The Distribution of Values of Artin L-Functions and Applications

vorgelegt von
Diplom-Mathematiker
Hartmut Bauer
aus Berlin

Vom Fachbereich 3 Mathematik
der Technischen Universität Berlin
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften
genehmigte Dissertation

Promotionsausschuss:

Vorsitzender: Prof. Dr. A. Bobenko
Berichter: Prof. Dr. M. E. Pohst
Berichter: Prof. Dr. F. Grunewald (Düsseldorf)
Zusätzlicher Gutachter: Prof. Dr. E. Friedman (Santiago de Chile)

Tag der wissenschaftlichen Aussprache: 15. Dezember 2000

Berlin 2000

D83

Contents

CHAPTER 1

Introduction

Artin L-series were introduced by Artin in his articles "Über eine neue Art von L-Reihen" (1923) and "Zur Theorie der L-Reihen mit allgemeinen Gruppencharakteren" (1930) [**3**]. In the proof of his "reciprocity law" Artin showed that in the case of an Abelian extension of number fields Artin L-functions are just Hecke L-functions. Therefore the theory of those functions did directly apply to Abelian Artin L-series. For example we know that Hecke L-functions with non-trivial character are entire functions. Artin's conjecture states the same for general Artin L-functions with non-trivial character. Since Brauer [**5**] it is known, that these functions have a meromorphic continuation to $\mathbb{C}$ and a functional equation like Hecke L-functions. However it is unknown if they have poles in the critical strip $0 < Re(s) < 1$. Artin's conjecture on the holomorphy of the Artin L-functions has inspired a lot of development in number theory [**21**], namely for example the Langlands' program to find a general reciprocity law for non-Abelian extensions of number fields. For the analogue of those Artin L-functions in the case of function fields Artin's conjecture is known to be true and this fact played a prominent role in Laurent Lafforgue's proof of the Langlands correspondence for function fields [**17**].
We report on the fundamentals of Artin L-series in the next chapter.

The study of the distribution of non-zero values of Riemann's Zeta-function starts with Harald Bohr's [**4**] work. He investigated the value distribution of the Zeta-function for $Re(s) > 1$. An exposition on this subject can be found in Titchmarsh's "The Theory of the Riemann Zeta-Function" [**30**]. The work of Voronin [**14**] extends this investigation to the investigation of the distribution of non-zero values in the strip $1/2 < Re(s) < 1$. He gets a quite new type of theorems, which are called "Universality" theorems in the literature. Generalizations of these theorems to other Dirichlet series exist, for example to Dedekind Zeta-functions [**26**] and to the Lerch Zeta-function [**10**]. Further generalizations are concerned with the joint distribution of non-zero values of Dirichlet L-functions ([**14**], [**11**]) and with the joint distribution of non-zero values of Lerch Zeta-functions [**18**].

Our approach generalizes the theorem on the joint distribution of non-zero values of Dirichlet L-series [**14**] to Artin L-series of an arbitrary normal extension $K/\mathbb{Q}$. It is unconditional, i.e. we do not presuppose Artin's conjecture to be true. It is more than a theorem on the joint distribution of non-zero values, since it states that we may approach jointly n arbitrary non-zero holomorphic functions by n Artin L-functions (Theorem 5.1).

To prove this result we need a mean value theorem. This theorem (Theorem 4.1) does not apply to Artin L-functions, since we do not know if they are holomorphic in the critical strip. However it is valid for Hecke L-functions and Dedekind Zeta-functions (Remark 4.1), because they only possess a limited number of poles. The method we use for this purpose is known as Carlson's method [**30**], and was applied to the k-th power moment of the Riemann Zeta-function.

A theorem of Davenport and Heilbronn [**8**] states that Hurwitz Zeta-functions and Zeta-functions attached to positive definite quadratic forms of discriminant d, such that the class number $h(d)$ is greater than 1, have zeros with $Re(s) > 1$. It was proved by Voronin [**14**], that those functions do have zeros in the strip $1/2 < Re(s) < 1$. We prove (Theorem 6.4) that this is true for every partial zeta-function attached to a class of a ray class group of any algebraic number field, provided that this group has cardinality greater than 1. This especially applies to the class group of a number field with class number greater than 1. The zeta-functions of every class of the class group of a number field have a functional equation like the Riemann Zeta-function [**16**, p.254], and therefore we have found other functions for which "the analogue of the Riemann hypothesis is false" [**30**, p.282]. If we take the generalized Riemann hypothesis for granted, then the sum of all these partial zeta-functions should have no zeros in the strip $1/2 < Re(s) < 1$, although each of its summands has infinitely many zeros. Thus these zeros must be at different places. We recall, that these zeta-functions play a prominent role in class field theory (Hasse [**13**], Stark [**28**]).

It is known, that the Dedekind Zeta-functions of different normal extensions differ. We show to which extend Zeta-functions of different normal extensions are really different (Theorem 6.6). The theorems in the last chapter are applications of Theorem 5.1 on Artin L-functions.

1. Notations

We use the big O-notation (Landau symbol) in the following way: By $f(t) = O(g(t))$ we mean that f is a function with the property $|f(t)| \leq Cg(t)$ for all t. The constant C depends only on f and g. By $f(t) = O_a(g(t))$ we emphasize, that $C > 0$ depends on a. The notation $vol(M)$ for some set $M \subset \mathbb{R}^n$ denotes the Lesbegue-measure of this set, which has volume 1 on the unit cube. $\Gamma(s)$ is the Gamma function [**2**]. The Greek letters Γ and γ are also used for curves in the complex plane or in $\mathbb{R}^n$. $Re(z)$ and $Im(z)$ are the real part and the complex part of $z \in \mathbb{C}$.

If $\alpha \in \mathbb{R}$ is a real number, then $\{\alpha\} := \alpha - [\alpha]$. $[\alpha]$ denotes the largest integer $n \in \mathbb{Z}$ with $n \leq \alpha$. $\gcd(a, b)$ is the greatest common divisor of integers in $\mathbb{Z}$ or of ideals, if defined. For a finite set M we denote by $\#M$ its cardinality. Algebraic number fields [**20**] are denoted by small or big Latin letters k, K, L. The Galois group of a normal extension K/k is denoted by $G(K/k)$. For a finite algebraic extension K/k we denote by $[K : k]$ its relative degree. The trace of an algebraic number α is denoted by $Trace(\alpha)$, its norm by $N(\alpha)$ or $N_{K/k}(\alpha)$, if relative to the subfield k. $\mathcal{O}_k$ denotes the ring of integers of the number field k. Ideals are denoted by $\mathfrak{a}$ or $\mathfrak{b}$. Those letters may also denote the modulus of a class group in the sense of class field theory [**13**]. The norm of an ideal $\mathfrak{a}$ is denoted by $N(\mathfrak{a})$. $\mathbb{P}$ is the set of all rational primes. $\mathbb{P}_k$ is the set of prime ideals of $\mathcal{O}_k$. The exponent k of the exact power p^k dividing a rational integer d, i.e. $\gcd(p^{k+1}, d) = p^k$, will be denoted by $v_p(d)$, i.e. $v_p(d) := k$. 1 denotes also the neutral element of a group. It may as well be used for the identity element of a Galois group G and for the character $\chi : G \to \mathbb{C}$ with $\chi(g) = 1 \in \mathbb{C}$ for all $g \in G$. The group of characters of an Abelian group G is denoted by G^*.

$GL_k(\mathbb{C})$ is the group of all $k \times k$-matrices, which have an inverse. For a matrix A we denote by $Tr(A)$ or $Trace(A)$ its trace and by $\det(A)$ its determinant. The restriction of a map $f : M \to T$ to a subset $U \subset M$ will be denoted by $f_{|U}$, i.e. $f_{|U} : U \to T$.

CHAPTER 2

Fundamentals

1. Linear Representation of Finite Groups and Artin L-Series

By a *class function* on a finite group G we mean a function $f : G \mapsto \mathbb{C}$ such that $f(\tau g \tau^{-1}) = f(g)$ for all $\tau, g \in G$. In other words: The value of a class function depends only on the conjugacy classes of the group.

DEFINITION 1. *Let G be a finite group, $\rho : G \mapsto GL_k(\mathbb{C})$ a group homomorphism. ρ is called a representation of G. Then $\chi : G \longmapsto \mathbb{C}$ with $\chi(g) := Trace(\rho(g))$ is called a character of G. The degree of this character is k.*

Obviously every character is a class function and the degree of a character is equal to $\chi(1)$.
We call this kind of characters also a *non-Abelian character* if we want to distinguish them from the usual *Abelian characters* of Abelian groups.

DEFINITION 2. *An irreducible representation of the group G is a group homomorphism $\rho : G \longrightarrow GL_k(\mathbb{C})$ that can not be decomposed into the direct sum of two representations. An irreducible character is the character of an irreducible representation.*

THEOREM 2.1. [**27**, p.18] *The irreducible characters of a finite group G form an orthonormal basis of the vector space of class functions on G with respect to the scalar product $\big(\chi, \psi\big) := \frac{1}{\#G} \sum_{g \in G} \chi(g)\overline{\psi(g)}$. The dimension of the vector space of class functions is equal to the number of conjugacy classes of G.*

Every character on a group G is the sum of (not necessarily different) irreducible characters of this group.

DEFINITION 3 (induced character).
Let U be a subgroup of the finite group G and χ a character of U. Then for every $g \in G$ we have the induced character of χ defined by

$$\chi^*(g) := \frac{1}{\#U} \sum_{v \in G} \chi(vgv^{-1})$$

where $\chi(a) := 0$ if $a \notin U$.

An induced character is a character in the sense of the above definition of characters.

THEOREM 2.2 (Frobenius reciprocity). [**27**, p.86] *Let U be a subgroup of G. If ψ is a class function on U and ϕ a class function on G, we have (with the scalar product above)*

$$\left(\psi, \phi_{|U}\right)_U = \left(\psi^*, \phi\right)_G.$$

THEOREM 2.3 (Brauer). [**23**, p.544] *Every character on a finite group G is a finite linear combination $\chi = \sum_l n_l \varphi_l^* - \sum_l m_l \psi_l^*$, where φ_l^* and ψ_l^* are induced from characters φ_l, ψ_l of degree 1 of subgroups of G and $n_l, m_l \in \mathbb{Z}^{\geq 0}$.*

Let K be a normal extension of k with Galois group $G(K/k)$. Denote by $I_{\mathfrak{P}}$ the inertia group and by $D_{\mathfrak{P}}$ the decomposition group of the Galois group $G(K/k)$ corresponding to the prime ideal $\mathfrak{P}$ with $\mathfrak{p} \subset \mathfrak{P}$, prime ideal $\mathfrak{p} \subset \mathcal{O}_k$ and $\mathfrak{P} \mid \mathfrak{p}\mathcal{O}_K$ ([**12**, p.33] and [**20**, p.98]).

If $I_{\mathfrak{P}} = \{1\}$, the Frobenius-Automorphism $\sigma := (\mathfrak{P}, K/k) \in D_{\mathfrak{P}}$ is defined by [**20**, p.108]

$$\sigma\alpha \equiv \alpha^{N(\mathfrak{p})} \mod \mathfrak{P}$$

for any $\alpha \in \mathcal{O}_K$ and $\mathfrak{p} \subset \mathfrak{P}$. If we exchange the prime ideal $\mathfrak{P}$ by the prime ideal $\mathfrak{P}'$ with $\mathfrak{p} \subset \mathfrak{P}'$, then the corresponding Frobenius-Automorphisms $(\mathfrak{P}, K/k)$ and $(\mathfrak{P}', K/k)$ are conjugate.

Let ρ be any finite linear representation of $G(K/k)$. Denote the character of ρ by χ. Set $L_{\mathfrak{p}}(s, \chi, K/k) := \det\left(E - N(\mathfrak{p})^{-s}\rho((\mathfrak{P}, K/k))\right)^{-1}$, where E is the unit matrix. Obviously this definition is independent of the choice of the prime ideal $\mathfrak{P}$ with $\mathfrak{p} \subset \mathfrak{P}$. Also it is clear that this definition depends only on χ and not on the specific representation ρ with $\chi(\sigma) = Tr(\rho(\sigma))$, since the value of a determinant is invariant under conjugation.

If $I_{\mathfrak{P}} \neq \{1\}$, then set $V^{I_{\mathfrak{P}}} := \{x \in \mathbb{C}^k \mid \forall_{\tau \in I_{\mathfrak{P}}} : \rho(\tau)(x) = x\}$. Then replace $E - N(\mathfrak{p})^{-s}\rho((\mathfrak{P}, K/k))$ in the definition of $L_{\mathfrak{p}}(s, \chi, K/k)$ by the restriction $E - N(\mathfrak{p})^{-s}\rho((\mathfrak{P}, K/k)) \mid_{V^{I_{\mathfrak{P}}}}$ to the subspace $V^{I_{\mathfrak{P}}}$.

We may write $\sigma_{\mathfrak{p}}$ instead of $(\mathfrak{P}, K/k)$, since class functions and $L_{\mathfrak{p}}(s, \chi, K/k)$ only depend on the conjugacy classes of a given group element. We write $L_p(s, \chi)$ for $L_p(s, \chi, K/\mathbb{Q})$.

Definition 4 (Artin L-Series). [**23**, p.540] *The Artin L-series of a character χ on the group $G(K/k)$ is defined by*

$$L(s, \chi, K/k) := \prod_{\mathfrak{p} \in \mathbb{P}_k} L_{\mathfrak{p}}(s, \chi, K/k) \text{ for all } s \in \mathbb{C} \text{ with } Re(s) > 1.$$

The function $L(s, \chi, K/k)$ has a meromorphic continuation to $\mathbb{C}$.

In [**3**, p.169] Artin defines the Artin L-series by its logarithm:

$$\log L(s, \chi, K/k) = \sum_{\mathfrak{p}^h} \frac{\chi(\mathfrak{p}^h)}{hN(\mathfrak{p})^{hs}} \text{ for } Re(s) > 1.$$

We do not describe the details of this definition. However we remark that the Dirichlet-coefficients $\frac{\chi(\mathfrak{p})}{h}$ of this Dirichlet series $\log L(s, \chi, K/k)$ are dominated by the Dirichlet-coefficients of $\chi(1) \log L(s, 1, K/k)$. According to the next theorem $L(s, 1, K/k)$ is identical with the Dedekind Zeta-function.

We write $L(s, \chi)$ for $L(s, \chi, K/\mathbb{Q})$. An Artin L-series $L(s, \chi, K/k)$ is called *primitive* if χ is an irreducible character of the Galois group of K/k.

Artin's conjecture says, that $L(s, \chi, K/k)$ is an entire function for all irreducible characters $\chi \neq 1$ [**23**, p.547]. However it is unproven until now. Therefore we do not know if Artin L-Series are entire or if they have poles in the critical strip $0 < Re(s) < 1$.

