  s := 0;;
gap> repeat
>        i := i + 1;  s := s + i^2;
>    until s > 200;
gap> s;
204        # first sum of the first i squares larger than 200
```

2.17 For

`'for <simple-var> in <list-expr> do <statements> od;'`

The `'for'` loop executes the statement sequence <statements> for every element of the list <list-expr>.

The statement sequence <statements> is first executed with <simple-var> bound to the first element of the list <list>, then with <simple-var> bound to the second element of <list> and so on. <simple-var> must be a simple variable, it must not be a list element selection `'<list-var>[<int-expr>]'` or a record component selection `'<record-var>.<ident>'`.

The execution of the `'for'` loop is exactly equivalent to the `'while'` loop

```
<loop-list> := <list>;
<loop-index> := 1;
while <loop-index> <= Length(<loop-list>) do
    <variable> := <loop-list>[<loop-index>];
    <statements>
    <loop-index> := <loop-index> + 1;
od;
```

with the exception that <loop-list> and <loop-index> are different variables for each `'for'` loop that do not interfere with each other.

The list <list> is very often a range.
`'for <variable> in [<from>..<to>] do <statements> od;'`
corresponds to the more common
`'for <variable> from <from> to <to>  do <statements> od;'`
in other programming languages.

```
gap> s := 0;;
gap> for i  in [1..100]  do
>        s := s + i;
> od;
gap> s;
5050
```

Note in the following example how the modification of the *list* in the loop body causes the loop body also to be executed for the new values

```
gap> l := [ 1, 2, 3, 4, 5, 6 ];;
gap> for i  in l  do
>        Print( i, " " );
>        if i mod 2 = 0  then Add( l, 3 * i / 2 );  fi;
> od;  Print( "\n" );
1 2 3 4 5 6 3 6 9 9
gap> l;
[ 1, 2, 3, 4, 5, 6, 3, 6, 9, 9 ]
```

Note in the following example that the modification of the *variable* that holds the list has no influence on the loop

```
gap> l := [ 1, 2, 3, 4, 5, 6 ];;
gap> for i  in l  do
>        Print( i, " " );
>        l := [];
> od;  Print( "\n" );
1 2 3 4 5 6
gap> l;
[  ]
```

2.18 Functions

```
function ( [ <arg-ident> \{, <arg-ident>\} ] )
    [ local   <loc-ident> \{, <loc-ident>\} ; ]
    <statements>
end
```

A function is in fact a literal and not a statement. Such a function literal can be assigned to a variable or to a list element or a record component. Later this function can be called as described in "Function Calls".

The following is an example of a function definition. It is a function to compute values of the Fibonacci sequence

```
gap> fib := function ( n )
>         local  f1,  f2,  f3,  i;
>         f1 := 1;  f2 := 1;
>         for i  in [3..n]  do
>             f3 := f1 + f2;
>             f1 := f2;
>             f2 := f3;
>         od;
>         return f2;
>     end;;
gap> List( [1..10], fib );
[ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 ]
```

Because for each of the formal arguments <arg-ident> and for each of the formal locals <loc-ident> a new variable is allocated when the function is called (see "Function Calls"), it is possible that a function calls itself. This is usually called *recursion*. The following is a recursive function that computes values of the Fibonacci sequence

```
gap> fib := function ( n )
>         if n < 3  then
>             return 1;
>         else
>             return fib(n-1) + fib(n-2);
>         fi;
>     end;;
gap> List( [1..10], fib );
[ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 ]
```

Note that the recursive version needs '2 *\ fib(<n>)-1' steps to compute 'fib(<n>)', while the iterative version of 'fib' needs only '<n>-2' steps. Both are not optimal however, the library function 'Fibonacci' only needs on the order of 'Log(<n>)' steps.

'<arg-ident> -> <expr>'

This is a shorthand for
'function (<arg-ident>) return <expr>; end'.
<arg-ident> must be a single identifier, i.e., it is not possible to write functions of several arguments this way. Also 'arg' is not treated specially, so it is also impossible to write functions that take a variable number of arguments this way.

The following is an example of a typical use of such a function

```
gap> Sum( List( [1..100], x -> x^2 ) );
338350
```

When a function <fun1> definition is evaluated inside another function <fun2>, GAP binds all the identifiers inside the function <fun1> that are identifiers of an argument or a local of <fun2> to the corresponding variable. This set of bindings is called the environment of the function <fun1>. When <fun1> is called, its body is executed in this environment. The following implementation of a simple stack uses this. Values can be pushed onto the stack and then later be popped off again. The interesting thing here is that the functions 'push' and 'pop' in the record returned by 'Stack' access the local variable 'stack' of 'Stack'. When 'Stack' is called a new variable for the identifier 'stack' is created. When the function definitions of 'push' and 'pop' are then evaluated (as part of the 'return' statement) each reference to 'stack' is bound to this new variable. Note also that the two stacks 'A' and 'B' do not interfere, because each call of 'Stack' creates a new variable for 'stack'.

```
gap> Stack := function ()
>         local   stack;
>         stack := [];
>         return rec(
>             push := function ( value )
>                 Add( stack, value );
>             end,
>             pop := function ()
>                 local   value;
>                 value := stack[Length(stack)];
>                 Unbind( stack[Length(stack)] );
>                 return value;
>             end
>         );
>     end;;
gap> A := Stack();;
gap> B := Stack();;
gap> A.push( 1 );  A.push( 2 );  A.push( 3 );
gap> B.push( 4 );  B.push( 5 );  B.push( 6 );
gap> A.pop();  A.pop();  A.pop();
3
2
1
gap> B.pop();  B.pop();  B.pop();
6
5
4
```

This feature should be used rarely, since its implementation in GAP is not very efficient.

2.19 Return

`'return;'`

In this form `'return'` terminates the call of the innermost function that is currently executing, and control returns to the calling function. An error is signalled if no function is currently executing. No value is returned by the function.

`'return <expr>;'`

In this form `'return'` terminates the call of the innermost function that is currently executing, and returns the value of the expression <expr>. Control returns to the calling function. An error is signalled if no function is currently executing.

Both statements can also be used in break loops. `'return;'` has the effect that the computation continues where it was interrupted by an error or the user hitting `'<ctr>C'`. `'return <expr>;'` can be used to continue execution after an error. What happens with the value <expr> depends on the particular error.

2.20 The Syntax in BNF

This section contains the definition of the GAP syntax in Backus-Naur form.

A BNF is a set of rules, whose left side is the name of a syntactical construct. Those names are enclosed in angle brackets and written in <italics>. The right side of each rule contains a possible form for that syntactic construct. Each right side may contain names of other syntactic constructs, again enclosed in angle brackets and written in <italics>, or character sequences that must occur literally; they are written in `'typewriter style'`.

Furthermore each righthand side can contain the following metasymbols written in *boldface*. If the right hand side contains forms separated by a pipe symbol (‖) this means that one of the possible forms can occur. If a part of a form is enclosed in square brackets ([ ]) this means that this part is optional, i.e. might be present or missing. If part of the form is enclosed in curly braces ({ }) this means that the part may occur arbitrarily often, or possibly be missing.

```
<Ident>        = ’a’‖...‖’z’‖’A’‖...‖’Z’‖’\_’ {’a’‖...‖’z’‖’A’‖...‖’Z’‖’0’‖...‖’9’‖’\_’}
<Var>          = <Ident>
               ‖  <Var> ’.’ <Ident>
               ‖  <Var> ’.’ ’(’ <Expr> ’)’
               ‖  <Var> ’[’ <Expr> ’]’
               ‖  <Var> ’\{’ <Expr> ’\}’
               ‖  <Var> ’(’ [ <Expr> { ’,’ <Expr> } ] ’)’
<List>         = ’[’ [ <Expr> ] {’,’ [ <Expr> ] } ’]’
               ‖  ’[’ <Expr> [, <Expr> ] ’..’ <Expr> ’]’
<Record>       = ’rec(’ [ <Ident> ’\:=’ <Expr> {’,’ <Ident> ’\:=’ <Expr> } ] ’)’
<Permutation>  = ’(’ <Expr> {’,’ <Expr> } ’)’ { ’(’ <Expr> {’,’ <Expr> } ’)’ }
<Function>     = ’function (’ [ <Ident> {’,’ <Ident> } ] ’)’
                  [ ’local’ <Ident> {’,’ <Ident> } ’;’ ]
                  <Statements>
                  ’end’
<Char>         = \verb’| <any character> ’|
<String>       = ’\"’ { <any character> } ’\"’
<Int>          = ’0’‖’1’‖...‖’9’ { ’0’‖’1’‖...‖’9’ }
<Atom>         = <Int>
               ‖  <Var>
               ‖  ’(’ <Expr> ’)’
               ‖  <Permutation>
               ‖  <Char>
               ‖  <String>
               ‖  <Function>
               ‖  <List>
               ‖  <Record>
<Factor>       = {’+’‖’-’} <Atom> [ ’\^’ {’+’‖’-’} <Atom> ]
<Term>         = <Factor> { ’\*’‖’/’‖’mod’ <Factor> }
<Arith>        = <Term> { ’+’‖’-’ <Term> }
<Rel>          = { ’not’ } <Arith> { =‖<>‖<‖>‖<=‖>=‖in <Arith> }
<And>          = <Rel> { ’and’ <Rel> }
<Log>          = <And> { ’or’ <And> }
<Expr>         = <Log>
               ‖  <Var> [ ’->’ <Log> ]
<Statement>    = <Expr>
               ‖  <Var> ’\:=’ <Expr>
               ‖  ’if’ <Expr> ’then’ <Statements>
                  { ’elif’ <Expr> ’then’ <Statements> }
                  [ ’else’                <Statements> ] ’fi’
               ‖  ’for’ <Var> ’in’ <Expr> ’do’ <Statements> ’od’
               ‖  ’while’ <Expr> ’do’ <Statements> ’od’
               ‖  ’repeat’ <Statements> ’until’ <Expr>
```

‖ 'return' [<Expr>]
‖ 'quit'
<Statements> = { <Statement> ';' }
‖ ';'

Chapter 3

Lists

Lists are the most important way to collect objects and treat them together. A *list* is a collection of elements. A list also implies a partial mapping from the integers to the elements. I.e., there is a first element of a list, a second, a third, and so on.

List constants are written by writing down the elements in order between square brackets '[', ']', and separating them with commas ','. An *empty list*, i.e., a list with no elements, is written as '[]'.

```
gap> [ 1, 2, 3 ];
[ 1, 2, 3 ]    # a list with three elements
gap> [ [], [ 1 ], [ 1, 2 ] ];
[ [  ], [ 1 ], [ 1, 2 ] ]    # a list may contain other lists
```

Usually a list has no holes, i.e., contain an element at every position. However, it is absolutely legal to have lists with holes. They are created by leaving the entry between the commas empty. Lists with holes are sometimes convenient when the list represents a mapping from a finite, but not consecutive, subset of the positive integers. We say that a list that has no holes is *dense*.

```
gap> l := [ , 4, 9,, 25,, 49,,,, 121 ];;
gap> l[3];
9
gap> l[4];
Error, List Element: <list>[4] must have a value
```

It is most common that a list contains only elements of one type. This is not a must though. It is absolutely possible to have lists whose elements are of different types. We say that a list whose elements are all of the same type is *homogeneous*.

```
gap> l := [ 1, E(2), Z(3), (1,2,3), [1,2,3], "What a mess" ];;
gap> l[1];  l[3];  l[5][2];
1
Z(3)
2
```

The first sections describe the functions that test if an object is a list and convert an object to a list (see ”IsList” and ”List”).

The next section describes how one can access elements of a list (see ”List Elements” and ”Length”).

The next sections describe how one can change lists (see ”List Assignment”, ”Add”, ”Append”, ”Identical Lists”, ”Enlarging Lists”).

The next sections describe the operations applicable to lists (see ”Comparisons of Lists” and ”Operations for Lists”).

The next sections describe how one can find elements in a list (see ”In”, ”Position”, ”PositionSorted”, ”PositionProperty”).

The next sections describe the functions that construct new lists, e.g., sublists (see ”Concatenation”, ”Flat”, ”Reversed”, ”Sublist”, ”Cartesian”).

The next sections describe the functions deal with the subset of elements of a list that have a certain property (see ”Number”, ”Collected”, ”Filtered”, ”ForAll”, ”ForAny”, ”First”).

The next sections describe the functions that sort lists (see ”Sort”, ”SortParallel”, ”Sortex”, ”Permuted”).

The next sections describe the functions to compute the product, sum, maximum, and minimum of the elements in a list (see ”Product”, ”Sum”, ”Maximum”, ”Minimum”, ”Iterated”).

The final section describes the function that takes a random element from a list (see ”RandomList”).

Lists are also used to represent sets, subsets, vectors, and ranges.

3.1 IsList

`'IsList( <obj> )'`

`'IsList'` returns `'true'` if the argument <obj>, which can be an arbitrary object, is a list and `'false'` otherwise. Will signal an error if <obj> is an unbound variable.