THEOREM 2.4. [**23**, p.544]

1. *$L(s,1,K/k)=\zeta_k(s)$.*
2. *If $k \subset K \subset L$ are Galois extensions of k, and χ is a character of $G(K/k)$, which may be viewed as a character of $G(L/k)$ by applying the restriction map, then* $L(s,\chi,K/k)=L(s,\chi,L/k)$.
3. *Let L/k be a Galois extension and K any subfield with $k \subset K \subset L$. Then for a character χ of $G(L/K)$ we have* $L(s,\chi,L/K)=L(s,\chi^*,L/k)$.
4. *$L(s,\chi+\psi,K/k)=L(s,\chi,K/k)L(s,\psi,K/k)$.*

REMARK 2.1. *All the proofs in the last Theorem are done for the Euler-factors $L_{\mathfrak{p}}(s,\chi,K/k)$. So these statements hold "locally":*

1. $L_{\mathfrak{p}}(s,1,K/k)=\left(1-N(\mathfrak{p})^{-s}\right)^{-1}$.
2. $L_{\mathfrak{p}}(s,\chi,K/k)=L_{\mathfrak{p}}(s,\chi,L/k)$.
3. $\prod_{\mathfrak{p}|\mathfrak{q}\mathcal{O}_K} L_{\mathfrak{p}}(s,\chi,L/K)=L_{\mathfrak{q}}(s,\chi^*,L/k)$.
4. $L_{\mathfrak{p}}(s,\chi+\psi,K/k)=L_{\mathfrak{p}}(s,\chi,K/k)L_{\mathfrak{p}}(s,\psi,K/k)$.

COROLLARY 2.1. [**23**, p.547]
If $k \subset K$ is a finite Galois extension with Galois group $G := G(K/k)$, then

$$\zeta_K(s)=\zeta_k(s)\prod_{\chi\neq 1} L(s,\chi,K/k)^{\chi(1)}.$$

Denote the conjugacy classes of the Galois group $G(K/\mathbb{Q})$ by $C_1,\dots,C_N$. Then

THEOREM 2.5 (Artin). [**3**, p.122] *Denote by $\pi(C_j,x)$ the number of rational primes $p \le x$ with $\sigma_p \in C_j$. Then*

$$\pi(x,C_j)=\frac{h_j}{k}\int_2^x \frac{dt}{\log t}+O(xe^{-a\log^{1/2}x})$$

where a is some positive constant, $k := \#G$ and $h_j := \#C_j$.

For a more effective and also unconditional version, see the article of Lagarias and Odlyzko [**15**].

2. Theorems from Complex Analysis and Hilbert Space Theory

A series $a_n, n \in \mathbb{N}$ of real numbers is called *conditionally convergent*, if $\sum\limits_{n\in\mathbb{N}} |a_n|$ is unbounded and $\sum\limits_{n\in\mathbb{N}} a_n$ converges for an appropriate rearrangement of the terms a_n. The following Theorem generalizes Riemann's Rearrangement Theorem, which states that a series of real numbers is conditionally convergent if and only if it can be rearranged such that its sum converges to an arbitrary preassigned real number.

THEOREM 2.6. [**14**, p.352] *Suppose that a series of vectors $\sum\limits_{n=1}^{\infty} u_n$ in a real Hilbert space $\mathcal{H}$ satisfies $\sum\limits_{n=1}^{\infty} \|u_n\|^2 < \infty$ and for every $e \in \mathcal{H}$ with $e \neq 0$ the series $\sum\limits_{n=1}^{\infty} \langle u_n, e\rangle$ converges conditionally. Then for any $v \in \mathcal{H}$ there is a permutation π of $\mathbb{N}$ such that $\sum\limits_{n=1}^{\infty} u_{\pi(n)} = v$ in the norm of $\mathcal{H}$.*

THEOREM 2.7 (Paley-Wiener). [**24**, p.13] *Let F be an entire function. Then the following statements are equivalent:*

(1) $\displaystyle\int_{-\infty}^{\infty} |F(x)|^2 dx < \infty$ *and* $\limsup\limits_{z\in\mathbb{C}} |F(z)e^{-(\sigma+\epsilon)|z|}| < \infty$ *for every* $\epsilon > 0$

(2) *there is a function* $f \in L^2(-\sigma,\sigma)$ *such that*
$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} f(u)e^{iuz} du$$

This theorem has the following consequence.

COROLLARY 2.2. *Suppose that an entire function $g \not\equiv 0$ has a series expansion $g(z) = \sum\limits_{n=0}^{\infty} \frac{a_n}{n!} z^n$ and the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ is bounded. Then for every $c > 1$ there is an unbounded sequence $\{u_k\}_{k\in\mathbb{N}}$ of positive real numbers such that $|g(u_k)| > \exp(-cu_k)$.*

In other words: The function $g(z)$ is not only bounded by $\exp(cz)$ from above but also in a certain sense from below.

<u>Proof</u>: Suppose the converse. Then we have $|g(u)| < A\exp(-cu)$ for some $A > 0$ and all positive real u. Then

$|g(u)\exp((1+\delta)u)| < A\exp(-\delta u)$ for $\delta := (c-1)/2$. Due to the conditions on the coefficients a_n we have $|g(-u)| < B\exp(u)$ for some $B > 0$ and positive real u. Therefore again $|g(-u)\exp(-(1+\delta)u)| < B\exp(-\delta u)$ and for the maximum C of A and B it follows $|g(u)\exp((1+\delta)u)| < C\exp(-\delta|u|)$ for all $u \in \mathbb{R}$.

Set $F(z) := g(z)\exp((1+\delta)z)$. Then $\limsup_{z\in\mathbb{C}} |F(z)e^{-(2+\delta+\epsilon)|z|}| < \infty$ for every $\epsilon > 0$. We have $|F(u)|^2 < C^2\exp(-2\delta|u|)$. Therefore condition (1) of the preceeding Theorem is satisfied. Then we have a function $f \in L^2(-(\delta+2), \delta+2)$ such that $F(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{iuz}du$. According to Plancherel's Theorem [**24**, p.2] we find that $f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} F(u)e^{-iux}du$ almost everywhere in $\mathbb{R}$. Since $|F(u)| < C\exp(-\delta|u|)$, the function defined by the integral is analytic in a strip near the real line. However the support of $f(x)$ lies inside a compact interval. Therefore the analytic function defined by $\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} F(u)e^{-iux}du$ is zero outside of this interval for real x. Therefore it must be zero everywhere, which is a contradiction to $g \not\equiv 0$. □

THEOREM 2.8 (Markov). [**1**, p.314] *Let P be a polynomial of degree $\leq n$ with real coefficients. Then* $\max_{|x|\leq 1}|P'(x)| \leq n^2 \max_{|x|\leq 1}|P(x)|$.

THEOREM 2.9. [**30**, p.303] *Suppose that $f(z)$ is holomorphic on $|z-z_0| \leq R$. Then for $|z-z_0| \leq R' < R$*

$$|f(z)|^2 \leq \frac{\int\limits_{|z-z_0|\leq R} |f(x+iy)|^2 dxdy}{\pi(R-R')^2}.$$

As an obvious consequence we get:

COROLLARY 2.3. *Suppose that $f_1, \dots, f_N$ are functions continuous on $|z-z_0| \leq R$ and holomorphic for all z with $|z-z_0| < R$. Suppose that for a sequence of holomorphic functions $\{\phi_{l,n}\}_{n\in\mathbb{N}}$ for $1 \leq l \leq N$*

$$\lim_{n\to\infty} \int\limits_{|z-z_0|\leq R} \sum_{l=1}^{N} |\phi_{l,n}(z) - f_l(z)|^2 dxdy = 0.$$

Then for every $\epsilon > 0$ there is a number $n_0 \in \mathbb{N}$ such that for a fixed $R' < R$ and all $n \geq n_0$, $|z-z_0| \leq R'$ and all $1 \leq l \leq N$

$$|f_l(z) - \phi_{l,n}(z)| < \epsilon.$$

Definition 5 (Hardy-space). *The vector space $\mathcal{H}_2$ of functions $f(s)$, which are analytic on the disc $|s| < R$ and with*

$$\lim_{r\to R} \int\limits_{|z|\le r} |f(z)|^2 dxdy \ < \infty$$

is a real Hilbert space with norm

$$\|f\|_2 := \big(\lim_{r\to R} \int\limits_{|z|\le r} |f(z)|^2 dxdy\big)^{1/2}$$

and scalar product

$$\langle f, g\rangle := \lim_{r\to R} Re \int\limits_{|z|\le r} f(z)\overline{g(z)} dxdy.$$

The general theory of such Hilbert spaces is developed in [**9**, p.257].

It is well known, that every function f analytic on $|s| < R$ has an convergent Taylor series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. This series is absolutely convergent and $\limsup\limits_{n\ge 0} |a_n|^{1/n} \le 1/R$. Likewise for $|z| = r < R$ we have $\sum\limits_{n,m=0}^{\infty} |a_n \overline{b}_m z^n \overline{z}^m| = \sum\limits_{n,m=0}^{\infty} |a_n||b_m| r^{n+m} = \sum\limits_{n=0}^{\infty} |a_n| r^n \sum\limits_{m=0}^{\infty} |b_m| r^m < \infty$ for every two functions analytic on $|z| < R$. Therefore

$$\int\limits_{|z|\le r} f(z)\overline{g(z)} dxdy = \sum_{n,m=0}^{\infty} a_n \overline{b}_m \int\limits_{|z|\le r} z^n \overline{z}^m dxdy$$

$$= \sum_{n,m=0}^{\infty} a_n \overline{b}_m \int\limits_0^{2\pi} \int\limits_0^{r} \rho^{n+m+1} e^{i(n-m)\varphi} d\rho d\varphi = \pi \sum_{n=0}^{\infty} a_n \overline{b}_n \frac{r^{2(n+1)}}{(n+1)}.$$

Therefore our space consists just of those functions with

$$\sum_{n=0}^{\infty} |a_n|^2 \frac{R^{2n}}{(n+1)} < \infty$$

and has the scalar product

$$\langle f, g\rangle = \pi R^2 \sum_{n=0}^{\infty} Re(a_n \overline{b}_n) \frac{R^{2n}}{(n+1)}.$$

Theorem 2.10 (Rouché). [**2**] *Let the curve γ be homologous to zero in a domain Ω and such that $n(\gamma, z)$ is either 0 or 1 for any point $z \in \Omega$ not on γ. Suppose that $f(z)$ and $g(z)$ are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ.*

We have $n(\gamma, z) := \frac{1}{2\pi i} \int\limits_{\gamma} \frac{1}{\zeta - z} d\zeta$.

Theorem 2.11. [**29**, p.304.(9.51)] *Suppose that $f(s)$ is regular and for some $A > 0$ and all $\sigma := Re(s) \geq \alpha$ we have $|f(s)| = O\big(|Im(s)|^A\big)$, whereas $\alpha \in \mathbb{R}$ is fixed. Suppose that for $\sigma > \sigma_0$ with some $\sigma_0 \in \mathbb{R}$*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} < \infty \text{ and } f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

If for $\sigma > \alpha$

$$\frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 dt$$

is bounded for $T \longrightarrow \infty$, then for $\sigma > \alpha$

$$\lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}$$

uniformly in every strip $\alpha < \sigma_1 \leq \sigma \leq \sigma_2$.

Lemma 2.1. [**30**, p.151] *Let $f(s) = \sum\limits_{n=1}^{\infty} a_n n^{-s}$ be absolutely convergent for $Re(s) > 1$. Then*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{\delta n} = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \Gamma(w - s) f(w) \delta^{s-w} dw$$

for $\delta > 0, c > 1, c > Re(s)$.

Lemma 2.2. [**30**, p.140] *$\delta > 0$ and $1/2 < \sigma < 1$. Then*

$$\sum_{0 < m < n < \infty} \frac{e^{-(m+n)\delta}}{m^\sigma n^\sigma log(n/m)} = O\big(\delta^{2\sigma - 2} \log \frac{1}{\delta}\big).$$

Theorem 2.12 (Phragmen-Lindelöf). [**16**, p.262] *Let $f(s)$ be holomorphic in the upper part of the strip: $a \leq \sigma \leq b$, and $t \geq t_1 > 0$. Assume that $f(s)$ is $O(e^{t^\alpha})$ with $1 \leq \alpha$, and $t \to \infty$ in this strip, and $f(s)$ is $O(t^M)$ for some real number $M \geq 0$, on the sides of the strip, namely $\sigma = a$ and $\sigma = b$. Then $f(s)$ is $O(t^M)$ in the strip. In particular, if f is bounded on the sides, then f is bounded on the strip.*

We state a consequence of Cauchy's integral formula.

Theorem 2.13. [**2**, p.122] *For any analytic function we have*

$$|f^{(n)}(0)| \leq \frac{n!}{r^n} \max_{|z|=r} |f(z)|$$

if f is continuous on $|z| \leq r$ and analytic in the disc $|z| < r$.

3. Theorems from Number Theory

Let $x \in \mathbb{R}^N, \gamma \subset \mathbb{R}^N$. The notation $x \in \gamma \mod \mathbb{Z}$ means that there is a vector $y \in \mathbb{Z}^N$ such that $x - y \in \gamma$. Fix a real number $\theta_0 \in \mathbb{R}$ and $\epsilon > 0$. We use the notation $|\theta_0 - \theta \mod \mathbb{Z}| < \epsilon$ to denote those numbers $\theta \in \mathbb{R}$ which have a representative number $\theta' \in \mathbb{R}$ such that $|\theta_0 - \theta'| < \epsilon$ and $\theta - \theta' \in \mathbb{Z}$.

Theorem 2.14. [**30**, p.301],[**14**] *Let $\alpha_1, \ldots, \alpha_N$ be real numbers which are $\mathbb{Q}$-linear independent, and let γ be a subregion of the unit cube of $\mathbb{R}^N$ with Jordan volume Γ. Denote by $I_\gamma(T)$ the measure of the set $\{t \mid t \in (0,T)$ and $(\alpha_1 t, \ldots, \alpha_N t) \in \gamma \mod \mathbb{Z}\}$. Then*

$$\lim_{T \longrightarrow \infty} \frac{I_\gamma(T)}{T} = \Gamma$$

A curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^N$ is said to be *uniformly distributed* $\mod \mathbb{Z}$ if for every parallelepiped $\Pi = \prod_{j=1}^{N} [a_j, b_j]$ with $a_j, b_j \in [0,1]$ for $1 \leq j \leq N$

$$\lim_{T \to \infty} \frac{vol\{t \mid t \in (0,T), \gamma(t) \in \Pi \mod \mathbb{Z}\}}{T} = \prod_{j=1}^{N} (b_j - a_j)$$

According to the preceeding Theorem 2.14 the curve $\gamma(t) := (\alpha_1 t, \ldots, \alpha_N t)$ is uniformly distributed.

THEOREM 2.15. [**14**, p.362] *Suppose that the curve* $\gamma(t) = \big(\gamma_1(t), \dots, \gamma_N(t)\big)$ *is uniformly distributed* $\mod \mathbb{Z}$ *and continuous as a function* $\mathbb{R}^{>0} \to \mathbb{R}^N$*. Let the function* F *be Riemann integrable on the unit cube in* $\mathbb{R}^N$*.*

Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\{\gamma_1(t)\}, \dots, \{\gamma_N(t)\})dt = \int_0^1 \cdots \int_0^1 F(x_1, \dots, x_N)dx_1 \cdots dx_N.$$

THEOREM 2.16. [**14**, p.362] *Suppose that* D *is a Jordan measurable and closed subregion of the unite cube in* $\mathbb{R}^N$*.* γ *is a continuous and uniformly distributed* $\mod \mathbb{Z}$ *curve.* Ω *is a family of complex-valued functions, which are uniformly bounded and equicontinuous on* D*.*

Then the following relation holds uniformly with respect to $F \in \Omega$*:*

$$\lim_{T\to\infty} \frac{1}{T} \int_{(0,T)\cap A_D} F(\{\gamma_1(t)\}, \dots, \{\gamma_N(t)\})dt = \int_D F(x_1, \dots, x_N)dx_1 \cdots dx_N$$

where $A_D := \{t \mid \gamma(t) \in D \mod \mathbb{Z}\}$.

CHAPTER 3

Fundamental Lemmata

Denote by $\mathbb{P}$ the set of rational primes.

Definition 6. *Suppose that*

$$F(s) = \prod_{p\in\mathbb{P}} f_p(p^{-s})$$

where $f_p(z)$ *is a rational function and the product converges absolutely for* $Re(s) > 1$.

Then for any finite set $M \subset \mathbb{P}$ *of primes and for any* $\theta \in \mathbb{R}^{\mathbb{P}}$ *we define*

$$F_M(s,\theta) := \prod_{p\in M} f_p(p^{-s}e^{-2\pi i\theta_p}).$$

This definition applies to Artin L-Series defined over $\mathbb{Q}$.
According to Definition 4 we have $L(s,\chi,K/\mathbb{Q}) = \prod_{p\in\mathbb{P}} L_p(s,\chi)$ for $Re(s) > 1$ with $L_p(s,\chi) = \det(E - \rho(\sigma_p)p^{-s} \mid_{V^{I_\mathfrak{P}}})^{-1}$. Then

$$f_p(z) = \det(E - \rho(\sigma_p)z \mid_{V^{I_\mathfrak{P}}}))^{-1}.$$

It is independent of the specific representation ρ of the character χ. Thus $L_M(s,\chi,\theta)$ is well defined for every Artin L-Series $L(s,\chi,K/\mathbb{Q})$ defined over $\mathbb{Q}$.

In the case of Hecke L-series $L(s,\chi)$ we have

$$f_p(p^{-s}) := \prod_{p\in\mathfrak{p}} \big(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\big)^{-1}$$

for the prime ideals $\mathfrak{p}$ lying above $p \in \mathbb{P}$. This is obviously a rational function in the argument p^{-s} since $N(\mathfrak{p}) = p^{f(\mathfrak{p}/p)}$ with $f(\mathfrak{p}/p) \in \mathbb{Z}^{\geq 1}$.

LEMMA 3.1. *Suppose that* $F_1(s),\dots,F_n(s)$ *are analytic functions which are represented by absolutely convergent products*

$$F_l(s) = \prod_{p\in\mathbb{P}} f_{p,l}(p^{-s})$$

for $Re(s) > 1$*,where* $f_{p,l}(z) = 1 + \sum\limits_{m=1}^{\infty} a_{p,l}^{(m)} z^m$ *are rational functions of* z *without poles in the disc* $|z| < 1$*. Set* $a_{d,l} := \prod\limits_{p\in\mathbb{P}} a_{p,l}^{(v_p(d))}$*. For all* $\epsilon > 0$ *there are constants* $c(\epsilon) > 0$ *with*

$$|a_{d,l}| \leq c(\epsilon) d^{\epsilon}.$$

Further suppose that they have an analytic continuation to the plane $Re(s) > 1 - 1/2k$ *with at most one simple pole at* $s = 1$ *for some* $k \geq 1$*.*

Assume that

$$\frac{1}{T}\int\limits_{-T}^{T} |F_l(\sigma + it)|^2 dt$$

is bounded for $\sigma \in (\alpha, 1)$ *and* $T \in \mathbb{R}^+$*, if* $\alpha \in (1 - \frac{1}{2k}, 1)$ *is fixed.*
Let $M_1 \subset M_2 \subset \dots$ *be finite sets of primes with* $\mathbb{P} = \bigcup_{j=1}^{\infty} M_j$*.*
Suppose $\lim\limits_{j->\infty} F_{l,M_j}(s,\theta_j) = f_l(s)$ *uniformly in* $|s - (1 - \frac{1}{4k})| \leq r < \frac{1}{4k}$ *for fixed* $r > 0$*.*

Then for any $\epsilon > 0$ *there exists a set* $A_\epsilon \subset \mathbb{R}$ *such that for all* $l = 1,\dots,n$ *and all* $t \in A_\epsilon$

$$\max_{|s-(1-\frac{1}{4k})|\leq r-\epsilon} |F_l(s+it) - f_l(s)| < \epsilon$$

and

$$\liminf_{T\longrightarrow\infty} \frac{vol(A_\epsilon \cap (0,T))}{T} > 0.$$

Corollary 3.1. *Let*

$$G_m(s) := \frac{\prod_{b=1}^{b=N_m} F_{m,b}(s)}{\prod_{b=1}^{b=N_m^*} F^*_{m,b}(s)} \text{ for } \; m = 1, \ldots, m_0.$$

Suppose that the functions $F_{m,b}(s), F^*_{m,b}(s)$ satisfy all the conditions of Lemma 3.1 for $m = 1, \ldots, m_0$ and $1 \le b \le N_m$ resp. $1 \le b \le N^*_m$. Assume that $\lim_{j\to\infty} G_{m,M_j}(s,\theta_j) = f_m(s)$ and $\lim_{j\to\infty} F_{m,b,M_j}(s,\theta_j) = f_{m,b}(s)$ uniformly in $|s-(1-(4k)^{-1})| \le r$. Under the further conditions that

$$\max_{m,b,|s|\le r} |f_{m,b}(s)| > 0$$

and

$$f_m(s) = \frac{\prod_{b=1}^{b=N_m} f_{m,b}(s)}{\prod_{b=1}^{b=N_m^*} f^*_{m,b}(s)} \text{ for } |s| \le r$$

we have:
For any $\epsilon > 0$ there is a set $B_\epsilon \subset \mathbb{R}$ such that for all $m = 1, \ldots, m_0$ and all $t \in B_\epsilon$

$$\max_{|s-(1-\frac{1}{4k})|\le r-\epsilon} |G_m(s+it) - f_m(s)| < \epsilon$$

and

$$\liminf_{T\longrightarrow\infty} \frac{vol(B_\epsilon \cap (0,T))}{T} > 0.$$

Proof: (of Lemma 3.1)
Notation: $D_{k,r} := \{s \in \mathbb{C} \mid |s-(1-(4k)^{-1})| \le r\}$.
$\|f(s)\|_r := \max_{s\in D_{k,r}} |f(s)|$.

Basically we follow the proof of Voronin [**14**, p.256]:
$F_{l,M_j}(s,\theta)$ depends continuously on the finite vector $(\theta_p)_{p\in M_j}$. Therefore there exists for all $\epsilon > 0$ a $\delta(\epsilon) > 0$ such that

$$\|F_{l,M_j}(s,\theta^{(1)}) - F_{l,M_j}(s,\theta^{(2)})\|_r \le \epsilon$$

if $|\theta^{(1)}_p - \theta^{(2)}_p| \le \delta$ for all $p \in M_j$.

According to the conditions of the Lemma we have for $j \in \mathbb{N}$ large enough and all l:

$$\|F_{l,M_j}(s,\theta_j) - f_l(s)\|_r < \epsilon$$

Therefore $|\theta_p - \theta_{p,j}| < \delta = \delta(j,\epsilon)$ for all $p \in M_j$ implies:

$$\|F_{l,M_j}(s,\theta) - f_l(s)\|_r < 2\epsilon$$

If $\theta(\tau) := \frac{\tau}{2\pi}\big(\log(p)\big)_{p\in\mathbb{P}}, \tau \in \mathbb{R}$, then by the definition of $F_{l,M}(s,\theta)$ we have the equality $F_{l,M}(s,\theta(\tau)) = F_{l,M}(s+i\tau,0)$. The symbol 0 in $F_{l,M}(s+i\tau,0)$ denotes the zero in the space $\mathbb{R}^{\mathbb{P}}$.

Hence if we have for all $p \in M_j$

$$|\tau\frac{\log p}{2\pi} - \theta_{j,p} \mod \mathbb{Z}| < \delta, \tag{1}$$

then

$$\|F_{l,M_j}(s+i\tau,0) - f_l(s)\|_r < 2\epsilon. \tag{2}$$

Let A_δ be the set of all τ satisfying (1) and $T_0 > 1$.

$$\text{Set } B := \frac{1}{T}\int\limits_{A_\delta\cap[T_0,T]}\int\limits_{D_{k,r}}\sum_{l=1}^{n}|F_l(s+i\tau) - F_{l,M_j}(s+i\tau,0)|^2 d\sigma dt d\tau$$

Set $Q := \mathbb{P}\cap(0,z]$, with $z > p$ for all $p \in M_j$. Then

$$B \leq 2(S_1 + S_2)$$

with

$$S_1 := \frac{1}{T}\int\limits_{A_\delta\cap[T_0,T]}\int\limits_{D_{k,r}}\sum_{l=1}^{n}|F_{l,Q}(s+i\tau,0) - F_{l,M_j}(s+i\tau,0)|^2 d\sigma dt d\tau$$

and

$$S_2 := \frac{1}{T}\int\limits_{A_\delta\cap[T_0,T]}\int\limits_{D_{k,r}}\sum_{l=1}^{n}|F_l(s+i\tau) - F_{l,Q}(s+i\tau,0)|^2 d\sigma dt d\tau.$$

To estimate S_1 notice, that
$|F_{l,Q}(s+i\tau,0) - F_{l,M_j}(s+i\tau,0)| = |F_{l,Q}(s,\theta(\tau)) - F_{l,M_j}(s,\theta(\tau))|$

Since the numbers $\log(p), p \in \mathbb{P}$ are linearly independent over $\mathbb{Q}$, the curve $\gamma(\tau) := \frac{\tau}{2\pi}(\log(p))_{2\leq p\leq z}$ is uniformly distributed $\mod \mathbb{Z}$. (Theorem 2.14).

For fixed z and M_j the family of functions $\{g_s\}_{s\in D_{k,r}}$

$$g_s(\theta) := |F_{l,Q}(s,\theta) - F_{l,M_j}(s,\theta)|^2$$

is uniformly bounded and equicontinuous in $(\theta_p)_{p\le z}$, and it depends only on $(\theta_p)_{p\le z} \mod \mathbb{Z}$.

Therefore because of Theorem 2.16

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{A_\delta\cap(T_0,T)} |F_{l,Q}(s,\theta(\tau)) - F_{l,M_j}(s,\theta(\tau))|^2 d\tau$$
$$= \int_{\mathcal{D}} |F_{l,Q}(s,\theta) - F_{l,M_j}(s,\theta)|^2 d\theta,$$

uniformly in $s \in D_{k,r}$. We have

$\mathcal{D} = \{(\theta_p)_{p\le z} \mid \forall p \in M_j : |\theta_p - \theta_{j,p} \mod \mathbb{Z}| < \delta \text{ and } \forall p \le z : 0 \le \theta_p \le 1\}$.

Because of $F_{l,Q}(s,\theta) = F_{l,M_j}(s,\theta)F_{l,Q\setminus M_j}(s,\theta)$ and equation (2) we have

$$\int_{\mathcal{D}} |F_{l,Q}(s,\theta(\tau)) - F_{l,M_j}(s,\theta(\tau))|^2 d\theta$$
$$\le (\max_l \|f_l\|_r + 2\epsilon)^2 \int\limits_{\mathcal{D}} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta$$

The functions $F_{l,Q\setminus M_j}(s,\theta) - 1$ do not depend on the variables θ_p for $p \in M_j$. So $\big(d\theta := \prod\limits_{\substack{p\in\mathbb{P}\\ p\le z}} d\theta_p,\ d\theta' := \prod\limits_{p\in Q\setminus M_j} d\theta_p$ and $d\theta'' := \prod\limits_{p\in M_j} d\theta_p\ \big)$

$$\int\limits_{\mathcal{D}} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta$$
$$\le \int\limits_{\mathcal{D}\cap(\theta_p)_{p\in M_j}} \Big(\int\limits_{\forall_{p\in Q\setminus M_j}:0\le\theta_p\le 1} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta'\Big) d\theta''$$
$$\le vol(\mathcal{D}) \int\limits_{\forall_{p\in Q\setminus M_j}:0\le\theta_p\le 1} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta'.$$

Since $F_{l,Q\setminus M_j}(s,\theta) = \prod_{p\in Q\setminus M_j} f_{l,p}\big(p^{-s}\exp(-2\pi i\theta_p)\big)$ we get

$$F_{l,Q\setminus M_j}(s,\theta) - 1 = \sum_{m>1} b_m(\theta) m^{-s}$$

where $b_m(\theta) = \prod_{p\in Q\setminus M_j} a_{p,l}^{(v_p(m))} e^{-2\pi i v_p(m)\theta_p}$ and so

$$|F_{l,Q\setminus M_j}(s,\theta) - 1|^2 = \sum_{m>1} |b_m(\theta)|^2 m^{-2Re(s)} + \sum_{m_1\neq m_2} b_{m_1}(\theta)\overline{b_{m_2}(\theta)} m_1^{-s}\overline{m_2^{-s}}.$$

Both series are absolutely convergent. Therefore the integration may be done term by term. Since the values of the $b_m(\theta)$ depend on θ, the integral over the second series is zero. The first series is independent of θ. Therefore

$$\int\limits_{\forall_{p\in Q\setminus M_j}:0\leq\theta_p\leq 1} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta' = \sum_{m>1} |b_m|^2 m^{-2Re(s)}$$

with $|b_m|^2 = \prod_{p\in Q\setminus M_j} |a_{p,l}^{(v_p(m))}|^2$. For an arbitrary small $\epsilon_1 > 0$ one has $|b^{(m)}|^2 \leq c(\epsilon_1) m^{\epsilon_1}$ because of the conditions on $a_{d,l}$ in the Lemma.