```
gap> IsList( [ 1, 3, 5, 7 ] );
true
gap> IsList( 1 );
false
```

3.2 List

`'List( <obj> )'`
`'List( <list>, <func> )'`

In its first form `'List'` returns the argument <obj>, which must be a list, a permutation, a string or a word, converted into a list. If <obj> is a list, it is simply returned. If <obj> is a permutation, `'List'` returns a list where the <i>-th element is the image of <i> under the permutation <obj>. If <obj> is a word, `'List'` returns a list where the <i>-th element is the <i>-th generator of the word, as a word of length 1.

```
gap> List( [1,2,3] );
[ 1, 2, 3 ]
gap> List( (1,2)(3,4,5) );
[ 2, 1, 4, 5, 3 ]
```

In its second form 'List' returns a new list, where each element is the result of applying the function <func>, which must take exactly one argument and handle the elements of <list>, to the corresponding element of the list <list>. The list <list> must not contain holes.

```
gap> List( [1,2,3], x->x^2 );
[ 1, 4, 9 ]
gap> List( [1..10], IsPrime );
[ false, true, true, false, true, false, true, false, false, false ]
```

Note that this function is called 'map' in Lisp and many other similar programming languages. This name violates the GAP rule that verbs are used for functions that change their arguments. According to this rule 'map' would change <list>, replacing every element with the result of the application <func> to this argument.

3.3 List Elements

'<list>[<pos>]'

The above construct evaluates to the <pos>-th element of the list <list>. <pos> must be a positive integer. List indexing is done with origin 1, i.e., the first element of the list is the element at position 1.

```
gap> l := [ 2, 3, 5, 7, 11, 13 ];;
gap> l[1];
2
gap> l[2];
3
gap> l[6];
13
```

If <list> does not evaluate to a list, or <pos> does not evaluate to a positive integer, or '<list>[<pos>]' is unbound an error is signalled. As usual you can leave the break loop with 'quit;'. On the other hand you can return a result to be used in place of the list element by 'return <expr>;'.

'<list>\{ <poss> \}'

The above construct evaluates to a new list <new> whose first element is '<list>[<poss>[1]]', whose second element is '<list>[<poss>[2]]', and so on. <poss> must be a dense list of positive integers, it need, however, not be sorted and may contain duplicate elements. If for any <i>, '<list>[<poss>[<i>]]' is unbound, an error is signalled.

```
gap> l := [ 2, 3, 5, 7, 11, 13, 17, 19 ];;
gap> l{[4..6]};
[ 7, 11, 13 ]
gap> l{[1,7,1,8]};
[ 2, 17, 2, 19 ]
```

The result is a new list, that is not identical to any other list. The elements of that list however are identical to the corresponding elements of the left operand (see "Identical Lists").

It is possible to nest such sublist extractions, as can be seen in the following example.

```
gap> m := [ [1,2,3], [4,5,6], [7,8,9], [10,11,12] ];;
gap> m{[1,2,3]}{[3,2]};
[ [ 3, 2 ], [ 6, 5 ], [ 9, 8 ] ]
gap> l := m{[1,2,3]};; l{[3,2]};
[ [ 7, 8, 9 ], [ 4, 5, 6 ] ]
```

Note the difference between the two examples. The latter extracts elements 1, 2, and 3 from <m> and then extracts the elements 3 and 2 from *this list*. The former extracts elements 1, 2, and 3 from <m> and then extracts the elements 3 and 2 from *each of those element lists*.

To be precise. With each selector `'[<pos>]'` or `'\{<poss>\}'` we associate a *level* that is defined as the number of selectors of the form `'\{<poss>\}'` to its left in the same expression. For example

```
        l[pos1]{poss2}{poss3}[pos4]{poss5}[pos6]
    level   0      0      1     1      1     2
```

Then a selector `'<list>[<pos>]'` of level <level> is computed as `'ListElement(<list>,<pos>,<level>)'`, where `'ListElement'` is defined as follows

```
ListElement := function ( list, pos, level )
    if level = 0  then
        return list[pos];
    else
        return List( list, elm -> ListElement(elm,pos,level-1) );
    fi;
end;
```

and a selector `'<list>\{<poss>\}'` of level <level> is computed as `'ListElements(<list>,<poss>,<level>)'`, where `'ListElements'` is defined as follows

```
ListElements := function ( list, poss, level )
    if level = 0  then
        return list{poss};
    else
        return List( list, elm -> ListElements(elm,poss,level-1) );
    fi;
end;
```

3.4 Length

'Length(<list>)'

'Length' returns the length of the list <list>. The *length* is defined as 0 for the empty list, and as the largest positive integer <index> such that '<list>[<index>]' has an assigned value for nonempty lists. Note that the length of a list may change if new elements are added to it or assigned to previously unassigned positions.

```
gap> Length( [] );
0
gap> Length( [ 2, 3, 5, 7, 11, 13, 17, 19 ] );
8
gap> Length( [ 1, 2,,, 5 ] );
5
```

For lists that contain no holes 'Length', 'Number' (see "Number"), and 'Size' return the same value. For lists with holes 'Length' returns the largest index of a bound entry, 'Number' returns the number of bound entries, and 'Size' signals an error.

3.5 List Assignment

'<list>[<pos>] \:= <object>;'

The list assignment assigns the object <object>, which can be of any type, to the list entry at the position <pos>, which must be a positive integer, in the list <list>. That means that accessing the <pos>-th element of the list <list> will return <object> after this assignment.

```
gap> l := [ 1, 2, 3 ];;
gap> l[1] := 3;;  l;                # assign a new object
[ 3, 2, 3 ]
gap> l[2] := [ 4, 5, 6 ];;  l;      # <object> may be of any type
[ 3, [ 4, 5, 6 ], 3 ]
gap> l[ l[1] ] := 10;;  l;          # <index> may be an expression
[ 3, [ 4, 5, 6 ], 10 ]
```

If the index <pos> is larger than the length of the list <list> (see "Length"), the list is automatically enlarged to make room for the new element. Note that it is possible to generate lists with holes that way.

```
gap> l[4] := "another entry";;  l; # <list> is enlarged
[ 3, [ 4, 5, 6 ], 10, "another entry" ]
gap> l[ 10 ] := 1;;  l;            # now <list> has a hole
[ 3, [ 4, 5, 6 ], 10, "another entry",,,,,, 1 ]
```

The function `'Add'` (see "Add") should be used if you want to add an element to the end of the list.

Note that assigning to a list changes the list. The ability to change an object is only available for lists and records (see "Identical Lists").

If <list> does not evaluate to a list, <pos> does not evaluate to a positive integer or <object> is a call to a function which does not return a value, for example `'Print'`, an error is signalled As usual you can leave the break loop with `'quit;'`. On the other hand you can continue the assignment by returning a list, an index or an object using `'return <expr>;'`.

`'<list>\{ <poss> \} \:= <objects>;'`

The list assignment assigns the object `'<objects>[1]'`, which can be of any type, to the list <list> at the position `'<poss>[1]'`, the object `'<objects>[2]'` to `'<list>[<poss>[2]]'`, and so on. <poss> must be a dense list of positive integers, it need, however, not be sorted and may contain duplicate elements. <objects> must be a dense list and must have the same length as <poss>.

```
gap> l := [ 2, 3, 5, 7, 11, 13, 17, 19 ];;
gap> l{[1..4]} := [10..13];;  l;
[ 10, 11, 12, 13, 11, 13, 17, 19 ]
gap> l{[1,7,1,10]} := [ 1, 2, 3, 4 ];; l;
[ 3, 11, 12, 13, 11, 13, 2, 19,, 4 ]
```

It is possible to nest such sublist assignments, as can be seen in the following example.

```
gap> m := [ [1,2,3], [4,5,6], [7,8,9], [10,11,12] ];;
gap> m{[1,2,3]}{[3,2]} := [ [11,12], [13,14], [15,16] ];;  m;
[ [ 1, 12, 11 ], [ 4, 14, 13 ], [ 7, 16, 15 ], [ 10, 11, 12 ] ]
```

The exact behaviour is defined in the same way as for list extractions (see "List Elements"). Namely with each selector `'[<pos>]'` or `'\{<poss>\}'` we associate a *level* that is defined as the number of selectors of the form `'\{<poss>\}'` to its left in the same expression. For example

```
    l[pos1]{poss2}{poss3}[pos4]{poss5}[pos6]
level   0      0      1     1      1     2
```

Then a list assignment `'<list>[<pos>]  \:=  <vals>;'` of level <level> is computed as `'ListAssignment(  <list>, <pos>,  <vals>, <level> )'`, where `'ListAssignment'` is defined as follows

```
ListAssignment := function ( list, pos, vals, level )
    local  i;
    if level = 0  then
        list[pos] := vals;
    else
        for i  in [1..Length(list)]  do
            ListAssignment( list[i], pos, vals[i], level-1 );
        od;
    fi;
end;
```

and a list assignment `'<list>\{<poss>\} \:= <vals>'` of level <level> is computed as `'ListAssignments( <list>, <poss>, <vals>, <level> )'`, where `'ListAssignments'` is defined as follows

```
ListAssignments := function ( list, poss, vals, level )
    local  i;
    if level = 0  then
        list{poss} := vals;
    else
        for i  in [1..Length(list)]  do
            ListAssignments( list[i], poss, vals[i], level-1 );
        od;
    fi;
end;
```

3.6 Add

`'Add( <list>, <elm> )'`

`'Add'` adds the element <elm> to the end of the list <list>, i.e., it is equivalent to the assignment `'<list>[  Length(<list>) + 1 ] \:=  <elm>'`. The list is automatically enlarged to make room for the new element. `'Add'` returns nothing, it is called only for its side effect.

Note that adding to a list changes the list. The ability to change an object is only available for lists and records (see "Identical Lists").

To add more than one element to a list use `'Append'` (see "Append").

```
gap> l := [ 2, 3, 5 ];;  Add( l, 7 );  l;
[ 2, 3, 5, 7 ]
```

3.7 Append

`'Append( <list1>, <list2> )'`

`'Append'` adds (see "Add") the elements of the list <list2> to the end of the list <list1>. <list2> may contain holes, in which case the corresponding entries in <list1> will be left unbound. `'Append'` returns nothing, it is called only for its side effect.

```
gap> l := [ 2, 3, 5 ];; Append( l, [ 7, 11, 13 ] );  l;
[ 2, 3, 5, 7, 11, 13 ]
gap> Append( l, [ 17,, 23 ] ); l;
[ 2, 3, 5, 7, 11, 13, 17,, 23 ]
```

Note that appending to a list changes the list. The ability to change an object is only available for lists and records (see "Identical Lists").

Note that `'Append'` changes the first argument, while `'Concatenation'` (see "Concatenation") creates a new list and leaves its arguments unchanged. As usual the name of the function that work destructively is a verb, but the name of the function that creates a new object is a substantive.

3.8 Identical Lists

With the list assignment (see "List Assignment", "Add", "Append") it is possible to change a list. The ability to change an object is only available for lists and records. This section describes the semantic consequences of this fact.

You may think that in the following example the second assignment changes the integer, and that therefore the above sentence, which claimed that only lists and records can be changed is wrong

```
i := 3;
i :=  i + 1;
```

But in this example not the *integer* `'3'` is changed by adding one to it. Instead the *variable* `'i'` is changed by assigning the value of `'i+1'`, which happens to be `'4'`, to `'i'`. The same thing happens in the following example

```
l := [ 1,  2 ];
l := [ 1, 2,  3 ];
```

The second assignment does not change the first list, instead it assigns a new list to the variable `'l'`. On the other hand, in the following example the list is changed by the second assignment.

```
l := [ 1,  2 ];
l[3] := 3;
```

To understand the difference first think of a variable as a name for an object. The important point is that a list can have several names at the same time. An assignment `'<var>   \:=   <list>;'` means in this interpretation that <var> is a name for the object <list>. At the end of the following example `'l2'` still has the value `'[ 1, 2 ]'` as this list has not been changed and nothing else has been assigned to it.

```
l1 := [ 1, 2 ];
l2 := l1;
l1 := [ 1, 2, 3 ];
```

But after the following example the list for which `'l2'` is a name has been changed and thus the value of `'l2'` is now `'[ 1, 2, 3 ]'`.