Set $\eta := 2r + \frac{1}{2k} - 1$. Then $\eta < 0$ since $r < \frac{1}{4k}$ and $k \geq 1$. Choose numbers $\epsilon_1 > 0$ and $\delta_1 > 0$ such that $\epsilon_2 := \epsilon_1 + \delta_1 + \eta < 0$. If M_j contains all primes smaller than y_j, then

$$\begin{aligned} & \sum_{m>1} |b^{(m)}|^2 m^{-2Re(s)} \\ \leq\ & c(\epsilon_1) \sum_{m>y_j} m^{\epsilon_1 - 2 + \frac{1}{2k} + 2r} \\ =\ & c(\epsilon_1) \sum_{m>y_j} m^{\epsilon_1 - 1 + \delta_1 + \frac{1}{2k} + 2r} m^{-1-\delta_1} \\ \leq\ & c(\epsilon_1) \sum_{m>y_j} m^{\epsilon_2} m^{-(1+\delta_1)} \\ \leq\ & c(\epsilon_1)\zeta(1+\delta_1) y_j^{\epsilon_2}. \end{aligned}$$

We have

$$\sum_{m>1} |b^{(m)}|^2 m^{-2Re(s)} \leq c(\epsilon_1)\zeta(1+\delta_1) y_j^{\epsilon_2}.$$

Then (δ_1 and ϵ_2 are fixed):

$$S_1 \leq n(\max_l \|f_l\|_r + 2\epsilon)^2 vol(\mathcal{D}) c(\epsilon_1)\zeta(1+\delta_1) y_j^{\epsilon_2}.$$

We choose a fixed j large enough (the choice of ϵ_2 and δ_1 depends only on r and k) such that

$$4n(\max_l \|f_l\|_r + 2)^2 c(\epsilon_1)\zeta(1+\delta_1)\frac{1}{\epsilon^2} < y_j^{-\epsilon_2}$$

This is possible since $\bigcup_{j=1}^{\infty} M_j = \mathbb{P}$ and $M_j \subset M_{j+1}$ and because we may choose δ sufficiently small such that $\epsilon = \epsilon(\delta, j) \leq 1$ in $\max_l \|f_l\|_r + 2\epsilon$. Then

$$S_1 < 1/4 \;\; vol(\mathcal{D})\epsilon^2.$$

From now on j is fixed, thus also $vol(\mathcal{D})$. Now we estimate S_2:

$$\begin{aligned} S_2 &= \frac{1}{T} \int\limits_{A_\delta \cap [T_0,T]} \int\limits_{D_{k,r}} \sum_{l=1}^{n} |F_l(s+i\tau) - F_{l,Q}(s+i\tau,0)|^2 d\sigma dt d\tau \\ &= \int\limits_{D_{k,r}} \sum_{l=1}^{n} \frac{1}{T} \int\limits_{A_\delta \cap [T_0,T]} |F_l(s+i\tau) - F_{l,Q}(s+i\tau,0)|^2 d\tau d\sigma dt. \end{aligned}$$

To cancel the pole at $s = 1$ we multiply by $\phi(s) = 1 - 2^{1-s}$. This function has a simple zero at $s = 1$. We get for $1 - \frac{1}{2k} < Re(s) \leq 1 - \frac{1}{4k} + r < 1$:

$0 < a(r) < |\phi(s)| < c(r)$ for some numbers $a(r), c(r) \in \mathbb{R}$.

This implies for $s \in D_{r,k}$:

$$\begin{aligned} & \frac{1}{T}\int\limits_{-T}^{T} |\phi(s+i\tau)F_l(s+i\tau) - \phi(s+i\tau)F_{l,Q}(s+i\tau,0)|^2 d\tau \\ \leq \;& c(r)\frac{1}{T}\int\limits_{-T}^{T} |F_l(s+i\tau) - F_{l,Q}(s+i\tau,0)|^2 d\tau. \end{aligned}$$

Since $\frac{1}{T}\int\limits_{-T}^{T} |F_j(\sigma+it)|^2 dt$ is bounded for $\sigma \in (\alpha, 1)$ with fixed $\alpha > 1 - \frac{1}{2k}$ and $T \in \mathbb{R}^+$, the same applies to the function

$$\phi(s+i\tau)F_l(s+i\tau) - \phi(s+i\tau)F_{l,Q}(s+i\tau,0).$$

So we can apply Theorem 2.11 to get:

$$\lim_{T\longrightarrow\infty}\frac{1}{2T}\int\limits_{-T}^{T}|\phi(s+i\tau)F_l(s+i\tau)-\phi(s+i\tau)F_{l,Q}(s+i\tau,0)|^2d\tau=\sum_{m=1}^{\infty}|c_m|^2m^{-2Re(s)},$$

where $\sum\limits_{m=1}^{\infty}c_m m^{-s}$ $(Re(s)>1)$ is the Dirichlet series of

$$\phi(s)(F_l(s)-F_{l,Q}(s,0)).$$

Therefore we have for z and $T>T(z)$ sufficiently large (remember $Q=\mathbb{P}\cap(0,z]$ and $z>y_j$)

$$\frac{1}{2T}\int\limits_{-T}^{T}|F_l(s+i\tau)-F_{l,Q}(s+i\tau,0)|^2d\tau\leq\frac{1}{8n}vol(\mathcal{D})\epsilon^2,$$

since in this case $c_m=0$ for all m with prime divisors less than z. Then

$$\begin{aligned}S_2 &= \frac{1}{T}\int\limits_{A_\delta\cap[T_0,T]}\int\limits_{D_{k,r}}\sum_{l=1}^{n}|F_l(s+i\tau)-F_{l,Q}(s+i\tau,0)|^2d\sigma dt d\tau\\ &\leq 2\int\limits_{D_{k,r}}\frac{1}{8}vol(\mathcal{D})\epsilon^2 d\sigma dt\\ &\leq \frac{1}{4}vol(\mathcal{D})\epsilon^2.\end{aligned}$$

This gives $(B\leq 2(S_1+S_2))$ for large T and z

$$B=\frac{1}{T}\int\limits_{A_\delta\cap[T_0,T]}\int\limits_{D_{k,r}}\sum_{l=1}^{n}|F_l(s+i\tau)-F_{l,M_j}(s+i\tau,0)|^2d\sigma dt d\tau\leq vol(\mathcal{D})\epsilon^2.$$

Remember that

$\mathcal{D}=\{(\theta_p)_{p\leq z}\mid\forall p\in M_j:|\theta_p-\theta_{j,p}\mod\mathbb{Z}|<\delta$ and $\forall p\leq z:0\leq\theta_p\leq 1\}$

and

$$A_\delta=\{\tau\mid|\tau\frac{\log p}{2\pi}-\theta_{j,p}\mod\mathbb{Z}|<\delta\},$$

where δ depends only on ϵ and j. We get because of Theorem 2.14

$$\lim_{T\to\infty}\frac{vol(A_\delta\cap[T_0,T])}{T}=vol(\mathcal{D})=(2\delta)^{\#M_j}>0. \tag{3}$$

Then for every T sufficiently large there is a set $Y \subset A_\delta \cap [T_0, T]$ with $vol(Y) > \frac{1}{4} vol(\mathcal{D}) T$ and for all $\tau \in Y$:

$$\int\limits_{D_{k,r}} \sum_{l=1}^{n} |F_l(s + i\tau) - F_{l,M_j}(s + i\tau, 0)|^2 d\sigma dt \leq 2\epsilon^2.$$

To see this define $Y := \{\tau \mid |g(\tau)| \leq 2\epsilon^2\} \cap A_\delta \cap [T_0, T]$ with $g(\tau) := \int\limits_{D_{k,r}} \sum\limits_{l=1}^{n} |F_l(s+i\tau) - F_{l,M_j}(s+i\tau, 0)|^2 d\sigma dt$. Denote its complement in $A_\delta \cap [T_0, T]$ by Y^c. Then

$$\frac{2\epsilon^2 vol(Y^c)}{T} \leq \frac{1}{T} \int_{Y^c} |g(\tau)| d\tau \leq \frac{1}{T} \int_{A_\delta \cap [T_0, T]} |g(\tau)| d\tau \leq vol(\mathcal{D}) \epsilon^2.$$

Therefore

$$2 \frac{vol(A_\delta \cap [T_0, T]) - vol(Y)}{T} \leq vol(\mathcal{D})$$

and with equation (3) we conclude $vol(Y) > \frac{1}{4} vol(\mathcal{D}) T$ for large T. Because of the definition of A_δ we have $\|F_{l,M_j}(s + i\tau, 0) - f_l(s)\|_r < 2\epsilon$. This gives

$$(\int\limits_{D_{k,r}} |F_l(s + i\tau) - f_l(s)|^2 d\sigma dt)^{1/2} \leq 4\epsilon.$$

As both functions F_l and f_l are holomorphic in the interior of $D_{k,r}$ for $s = \sigma + it$, continuous on the border of $D_{k,r}$ and ϵ is arbitrary, the Lemma follows from Theorem 2.9.

□

Recall the definition of $L_M(s, \chi, \theta)$ and $L_p(s, \chi, \theta)$ at the beginning of this chapter.

LEMMA 3.2. *Let $\chi_1, \ldots, \chi_n$ be linearly independent non-Abelian characters of $G := G(K/\mathbb{Q})$, where K is a finite normal algebraic extension of $\mathbb{Q}$. Let $k := \#G$ and $0 < r < \frac{1}{4k}$.*
Suppose that $f_1(s), \ldots, f_n(s)$ are analytic for $|s| < r$ and continuous for $|s| \leq r$ and not zero on the disc $|s| \leq r$. Then for every pair $\epsilon > 0$ and $y \in \mathbb{R}^+$ there exists a finite set of primes M containing all primes smaller than y and $\theta \in \mathbb{R}^{\mathbb{P}}$ such that:

$$\max_{j=1}^{n} \max_{|s| \leq r} |L_M(s + 1 - \frac{1}{4k}, \chi_j, \theta) - f_j(s)| < \epsilon.$$

Proof:

Choose $\gamma > 1$ such that $\gamma^2 r < \frac{1}{4k}$ and

$$\forall_j : \max_{|s|\le r} |f_j(s) - f_j(s/\gamma^2)| < \epsilon/2.$$

Because $f_j(s) \neq 0$ we can write

$$f_j(\frac{s}{\gamma^2}) = \exp(g_j(s)) \text{ for some } g_j(s) \text{ analytic in } |s| < \gamma^2 r.$$

Hence it is sufficient to prove the Lemma for the logarithms of the functions.
Remember that the Euler-factors (all but finitely many) of Artin L-series $L(s, \chi_j)$ are defined by $1/\det(E_{k_j} - \rho_j(\sigma_p)p^{-s})$, where σ_p is one of the conjugate Frobenius-Automorphisms over $p \in \mathbb{P}$ and
$\rho_j : G \to GL_{k_j}(\mathbb{C})$ is a representation of G with $\chi_j(\sigma) = \mathrm{trace}(\rho_j(\sigma))$ for $\sigma \in G$.
For the Euler-factors of $L_M(s', \chi_j, \theta)$ we get:

$$\log L_p(s', \chi_j, \theta) = \frac{\mathrm{trace}(\rho_\mathrm{j}(\sigma_\mathrm{p}))\exp(-2\pi \mathrm{i}\theta_\mathrm{p})}{\mathrm{p}^{\mathrm{s}'}} + \sum_{m\ge 2} a_{m,p}p^{-ms'}.$$

The first term is equal to $\frac{\chi_j(\sigma_p)\exp(-2\pi i\theta_p)}{p^{s'}}$. Therefore

$$\log L_M(s', \chi_j, \theta) = \sum_{p\in M} \frac{\chi_j(\sigma_p)e^{(-2\pi i\theta_p)}}{p^{s'}} + \sum_{p\in M}\sum_{m\ge 2} a_{m,p}p^{-ms'}.$$

The second term is a uniformly and absolutely convergent series for all primes in $\mathbb{Q}$, since its coefficients are dominated by the coefficients of $\chi_j(1)\log\zeta(s)$ as remarked on page 9.

We define a real Hilbert space $\mathcal{H}_n^{(R)}$ of vectors of functions holomorphic on the disc $|s| < R$. The scalar product is (always $R' < R$)

$$\langle (h_j)_{j=1}^n, (f_j)_{j=1}^n \rangle := \lim_{R'\to R} Re \int\limits_{|s|\le R'} \sum_{j=1}^n f_j(s)\overline{h_j(s)}d\sigma dt.$$

The functions h_j and f_j, $j = 1, \dots n$ are holomorphic in $|s| < R$ and satisfy (setting $g := h_j$ or $g := f_j$),

$$\lim_{R'\to R} \int\limits_{|s|\le R'} |g(s)|^2 d\sigma dt < \infty.$$

This Hilbert space is n times the product of the Hilbert space $\mathcal{H}_2$ (Def. 5, p.13).

Set $R := \gamma r$ ($\gamma > 1$) and $\eta_p(s) := \left(\frac{\chi_j(\sigma_p)\exp(-2\pi i\theta_p)}{p^{s'}}\right)_{j=1}^{n}$, where $s' = s + 1 - \frac{1}{4k}$ with $|s| \leq R$.

Denote the different conjugacy classes of the group G by $C_1, \ldots, C_N$. Obviously $n \leq N$, since N is the dimension of the vector space of class functions on G.

Denote the different prime classes by $\mathbb{P}_j := \{p \mid \sigma_p \in C_j\}$.
To define θ: In the natural order of each set $\mathbb{P}_j \subset \mathbb{Z}$ denote the primes $p \in \mathbb{P}_j$ by $p_{j,l}$ such that $p_{j,1} < p_{j,2} < p_{j,3} \ldots < p_{j,l} < p_{j,l+1} < \ldots$.
Set $\theta_{p_{j,l}} := \frac{l}{4}$. Thereby θ_p is defined for all but finitely many primes $p \in \mathbb{P}$. For the primes ramified in K set $\theta_p := 0$.

We will use Theorem 2.6 on conditionally convergent series in real Hilbert spaces.

We only need to show that the series $\eta_p, p \in \mathbb{P}$ fulfills the conditions of this Theorem:

$$\sum_{p\in\mathbb{P}} \|\eta_p\|^2 \leq C \sum_{p\in\mathbb{P}} p^{\frac{1}{2k}-2+2R} < \infty \text{ with } C = n \max_{j=1}^{n}\{\chi_j(1)^2\}.$$

(obviously $\frac{1}{2k} - 2 + 2R < -1$)
For e (as in Theorem 2.6) we can choose any $\varphi(s) \in \mathcal{H}_n^R$ with $\|\varphi\| := \langle\varphi, \varphi\rangle^{1/2} = 1$.
Now we have to show that

$$\sum_{p\in\mathbb{P}} \langle\eta_p, \varphi\rangle$$

is conditionally convergent, or equivalently:
$\lim_{p\to\infty} \langle\eta_p, \varphi\rangle = 0$ and there exist two sets of primes $\mathbb{P}_+$ and $\mathbb{P}_-$ such that
$\forall_{p\in\mathbb{P}_+} : \langle\eta_p, \varphi\rangle > 0, \; \sum_{p\in\mathbb{P}_+} \langle\eta_p, \varphi\rangle = \infty$, and

$\forall_{p\in\mathbb{P}_-} : \langle\eta_p, \varphi\rangle < 0, \; \sum_{p\in\mathbb{P}_-} \langle\eta_p, \varphi\rangle = -\infty$.