```
l1 := [ 1, 2 ];
l2 := l1;
l1[3] := 3;
```

We shall say that two lists are *identical* if changing one of them by a list assignment also changes the other one. This is slightly incorrect, because if *two* lists are identical, there are actually only two names for *one* list. However, the correct usage would be very awkward and would only add to the confusion. Note that two identical lists must be equal, because there is only one list with two different names. Thus identity is an equivalence relation that is a refinement of equality.

Let us now consider under which circumstances two lists are identical.

If you enter a list literal than the list denoted by this literal is a new list that is not identical to any other list. Thus in the following example `'l1'` and `'l2'` are not identical, though they are equal of course.

```
l1 := [ 1, 2 ];
l2 := [ 1, 2 ];
```

Also in the following example, no lists in the list `'l'` are identical.

```
l := [];
for i  in [1..10]  do l[i] := [ 1, 2 ];  od;
```

If you assign a list to a variable no new list is created. Thus the list value of the variable on the left hand side and the list on the right hand side of the assignment are identical. So in the following example `'l1'` and `'l2'` are identical lists.

```
l1 := [ 1, 2 ];
l2 := l1;
```

If you pass a list as argument, the old list and the argument of the function are identical. Also if you return a list from a function, the old list and the value of the function call are identical. So in the following example `'l1'` and `'l2'` are identical list

```
l1 := [ 1, 2 ];
f := function ( l )  return l;  end;
l2 := f( l1 );
```

The functions `'Copy'` and `'ShallowCopy'` (see "Copy" and "ShallowCopy") accept a list and return a new list that is equal to the old list but that is *not* identical to the old list. The difference between `'Copy'` and `'ShallowCopy'` is that in the case of `'ShallowCopy'` the corresponding elements of the new and the old lists will be identical, whereas in the case of `'Copy'` they will only be equal. So in the following example `'l1'` and `'l2'` are not identical lists.

```
l1 := [ 1, 2 ];
l2 := Copy( l1 );
```

If you change a list it keeps its identity. Thus if two lists are identical and you change one of them, you also change the other, and they are still identical afterwards. On the other hand, two lists that are not identical will never become identical if you change one of them. So in the following example both `'l1'` and `'l2'` are changed, and are still identical.

```
l1 := [ 1, 2 ];
l2 := l1;
l1[1] := 2;
```

3.9 IsIdentical

`'IsIdentical( <l>, <r> )'`

`'IsIdentical'` returns `'true'` if the objects <l> and <r> are identical. Unchangeable objects are considered identical if the are equal. Changeable objects, i.e., lists and records, are identical if changing one of them by an assignment also changes the other one, as described in "Identical Lists".

```
gap> IsIdentical( 1, 1 );
true
gap> IsIdentical( 1, () );
false
gap> l := [ \verb|'h'|, \verb|'a'|, \verb|'l'|, \verb|'l'|, \verb|'o'| ];;
gap> l = "hallo";
true
gap> IsIdentical( l, "hallo" );
false
```

3.10 Enlarging Lists

The previous section (see "List Assignment") told you (among other things), that it is possible to assign beyond the logical end of a list, automatically enlarging the list. This section tells you how this is done.

It would be extremly wasteful to make all lists large enough so that there is room for all assignments, because some lists may have more than 100000 elements, while most lists have less than 10 elements.

On the other hand suppose every assignment beyond the end of a list would be done by allocating new space for the list and copying all entries to the new space. Then creating a list of 1000 elements by assigning them in order, would take half a million copy operations and also create a lot of garbage that the garbage collector would have to reclaim.

So the following strategy is used. If a list is created it is created with exactly the correct size. If a list is enlarged, because of an assignment beyond the end of the list, it is enlarged by at least `'<length>/8 + 4'` entries. Therefore the next assignments beyond the end of the list do not need to enlarge the list. For example creating a list of 1000 elements by assigning them in order, would now take only 32 enlargements.

The result of this is of course that the *physical length*, which is also called the size, of a list may be different from the *logical length*, which is usually called simply the length of the list. Aside from the implications for the performance you need not be aware of the physical length. In fact all you can ever observe, for example by calling 'Length' is the logical length.

Suppose that 'Length' would have to take the physical length and then test how many entries at the end of a list are unassigned, to compute the logical length of the list. That would take too much time. In order to make 'Length', and other functions that need to know the logical length, more efficient, the length of a list is stored along with the list.

A note aside. In the previous version 2.4 of GAP a list was indeed enlarged every time an assignment beyond the end of the list was performed. To deal with the above inefficiency the following hacks where used. Instead of creating lists in order they were usually created in reverse order. In situations where this was not possible a dummy assignment to the last position was performed, for example

```
l := [];
l[1000] := "dummy";
l[1] := first_value();
for i  from 2  to 1000  do l[i] := next_value(l[i-1]);  od;
```

3.11 Comparisons of Lists

'<list1> = <list2>'
'<list1> <> <list2>'

The equality operator '=' evaluates to 'true' if the two lists <list1> and <list2> are equal and 'false' otherwise. The inequality operator '<>' evaluates to 'true' if the two lists are not equal and 'false' otherwise. Two lists <list1> and <list2> are equal if and only if for every index <i>, either both entries '<list1>[<i>]' and '<list2>[<i>]' are unbound, or both are bound and are equal, i.e., '<list1>[<i>] = <list2>[<i>]' is 'true'.

```
gap> [ 1, 2, 3 ] = [ 1, 2, 3 ];
true
gap> [ , 2, 3 ] = [ 1, 2, ];
false
gap> [ 1, 2, 3 ] = [ 3, 2, 1 ];
false
```

'<list1> <\ <list2>','<list1> <= <list2>''<list1> > <list2>','<list1> >= <list2>'

The operators '<', '<=', '>' and '>=' evaluate to 'true' if the list <list1> is less than, less than or equal to, greater than, or greater than or equal to the list <list2> and to 'false' otherwise. Lists are ordered lexicographically, with unbound entries comparing very small. That means the following. Let <i> be the smallest positive integer <i>, such that neither both entries '<list1>[<i>]'

and '<list2>[<i>]' are unbound, nor both are bound and equal. Then <list1> is less than <list2> if either '<list1>[<i>]' is unbound (and '<list2>[<i>]' is not) or both are bound and '<list1>[<i>] <\ <list2>[<i>]' is 'true'.

```
gap> [ 1, 2, 3, 4 ] < [ 1, 2, 4, 8 ];
true    # \verb|'<list1>[3] <\ <list2>[3]'|
gap> [ 1, 2, 3 ] < [ 1, 2, 3, 4 ];
true    # \verb|'<list1>[4]'| is unbound and therefore very small
gap> [ 1, , 3, 4 ] < [ 1, 2, 3 ];
true    # \verb|'<list1>[2]'| is unbound and therefore very small
```

You can also compare objects of other types, for example integers or permutations with lists. Of course those objects are never equal to a list. Records are greater than lists, objects of every other type are smaller than lists.

```
gap> 123 < [ 1, 2, 3 ];
true
gap> [ 1, 2, 3 ] < rec( a := 123 );
true
```

3.12 Operations for Lists

'<list> *\ <obj>'
'<obj> *\ <list>'

The operator '*' evaluates to the product of list <list> by an object <obj>. The product is a new list that at each position contains the product of the corresponding element of <list> by <obj>. <list> may contain holes, in which case the result will contain holes at the same positions.

The elements of <list> and <obj> must be objects of the following types; integers, rationals, cyclotomics, elements of a finite field, permutations, matrices, words in abstract generators, or words in solvable groups.

```
gap> [ 1, 2, 3 ] * 2;
[ 2, 4, 6 ]
gap> 2 * [ 2, 3,, 5,, 7 ];
[ 4, 6,, 10,, 14 ]
gap> [ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ] * (1,4);
[ (1,4), (1,4)(2,3), (1,2,4), (1,2,3,4), (1,3,2,4), (1,3,4) ]
```

Many more operators are available for vectors and matrices, which are also represented by lists.

3.13 In

`'<elm> in <list>'`

The `'in'` operator evaluates to `'true'` if the object <elm> is an element of the list <list> and to `'false'` otherwise. <elm> is an element of <list> if there is a positive integer <index> such that `'<list>[<index>]=<elm>'` is `'true'`. <elm> may be an object of an arbitrary type and <list> may be a list containing elements of any type.

It is much faster to test for membership for sets, because for sets, which are always sorted, `'in'` can use a binary search, instead of the linear search used for ordinary lists. So if you have a list for which you want to perform a large number of membership tests you may consider converting it to a set with the function `'Set'` (see "Set").

```
gap> 1 in [ 2, 2, 1, 3 ];
true
gap> 1 in [ 4, -1, 0, 3 ];
false
gap> s := Set([2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32]);;
gap> 17 in s;
false       # uses binary search and only 4 comparisons
gap> 1 in [ "This", "is", "a", "list", "of", "strings" ];
false
gap> [1,2] in [ [0,6], [0,4], [1,3], [1,5], [1,2], [3,4] ];
true
```

`'Position'` (see "Position") and `'PositionSorted'` (see "PositionSorted") allow you to find the position of an element in a list.

3.14 Position

`'Position( <list>, <elm> )'`

`'Position'` returns the position of the element <elm>, which may be an object of any type, in the list <list>. If the element is not in the list the result is `'false'`. If the element appears several times, the first position is returned.

It is much faster to search for an element in a set, because for sets, which are always sorted, `'Position'` can use a binary search, instead of the linear search used for ordinary lists. So if you have a list for which you want to perform a large number of searches you may consider converting it to a set with the function `'Set'` (see "Set").

```
gap> Position( [ 2, 2, 1, 3 ],  1 );
3
gap> Position( [ 2, 1, 1, 3 ], 1 );
2
```

```
gap> Position( [ 4, -1, 0, 3 ],  1 );
false
gap> s := Set([2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32]);;
gap> Position( s,  17 );
false       # uses binary search and only 4 comparisons
gap> Position( [ "This", "is", "a", "list", "of", "strings" ],  1 );
false
gap> Position( [ [0,6], [0,4], [1,3], [1,5], [1,2], [3,4] ],  [1,2] );
5
```

The `'in'` operator (see "In") can be used if you are only interested to know whether the element is in the list or not. `'PositionSorted'` (see "PositionSorted") can be used if the list is sorted. `'PositionProperty'` (see "PositionProperty") allows you to find the position of an element that satisfies a certain property in a list.

3.15 PositionSorted

`'PositionSorted( <list>, <elm> )'`

`'PositionSorted( <list>, <elm>, <func> )'`

In the first form `'PositionSorted'` returns the position of the element <elm>, which may be an object of any type, with respect to the sorted list <list>.

In the second form `'PositionSorted'` returns the position of the element <elm>, which may be an object of any type with respect to the list <list>, which must be sorted with respect to <func>. <func> must be a function of two arguments that returns `'true'` if the first argument is less than the second argument and `'false'` otherwise.

`'PositionSorted'` returns <pos> such that `'<list>[<pos>-1] < <elm>'` and `'<elm> <= <list>[<pos>]'`. That means, if <elm> appears once in <list>, its position is returned. If <elm> appears several times in <list>, the position of the first occurrence is returned. If <elm> is not an element of <list>, the index where <elm> must be inserted to keep the list sorted is returned.

```
gap> PositionSorted( [1,4,5,5,6,7], 0 );
1
gap> PositionSorted( [1,4,5,5,6,7], 2 );
2
gap> PositionSorted( [1,4,5,5,6,7], 4 );
2
gap> PositionSorted( [1,4,5,5,6,7], 5 );
3
gap> PositionSorted( [1,4,5,5,6,7], 8 );
7
```

`'Position'` (see "Position") is another function that returns the position of an element in a list. `'Position'` accepts unsorted lists, uses linear instead of binary search and returns `'false'` if <elm> is not in <list>.

3.16 PositionProperty

`'PositionProperty( <list>, <func> )'`

`'PositionProperty'` returns the position of the first element in the list <list> for which the unary function <func> returns `'true'`. <list> must not contain holes. If <func> returns `'false'` for all elements of <list> `'false'` is returned. <func> must return `'true'` or `'false'` for every element of <list>, otherwise an error is signalled.

```
gap> PositionProperty( [10^7..10^8], IsPrime );
20
gap> PositionProperty( [10^5..10^6],
>                       n -> not IsPrime(n) and IsPrimePowerInt(n) );
490
```

`'First'` (see "First") allows you to extract the first element of a list that satisfies a certain property.

3.17 Concatenation

`'Concatenation( <list1>, <list2>.. )'`
`'Concatenation( <list> )'`

In the first form `'Concatenation'` returns the concatenation of the lists <list1>, <list2>, etc. The *concatenation* is the list that begins with the elements of <list1>, followed by the elements of <list2> and so on. Each list may also contain holes, in which case the concatenation also contains holes at the corresponding positions.

```
gap> Concatenation( [ 1, 2, 3 ], [ 4, 5 ] );
[ 1, 2, 3, 4, 5 ]
gap> Concatenation( [2,3,,5,,7], [11,,13,,,,17,,19] );
[ 2, 3,, 5,, 7, 11,, 13,,,, 17,, 19 ]
```

In the second form <list> must be a list of lists <list1>, <list2>, etc, and `'Concatenation'` returns the concatenation of those lists.

```
gap> Concatenation( [ [1,2,3], [2,3,4], [3,4,5] ] );
[ 1, 2, 3, 2, 3, 4, 3, 4, 5 ]
```

The result is a new list, that is not identical to any other list. The elements of that list however are identical to the corresponding elements of the argument lists (see "Identical Lists").

Note that `'Concatenation'` creates a new list and leaves it arguments unchanged, while `'Append'` (see "Append") changes its first argument. As usual the name of the function that works destructively is a verb, but the name of the function that creates a new object is a substantive.

`'Set(Concatenation(<set1>,<set2>..))'` (see "Set") is a way to compute the union of sets, however, `'Union'` is more efficient.

3.18 Flat

'Flat(<list>)'

'Flat' returns the list of all elements that are contained in the list <list> or its sublists. That is, 'Flat' first makes a new empty list <new>. Then it loops over the elements <elm> of <list>. If <elm> is not a list it is added to <new>, otherwise 'Flat' appends 'Flat(<elm>)' to <new>.

```
gap> Flat( [ 1, [ 2, 3 ], [ [ 1, 2 ], 3 ] ] );
[ 1, 2, 3, 1, 2, 3 ]
gap> Flat( [ ] );
[  ]
```

3.19 Reversed

'Reversed(<list>)'

'Reversed' returns a new list that contains the elements of the list <list>, which must not contain holes, in reverse order. The argument list is unchanged.

```
gap> Reversed( [ 1, 4, 5, 5, 6, 7 ] );
[ 7, 6, 5, 5, 4, 1 ]
```

The result is a new list, that is not identical to any other list. The elements of that list however are identical to the corresponding elements of the argument list (see "Identical Lists").

3.20 Sublist

'Sublist(<list>, <inds>)'

'Sublist' returns a new list in which the <i>-th element is the element '<list>[<inds>[<i>]]', of the list <list>. <inds> must be a list of positive integers without holes, it need, however, not be sorted and may contains duplicate elements. If '<list>[<inds>[<i>]]' is unbound for an <i>, an error is signalled.

```
gap> Sublist( [ 2, 3, 5, 7, 11, 13, 17, 19 ], [4..6] );
[ 7, 11, 13 ]
gap> Sublist( [ 2, 3, 5, 7, 11, 13, 17, 19 ], [1,7,1,8] );
[ 2, 17, 2, 19 ]
```

The result is a new list, that is not identical to any other list. The elements of that list however are identical to the corresponding elements of the argument list (see "Identical Lists").

'Filtered' (see "Filtered") allows you to extract elements from a list according to a predicate.

'Sublist' has been made obsolete by the introduction of the construct '<list>\{ <inds> \}' (see "List Elements").

3.21 Cartesian

'Cartesian(<list1>, <list2>..)'

'Cartesian(<list>)'

In the first form 'Cartesian' returns the cartesian product of the lists <list1>, <list2>, etc.

In the second form <list> must be a list of lists <list1>, <list2>, etc., and 'Cartesian' returns the cartesian product of those lists.

The *cartesian product* is a list <cart> of lists <tup>, such that the first element of <tup> is an element of <list1>, the second element of <tup> is an element of <list2>, and so on. The total number of elements in <cart> is the product of the lengths of the argument lists. In particular <cart> is empty if and only if at least one of the argument lists is empty. Also <cart> contains duplicates if and only if no argument list is empty and at least one contains duplicates.

The last index runs fastest. That means that the first element <tup1> of <cart> contains the first element from <list1>, from <list2> and so on. The second element <tup2> of <cart> contains the first element from <list1>, the first from <list2>, an so on, but the last element of <tup2> is the second element of the last argument list. This implies that <cart> is a set if and only if all argument lists are sets.

```
gap> Cartesian( [1,2], [3,4], [5,6] );
[ [ 1, 3, 5 ], [ 1, 3, 6 ], [ 1, 4, 5 ], [ 1, 4, 6 ], [ 2, 3, 5 ],
  [ 2, 3, 6 ], [ 2, 4, 5 ], [ 2, 4, 6 ] ]
gap> Cartesian( [1,2,2], [1,1,2] );
[ [ 1, 1 ], [ 1, 1 ], [ 1, 2 ], [ 2, 1 ], [ 2, 1 ], [ 2, 2 ],
  [ 2, 1 ], [ 2, 1 ], [ 2, 2 ] ]
```

The function 'Tuples' computes the <k>-fold cartesian product of a list.

3.22 Number

'Number(<list>)'

'Number(<list>, <func>)'

In the first form 'Number' returns the number of bound entries in the list <list>.

For lists that contain no holes 'Number', 'Length' (see "Length"), and 'Size' return the same value. For lists with holes 'Number' returns the number of bound entries, 'Length' returns the largest index of a bound entry, and 'Size' signals an error.

'Number' returns the number of elements of the list <list> for which the unary function <func> returns 'true'. If an element for which <func> returns 'true' appears several times in <list> it will also be counted several times. <func> must return either 'true' or 'false' for every element of <list>, otherwise an error is signalled.

```
gap> Number( [ 2, 3, 5, 7 ] );
4
gap> Number( [, 2, 3,, 5,, 7,,,, 11 ] );
5
gap> Number( [1..20], IsPrime );
8
gap> Number( [ 1, 3, 4, -4, 4, 7, 10, 6 ], IsPrimePowerInt );
4
gap> Number( [ 1, 3, 4, -4, 4, 7, 10, 6 ],
>            n -> IsPrimePowerInt(n) and n mod 2 <> 0 );
2
```

'Filtered' (see ”Filtered”) allows you to extract the elements of a list that have a certain property.

3.23 Collected

'Collected(<list>)'

'Collected' returns a new list <new> that contains for each different element <elm> of <list> a list of two elements, the first element is <elm> itself, and the second element is the number of times <elm> appears in <list>. The order of those pairs in <new> corresponds to the ordering of the elements <elm>, so that the result is sorted.

```
gap> Factors( Factorial( 10 ) );
[ 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 5, 5, 7 ]
gap> Collected( last );
[ [ 2, 8 ], [ 3, 4 ], [ 5, 2 ], [ 7, 1 ] ]
gap> Collected( last );
[ [ [ 2, 8 ], 1 ], [ [ 3, 4 ], 1 ], [ [ 5, 2 ], 1 ], [ [ 7, 1 ], 1 ] ]
```

3.24 Filtered

'Filtered(<list>, <func>)'

'Filtered' returns a new list that contains those elements of the list <list> for which the unary function <func> returns 'true'. The order of the elements in the result is the same as the order of the corresponding elements of <list>. If an element, for which <func> returns 'true' appears several times in <list> it will also appear the same number of times in the result. <list> may contain holes, they are ignored by 'Filtered'. <func> must return either 'true' or 'false' for every element of <list>, otherwise an error is signalled.

```
gap> Filtered( [1..20], IsPrime );
[ 2, 3, 5, 7, 11, 13, 17, 19 ]
gap> Filtered( [ 1, 3, 4, -4, 4, 7, 10, 6 ], IsPrimePowerInt );
```

```
[ 3, 4, 4, 7 ]
gap> Filtered( [ 1, 3, 4, -4, 4, 7, 10, 6 ],
>              n -> IsPrimePowerInt(n) and n mod 2 <> 0 );
[ 3, 7 ]
```

The result is a new list, that is not identical to any other list. The elements of that list however are identical to the corresponding elements of the argument list (see "Identical Lists").

'Sublist' (see "Sublist") allows you to extract elements of a list according to indices given in another list.

3.25 ForAll

'ForAll(<list>, <func>)'

'ForAll' returns 'true' if the unary function <func> returns 'true' for all elements of the list <list> and 'false' otherwise. <list> may contain holes. <func> must return either 'true' or 'false' for every element of <list>, otherwise an error is signalled.

```
gap> ForAll( [1..20], IsPrime );
false
gap> ForAll( [2,3,4,5,8,9], IsPrimePowerInt );
true
gap> ForAll( [2..14], n -> IsPrimePowerInt(n) or n mod 2 = 0 );
true
```

'ForAny' (see "ForAny") allows you to test if any element of a list satisfies a certain property.

3.26 ForAny

'ForAny(<list>, <func>)'

'ForAny' returns 'true' if the unary function <func> returns 'true' for at least one element of the list <list> and 'false' otherwise. <list> may contain holes. <func> must return either 'true' or 'false' for every element of <list>, otherwise 'ForAny' signals an error.

```
gap> ForAny( [1..20], IsPrime );
true
gap> ForAny( [2,3,4,5,8,9], IsPrimePowerInt );
true
gap> ForAny( [2..14],
>    n -> IsPrimePowerInt(n) and n mod 5 = 0 and not IsPrime(n) );
false
```

'ForAll' (see "ForAll") allows you to test if all elements of a list satisfies a certain propertie.

3.27 First

'First(<list>, <func>)'

'First' returns the first element of the list <list> for which the unary function <func> returns 'true'. <list> may contain holes. <func> must return either 'true' or 'false' for every element of <list>, otherwise an error is signalled. If <func> returns 'false' for every element of <list> an error is signalled.

```
gap> First( [10^7..10^8], IsPrime );
10000019
gap> First( [10^5..10^6],
>           n -> not IsPrime(n) and IsPrimePowerInt(n) );
100489
```

'PositionProperty' (see "PositionProperty") allows you to find the position of the first element in a list that satisfies a certain property.

3.28 Sort

'Sort(<list>)'
'Sort(<list>, <func>)'

'Sort' sorts the list <list> in increasing order, using shellsort. In the first form 'Sort' uses the operator '<' to compare the elements. In the second form 'Sort' uses the function <func> to compare elements. This function must be a function taking two arguments that returns 'true' if the first is strictly smaller than the second and 'false' otherwise.

'Sort' does not return anything, since it changes the argument <list>. Use 'ShallowCopy' (see "ShallowCopy") if you want to keep <list>. Use 'Reversed' (see "Reversed") if you want to get a new list sorted in decreasing order.

It is possible to sort lists that contain multiple elements which compare equal. It is not guaranteed that those elements keep their relative order, i.e., 'Sort' is not stable.

```
gap> list := [ 5, 4, 6, 1, 7, 5 ];;  Sort( list );  list;
[ 1, 4, 5, 5, 6, 7 ]
gap> list := [ [0,6], [1,2], [1,3], [1,5], [0,4], [3,4] ];;
gap> Sort( list, function(v,w) return v*v < w*w; end ); list;
[ [ 1, 2 ], [ 1, 3 ], [ 0, 4 ], [ 3, 4 ], [ 1, 5 ], [ 0, 6 ] ]
#  sorted according to the Euclidian distance from [0,0]
gap> list := [ [0,6], [1,3], [3,4], [1,5], [1,2], [0,4], ];;
gap> Sort( list, function(v,w) return v[1] < w[1]; end ); list;
[ [ 0, 6 ], [ 0, 4 ], [ 1, 3 ], [ 1, 5 ], [ 1, 2 ], [ 3, 4 ] ]
# note the random order of the elements with equal first component
```

`'SortParallel'` (see "SortParallel") allows you to sort a list and apply the exchanges that are necessary to another list in parallel. `'Sortex'` (see "Sortex") sorts a list and returns the sorting permutation.

3.29 SortParallel

`'SortParallel( <list1>, <list2> )'`
`'SortParallel( <list1>, <list2>, <func> )'`

`'SortParallel'` sorts the list <list1> in increasing order just as `'Sort'` (see "Sort") does. In parallel it applies the same exchanges that are necessary to sort <list1> to the list <list2>, which must of course have at least as many elements as <list1> does.

```
gap> list1 := [ 5, 4, 6, 1, 7, 5 ];;
gap> list2 := [ 2, 3, 5, 7, 8, 9 ];;
gap> SortParallel( list1, list2 );
gap> list1;
[ 1, 4, 5, 5, 6, 7 ]
gap> list2;
[ 7, 3, 2, 9, 5, 8 ]    # \verb|'[ 7, 3, 9, 2, 5, 8 ]'| is also possible
```

`'Sortex'` (see "Sortex") sorts a list and returns the sorting permutation.

3.30 Sortex

`'Sortex( <list> )'`

`'Sortex'` sorts the list <list> and returns the permutation that must be applied to <list> to obtain the sorted list.