We compute:

$$
\begin{aligned}
\langle \eta_p, \varphi \rangle &= \lim_{R' \to R} Re \int\limits_{|s| \le R'} \sum_{j=1}^{n} \eta_{p,j}(s) \overline{\varphi_j(s)} d\sigma dt \\
&= \lim_{R' \to R} Re \int\limits_{|s| \le R'} \sum_{j=1}^{n} \chi_j(\sigma_p) e^{-2\pi i \theta_p} p^{-s'} \overline{\varphi_j(s)} d\sigma dt \\
&= \lim_{R' \to R} Re \big(e^{-2\pi i \theta_p} \int\limits_{|s| \le R'} p^{-(s+1-\frac{1}{4k})} \big(\sum_{j=1}^{n} \chi_j(\sigma_p) \overline{\varphi_j(s)} \big) d\sigma dt \big).
\end{aligned}
$$

It follows that

$$\lim_{p \to \infty} |\langle \eta_p, \varphi \rangle| = 0.$$

Since the characters χ_j are linearly independent and $\varphi \neq 0$, there is a conjugacy class C_l in G such that $\varphi_0(s) := \sum_{j=1}^{n} \chi_j(\sigma_p) \overline{\varphi_j(s)} \not\equiv 0$ for all $\sigma_p \in C_l$.

As the functions φ_j are holomorphic in the disc $|s| < R$, we have

$$\varphi_0(s) = \sum_{m=0}^{\infty} \overline{\alpha_m s^m}.$$

For $p \in C_l$ we get

$$
\begin{aligned}
\langle \eta_p, \varphi \rangle &= \lim_{R' \to R} Re \big(e^{-2\pi i \theta_p} \int\limits_{|s| \le R'} \exp\big(-\log(p)(s+1-\frac{1}{4k}) \big) \varphi_0(s) d\sigma dt \\
&= Re \big(e^{-2\pi i \theta_p} \Delta(\log p) \big).
\end{aligned}
$$

Here $\Delta(x) := \lim_{R' \to R} \int_{|s| \le R'} \exp\big(-x(s+1-\frac{1}{4k}) \big) \varphi_0(s) d\sigma dt$.

Therefore

$$
\begin{aligned}
\Delta(x) &= \exp\big(-x(1-\frac{1}{4k})\big)\lim_{R'\to R}\int\limits_{|s|\le R'}\exp(-xs)\varphi_0(s)d\sigma dt\\
&= \pi R^2\exp\big(-x(1-\frac{1}{4k})\big)\sum_{m=0}^{\infty}\frac{(-1)^m\overline{\alpha}_m(xR^2)^m}{(m+1)!}.
\end{aligned}
$$

We have

$$
\|\varphi_0\|^2 = \lim_{R'\to R}\int\limits_{|s|\le R'}|\varphi_0|^2 d\sigma dt = \pi R^2\sum_{m=0}^{\infty}\frac{|\alpha_m|^2R^{2m}}{m+1}.
$$

Using the continuous linear mapping $L((f_j)_{j=1}^n) := \sum\limits_{j=1}^{n}\overline{\chi_j(C_l)}f_j$ we get

$$
\|\varphi_0\|^2 = \|L(\varphi)\|^2 \le \|L\|^2\|\varphi\|^2 = \|L\|^2 < \infty.
$$

This gives:

$$
\pi R^2\sum_{m=0}^{\infty}\frac{|\alpha_m|^2R^{2m}}{m+1} = \|\varphi_0\|^2 \le \|L\|^2.
$$

Setting $\beta_m := (-1)^m R^m\overline{\alpha}_m/(m+1)$ we get $\sum\limits_{m=0}^{\infty}|\beta_m|^2 \le \|L\|^2/(\pi R^2)$, which gives us an upper bound for all $|\beta_m|$.
Set

$$
F(u) := \sum_{m=0}^{\infty}\frac{\beta_m}{m!}u^m.
$$

$F(u)$ is an entire function and $F\not\equiv 0$ since $\varphi_0\ne 0$. For any $\delta>0$ there is a sequence of positive real numbers with $u_n\longrightarrow\infty$ such that

$$
|F(u_n)| > \exp\big(-(1+2\delta)u_n\big).
$$

This is a consequence of Corollary 2.2. We have
$\Delta(x) = \pi R^2\exp\big(-x(1-\frac{1}{4k})\big)F(xR)$. Set $x_n := u_n/R$. Then

$$
|\Delta(x_n)| > \exp\big(-(1-\delta_0)x_n\big)
$$

for $\delta_0>0$ sufficiently small and x_n sufficiently large.

As a consequence we find subintervals I_n of $[x_n - 1, x_n + 1]$ of length greater than $\frac{1}{2x_n^8}$ in which one of the inequalities

$$
\begin{aligned}
|Re\Delta(x)| &> \frac{e^{-(1-\delta_0)x}}{200} \text{ or} \qquad (4)\\
|Im\Delta(x)| &> \frac{e^{-(1-\delta_0)x}}{200} \qquad (5)
\end{aligned}
$$

holds.

To prove this we approximate Δ by polynomials. Set $N := [x_n] + 1$. Let B be an upper bound for the $|\beta_m|$. This gives $|F(xR)| \leq Be^{xR}$. For $x \in [x_n - 1, x_n + 1]$ we have (remember $R < \gamma^2 r < 1/4k$)

$$
\begin{aligned}
|\sum_{m=N^2}^{\infty} \frac{\beta_m}{m!}(xR)^m| &\leq B \sum_{m=N^2}^{\infty} \frac{1}{m!}(xR)^m \leq B\frac{(xR)^{N^2}}{N^2!} \sum_{m=0}^{\infty} \frac{1}{m!}(xR)^m \\
&\leq B\frac{N^{N^2}}{N^2!}e^N \leq B\big(\frac{N}{N^2/e}\big)^{N^2} e^N \leq B\frac{e^{N^2+N}}{N^{N^2}} \leq e^{-2x_n}
\end{aligned}
$$

if x_n is sufficiently large.
For $x \in [x_n - 1, x_n + 1]$ we also have $\big((1 - \frac{1}{4k}) < 1\ \big)$.

$$
\sum_{N^2=m}^{\infty} \frac{\big(-(1-\frac{1}{4k})x\big)^m}{m!} \leq e^{-2x_n}
$$

Hence $F(xR) = P_1(x) + O(e^{-2x_n})$ and $\exp\left(-(1-\frac{1}{4k})x\right) = P_2(x) + O(e^{-2x_n})$, where P_1 and P_2 are polynomials of degree $N^2 - 1$. This gives $\Delta(x) = P_n(x) + O(e^{-x_n})$ for all $N = [x_n] + 1$ and $x \in [x_n - 1, x_n + 1]$, where $P_n(x)$ is a polynomial of degree less than N^4.

Thus we also have $Re\Delta(x) = Re(P_n(x)) + O(e^{-x_n})$ and $Im\Delta(x) = Im(P_n(x)) + O(e^{-x_n})$. However if $x \in \mathbb{R}$, then $Re(P_n(x))$ and $Im(P_n(x))$ are polynomials with real coefficients.
We may suppose that either $|Re\Delta(x_n)| > \frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$ or $|Im\Delta(x_n)| > \frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$, since $|\Delta(x_n)| > \exp\big(-(1-\delta_0)x_n\big)$.

Suppose that $|Re\Delta(x_n)| > \frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$. Denote the polynomial $Re(P_n(x))$ again by $P_n(x)$. Since $|Re\Delta(x_n)| > \frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$ we have $\frac{1}{4}e^{-(1-\delta_0)x_n} \leq |P_n(x_n)|$ for large n. Set $a := \max_{|x-x_n|\leq 1} |P_n(x)|$. Then there exists a $\xi \in [x_n - 1, x_n + 1]$ such that $a = |P_n(\xi)|$. There exists a $\kappa \in (\xi, x)$ or $\kappa \in (x, \xi)$ such that
$|P_n(\xi) - P_n(x)| = |P_n'(\kappa)(x - \xi)|$. Set $\tau := N^8|\xi - x|$. Then because of Theorem 2.8 we have $|P_n(\xi) - P_n(x)| \leq \tau a$. If $\tau \leq 1/2$ then

$|1-\frac{P_n(x)}{P_n(\xi)}|\leq 1/2$, therefore $|P_n(x)|\geq \frac{a}{2}\geq \frac{|P_n(x_n)|}{2}\geq \frac{1}{8}e^{-(1-\delta_0)x_n}$ for all x with $|x-\xi|\leq \frac{1}{2N^8}$. It follows that

$$|Re\Delta(x)|\geq \frac{1}{16}e^{-(1-\delta_0)x_n}\geq \frac{1}{16e^2}e^{-(1-\delta_0)x}\geq \frac{1}{200}e^{-(1-\delta_0)x}$$

for large n and $|x-\xi|\leq \frac{1}{2N^8}$.

The same argumentation applies to $Im\Delta(x)$ if
$|Im\Delta(x_n)|>\frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$.

In the natural order of the set $\mathbb{P}_l$ we have for $p_r\in\mathbb{P}_l$, and $p_1<p_2<\ldots<p_r<\ldots$ that $\theta_{p_r}=r/4$ by the definition of θ. Thus we get $e^{-2\pi i\theta_{p_r}}=(-i)^r$. Therefore

$$\langle\eta_{p_r},\varphi\rangle = Re\big((-i)^r\Delta(\log(p_r))\big).$$

One of the inequalities (4), (5) is satisfied infinitely often. Consider the interval $I_n:=[\alpha,\alpha+\beta]$ such that on I_n one of the inequalities
$|Im(\Delta(x))|\geq \frac{1}{200}e^{-(1-\delta_0)x}$ or $|Re(\Delta(x))|\geq \frac{1}{200}e^{-(1-\delta_0)x}$ holds and
$\beta\geq \frac{1}{2x_n^8}$.

According to Theorem 2.5 the number of primes $p\in\mathbb{P}_l$ for which $\log p\in I_n$ is ($h_l:=\#C_l$):

$$\begin{aligned}\pi(e^{\alpha+\beta},C_l)-\pi(e^{\alpha},C_l) &= \frac{h_l}{k}\int\limits_{e^{\alpha}}^{e^{\alpha+\beta}}\frac{dt}{\log t}+O(e^{\alpha+\beta}e^{-a\alpha^{1/2}})\\ &\geq \frac{h_l}{k}e^{\alpha}\Bigg(\frac{e^{\beta}-1}{\alpha+\beta}+O(\frac{e^{\beta}}{e^{a\alpha^{1/2}}})\Bigg).\end{aligned}$$

Since $2\geq\beta\geq\frac{1}{2x_n^8}$, we get $e^{\beta}-1\geq\frac{1}{2x_n^8}$ and $\frac{e^{\beta}-1}{\alpha+\beta}\geq\frac{e^{\beta}-1}{x_n+2}\geq\frac{1}{2x_n^9+4x_n^8}$. Next $\frac{e^{\beta}}{e^{a\alpha^{1/2}}}\leq\frac{e^2}{e^{a\sqrt{x_n-1}}}$ and $e^{\alpha}\geq e^{x_n}/e$. Thus for x_n sufficiently large we get

$$\pi(e^{\alpha+\beta},C_l)-\pi(e^{\alpha},C_l)\geq\frac{h_l}{k}\frac{e^{x_n}}{x_n^{10}}.$$

The number of primes p with $\log p\in I_n$ and $\exp(-2\pi i\theta_p)=1$, $\exp(-2\pi i\theta_p)=-1, \exp(-2\pi i\theta_p)=i$, or $\exp(-2\pi i\theta_p)=-i$ is therefore greater than $\frac{h_l}{k}\frac{e^{x_n}}{4x_n^{10}}$.

Therefore

$$\sum_{\substack{p\in\mathbb{P}_l,\log p\in I_n\\ \mathrm{Re}(e^{-2\pi i\theta_p}\Delta(\log p))>0}}\langle\eta_p,\varphi\rangle > c_1e^{\delta_0x_n/2}$$

for some positive constant c_1. The same holds for a subset of primes with $\mathrm{Re}(e^{-2\pi i\theta_p}\Delta(\log p)) < 0$. The sum is less than $-c_1 e^{\delta_0 x_n/2}$.
As $x_n \to \infty$ the corresponding series diverge to $+\infty$ and $-\infty$.

The rest of the proof is a consequence of Corollary 2.3:
$R/\gamma = r < R$. According to Theorem 2.6 we can order $\mathbb{P}$ such that we get a sequence of finite subsets $M_n \subset \mathbb{P}$ with $M_n \subset M_{n+1}$, $\bigcup_{n\in\mathbb{N}} M_n = \mathbb{P}$ and uniformly in $|s| \leq r$ $\lim_{n\to\infty} \log L_{M_n}(z, \chi_j, \theta) = g_j(s)$ for $z = s+1-\frac{1}{4k}$.
Therefore

$$|f_j(s)-L_M(s+1-\frac{1}{4k}, \chi_j, \theta)| \leq |f_j(s)-f_j(s/\gamma^2))|+|e^{g_j(s)}-L_M(z, \chi_j, \theta)| < \epsilon$$

for some $n \in \mathbb{N}$ sufficiently large, $|s| \leq r$ and $M := M_n$. Because of $\bigcup_{n\in\mathbb{N}} M_n = \mathbb{P}$ we may choose $n \in \mathbb{N}$ such that all primes less than a given $y \in \mathbb{R}_+$ are contained in $\mathbb{P}$. $\square$

REMARK 3.1. *In the preceeding Lemma we may replace the set $\mathbb{P}$ by $\mathbb{P} \setminus \{p_1, \ldots, p_d\}$, where $p_1, \ldots, p_d$ are primes. The set M may be replaced by a finite set of primes $M \subset \mathbb{P} \setminus \{p_1, \ldots, p_d\}$ containing all primes smaller than y. Also we may replace for a finite number of primes the factors $L_p(s, \chi)$ by different Euler-factors satisfying the conditions of Lemma 3.1 and its Corollary 3.1.*

The proof of the Remark is obvious because of the proof of the Lemma, since it was proved that the series η_p is conditionally convergent and this persists if we only change a finite number of the η_p.