```
gap> list1 := [ 5, 4, 6, 1, 7, 5 ];;
gap> list2 := Copy( list1 );;
gap> perm := Sortex( list1 );
(1,3,5,6,4)
gap> list1;
[ 1, 4, 5, 5, 6, 7 ]
gap> Permuted( list2, perm );
[ 1, 4, 5, 5, 6, 7 ]
```

`'Permuted'` (see "Permuted") allows you to rearrange a list according to a given permutation.

3.31 Permuted

'Permuted(<list>, <perm>)'

'Permuted' returns a new list <new> that contains the elements of the list <list> permuted according to the permutation <perm>. That is '<new>[<i>\^<perm>] = <list>[<i>]'.

```
gap> Permuted( [ 5, 4, 6, 1, 7, 5 ], (1,3,5,6,4) );
[ 1, 4, 5, 5, 6, 7 ]
```

'Sortex' (see "Sortex") allows you to compute the permutation that must be applied to a list to get the sorted list.

3.32 Product

'Product(<list>)'
'Product(<list>, <func>)'

In the first form 'Product' returns the product of the elements of the list <list>, which must have no holes. If <list> is empty, the integer 1 is returned.

In the second form 'Product' applies the function <func> to each element of the list <list>, which must have no holes, and multiplies the results. If the <list> is empty, the integer 1 is returned.

```
gap> Product( [ 2, 3, 5, 7, 11, 13, 17, 19 ] );
9699690
gap> Product( [1..10], x->x^2 );
13168189440000
gap> Product( [ (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) ] );
(1,4)(2,3)
```

'Sum' (see "Sum") computes the sum of the elements of a list.

3.33 Sum

'Sum(<list>)'
'Sum(<list>, <func>)'

In the first form 'Sum' returns the sum of the elements of the list <list>, which must have no holes. If <list> is empty 0 is returned.

In the second form 'Sum' applies the function <func> to each element of the list <list>, which must have no holes, and sums the results. If the <list> is empty 0 is returned.

```
gap> Sum( [ 2, 3, 5, 7, 11, 13, 17, 19 ] );
77
gap> Sum( [1..10], x->x^2 );
385
gap> Sum( [ [1,2], [3,4], [5,6] ] );
[ 9, 12 ]
```

'`Product`' (see "Product") computes the product of the elements of a list.

3.34 Maximum

'`Maximum( <obj1>, <obj2>.. )`'
'`Maximum( <list> )`'

'`Maximum`' returns the maximum of its arguments, i.e., that argument obj_i for which $obj_k <= obj_i$ for all k. In its second form '`Maximum`' takes a list <list> and returns the maximum of the elements of this list.

Typically the arguments or elements of the list respectively will be integers, but actually they can be objects of an arbitrary type. This works because any two objects can be compared using the '`<`' operator.

```
gap> Maximum( -123, 700, 123, 0, -1000 );
700
gap> Maximum( [ -123, 700, 123, 0, -1000 ] );
700
gap> Maximum( [ 1, 2 ], [ 0, 15 ], [ 1, 5 ], [ 2, -11 ] );
[ 2, -11 ]        # lists are compared elementwise
```

3.35 Minimum

'`Minimum( <obj1>, <obj2>.. )`'
'`Minimum( <list> )`'

'`Minimum`' returns the minimum of its arguments, i.e., that argument obj_i for which $obj_i <= obj_k$ for all k. In its second form '`Minimum`' takes a list <list> and returns the minimum of the elements of this list.

Typically the arguments or elements of the list respectively will be integers, but actually they can be objects of an arbitrary type. This works because any two objects can be compared using the '`<`' operator.

```
gap> Minimum( -123, 700, 123, 0, -1000 );
-1000
gap> Minimum( [ -123, 700, 123, 0, -1000 ] );
```

```
-1000
gap> Minimum( [ 1, 2 ], [ 0, 15 ], [ 1, 5 ], [ 2, -11 ] );
[ 0, 15 ]        # lists are compared elementwise
```

3.36 Iterated

'Iterated(<list>, <f>)'

'Iterated' returns the result of the iterated application of the function <f>, which must take two arguments, to the elements of <list>. More precisely 'Iterated' returns the result of the following application, '<f>(..<f>(<f>(<list>[1], <list>[2]), <list>[3]),..,<list>[<n>])'.

```
gap> Iterated( [ 126, 66, 105 ], Gcd );
3
```

3.37 RandomList

'RandomList(<list>)'

'RandomList' returns a random element of the list <list>. The results are equally distributed, i.e., all elements are equally likely to be selected.

```
gap> RandomList( [1..200] );
192
gap> RandomList( [1..200] );
152
gap> RandomList( [ [ 1, 2 ], 3, [ 4, 5 ], 6 ] );
[ 4, 5 ]
```

'RandomSeed(<n>)'

'RandomSeed' seeds the pseudo random number generator 'RandomList'. Thus to reproduce a computation exactly you can call 'RandomSeed' each time before you start the computation. When GAP is started the pseudo random number generator is seeded with 1.

```
gap> RandomSeed(1);  RandomList([1..100]);  RandomList([1..100]);
96
76
gap> RandomSeed(1);  RandomList([1..100]);  RandomList([1..100]);
96
76
```

'RandomList' is called by all random functions for domains.

Chapter 4

Sets

A very important mathematical concept, maybe the most important of all, are sets. Mathematically a *set* is an abstract object such that each object is either an element of the set or it is not. So a set is a collection like a list, and in fact GAP uses lists to represent sets. Note that this of course implies that GAP only deals with finite sets.

Unlike a list a set must not contain an element several times. It simply makes no sense to say that an object is twice an element of a set, because an object is either an element of a set, or it is not. Therefore the list that is used to represent a set has no duplicates, that is, no two elements of such a list are equal.

Also unlike a list a set does not impose any ordering on the elements. Again it simply makes no sense to say that an object is the first or second etc. element of a set, because, again, an object is either an element of a set, or it is not. Since ordering is not defined for a set we can put the elements in any order into the list used to represent the set. We put the elements sorted into the list, because this ordering is very practical. For example if we convert a list into a set we have to remove duplicates, which is very easy to do after we have sorted the list, since then equal elements will be next to each other.

In short sets are represented by sorted lists without holes and duplicates in GAP. Such a list is in this document called a proper set. Note that we guarantee this representation, so you may make use of the fact that a set is represented by a sorted list in your functions.

In some contexts, we also want to talk about multisets. A *multiset* is like a set, except that an element may appear several times in a multiset. Such multisets are represented by sorted lists with holes that may have duplicates.

The first section in this chapter describes the functions to test if an object is a set and to convert objects to sets (see "IsSet" and "Set").

The next section describes the function that tests if two sets are equal (see "SetIsEqual").

The next sections describe the destructive functions that compute the standard set operations for sets (see "SetAdd", "SetRemove", "SetUnite", "SetIntersect", and "SetSubtract").

The last section tells you more about sets and their internal representation (see "More about Sets").

All set theoretic functions, especially 'Intersection' and 'Union', also accept sets as arguments. Thus all functions described in the chapter Domains in the GAP-manual are applicable to sets (see "Set Functions for Sets").

Since sets are just a special case of lists, all the operations and functions for lists, especially the membership test (see "In"), can be used for sets just as well.

4.1 IsSet

'IsSet(<obj>)'

'IsSet' returns 'true' if the object <obj> is a set and 'false' otherwise. An object is a set if it is a sorted lists without holes or duplicates. Will cause an error if evaluation of <obj> is an unbound variable.

```
gap> IsSet( [] );
true
gap> IsSet( [ 2, 3, 5, 7, 11 ] );
true
gap> IsSet( [, 2, 3,, 5,, 7,,,, 11 ] );
false       # this list contains holes
gap> IsSet( [ 11, 7, 5, 3, 2 ] );
false       # this list is not sorted
gap> IsSet( [ 2, 2, 3, 5, 5, 7, 11, 11 ] );
false       # this list contains duplicates
gap> IsSet( 235711 );
false       # this argument is not even a list
```

4.2 Set

'Set(<list>)'

'Set' returns a new proper set, which is represented as a sorted list without holes or duplicates, containing the elements of the list <list>.

'Set' returns a new list even if the list <list> is already a proper set, in this case it is equivalent to 'ShallowCopy' (see "ShallowCopy"). Thus the result is a new list that is not identical to any other list. The elements of the result are however identical to elements of <list>. If <list> contains equal elements, it is not specified to which of those the element of the result is identical (see "Identical Lists").

```
gap> Set( [3,2,11,7,2,,5] );
[ 2, 3, 5, 7, 11 ]
gap> Set( [] );
[  ]
```

4.3 SetIsEqual

'SetIsEqual(<list1>, <list2>)'

'SetIsEqual' returns 'true' if the two lists <list1> and <list2> are equal *when viewed as sets*, and 'false' otherwise. <list1> and <list2> are equal if every element of <list1> is also an element of <list2> and if every element of <list2> is also an element of <list1>.

If both lists are proper sets then they are of course equal if and only if they are also equal as lists. Thus 'SetIsEqual(<list1>, <list2>)' is equivalent to 'Set(<list1>) = Set(<list2>)' (see "Set"), but the former is more efficient.

```
gap> SetIsEqual( [2,3,5,7,11], [11,7,5,3,2] );
true
gap> SetIsEqual( [2,3,5,7,11], [2,3,5,7,11,13] );
false
```

4.4 SetAdd

'SetAdd(<set>, <elm>)'

'SetAdd' adds <elm>, which may be an elment of an arbitrary type, to the set <set>, which must be a proper set, otherwise an error will be signalled. If <elm> is already an element of the set <set>, the set is not changed. Otherwise <elm> is inserted at the correct position such that <set> is again a set afterwards.

```
gap> s := [2,3,7,11];;
gap> SetAdd( s, 5 );  s;
[ 2, 3, 5, 7, 11 ]
gap> SetAdd( s, 13 );  s;
[ 2, 3, 5, 7, 11, 13 ]
gap> SetAdd( s, 3 );  s;
[ 2, 3, 5, 7, 11, 13 ]
```

'SetRemove' (see "SetRemove") is the counterpart of 'SetAdd'.

4.5 SetRemove

'SetRemove(<set>, <elm>)'

'SetRemove' removes the element <elm>, which may be an object of arbitrary type, from the set <set>, which must be a set, otherwise an error will be signalled. If <elm> is not an element of <set> nothing happens. If <elm> is an element it is removed and all the following elements in the list are moved one position forward.

```
gap> s := [ 2, 3, 4, 5, 6, 7 ];;
gap> SetRemove( s, 6 );
gap> s;
[ 2, 3, 4, 5, 7 ]
gap> SetRemove( s, 10 );
gap> s;
[ 2, 3, 4, 5, 7 ]
```

'SetAdd' (see "SetAdd") is the counterpart of 'SetRemove'.

4.6 SetUnite

'SetUnite(<set1>, <set2>)'

'SetUnite' unites the set <set1> with the set <set2>. This is equivalent to adding all the elements in <set2> to <set1> (see "SetAdd"). <set1> must be a proper set, otherwise an error is signalled. <set2> may also be list that is not a proper set, in which case 'SetUnite' silently applies 'Set' to it first (see "Set"). 'SetUnite' returns nothing, it is only called to change <set1>.

```
gap> set := [ 2, 3, 5, 7, 11 ];;
gap> SetUnite( set, [ 4, 8, 9 ] );  set;
[ 2, 3, 4, 5, 7, 8, 9, 11 ]
gap> SetUnite( set, [ 16, 9, 25, 13, 16 ] );  set;
[ 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 25 ]
```

The function 'UnionSet' (see "Set Functions for Sets") is the nondestructive counterpart to the destructive procedure 'SetUnite'.

4.7 SetIntersect

'SetIntersect(<set1>, <set2>)'

'SetIntersect' intersects the set <set1> with the set <set2>. This is equivalent to removing all the elements that are not in <set2> from <set1> (see "SetRemove"). <set1> must be a set, otherwise an error is signalled. <set2> may be a list that is not a proper set, in which case 'SetIntersect' silently applies 'Set' to it first (see "Set"). 'SetIntersect' returns nothing, it is only called to change <set1>.

```
gap> set := [ 2, 3, 4, 5, 7, 8, 9, 11, 13, 16 ];;
gap> SetIntersect( set, [ 3, 5, 7, 9, 11, 13, 15, 17 ] );  set;
[ 3, 5, 7, 9, 11, 13 ]
gap> SetIntersect( set, [ 9, 4, 6, 8 ] );  set;
[ 9 ]
```

The function 'IntersectionSet' (see "Set Functions for Sets") is the nondestructive counterpart to the destructive procedure 'SetIntersect'.

4.8 SetSubtract

'SetSubtract(<set1>, <set2>)'

'SetSubtract' subtracts the set <set2> from the set <set1>. This is equivalent to removing all the elements in <set2> from <set1> (see "SetRemove"). <set1> must be a proper set, otherwise an error is signalled. <set2> may be a list that is not a proper set, in which case 'SetSubtract' applies 'Set' to it first (see "Set"). 'SetSubtract' returns nothing, it is only called to change <set1>.

```
gap> set := [ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 ];;
gap> SetSubtract( set, [ 6, 10 ] );  set;
[ 2, 3, 4, 5, 7, 8, 9, 11 ]
gap> SetSubtract( set, [ 9, 4, 6, 8 ] );  set;
[ 2, 3, 5, 7, 11 ]
```

The function 'Difference' is the nondestructive counterpart to destructive the procedure 'SetSubtract'.

4.9 Set Functions for Sets

As was already mentioned in the introduction to this chapter all domain functions also accept sets as arguments. Thus all functions described in the chapter Domains in the GAP-manual are applicable to sets. This section describes those functions where it might be helpful to know the implementation of those functions for sets.

'IsSubset(<set1>, <set2>)'

This is implemented by 'SetIsSubset', which you can call directly to save a little bit of time. Either argument to 'SetIsSubset' may also be a list that is not a proper set, in which case 'IsSubset' silently applies 'Set' (see "Set") to it first.

'Union(<set1>, <set2>)'

This is implemented by 'UnionSet', which you can call directly to save a little bit of time. Note that 'UnionSet' only accepts two sets, unlike 'Union', which accepts several sets or a list of sets. The result of 'UnionSet' is a new set, represented as a sorted list without holes or duplicates. Each argument to 'UnionSet' may also be a list that is not a proper set, in which case 'UnionSet' silently applies 'Set' (see "Set") to this argument. 'UnionSet' is implemented in terms of its destructive counterpart 'SetUnite' (see "SetUnite").

'Intersection(<set1>, <set2>)'

This is implemented by 'IntersectionSet', which you can call directly to save a little bit of time. Note that 'IntersectionSet' only accepts two sets, unlike 'Intersection', which accepts several sets or a list of sets. The result of 'IntersectionSet' is a new set, represented as a sorted list without holes or duplicates. Each argument to 'IntersectionSet' may also be a list that is not a proper set, in which case 'IntersectionSet' silently applies 'Set' (see "Set") to this argument. 'IntersectionSet' is implemented in terms of its destructive counterpart 'SetIntersect' (see "SetIntersect").

The result of 'IntersectionSet' and 'UnionSet' is always a new list, that is not identical to any other list. The elements of that list however are identical to the corresponding elements of <set1>. If <set1> is not a proper list it is not specified to which of a number of equal elements in <set1> the element in the result is identical (see "Identical Lists").

4.10 More about Sets

In the previous section we defined a proper set as a sorted list without holes or duplicates. This representation is not only nice to use, it is also a good internal representation supporting efficient algorithms. For example the 'in' operator can use binary instead of a linear search since a set is sorted. For another example 'Union' only has to merge the sets.

However, all those set functions also allow lists that are not proper sets, silently making a copy of it and converting this copy to a set. Suppose all the functions would have to test their arguments every time, comparing each element with its successor, to see if they are proper sets. This would chew up most of the performance advantage again. For example suppose 'in' would have to run over the whole list, to see if it is a proper set, so it could use the binary search. That would be ridiculous.

To avoid this a list that is a proper set may, but need not, have an internal flag set that tells those functions that this list is indeed a proper set. Those functions do not have to check this argument then, and can use the more efficient algorithms. This section tells you when a proper set obtains this flag, so you can write your functions in such a way that you make best use of the algorithms.

The results of 'Set', 'Difference', 'Intersection' and 'Union' are known to be sets by construction, and thus have the flag set upon creation.

If an argument to 'IsSet', 'SetIsEqual', 'IsSubset', 'Set', 'Difference', 'Intersection' or 'Union' is a proper set, that does not yet have the flag set, those functions will notice that and set the flag for this set. Note that 'in' will use linear search if the right operand does not have the flag set, will therefore not detect if it is a proper set and will, unlike the functions above, never set the flag.

If you change a proper set, that does have this flag set, by assignment, 'Add' or 'Append' the set will generally lose it flag, even if the change is such that the resulting list is still a proper set. However if the set has more than 100 elements and the value assigned or added is not a list and not a record and the resulting list is still a proper set than it will keep the flag. Note that changing a list that is not a proper set will never set the flag, even if the resulting list is a proper set. Such a set will obtain the flag only if it is passed to a set function.

Suppose you have built a proper set in such a way that it does not have the flag set, and that you now want to perform lots of membership tests. Then you should call 'IsSet' with that set as an argument. If it is indeed a proper set 'IsSet' will set the flag, and the subsequent 'in' operations will use the more efficient binary search. You can think of the call to 'IsSet' as a hint to GAP that this list is a proper set.

There is no way you can set the flag for an ordinary list without going through the checking in 'IsSet'. The internal functions depend so much on the fact that a list with this flag set is indeed sorted and without holes and duplicates that the risk would be too high to allow setting the flag without such a check.

Chapter 5

Records

Records are next to lists the most important way to collect objects together. A record is a collection of *components*. Each component has a unique *name*, which is an identifier that distinguishes this component, and a *value*, which is an object of arbitrary type. We often abbreviate *value of a component* to *element*. We also say that a record *contains* its elements. You can access and change the elements of a record using its name.

Record literals are written by writing down the components in order between `'rec('` and `')'`, and separating them by commas `','`. Each component consists of the name, the assignment operator `'\:='`, and the value. The *empty record*, i.e., the record with no components, is written as `'rec()'`.

```
gap> rec( a := 1, b := "2" );    # a record with two components
rec(
  a := 1,
  b := "2" )
gap> rec( a := 1, b := rec( c := 2 ) );    # record may contain records
rec(
  a := 1,
  b := rec(
      c := 2 ) )
```

Records usually contain elements of various types, i.e., they are usually not homogeneous like lists.

The first section in this chapter tells you how you can access the elements of a record (see "Accessing Record Elements").

The next sections tell you how you can change the elements of a record (see "Record Assignment" and "Identical Records").

The next sections describe the operations that are available for records (see "Comparisons of Records", "Operations for Records", "In for Records", and "Printing of Records").

The next section describes the function that tests if an object is a record (see "IsRec").

The next sections describe the functions that test whether a record has a component with a given name, and delete such a component (see "IsBound" and "Unbind"). Those functions are also applicable to lists.

The final sections describe the functions that create a copy of a record (see ”Copy” and ”Shallow-Copy”). Again those functions are also applicable to lists.

5.1 Accessing Record Elements

`'<rec>.<name>'`

The above construct evaluates to the value of the record component with the name <name> in the record <rec>. Note that the <name> is not evaluated, i.e., it is taken literal.

```
gap> r := rec( a := 1, b := 2 );;
gap> r.a;
1
gap> r.b;
2
```

`'<rec>.(<name>)'`

This construct is similar to the above construct. The difference is that the second operand <name> is evaluated. It must evaluate to a string or an integer otherwise an error is signalled. The construct then evaluates to the element of the record <rec> whose name is, as a string, equal to <name>.

```
gap> old := rec( a := 1, b := 2 );;
gap> new := rec();
rec(
   )
gap> for i  in RecFields( old )  do
>        new.(i) := old.(i);
>    od;
gap> new;
rec(
  a := 1,
  b := 2 )
```

If <rec> does not evaluate to a record, or if <name> does not evaluate to a string, or if `'<rec>.<name>'` is unbound, an error is signalled. As usual you can leave the break loop with `'quit;'`. On the other hand you can return a result to be used in place of the record element by `'return <expr>;'`.

5.2 Record Assignment

`'<rec>.<name> \:= <obj>;'`

The record assignment assigns the object <obj>, which may be an object of arbitrary type, to the record component with the name <name>, which must be an identifier, of the record <rec>. That means that accessing the element with name <name> of the record <rec> will return <obj> after this assignment. If the record <rec> has no component with the name <name>, the record is automatically extended to make room for the new component.

```
gap> r := rec( a := 1, b := 2 );;
gap> r.a := 10;;  r;
rec(
  a := 10,
  b := 2 )
gap> r.c := 3;;  r;
rec(
  a := 10,
  b := 2,
  c := 3 )
```

The function `'IsBound'` (see "IsBound") can be used to test if a record has a component with a certain name, the function `'Unbind'` (see "Unbind") can be used to remove a component with a certain name again.

Note that assigning to a record changes the record. The ability to change an object is only available for lists and records (see "Identical Records").

`'<rec>.(<name>) := <obj>;'`

This construct is similar to the above construct. The difference is that the second operand <name> is evaluated. It must evaluate to a string or an integer otherwise an error is signalled. The construct then assigns <obj> to the record component of the record <rec> whose name is, as a string, equal to <name>.

If <rec> does not evaluate to a record, <name> does not evaluate to a string, or <obj> is a call to a function that does not return a value, e.g., `'Print'`, an error is signalled. As usual you can leave the break loop with `'quit;'`. On the other hand you can continue the assignment by returning a record in the first case, a string in the second, or an object to be assigned in the third, using `'return <expr>;'`.

5.3 Identical Records

With the record assignment (see "Record Assignment") it is possible to change a record. The ability to change an object is only available for lists and records. This section describes the semantic consequences of this fact.

You may think that in the following example the second assignment changes the integer, and that therefore the above sentence, which claimed that only records and lists can be changed, is wrong.

```
i := 3;
i :=  i + 1;
```

But in this example not the *integer* '3' is changed by adding one to it. Instead the *variable* 'i' is changed by assigning the value of 'i+1', which happens to be '4', to 'i'. The same thing happens in the following example

```
r := rec( a := 1 );
r := rec( a := 1, b := 2 );
```

The second assignment does not change the first record, instead it assigns a new record to the variable 'r'. On the other hand, in the following example the record is changed by the second assignment.

```
r := rec( a := 1 );
r.b := 2;
```

To understand the difference first think of a variable as a name for an object. The important point is that a record can have several names at the same time. An assignment '<var> \:= <record>;' means in this interpretation that <var> is a name for the object <record>. At the end of the following example 'r2' still has the value 'rec(a \:= 1)' as this record has not been changed and nothing else has been assigned to 'r2'.

```
r1 := rec( a := 1 );
r2 := r1;
r1 := rec( a := 1, b := 2 );
```

But after the following example the record for which 'r2' is a name has been changed and thus the value of 'r2' is now 'rec(a \:= 1, b \:= 2)'.

```
r1 := rec( a := 1 );
r2 := r1;
r1.b := 2;
```

We shall say that two records are *identical* if changing one of them by a record assignment also changes the other one. This is slightly incorrect, because if *two* records are identical, there are actually only two names for *one* record. However, the correct usage would be very awkward and would only add to the confusion. Note that two identical records must be equal, because there is only one records with two different names. Thus identity is an equivalence relation that is a refinement of equality.

Let us now consider under which circumstances two records are identical.

If you enter a record literal then the record denoted by this literal is a new record that is not identical to any other record. Thus in the following example 'r1' and 'r2' are not identical, though they are equal of course.

```
r1 := rec( a := 1 );
r2 := rec( a := 1 );
```

Also in the following example, no records in the list `'l'` are identical.

```
l := [];
for i  in [1..10]  do
    l[i] := rec( a := 1 );
od;
```

If you assign a record to a variable no new record is created. Thus the record value of the variable on the left hand side and the record on the right hand side of the assignment are identical. So in the following example `'r1'` and `'r2'` are identical records.

```
r1 := rec( a := 1 );
r2 := r1;
```

If you pass a record as argument, the old record and the argument of the function are identical. Also if you return a record from a function, the old record and the value of the function call are identical. So in the following example `'r1'` and `'r2'` are identical record

```
r1 := rec( a := 1 );
f := function ( r )  return r;  end;
r2 := f( r1 );
```

The functions `'Copy'` and `'ShallowCopy'` (see "Copy" and "ShallowCopy") accept a record and return a new record that is equal to the old record but that is *not* identical to the old record. The difference between `'Copy'` and `'ShallowCopy'` is that in the case of `'ShallowCopy'` the corresponding elements of the new and the old records will be identical, whereas in the case of `'Copy'` they will only be equal. So in the following example `'r1'` and `'r2'` are not identical records.