CHAPTER 4

A Mean Value Theorem

THEOREM 4.1. *Assume that a Dirichlet series* $\sum\limits_{n=1}^{\infty} a_n n^{-s}$ *satisfies* $a_n = O_\epsilon(n^\epsilon)$ *for every* $\epsilon > 0$. *Suppose that this series converges for* $Re(s) > 1$ *absolutely and can be analytically continued to the complex plane and has no pole except a simple pole at* $s = 1$. *Denote this function by* $f(s)$. *Suppose further that* $|f(s)|^2 = O(|t|^M)$ *for some* $M := M(a_0, b_0) \in \mathbb{R}$ *and* $s = \sigma + it$ *where* $|t| \geq 1$ *and* $\sigma \in [a_0, b_0]$ *with* $a_0, b_0 \in \mathbb{R}$ *and* $a_0 < 0$, $b_0 > 1$. *Then*

$$\frac{1}{T}\int\limits_{-T}^{T} |f(s+it)|^2 dt$$

is bounded for every s *with* $Re(s) > \max\{1-\frac{1}{M+1}, 1/2\}$. *We can choose* $M = \inf\{m : |f(s)|^2 = O(|t|^m)\}$.

<u>Proof</u>: Obviously there is a $\xi > 0$ such that

$$\frac{1}{T}\int\limits_{-T}^{T} |f(s+it)|^2 dt = O(T^\xi)$$

(take for example $\xi := M$).

Set $\mu := \inf\{M : |f(s)|^2 = O(|t|^M)\}$.

Using Lemma 2.1, we get for $Re(s) > 1$, $(\delta > 0, c > 1, c > \sigma)$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-\delta n} = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \Gamma(w-s) f(w) \delta^{s-w} dw.$$

Because of the condition $a_n = O_\epsilon(n^\epsilon)$ the series on the left side of the equation is absolutely convergent for all $Re(s) > 0$ and therefore it is

a holomorphic function in this plane. Using Stirling's formula on the Γ-function we get $|\Gamma(s)| \leq C_{[a,b]}|t|^{\sigma-1/2}\exp(-\frac{\pi}{2}|t|)$, where $s = \sigma + it$ and $\sigma \in [a,b]$ for every interval $[a,b]$.
Therefore and because of $|f(s)|^2 = O(|t|^M)$ the function $\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw$ is an analytic function for all $c > 0$ and $b_0 > Re(s) > 0$. If $\sigma > \alpha > \sigma - 1$, we have

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw =$$
$$\frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw + f(s) + \mathrm{Res}_{w=1}\Gamma(w-s)f(w)\delta^{s-w}$$

because of the Residue Theorem. Set $B := \mathrm{Res}_{s=1}f(s)$. Then we find for f the expression

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}e^{-\delta n} - \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw - B\Gamma(1-s)\delta^{s-1},$$

where $Re(s) \geq 1/2$, $\sigma > \alpha > \sigma - 1$.

Set $Z_1 := \sum\limits_{n=1}^{\infty}\frac{a_n}{n^s}e^{-\delta n}$ and $Z_2 := \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw$.

We have $Z_3 := B\Gamma(1-s)\delta^{s-1} = O(|t|^{1-\sigma-1/2}e^{-\frac{\pi}{2}|t|}\delta^{\sigma-1})$. This implies $B\Gamma(1-s)\delta^{s-1} = O(\delta^{\sigma-1}e^{-\frac{\pi}{2}|t|})$, if $|t| \geq 1$ and $1/2 \leq \sigma \leq 1$.
For $x, y \in \mathbb{C}$ we have $|x+y|^2 \leq 2(|x|^2 + |y|^2)$, therefore $|Z_1 + Z_2 + Z_3|^2 \leq 4(|Z_1|^2 + |Z_2|^2 + |Z_3|^2)$.
If $\sigma \geq a > 1/2$, then using Lemma 2.2, we get

$$\begin{aligned}\int\limits_{T/2}^{T}|Z_1|^2dt &= O(T\sum_{m=1}^{\infty}\frac{|a_m|^2}{m^{2a}}e^{-2\delta m}) + O\Big(\sum_{m\neq n}\frac{|a_m||a_n|e^{-(m+n)\delta}}{m^\sigma n^\sigma|\log(m/n)|}\Big)\\ &= O_a(T) + O(\delta^{2\sigma-2-\epsilon})\end{aligned}$$

for some small $\epsilon > 0$ $\big($since $a_n = O(n^\epsilon)\big)$.

Set $w := \alpha + iv$. We obtain

$$
\begin{aligned}
|Z_2| &\le \frac{\delta^{\sigma-\alpha}}{2\pi}\int\limits_{-\infty}^{\infty}|\Gamma(w-s)f(s)|dv\\
&\le \frac{\delta^{\sigma-\alpha}}{2\pi}\Big(\int\limits_{-\infty}^{\infty}|\Gamma(w-s)|dv\int\limits_{-\infty}^{\infty}|\Gamma(w-s)f^2(w)|dv\Big)^{1/2}.
\end{aligned}
$$

Since the first integral is just an integral over the Γ-function, it is bounded. Assume $T \ge |t|$ (recall that $s=\sigma+it$). Set $I_T := (-\infty, 2T] \cup [2T, \infty)$:

$$
\int\limits_{I_T}|\Gamma(w-s)f^2(w)|dv = O\Big(\int\limits_{I_T}e^{-\frac{\pi}{2}|v-t|}|v-t|^{-1/2}|v|^M dv\Big) = O\big(e^{-\frac{\pi}{3}T}\big).
$$

Hence

$$
\begin{aligned}
\int\limits_{T/2}^{T}|Z_2|^2dt &= O\Big(\delta^{2\sigma-2\alpha}\frac{T}{2}O(e^{-\frac{\pi}{3}T}) + \delta^{2\sigma-2\alpha}\int\limits_{-2T}^{2T}|f(w)|^2(\int\limits_{T/2}^{T}|\Gamma(w-s)|dt)dv\Big)\\
&= O(\delta^{2\sigma-2\alpha}) + O\Big(\delta^{2\sigma-2\alpha}\int\limits_{-2T}^{2T}|f(w)|^2dv\Big) = O\big(\delta^{2\sigma-2\alpha}T^{1+M}\big).
\end{aligned}
$$

For Z_3 we get

$$
\int\limits_{T/2}^{T}|Z_3|^2dt = O\Bigg(\delta^{2(\sigma-1)}\int\limits_{T/2}^{T}\exp(-\frac{2\pi}{2}|t|)dt\Bigg) = O(\delta^{2(\sigma-1)}).
$$

This gives ($M=\mu+\epsilon$):

$$
\int\limits_{T/2}^{T}|f(s)|^2dt = O_a(T) + O(\delta^{2\sigma-2-\epsilon}) + O(\delta^{2\sigma-2\alpha}T^{1+\mu+\epsilon}) + O(\delta^{2(\sigma-1)}).
$$

Set $\delta := T^{-\frac{\gamma}{2}}$ with $\gamma := \frac{\epsilon+\mu}{1-\alpha}$. Then $\gamma > 0$ and $\delta > 0$ is well defined. For $\sigma > \max\{1-\frac{1-\alpha}{\mu+1+\epsilon}, a, 1/2\}$ we get

$\delta^{2(\sigma-2)-\epsilon} = O(T)$, $\delta^{2(\sigma-2)} = O(T)$ and $\delta^{2\sigma-2\alpha}T^{1+\mu+\epsilon} = O(T)$.
Taking the limits $\alpha \to 0$ and $\epsilon \to 0$, we get

$$\int_{T/2}^{T} |f(s)|^2 dt = O_a(T)$$

for $\sigma > \max\{1 - \frac{1}{\mu+1}, a\}$.

Adding up $\int_{T/2}^{T} |f(s)|^2 dt + \int_{T/4}^{T/2} |f(s)|^2 dt + \int_{T/8}^{T/4} |f(s)|^2 dt + \ldots$ gives $\int_{1}^{T} |f(s)|^2 dt = O_a(T)$ and analogously $\int_{-T}^{1} |f(s)|^2 dt = O_a(T)$ for the fixed $a > 1/2$.

Since $a > 1/2$ can be chosen arbitrary, we have
$Re(s) > \max\{1 - \frac{1}{\mu+1}, 1/2\}$ as a sufficient condition for
$\frac{1}{T} \int_{-T}^{T} |f(s+it)|^2 dt$ to be bounded. □

REMARK 4.1. For Hecke L-series over a field k with $\mathbb{Q} \subset k \subset K$, where K is a finite normal extension of $\mathbb{Q}$, the conditions of Theorem 4.1 are satisfied with $M = [K : \mathbb{Q}]$.

Proof: Denote the Dirichlet-coefficients of the Hecke-L-series $L(s,\chi)$ by $a_n(\chi)$ and the Dirichlet coefficients of the Dedekind Zeta-function $\zeta_k(s)$ by a_n. Then we have $|a_n(\chi)| \leq a_n$, where a_n is the number of ideals of norm n in the ring of integers of k. Therefore we have $|a_n| = O_\epsilon(n^\epsilon)$ [**22**, p.152].

Every Hecke L-series satisfies a functional equation.
$\Lambda(s,\chi) := C^s \Gamma(\frac{s+1}{2})^a \Gamma(\frac{s}{2})^{r_1 - a} \Gamma(s)^{r_2} L(s,\chi)$,
where r_1 is the number of real embeddings of k, r_2 the number of complex embeddings of k, a is the number of infinite places of the conductor of χ and $C \in \mathbb{R}^{>0}$ is a constant. Then $r_1 + 2r_2 = [k : \mathbb{Q}] \leq [K : \mathbb{Q}]$. We have $\Lambda(s,\chi) = W\Lambda(1-s,\chi)$, where W is a root of unity. $L(s,\chi)$ is a holomorphic function for all $s \in \mathbb{C}$ if $L(s,\chi) \neq \zeta_k(s)$. If $L(s,\chi) = \zeta_k(s)$ there is a simple pole at $s = 1$.

According to a Theorem of Lavrik [**19**, (p.133: Lemma 2.1)] we have:

$\Lambda(s,\chi) = \frac{c}{s(1-s)} + \sum\limits_{n=1}^{\infty}(a_n f(\frac{C}{n},s) + W\bar{a}_n f(\frac{C}{n},1-s))$, where c is a constant for ζ_k and zero in all other cases.

We have $f(x,s) = \frac{1}{2\pi i}\int\limits_{\delta-i\infty}^{\delta+i\infty} x^z \Gamma(\frac{z+1}{2})^a \Gamma(\frac{z}{2})^{r_1-a}\Gamma(z)^{r_2}\frac{dz}{z-s}$, where $\delta \in \mathbb{R}$ and $\delta > \max\{\mathrm{Re}(s),0\}$. If we take $\delta > \max\{\mathrm{Re}(s)+1,0\}$, then $|f(x,s)| \leq \frac{x^\delta}{2\pi}\int\limits_{-\infty}^{\infty}|\Gamma(\frac{\delta+it+1}{2})|^a|\Gamma(\frac{\delta+it}{2})|^{r_1-a}|\Gamma(\delta+it)|^{r_2}dt = C_\delta x^\delta$.

This means for $Re(s) \in [-1,2]$ that for some constant C'_δ and $\delta > 3$ we have $|\Lambda(s,\chi)| \leq C'_\delta 2\sum\limits_{n\in\mathbb{N}}|a_n(\chi)|\frac{1}{n^\delta}$.

Therefore $|\Lambda(s,\chi)| \leq 2C^4C'_4\zeta_k(4)$. The same holds for ζ_k if we suppose that $|Im(s)|$ is large enough, such that we can ignore $\frac{c}{s(1-s)}$. Because of the well known properties of the Γ-function we therefore get $L(s,\chi) = O(\exp(A|t|)$ and $\zeta_k(s) = O(\exp(A|t|)$ for every fixed strip $Re(s) \in [a,b]$, $Im(s) = t$ and some $A \in \mathbb{R}^{>0}$. To apply the Phragmen-Lindelöf-principle 2.12, we must show that $L(s,\chi) = O(|t|^M)$ on the borders $Re(s) = -\epsilon$ and $Re(s) = 1+\epsilon$ for large $t = Im(s)$ and every fixed small $\epsilon > 0$. This would imply that $L(s,\chi) = O(|t|^M)$ for all $Re(s) \in [-\epsilon, 1+\epsilon]$ and $|Im(s)| = |t| > 1$.

The series $L(s,\chi)$ and $\zeta_k(s)$ converge absolutely for all s with $Re(s) = 1+\epsilon$ and we have $|L(s,\chi)| \leq \zeta_k(1+\epsilon)$ and $|\zeta_k(s)| \leq \zeta_k(1+\epsilon)$. This is an absolute constant independent of $Im(s) = t$. Using the functional equation we find that $|L(s,\chi)| = O_\epsilon\big(g(|t|)\big)$ and $|\zeta_k(s)| = O_\epsilon\big(g(|t|)\big)$ for s with $Re(s) = -\epsilon$, where

$$g(|t|) = \frac{|\Gamma(\frac{1-s+1}{2})^a\Gamma(\frac{1-s}{2})^{r_1-a}\Gamma(1-s)^{r_2}|}{|\Gamma(\frac{s+1}{2})^a\Gamma(\frac{s}{2})^{r_1-a}\Gamma(s)^{r_2}|}.$$

Stirling's formula gives $|\Gamma(s)| = O\big(|t|^{\sigma-1/2}\exp(-\frac{\pi}{2}|t|)\big)$, where the constant in the big O depends only on the interval $\sigma \in [a,b]$ with $s = \sigma+it$. Therefore $g(|t|) = O\big(|t|^{r_1\frac{1-2\sigma}{2}}|t|^{r_2(1-2\sigma)}\big) = O\big(|t|^{(1-2\sigma)[k:\mathbb{Q}]/2}\big)$ follows. We have $Re(s) = -\epsilon$. Thus we get in the strip $\sigma \in [-\epsilon, 1+\epsilon]$ $L(s,\chi) = O(|t|^{M_\epsilon})$ and $\zeta_k(s) = O(|t|^{M_\epsilon})$ with $M_\epsilon = (1+2\epsilon)\frac{[k:\mathbb{Q}]}{2}$. The infimum is obviously $[k:\mathbb{Q}]/2$. $\square$

CHAPTER 5

Main Theorem

We prove the following statement on Artin L-functions over $\mathbb{Q}$:

THEOREM 5.1. *Let K be a finite Galois-extension of $\mathbb{Q}$ and $\chi_1, \dots, \chi_n$ linearly independent characters of the group $G := G(K/\mathbb{Q})$. Let $k := \#G$ and $f_1(s), \dots, f_n(s)$ be holomorphic functions on $|s| < r$ and continuous on $|s| \le r$, where r is a fixed number with $0 < r < \frac{1}{4k}$. Further suppose $f_j(s) \neq 0$ on $|s| \le r$.*

Then for every $\epsilon > 0$ there is a set $A_\epsilon \subset \mathbb{R}$ such that

$$\liminf_{T \longrightarrow \infty} \frac{vol(A_\epsilon \cap (0,T))}{T} > 0$$

and for $j = 1, \dots, n$

$$\forall_{t \in A_\epsilon} \forall_{|s| \le r} : |L(s + 1 - \frac{1}{4k} + it, \chi_j, K/\mathbb{Q}) - f_j(s)| < \epsilon,$$

where $L(z, \chi_j, K/\mathbb{Q})$ denotes the Artin L-function corresponding to the non-Abelian character χ_j.