```
r1 := rec( a := 1 );
r2 := Copy( r1 );
```

If you change a record it keeps its identity. Thus if two records are identical and you change one of them, you also change the other, and they are still identical afterwards. On the other hand, two records that are not identical will never become identical if you change one of them. So in the following example both `'r1'` and `'r2'` are changed, and are still identical.

```
r1 := rec( a := 1 );
r2 := r1;
r1.b := 2;
```

5.4 Comparisons of Records

'<rec1> = <rec2>'

'<rec1> <> <rec2>'

The equality operator '=' returns 'true' if the record <rec1> is equal to the record <rec2> and 'false' otherwise. The inequality operator '<>' returns 'true' if the record <rec1> is not equal to <rec2> and 'false' otherwise.

Usually two records are considered equal, if for each component of one record the other record has a component of the same name with an equal value and vice versa. You can also compare records with other objects, they are of course different, unless the record has a special comparison function (see below) that says otherwise.

```
gap> rec( a := 1, b := 2 ) = rec( b := 2, a := 1 );
true
gap> rec( a := 1, b := 2 ) = rec( a := 2, b := 1 );
false
gap> rec( a := 1 ) = rec( a := 1, b := 2 );
false
gap> rec( a := 1 ) = 1;
false
```

However a record may contain a special 'operations' record that contains a function that is called when this record is an operand of a comparison. The precise mechanism is as follows. If the operand of the equality operator '=' is a record, and if this record has an element with the name 'operations' that is a record, and if this record has an element with the name '=' that is a function, then this function is called with the operands of '=' as arguments, and the value of the operation is the result returned by this function. In this case a record may also be equal to an object of another type if this function says so. It is probably not a good idea to define a comparison function in such a way that the resulting relation is not an equivalence relation, i.e., not reflexive, symmetric, and transitive. Note that there is no corresponding '<>' function, because '<left> <> <right>' is implemented as 'not <left> = <right>'.

The following example shows one piece of the definition of residue classes, using record operations. Of course this is far from a complete implementation. Note that the '=' must be quoted, so that it is taken as an identifier (see "Identifiers").

```
gap> ResidueClassOps := rec( );;
gap> ResidueClassOps.\= := function ( l, r )
>   return (l.modulus = r.modulus)
>     and (l.representative-r.representative) mod l.modulus = 0;
> end;;
gap> ResidueClass := function ( representative, modulus )
>   return rec(
```

```
>     representative := representative,
>     modulus        := modulus,
>     operations     := ResidueClassOps );
> end;;
gap> l := ResidueClass( 13, 23 );;
gap> r := ResidueClass( -10, 23 );;
gap> l = r;
true
gap> l = ResidueClass( 10, 23 );
false
```

'<rec1> < <rec2>'

'<rec1> <= <rec2>'

'<rec1> > <rec2>'

'<rec1> >= <rec2>'

The operators '<', '<=', '>', and '>=' evaluate to 'true' if the record <rec1> is less than, less than or equal to, greater than, and greater than or equal to the record <rec2>, and to 'false' otherwise.

To compare records we imagine that the components of both records are sorted according to their names. Then the records are compared lexicographically with unbound elements considered smaller than anything else. Precisely one record <rec1> is considered less than another record <rec2> if <rec2> has a component with name <name2> and either <rec1> has no component with this name or '<rec1>.<name2> < <rec2>.<name2>' and for each component of <rec1> with name '<name1> < <name2>' <rec2> has a component with this name and '<rec1>.<name1> = <rec2>.<name1>'. Records may also be compared with objects of other types, they are greater than anything else, unless the record has a special comparison function (see below) that says otherwise.

```
gap> rec( a := 1, b := 2 ) < rec( b := 2, a := 1 );
false    # they are equal
gap> rec( a := 1, b := 2 ) < rec( a := 2, b := 0 );
true    # the \verb|'a'| elements are compared first and 1 is less than 2
gap> rec( a := 1 ) < rec( a := 1, b := 2 );
true    # unbound is less than 2
gap> rec( a := 1 ) < rec( a := 0, b := 2 );
false    # the \verb|'a'| elements are compared first and 0 is less than 1
gap> rec( b := 1 ) < rec( b := 0, a := 2 );
true    # the \verb|'a'|-s are compared first and unbound is less than 2
gap> rec( a := 1 ) < 1;
false    # other objects are less than records
```

However a record may contain a special 'operations' record that contains a function that is called when this record is an operand of a comparison. The precise mechanism is as follows. If the operand of the equality operator '<' is a record, and if this record has an element with the name 'operations' that is a record, and if this record has an element with the name `'<'` that is a function, then this function is called with the operands of `'<'` as arguments, and the value of the operation is the result returned by this function. In this case a record may also be smaller than an object of another type if this function says so. It is probably not a good idea to define a comparison function in such a way that the resulting relation is not an ordering relation, i.e., not antisymmetric, and transitive. Note that there are no corresponding `'<='`, `'>'`, and `'>='` functions, since those operations are implemented as `'not <right> < <left>'`, `'<right> < <left>'`, and `'not <left> < <right>'` respectively.

The following example shows one piece of the definition of residue classes, using record operations. Of course this is far from a complete implementation. Note that the `'<'` must be quoted, so that it is taken as an identifier (see "Identifiers").

```
gap> ResidueClassOps := rec( );;
gap> ResidueClassOps.< := function ( l, r )
>   if l.modulus <> r.modulus  then
>     Error("<l> and <r> must have the same modulus");
>   fi;
>   return   l.representative mod l.modulus
>          < r.representative mod r.modulus;
> end;;
gap> ResidueClass := function ( representative, modulus )
>   return rec(
>     representative := representative,
>     modulus        := modulus,
>     operations     := ResidueClassOps );
> end;;
gap> l := ResidueClass( 13, 23 );;
gap> r := ResidueClass( -1, 23 );;
gap> l < r;
true    # 13 is less than 22
gap> l < ResidueClass( 10, 23 );
false    # 10 is less than 13
```

5.5 Operations for Records

Usually no operations are defined for record. However a record may contain a special `'operations'` record that contains functions that are called when this record is an operand of a binary operation. This mechanism is detailed below for the addition.

`'<obj> + <rec>'`, `'<rec> + <obj>'`

If either operand is a record, and if this record contains an element with name `'operations'` that is a record, and if this record in turn contains an element with the name `'+'` that is a function, then this function is called with the two operands as arguments, and the value of the addition is the value returned by that function. If both operands are records with such a function `'<rec>.operations.+'`, then the function of the *right* operand is called. If either operand is a record, but neither operand has such a function `'<rec>.operations.+'`, an error is signalled.

`'<obj>  -  <rec>'`, `'<rec>  -  <obj>'`

`'<obj> * <rec>'`, `'<rec> * <obj>'`

`'<obj>  /  <rec>'`, `'<rec>  /  <obj>'`

`'<obj> mod <rec>'`, `'<rec> mod <obj>'`

`'<obj> ^ <rec>'`, `'<rec> ^ <obj>'`

This is evaluated similar, but the functions must obviously be called `'-'`, `'*'`, `'/'`, `'mod'`, `'^'` respectively.

The following example shows one piece of the definition of a residue classes, using record operations. Of course this is far from a complete implementation. Note that the `'*'` must be quoted, so that it is taken as an identifier (see "Identifiers").

```
gap> ResidueClassOps := rec( );;
gap> ResidueClassOps.* := function ( l, r )
>   if l.modulus <> r.modulus  then
>     Error("<l> and <r> must have the same modulus");
>   fi;
>   return rec(
>     representative := (l.representative * r.representative)
>                       mod l.modulus,
>     modulus        := l.modulus,
>     operations     := ResidueClassOps );
> end;;
gap> ResidueClass := function ( representative, modulus )
>   return rec(
>     representative := representative,
>     modulus        := modulus,
>     operations     := ResidueClassOps );
> end;;
gap> l := ResidueClass( 13, 23 );;
gap> r := ResidueClass( -1, 23 );;
gap> s := l * r;
rec(
  representative := 10,
  modulus := 23,
  operations := rec(
      * := function ( l, r ) ... end ) )
```

5.6 In for Records

`'<element> in <rec>'`

Usually the membership test is only defined for lists. However a record may contain a special 'operations' record, that contains a function that is called when this record is the right operand of the 'in' operator. The precise mechanism is as follows.

If the right operand of the 'in' operator is a record, and if this record contains an element with the name 'operations' that is a record, and if this record in turn contains an element with the name 'in' that is a function, then this function is called with the two operands as arguments, and the value of the membership test is the value returned by that function. The function should of course return 'true' or 'false'.

The following example shows one piece of the definition of residue classes, using record operations. Of course this is far from a complete implementation. Note that the 'in' must be quoted, so that it is taken as an identifier (see "Identifiers").

```
gap> ResidueClassOps := rec( );;
gap> ResidueClassOps.\in := function ( l, r )
>   if IsInt( l )  then
>     return (l - r.representative) mod r.modulus = 0;
>   else
>     return false;
>   fi;
> end;;
gap> ResidueClass := function ( representative, modulus )
>   return rec(
>     representative := representative,
>     modulus        := modulus,
>     operations     := ResidueClassOps );
> end;;
gap> l := ResidueClass( 13, 23 );;
gap> -10 in l;
true
gap> 10 in l;
false
```

5.7 Printing of Records

'Print(<rec>)'

If a record is printed by 'Print' or by the main loop, it is usually printed as record literal, i.e., as a sequence of components, each in the format '<name> \:= <value>', separated by commas and enclosed in 'rec(' and ')'.

```
gap> r := rec();;  r.a := 1;;  r.b := 2;;
gap> r;
rec(
```

```
  a := 1,
  b := 2 )
```

But if the record has an element with the name 'operations' that is a record, and if this record has an element with the name 'Print' that is a function, then this function is called with the record as argument. This function must print whatever the printed representation of the record should look like.

The following example shows one piece of the definition of residue classes, using record operations. Of course this is far from a complete implementation. Note that it is typical for records that mimic group elements to print as a function call that, when evaluated, will create this group element record.

```
gap> ResidueClassOps := rec( );;
gap> ResidueClassOps.Print := function ( r )
>   Print( "ResidueClass( ",
>               r.representative mod r.modulus, ", ",
>               r.modulus, " )" );
> end;;
gap> ResidueClass := function ( representative, modulus )
>   return rec(
>     representative := representative,
>     modulus        := modulus,
>     operations     := ResidueClassOps );
> end;;
gap> l := ResidueClass( 33, 23 );
ResidueClass( 10, 23 )
```

5.8 IsRec

'IsRec(<obj>)'

'IsRec' returns 'true' if the object <obj>, which may be an object of arbitrary type, is a record, and 'false' otherwise. Will signal an error if <obj> is a variable with no assigned value.

```
gap> IsRec( rec( a := 1, b := 2 ) );
true
gap> IsRec( IsRec );
false
```

5.9 IsBound

'IsBound(<rec>.<name>)'
'IsBound(<list>[<n>])'

In the first form 'IsBound' returns 'true' if the record <rec> has a component with the name <name>, which must be an ident and 'false' otherwise. <rec> must evaluate to a record, otherwise an error is signalled.

In the second form 'IsBound' returns 'true' if the list <list> has a element at the position <n>, and 'false' otherwise. <list> must evaluate to a list, otherwise an error is signalled.

```
gap> r := rec( a := 1, b := 2 );;
gap> IsBound( r.a );
true
gap> IsBound( r.c );
false
gap> l := [  , 2, 3,  , 5,  , 7,  ,  ,   , 11 ];;
gap> IsBound( l[7] );
true
gap> IsBound( l[4] );
false
gap> IsBound( l[101] );
false
```

Note that 'IsBound' is special in that it does not evaluate its argument, otherwise it would always signal an error when it is supposed to return 'false'.

5.10 Unbind

'Unbind(<rec>.<name>)'
'Unbind(<list>[<n>])'

In the first form 'Unbind' deletes the component with the name <name> in the record <rec>. That is, after execution of 'Unbind', <rec> no longer has a record component with this name. Note that it is not an error to unbind a nonexisting record component. <rec> must evaluate to a record, otherwise an error is signalled.

In the second form 'Unbind' deletes the element at the position <n> in the list <list>. That is, after execution of 'Unbind', <list> no longer has an assigned value at the position <n>. Note that it is not an error to unbind a nonexisting list element. <list> must evaluate to a list, otherwise an error is signalled.

```
gap> r := rec( a := 1, b := 2 );;
gap> Unbind( r.a );  r;
rec(
  b := 2 )
gap> Unbind( r.c );  r;
rec(
  b := 2 )
```

```
gap> l := [  , 2, 3,  5,  , 7,  ,  ,  , 11 ];;
gap> Unbind( l[3] );  l;
[ , 2,, 5,, 7,,,, 11 ]
gap> Unbind( l[4] );  l;
[ , 2,,,, 7,,,, 11 ]
```

Note that 'Unbind' does not evaluate its argument, otherwise there would be no way for 'Unbind' to tell which component to remove in which record, because it would only receive the value of this component.

5.11 Copy

'Copy(<obj>)'

'Copy' returns a copy <new> of the object <obj>. You may apply 'Copy' to objects of any type, but for objects that are not lists or records 'Copy' simply returns the object itself.

For lists and records the result is a *new* list or record that is *not identical* to any other list or record (see "Identical Lists" and "Identical Records"). This means that you may modify this copy <new> by assignments (see "List Assignment" and "Record Assignment") or by adding elements to it (see "Add" and "Append"), without modifying the original object <obj>.

```
gap> list1 := [ 1, 2, 3 ];;
gap> list2 := Copy( list1 );
[ 1, 2, 3 ]
gap> list2[1] := 0;;  list2;
[ 0, 2, 3 ]
gap> list1;
[ 1, 2, 3 ]
```

That 'Copy' returns the object itself if it is not a list or a record is consistent with this definition, since there is no way to change the original object <obj> by modifying <new>, because in fact there is no way to change the object <new>.

'Copy' basically executes the following code for lists, and similar code for records.

```
new := [];
for i  in [1..Length(obj)]  do
    if IsBound(obj[i])  then
        new[i] := Copy( obj[i] );
    fi;
od;
```

Note that 'Copy' recursively copies all elements of the object <obj>. If you only want to copy the top level use 'ShallowCopy' (see "ShallowCopy").

```
gap> list1 := [ [ 1, 2 ], [ 3, 4 ] ];;
gap> list2 := Copy( list1 );
[ [ 1, 2 ], [ 3, 4 ] ]
gap> list2[1][1] := 0;;  list2;
[ [ 0, 2 ], [ 3, 4 ] ]
gap> list1;
[ [ 1, 2 ], [ 3, 4 ] ]
```

The above code is not entirely correct. If the object <obj> contains a list or record twice this list or record is not copied twice, as would happen with the above definition, but only once. This means that the copy <new> and the object <obj> have exactly the same structure when view as a general graph.

```
gap> sub := [ 1, 2 ];;  list1 := [ sub, sub ];;
gap> list2 := Copy( list1 );
[ [ 1, 2 ], [ 1, 2 ] ]
gap> list2[1][1] := 0;;  list2;
[ [ 0, 2 ], [ 0, 2 ] ]
gap> list1;
[ [ 1, 2 ], [ 1, 2 ] ]
```

5.12 ShallowCopy

'ShallowCopy(<obj>)'

'ShallowCopy' returns a copy of the object <obj>. You may apply 'ShallowCopy' to objects of any type, but for objects that are not lists or records 'ShallowCopy' simply returns the object itself.

For lists and records the result is a *new* list or record that is *not identical* to any other list or record (see "Identical Lists" and "Identical Records"). This means that you may modify this copy <new> by assignments (see "List Assignment" and "Record Assignment") or by adding elements to it (see "Add" and "Append"), without modifying the original object <obj>.

```
gap> list1 := [ 1, 2, 3 ];;
gap> list2 := ShallowCopy( list1 );
[ 1, 2, 3 ]
gap> list2[1] := 0;;  list2;
[ 0, 2, 3 ]
gap> list1;
[ 1, 2, 3 ]
```

That 'ShallowCopy' returns the object itself if it is not a list or a record is consistent with this definition, since there is no way to change the original object <obj> by modifying <new>, because in fact there is no way to change the object <new>.

'ShallowCopy' basically executes the following code for lists, and similar code for records.

```
new := [];
for i  in [1..Length(obj)]  do
    if IsBound(obj[i])  then
        new[i] := obj[i];
    fi;
od;
```

Note that 'ShallowCopy' only copies the top level. The subobjects of the new object <new> are identical to the corresponding subobjects of the object <obj>. If you want to copy recursively use 'Copy' (see ”Copy”).

5.13 RecFields

'RecFields(<rec>)'

'RecFields' returns a list of strings corresponding to the names of the record components of the record <rec>.

```
gap> r := rec( a := 1, b := 2 );;
gap> RecFields( r );
[ "a", "b" ]
```

Note that you cannot use the string result in the ordinary way to access or change a record component. You must use the '<rec>.(<name>)' construct (see ”Accessing Record Elements” and ”Record Assignment”).

Bibliography

[AK99] Vincenzo Acciaro and Jürgen Klüners. Computing Automorphisms of Abelian Number Fields. *Math. Comp.*, 68(227):1179–1186, 1999. 642, 644, 645

[Bai96] Georg Baier. Zum Round 4 Algorithmus. Diplomarbeit, Technische Universität Berlin, 1996. 678, 685

[BH96] Yuri Bilu and Guillaume Hanrot. Solving Thue equations of high degree. *J. Number Th.*, 60:373–392, 1996. 873

[Bli14] H. F. Blichfeldt. A new principle in the geometry of numbers, with some applications. *Trans. Amer. Math. Soc.*, 15:227–235, 1914. 421

[BN96] Johannes Buchmann and Stefan Neis. Algorithms for linear algebra problems over principal ideal rings. *Technical Report*, 1996. 584

[BP91] Wieb Bosma and Michael E. Pohst. Computations with finitely generated modules over Dedekind domains. In Stephen M. Watt, editor, *Proceedings ISSAC'91*, pages 151–156, 1991. 619

[CDO96] Henri Cohen, Francisco Diaz y Diaz, and Michel Olivier. Computing ray class groups, conductors and discriminants. In Henri Cohen, editor, *ANTS II*, volume 1122 of *LNCS*, pages 52–59. Springer-Verlag, 1996. 337, 845

[CDO97] Henri Cohen, Francisco Diaz y Diaz, and Michel Olivier. Computing ray class groups, conductors and discriminants. *Submitted to Math. Comp.*, 1997. 337, 845

[CHM98] John H. Conway, Alexander Hulpke, and John McKay. On transitive permutation groups. *LMS Journal of Computation and Mathematics*, 1:1–8, 1998. 405

[CM94] David Casperson and John McKay. Symmetric Functions, m-Sets, and Galois Groups. *Math. Comp.*, 63:749–757, 1994. 408, 409, 411, 412

[Coh95] Henri Cohen. *A Course in Computational Algebraic Number Theory*. Springer, 1995. 760

[Coh96] Henri Cohen. Hermite and Smith normal form algorithms over Dedekind domains. *Math. Comp.*, 65:1681–1699, 1996. 453, 619, 623

[Dab95] Mario Daberkow. *Über die Bestimmung der ganzen Elemente in Ra-dikalerweiterungen algebraischer Zahlkörper*. Dissertation, Technische Universität Berlin, 1995. 674, 675, 676, 889

[Fie97] Claus Fieker. *Über relative Normgleichungen in algebraischen Zahl-körpern*. Dissertation, Technische Universität Berlin, 1997. 683

[Fin84] Ulrich Fincke. *Ein Ellipsoidverfahren zur Lösung von Normgleichungen in algebraischen Zahlkörpern*. Dissertation, Heinrich-Heine-Universität Düsseldorf, 1984. 683

[Fri97] Carsten Friedrichs. Berechnung relativer Ganzheitsbasen mit dem Round-2-Algorithmus. Diplomarbeit, Technische Universität Berlin, 1997. 637, 678, 685

[Gei97] Katharina Geißler. Zur Berechnung von Galoisgruppen. Diplomarbeit, Technische Universität Berlin, 1997. 392

[GPP93] Istvan Gaál, Attila Pethő, and Michael E. Pohst. On the resolution of index form equations in quartic number fields. *J. Symbolic Comp.*, 16:563–584, 1993. 668

[GPP96] Istvan Gaál, Attila Pethő, and Michael E. Pohst. Simultaneous representation of integers by a pair of ternary quadratic forms – with an application to index form equations in quartic number fields. *J. Number Th.*, 57:90–104, 1996. 668

[GS89] Istvan Gaál and Nicole Schulte. Computing all power integral bases of cubic fields. *Math. Comp.*, 53:689–696, 1989. 668

[Has63] Helmut Hasse. *Zahlentheorie*. Akademie Verlag, Berlin, 1963. 846

[Has70] Helmut Hasse. *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper*. Physica Verlag, Berlin, 1970. 828

[Heß96] Florian Heß. Zur Klassengruppenberechnung in algebraischen Zahlkörpern. Diplomarbeit, Technische Universität Berlin, 1996. 466

[Heß99] Florian Heß. *Zur Divisorenklassengruppenberechnung in globalen Funktionenkörpern*. Dissertation, Technische Universität Berlin, 1999. 61, 62, 63, 66, 94, 103, 107

[Hop98] Andreas Hoppe. *Normal forms over Dedekind domains, efficient implementation in the computer algebra system KANT*. Dissertation, Technische Universität Berlin, 1998. 607, 608, 611, 623

[Klü95] Jürgen Klüners. Über die Berechnung von Teilkörpern algebraischer Zahlkörper. Diplomarbeit, Technische Universität Berlin, 1995. 403, 714

[Klü97] Jürgen Klüners. *Über die Berechnung von Automorphismen und Teilkörpern algebraischer Zahlkörper*. Dissertation, Technische Universität Berlin, 1997. 403, 642, 644, 645, 714

[KP97] Jürgen Klüners and Michael E. Pohst. On computing subfields. *J. Symbolic Comp.*, 24(3):385–397, 1997. 403, 714

[Lan87] Serge Lang. *Elliptic Functions*. Springer-Verlag, 1987. 881

[Maa49] Hans Maass. Über eine neue art von nichtanalytischen automorphen funktionen und die bestimmung dirichletscher reihen durch funktionalgleichungen. *Math. Ann*, 121:141–183, 1949. 302

[Nag69] Trygve Nagell. Sur un type particuliar d'unités algebriqués. *Ark. Mat.*, 8:163–184, 1969. 664, 724

[Pau96] Sebastian Pauli. Zur Berechnung von Strahlklassengruppen. Diplomarbeit, Technische Universität Berlin, 1996. 337, 826, 839, 843, 846

[PP98] Sebastian Pauli and Michael E. Pohst. On the computation of the multiplicative group of residue class rings. *Submitted to Math. Comp.*, 1998. 337, 839, 843

[PZ89] Michael E. Pohst and Hans Zassenhaus. *Algorithmic Algebraic Number Theory*. Cambridge University Press, 1989. 665

[S+97] Martin Schönert et al. GAP 3.4, patchlevel 4. School of Mathematical and Computational Sciences, University of St.Andrews, Scotland, 1997. 392, 405, 407, 413

[Sch90a] Reinhard Schertz. Über die Nenner normierter Teilwerte der Weierstraß schen $\wp$–Funktion. *J. Number Th.*, 34:229–234, 1990. 479

[Sch90b] Reinhard Schertz. Zur expliziten Berechnung von Ganzheitsbasen in Strahlklassenkörpern über imaginär- quadratischen Zahlkörpern. *J. Number Th.*, 34:41–53, 1990. 282, 477, 479

[Sch96] Martin Schörnig. *Untersuchung konstruktiver Probleme in globalen Funktionenkörpern*. Dissertation, Technische Universität Berlin, 1996. 114, 175, 184, 190, 234, 244

[SM85] Leonhard H. Soicher and John McKay. Computing Galois Groups over the rationals. *J. Number Th.*, 20:273–281, 1985. 408, 409, 411, 412

[Sta73] Richard P. Stauduhar. The determination of Galois Groups. *Math. Comp.*, 27:981–996, 1973. 392

[Str84] Roelof J. Stroeker. How to solve a diophantine equation. *Amer. Math. Monthly*, 91:385–392, 1984. 506

[TdW89] Nikos Tzanakis and Benjamin M. M. de Weger. On the practical solution of the thue equation. *J. Number Th.*, 31:99–132, 1989. 506

[Wil] Klaus Wildanger. Computing all integral points on Mordell's equation $y^2 = x^3 + k$ by solving cubic index form equations. Submitted to Acta Arith. 506

[Wil93] Klaus Wildanger. Über Grundeinheitenberechnung in algebraischen Zahlkörpern. Diplomarbeit, Heinrich-Heine-Universität Düsseldorf, 1993. 726, 727, 731

Index

Packages

A

B

C

E

F

G

H

I

M

N

O

P

Q

R

S

T

U

V

W

Z