Proof: The Theorem 2.3 of Brauer states that every character is a finite linear combination $\chi = \sum\limits_l n_l \varphi_l^* - \sum\limits_l m_l \psi_l^*$, where φ_l^*, and ψ_l^* are induced from characters φ_l, ψ_l of degree 1 of subgroups of G. According to Theorem 2.4 we get that $L(z, \chi, K/\mathbb{Q}) = \prod\limits_{l=1}^{m_1} L(z, \varphi_l)^{n_l} \Big/ \prod\limits_{l=1}^{n_1} L(z, \psi_l)^{m_l}$, where the series $L(z, \varphi_l)$ and $L(z, \psi_l)$ are Hecke-L-series over number fields contained in K. These are entire functions with the only exception of the Dedekind ζ-functions which have a simple pole at $z = 1$. Therefore one of the conditions of Lemma 3.1 is satisfied by Remark 4.1: The mean values $\frac{1}{T} \int\limits_{-T}^{T} |f(\sigma + it)|^2 dt$ are bounded even for $\sigma > 1 - \frac{1}{k+1}$, where $k = [K : \mathbb{Q}]$ and $f(z)$ is a Hecke L-function of a number field contained in K. Obviously $1 - \frac{1}{k+1} \le 1 - \frac{1}{2k}$. For the Dirichlet-coefficients

$a_n(\chi)$ of Hecke L-functions we have: $|a_n(\chi)| = O_\epsilon(n^\epsilon)$.
We have to show that the conditions in Corollary 3.1 are fulfilled.

We notice Theorem 2.4 and its Remark. If the characters $\chi_1, \dots, \chi_n$ are not yet a basis of the class functions of G, then add some more characters (for example from the set of irreducible characters of G). Choose additional holomorphic functions f_j, for example constants $\neq 0$, which then satisfy the conditions of Lemma 3.2.
As we now have a basis of class functions, every character χ_l^*, ψ_l^* can be expressed as a linear combination of this basis.

Choose $\gamma > 1$ such that $\gamma^2 r < \frac{1}{4k}$ and $\max\limits_{|s|\leq r} |f_j(s) - f_j(\frac{s}{\gamma^2})| < \epsilon/2$ for $j = 1, \dots, n$. Apply Lemma 3.2 for the functions $f_j(\frac{s}{\gamma^2})$ and $|s| \leq r\gamma$.
Now choose a sequence $\epsilon_m := 1/m$, $y_m := \max M_{m-1} + 1$ $(y_0 := 1)$, $\theta_m = (\theta_{m,p})_{p\in\mathbb{P}} \in \mathbb{R}^{\mathbb{P}}$ and $M_m \subset \mathbb{P}$ such that Lemma 3.2 with $\epsilon = \epsilon_m$, $y = y_m$ and $M = M_m$ is satisfied. $M_m \subset M_{m+1}$ is a consequence.
The series expansion of the logarithm gives

$$L_{M_m}(s + 1 - \frac{1}{4k}, \theta_m, \chi_j) =$$
$$\sum_{p\in M_m} \frac{\chi_j(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}} + \sum_{p\in M_m, \kappa\geq 2} a_p(\chi_j, \theta_m, \kappa)p^{-\kappa(s+1-\frac{1}{4k})}.$$

Then because of $\lim\limits_{m\to\infty} L_{M_m}(s+1-\frac{1}{4k}, \theta_m, \chi_j) = f_j(\frac{s}{\gamma^2})$ uniformly in $|s| \leq r\gamma$ and $f_j(\frac{s}{\gamma^2}) \neq 0$, we get for the logarithms of these functions:

$$\lim_{m\to\infty} \left(\sum_{p\in M_m} \frac{\chi_j(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}} + \sum_{p\in M_m, \kappa\geq 2} a_p(\chi_j, \theta_m, \kappa)p^{-\kappa(s+1-\frac{1}{4k})} \right) = \log f_j(\frac{s}{\gamma^2}),$$

where the second sum represents an absolutely convergent series for all $p \in \mathbb{P}$:

$$\sum_{p\in\mathbb{P}, \kappa\geq 2} |a_p(\chi_j, \theta_m, \kappa)p^{-\kappa(s+1-\frac{1}{4k})}| = \sum_{p\in\mathbb{P}, \kappa\geq 2} |a_p(\chi_j, \kappa)|p^{-\kappa(Re(s)+1-\frac{1}{4k})} < \infty,$$

since $|a_p(\chi_j, \kappa)| \leq \chi_j(1)a_p(1, \kappa) = \frac{\chi_j(1)}{\kappa}$ as remarked on page 9. Therefore

$$\lim_{m\to\infty} \sum_{p\in M_m} \frac{\chi_j(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}}$$

converges uniformly in $|s| \leq r\gamma$ to an analytic function for every χ_j.
For every character $\chi := \chi_j$ we have (Theorem 2.4 and Remark)

$$L_{M_m}(s+1-\frac{1}{4k},\theta_m,\chi)=\frac{\prod_{l=1}^{m_1}L_{M_m}(s+1-\frac{1}{4k},\theta_m,\varphi_l^*)}{\prod_{l=1}^{n_1}L_{M_m}(s+1-\frac{1}{4k},\theta_m,\psi_l^*)}.$$

The last statement uses essentially $L_p(s,\theta,\phi_1+\phi_2)=L_p(s,\theta,\phi_1)L_p(s,\theta,\phi_2)$, which is a consequence of the definitions.

$$\begin{aligned}&\log\left(L_{M_m}(s+1-\frac{1}{4k},\theta_m,\varphi_l^*)\right)\\ =&\sum_{p\in M_m}\frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}}+\sum_{p\in M_m,\kappa\geq 2}a_p(\varphi_j^*,\theta_m,\kappa)p^{-\kappa(s+1-\frac{1}{4k})}.\end{aligned}$$

Since the series $\sum\limits_{p\in\mathbb{P},\kappa\geq 2}a_p(\varphi_j^*,\theta_m,\kappa)p^{-\kappa(s+1-\frac{1}{4k})}$ is absolutely convergent, we only have to show the convergence of

$$\lim_{m\to\infty}\sum_{p\in M_m}\frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}}.$$

However since φ_l^* is a class function on G and $\chi_1,\dots,\chi_k$ is a basis of the class functions we get a linear combination $\varphi_l^*=\sum_{j=1}^k r_{j,l}\chi_j$ and therefore

$$\lim_{m\to\infty}\sum_{p\in M_m}\frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}}=\sum_{j=1}^k r_{j,l}\big(\lim_{m\to\infty}\sum_{p\in M_m}\frac{\chi_j(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}}\big).$$

converges uniformly in $|s|\leq r\gamma$. This proves that

$\lim\limits_{m\to\infty}\log L_{M_m}(s+1-\dfrac{1}{4k},\theta_m,\varphi_l^*)$ and $\lim\limits_{m\to\infty}\log L_{M_m}(s+1-\dfrac{1}{4k},\theta_m,\psi_l^*)$ converge uniformly on $|s|\leq r\gamma$ to some analytic functions $g_{\varphi_l^*}(\frac{s}{\gamma^2}),g_{\psi_l^*}(\frac{s}{\gamma^2})$.
Thus it is clear that the functions $L_{M_m}(s+1-\frac{1}{4k},\theta_m,\varphi_l^*)$ and $L_{M_m}(s+1-\frac{1}{4k},\theta_m,\psi_l^*)$ converge to some holomorphic functions $f_{\varphi_l^*}(\frac{s}{\gamma^2})$ and $f_{\psi_l^*}(\frac{s}{\gamma^2})$ with $f_{\varphi_l^*}\neq 0$ and $f_{\psi_l^*}(s)\neq 0$ on $|s|\leq r\gamma$.
Therefore the conditions in Corollary 3.1 are fulfilled and we find a set A_ϵ such that $\liminf\limits_{T\longrightarrow\infty}\frac{vol(A_\epsilon\cap(0,T))}{T}>0$ and for $|s|\leq r\gamma-\epsilon/2$ and $t\in A_\epsilon$

$$|L(s+1-\frac{1}{4k}+it,\chi_j,K/\mathbb{Q})-f_j(\frac{s}{r\gamma^2})|<\epsilon/2.$$

If $\epsilon>0$ is chosen sufficiently small, such that $r\leq r\gamma-\epsilon/2$, then for $t\in A_\epsilon$, $|s|\leq r$ and $j=1,\dots,n$

$$|L(s+1-\frac{1}{4k}+it,\chi_j,K/\mathbb{Q})-f_j(s)|<\epsilon.$$

□

Remark 5.1. *As in Remark 3.1 we may again replace* $\mathbb{P}$ *by some set* $\mathbb{P} \setminus \{p_1, \dots, p_d\}$ *with primes* $p_1, \dots, p_d$. *Thus the statement of the last Theorem remains true if we replace the Artin L-Series by those series, where the Euler product is just extended over the set* $\mathbb{P} \setminus \{p_1, \dots, p_d\}$. *These Artin L-Series just differ by a a finite product* $\prod_{j=1}^{d} L_{p_j}(s, \chi)$ *from the original ones.*
We may even change a finite number of Euler-factors to get the same result.

This follows from Remark 3.1.

CHAPTER 6

Consequences

1. Artin L-Series

We know from Artin [**3**, p.122], that there is no multiplicative relation of the form $\prod_j L(s,\chi_j)^{c_j} = 1$ between the primitive Artin L-series of a normal extension of $\mathbb{Q}$.

THEOREM 6.1. *Suppose that for a continuous function $f : \mathbb{C}^k \longrightarrow \mathbb{C}$ and the primitive Artin L-series $L(s,\chi_j)$ there is a relation of the form*

$$f(L(s,\chi_1),\dots,L(s,\chi_k)) = 0$$

for all $s \in \mathbb{C}$ where these Artin L-series are defined.
Then we have $f \equiv 0$.

Proof: Suppose that $f \not\equiv 0$. Then there is an open set $U \subset \mathbb{C}^k$ such that $f(z) \neq 0$ for all $z \in U$. Because U is open we may find a point $a \in U$ with $a_j \neq 0$ for $j = 1,\dots,k$. According to Theorem 5.1 there is a complex number $s \in \mathbb{C}$ such that $|L(s,\chi_j) - a_j| < \epsilon$ for every $\epsilon > 0$ and all $j = 1,\dots,k$. So we may suppose that the point $b \in \mathbb{C}^k$ with $b_j := L(s,\chi_j)$ is contained in U, and therefore $f(L(s,\chi_1),\dots,L(s,\chi_k)) \neq 0$, contradicting the assumption of the Theorem. $\square$

THEOREM 6.2. *Let $\chi_1,\dots,\chi_n$ be linearly independent characters of the Galois group of a normal extension $K/\mathbb{Q}$. Then the map $\gamma : \mathbb{R} \to \mathbb{C}^{n(m+1)}$ given by*

$$\gamma(t) = \Big(L(\sigma{+}it,\chi_1), L'(\sigma{+}it,\chi_1),\dots,L^{(m)}(\sigma{+}it,\chi_1),\dots,L^{(m)}(\sigma{+}it,\chi_n)\Big)$$

is everywhere dense in $\mathbb{C}^{n(m+1)}$, if $1 - \frac{1}{2[K:\mathbb{Q}]} < \sigma < 1$. For a given point $a \in \mathbb{C}^{n(m+1)}$ the set of numbers $t \in \mathbb{R}^{>0}$ such that $\|\gamma(t) - a\| < \epsilon$ for some fixed $\epsilon > 0$ is unbounded.

This is a straight forward generalization of Voronin's Theorem [**14**, p.270] on Dirichlet L-functions.

Proof: Let $(a_0(\chi_1),\dots,a_m(\chi_1),a_0(\chi_2),\dots,a_m(\chi_2),\dots,a_m(\chi_n))$ be any point in $\mathbb{C}^{n(m+1)}$. If one of the $a_0(\chi_j)$ is zero, then replace it by some $b_0(\chi_j)\neq 0$ with $|b_0(\chi_j)-a_0(\chi_j)|<\epsilon$ for some small $\epsilon>0$.
Because of Theorem 2.13 we only need to approximate the polynomials $p_{\chi_j}(s):=a_0(\chi_j)+\frac{a_1(\chi_j)}{1!}s+\dots+\frac{a_k(\chi_j)}{k!}s^k$ simultaneously by $L(s+\sigma+it,\chi_j)$ for appropriate values $t\in\mathbb{R}$. This is possible because of Theorem 5.1. Take r small enough such that $p_{\chi_j}(s)\neq 0$ on the disc $|s|\leq r$ and such that $1-\frac{1}{2[K:\mathbb{Q}]}<\sigma+Re(s)<1$ for all $|s|\leq r$. We know by Theorem 5.1 that the set A_ϵ is unbounded since

$$\liminf_{T\longrightarrow\infty}\frac{vol(A_\epsilon\cap(0,T))}{T}>0.$$

□

For general Artin L-functions $L(s,\phi,K/k)$ with normal extension K/k, Galois group $G(K/k)$ and $k\neq\mathbb{Q}$ we do not get a joint "universality" theorem for linearly independent characters like Theorem 5.1. For example let k be a quadratic number field with class number divisible by a prime $p>2$ and H_k its Hilbert class field. Then the irreducible characters of $G(H_k/k)$ are all of degree one and also Abelian. We may get $L(s,\chi,H_k/k)=L(s,\overline{\chi},H_k/k)$ [**3**, p.122/123].

THEOREM 6.3. *Let K/k be an arbitrary normal extension with Galois group $G(K/k)$ and ϕ an arbitrary character on $G(K/k)$. Let L with $K\subset L$ be a normal extension of $\mathbb{Q}$ and set $\kappa:=[L:\mathbb{Q}]$. r with $0<r<\frac{1}{4\kappa}$ is fixed. Let $f(s)$ be any function, which is holomorphic on $|s|<r$, continuous for $|s|\leq r$ and $f(s)\neq 0$ for $|s|\leq r$. Then we get a set $A_\epsilon\subset\mathbb{R}$ such that*

$$\liminf_{T\longrightarrow\infty}\frac{vol(A_\epsilon\cap(0,T))}{T}>0$$

and

$$\forall_{t\in A_\epsilon}\forall_{|s|\leq r}:|L(s+1-\frac{1}{4\kappa}+it,\phi,K/k)-f(s)|<\epsilon$$

for the Artin L-function $L(s,\phi,K/k)$.

Proof: Since $L/\mathbb{Q}$ is a normal extension, the same holds for L/k. Therefore we may use Theorem 2.4 (2). We get $L(s,\phi,K/k)=L(s,\phi,L/k)$. The group $G(L/k)$ is a subgroup of $G(L/\mathbb{Q})$. Because of Theorem 2.4 (3) we get $L(s,\phi,L/k)=L(s,\phi^*,L/\mathbb{Q})$. Since ϕ^* is a character of $G(L/\mathbb{Q})$, we have $\phi^*=\sum_{j=1}^{n}m_j\phi_j$, where the ϕ_j, $j=1,\dots,n$, are the irreducible characters of $G(L/\mathbb{Q})$, $m_j\in\mathbb{Z}^{\geq 0}$ and one $m_j\geq 1$. Let this be m_1, i.e. $m_1\geq 1$. We may apply Theorem 5.1 to $L(s,m_j\phi_j,L/\mathbb{Q})$ for those m_j with $m_j \neq 0$. Set $f_1(s) := f(s)$ and $f_j(s) := 1$ for $2 \leq j \leq n$. According to Theorem 2.4 (4) we get

$$L(s, \phi, K/k) = \prod_{\substack{j=1 \\ m_j \neq 0}}^{n} L(s, m_j\phi_j, L/\mathbb{Q}).$$

Therefore the last theorem is a consequence of Theorem 5.1 applied to the Artin L-functions $L(s, m_j\phi_j, L/\mathbb{Q})$ with $m_j \neq 0$. □

2. Zeros of Zeta-Functions

Davenport and Heilbronn [**8**] showed that the ζ-function of an ideal class of a complex quadratic number field has infinitely many zeros in the region $Re(s) > 1$, provided that this number field has class number greater than 1. Voronin proved that these ζ-functions have infinitely many zeros in the strip $1/2 < Re(s) < 1$ [**14**, p.283]. We generalize this result to arbitrary partial ζ-functions attached to any class group of an arbitrary number field, provided that this class group has cardinality greater than 1.

Suppose that $G := I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ is a class group of an arbitrary number field k in the sense of class field theory ([**13**, I; p.4] or [**12**, p.63]). $I^{(\mathfrak{f})}$ is the group of fractional ideals of $\mathcal{O}_k$ prime to $\mathfrak{f}$. $H_{\mathfrak{f}} \subset I^{(\mathfrak{f})}$ is a subgroup with conductor $\mathfrak{f}$ containing the ray of principal ideals
$S_{\mathfrak{f}} := \{\alpha\mathcal{O}_k \mid \alpha \in k \text{ and } \alpha \equiv 1 \mod {}^*\mathfrak{f}\}$.
We remember the definition of the zeta-function of an ideal class $\mathcal{A} \in G$ [**12**, p.100]:

$$\zeta(s, \mathcal{A}) := \sum_{\substack{\mathfrak{a} \in \mathcal{A} \\ \mathfrak{a} \subset \mathcal{O}_k}} \frac{1}{N(\mathfrak{a})^s}, \text{ where } Re(s) > 1.$$

This function may be continued to the entire complex plane $\mathbb{C}$ and has a simple pole at $s = 1$. We get for Hecke L-functions with the Abelian character χ of G [**12**, p.87]

$$L(s, \chi) = \sum_{\mathcal{A} \in G} \chi(\mathcal{A})\zeta(s, \mathcal{A}),$$

where $\chi(\mathcal{A}) := \chi(\mathfrak{a})$ for some $\mathfrak{a} \in \mathcal{A}$. We may extend the sum defining the Hecke L-function to a sum over all ideals prime to the conductor of χ and, like in the definition of Artin L-series, to all ideals. However since we wish to use the formula $\zeta(s, \mathcal{A}) = \frac{1}{\#G} \sum_{\chi \in G^*} \overline{\chi(\mathfrak{a})} L(s, \chi)$ with

$\mathfrak{a} \in \mathcal{A}$, we must presuppose that the integral ideals in the classes $\mathcal{A}$ are prime to $\mathfrak{f}$.

THEOREM 6.4. *Suppose that k is a number field, $\mathcal{O}_k$ its ring of integers. Let $H_\mathfrak{f}$ be an ideal group with conductor $\mathfrak{f}$, and $I^{(\mathfrak{f})}$ the group of fractional ideals of $\mathcal{O}_k$ prime to $\mathfrak{f}$. If $I^{(\mathfrak{f})}/H_\mathfrak{f}$ contains more than one ideal class, then the partial ζ-function $\zeta(s, \mathcal{A})$ over any of those classes $\mathcal{A} \in I^{(\mathfrak{f})}/H_\mathfrak{f}$ has infinitely many zeros in the strip $1/2 < Re(s) < 1$.*

If $T > 0$ is sufficiently large, then there is a number $c > 0$ such that there are at least cT zeros of $\zeta(s, \mathcal{A})$ in the region with $1/2 < Re(s) < 1$ and $|Im(s)| < T$.

Proof: From class field theory we know that there is a unique Abelian extension L of k with Galois group $G(L/k)$ and a unique isomorphism $I^{(\mathfrak{f})}/H_\mathfrak{f} \longrightarrow G(L/k)$, called the Artin-Isomorphism. Using this isomorphism Artin proved that every Abelian Artin L-series is a Hecke L-series and vice versa [**3**, p.131, p.171].

So we may proceed by proving our theorem on those Artin L-series attached to $G(L/k)$. There is a unique normal extension $K/\mathbb{Q}$ with $L \subset K$. Every irreducible character χ of $G(L/k)$ may be regarded as a character of $G(K/k)$ by applying the restriction map
$\sigma \in G(K/k) \mapsto \sigma_{|L} \in G(L/k)$, i.e. $\sigma \in G(K/k) \mapsto \chi(\sigma_{|L}) \in \mathbb{C}$.
According to Theorem 2.4 (2) we know that $L(s, \chi, K/k) = L(s, \chi, L/k)$.
Further $G(K/k) \subset G(K/\mathbb{Q})$ is a subgroup of $G(K/\mathbb{Q})$. Once again because of Theorem 2.4 (3) we find $L(s, \chi, K/k) = L(s, \chi^*, K/\mathbb{Q})$. The group of all different characters of $I^{(\mathfrak{f})}/H_\mathfrak{f}$ are linearly independent, as well as the characters of $G(L/k)$. The same does not necessarily apply to the induced characters χ^* of the group $G(K/\mathbb{Q})$. However we may prove that the dimension of the subspace spanned by these induced characters is larger than 1:

Suppose that $\chi \neq 1$ is an irreducible character of $G(L/k)$, that is an irreducible character of $G(K/k)$ if we apply the restriction map. Because of Theorem 2.2 we know that $\left(\chi^*, 1\right)_{G(K/\mathbb{Q})} = \left(\chi, 1_{|G(K/k)}\right)_{G(K/k)} = \left(\chi, 1\right)_{G(K/k)} = 0$. The last equation is obvious since $\chi \neq 1$ are both irreducible characters of $G(K/k)$ (they both have degree 1). If we denote the irreducible characters of $G(K/\mathbb{Q})$ by $\phi_1 := 1, \phi_2, \ldots, \phi_h$, then for every nontrivial character of $G(L/k)$ we have $\chi^* = \sum_{j=2}^{h} m_j \phi_j$ with $m_j \in \mathbb{Z}^{\geq 0}$. For the induced character 1^* of the trivial character

$1 \in G^*(L/k)$ we get $\left(1^*, 1\right)_{G(K/\mathbb{Q})} = \left(1, 1_{|G(K/k)}\right)_{G(K/k)} = \left(1, 1\right)_{G(K/k)} = 1$. Therefore we have $1^* = \phi_1 + \sum\limits_{j=2}^{h} n_j \phi_j$ with $n_j \in \mathbb{Z}^{\geq 0}$.

So we get

$$L(s, 1) = L(s, 1^*, K/\mathbb{Q}) = L(s, \phi_1, K/\mathbb{Q}) \prod_{j=2}^{h} L(s, \phi_j, K/\mathbb{Q})^{n_j}$$

and for the non-trivial Abelian characters

$$L(s, \chi) = \prod_{j=2}^{h} L(s, \phi_j, K/\mathbb{Q})^{m_j}.$$

Since the irreducible characters ϕ_j are linearly independent, we may apply Theorem 5.1 and Remark 5.1 to $L(s, \phi_1, K/\mathbb{Q})$ and $L(s, \phi_j, K/\mathbb{Q})$ with $2 \leq j \leq h$. Set $\kappa := \#G(K/\mathbb{Q})$. We may therefore find for every $\epsilon_1 > 0$ a set A_{ϵ_1} such that for every $t \in A_{\epsilon_1}$ and for fixed $r < \frac{1}{4\kappa}$ we get $|L(s + it + 1 - \frac{1}{4\kappa}, \phi_j, K/\mathbb{Q}) - 1| < \epsilon_1$ for all $2 \leq j \leq h$ and $|L(s + it + 1 - \frac{1}{4\kappa}, \phi_1, K/\mathbb{Q}) - \big(s - \sum\limits_{\chi \neq 1} \overline{\chi(\mathfrak{a})}\big)| < \epsilon_1$, if $|s| \leq r$. We have $s - \sum\limits_{\chi \neq 1} \overline{\chi(\mathfrak{a})} \neq 0$ for $|s| < 1/2$ since $\sum\limits_{\chi \neq 1} \overline{\chi(\mathfrak{a})} = -1$, if $\mathfrak{a}$ is not in the principal class of $I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ and $\sum\limits_{\chi \neq 1} \overline{\chi(\mathfrak{a})} \geq 1$ if $\mathfrak{a} \in H_{\mathfrak{f}}$ [**12**, p.86].

This gives $|L(s + it + 1 - \frac{1}{4\kappa}, 1) - \big(s - \sum\limits_{\chi \neq 1} \overline{\chi(\mathfrak{a})}\big)| < \epsilon$ and for $\chi \neq 1$ $|L(s + it + 1 - \frac{1}{4\kappa}, \chi) - 1| < \epsilon$ for all $|s| \leq r$ and all $t \in A_\epsilon$ in some set A_ϵ.

Take some integral ideal $\mathfrak{a}$ from the class $\mathcal{A}$. We have $\zeta(s, \mathcal{A}) = \frac{1}{\#G} \sum\limits_{\chi \in G^*} \overline{\chi(\mathfrak{a})} L(s, \chi)$. Therefore $|\zeta(s + it + 1 - \frac{1}{4\kappa}, \mathcal{A}) - \frac{s}{\#G}| < \epsilon$ for all $|s| \leq r$ and all $t \in A_\epsilon$ as a result of the preceeding.

Suppose that $\epsilon < \frac{r}{\#G}$. Then

$$|\zeta(s + it + 1 - \frac{1}{4\kappa}, \mathcal{A}) - \frac{s}{\#G}| < |\frac{s}{\#G}|$$

on the circle $|s| = r$. Inside the disc $|s| < r$ there is exactly one zero of the function $s \mapsto s$. According to Theorem 2.10 we obtain the same number of zeros for the function $\zeta(s + it + 1 - \frac{1}{4\kappa}, \mathcal{A})$ in the disc $|s| < r$

and every fixed $t \in A_\epsilon$. Noting that $c := \liminf_{T \longrightarrow \infty} \frac{vol(A_\epsilon \cap (0,T))}{T} > 0$ we have completed the proof. $\square$

Suppose that a number field k has class number greater than 1. Its signature is r_1, r_2 and its degree $N := [k : \mathbb{Q}]$. Denote its discriminant by D_k and its different by $\mathfrak{D}_k$. Set $Z(s, \mathcal{A}) := \sqrt{\frac{D_k}{4^{r_2}\pi^N}}^{\,s} \Gamma(\frac{s}{2})^{r_1}\Gamma(s)^{r_2}\zeta(s, \mathcal{A})$, where $\mathcal{A}$ is an arbitrary class of the class group of $\mathcal{O}_k$. Denote by $\mathcal{A}'$ the class with the property $\mathcal{A}\mathcal{A}' = \mathfrak{D}_k$ in the class group. The function $Z(s, \mathcal{A})$ has the following well known functional equation: $Z(s, \mathcal{A}) = Z(1 - s, \mathcal{A}')$ [**16**, p.254]. Because of the preceeding theorem this ζ-function has zeros in the strip $\frac{1}{2} < Re(s) < 1$.

3. Dedekind Zeta-Functions and Hecke L-Functions

THEOREM 6.5. *Let $K_1, \ldots, K_r$ be finite normal extensions of $\mathbb{Q}$ with $K_i \cap K_j = \mathbb{Q}$ for $i \neq j$.*
If for a continuous function $f(x_1, \ldots, x_r)$ the equation

$$\forall_{s \in \mathbb{C}\setminus\{0\}} f(\zeta_{K_1}(s), \ldots, \zeta_{K_r}(s)) = 0$$

holds, then

$$f \equiv 0.$$

Proof: We have according to Corollary 2.1

$$\zeta_K(s) = \zeta(s) \prod_{\chi \neq 1} L(s, \chi)^{\chi(1)},$$

where the product is taken over all non-trivial irreducible characters of the Galois group of the normal extension $K/\mathbb{Q}$.
These characters χ and the character $1 = \mathrm{id}_{G(K/\mathbb{Q})}$ are a basis of the class functions on the group $G := G(K/\mathbb{Q})$.

Let K be the smallest field that contains all $K_1, \ldots, K_r$. K is a finite normal extension of $\mathbb{Q}$. The corresponding irreducible characters of $G(K_j/\mathbb{Q})$ may be regarded as characters of $G(K/\mathbb{Q})$ by using the restriction maps $\sigma \in G(K/\mathbb{Q}) \mapsto \sigma_{|K_j} \in G(K_j/\mathbb{Q})$. Since $G(K/\mathbb{Q}) \cong \prod_j^r G(K_j/\mathbb{Q})$ is a direct product, they are linearly independent. Let $a \in \mathbb{C}^r$ be any point for which $f(a_1, \ldots, a_r) \neq 0$, then there is an open subset $U \subset \mathbb{C}^r$ containing a, on which $f(x_1, \ldots, x_r) \neq 0$ for all $x \in U$. Therefore we may suppose that $a_j \neq 0$. According to Theorem 5.1 we find for every $\epsilon > 0$ a value $s \in \mathbb{C}$, such that

$\max_{j=1}^{r} |\zeta_{K_j}(s) - a_j| < \epsilon$, that is $(\zeta_{K_1}(s), \dots, \zeta_{K_r}(s)) \in U$ for small ϵ. This completes the proof. $\square$

In general we cannot prove that for different Galois extensions K_j of $\mathbb{Q}$ the corresponding Dedekind-ζ-functions are algebraically independent. For example let $K := \mathbb{Q}(\xi)$ be the field, where ξ is a primitive 8th root of unity. This extension has 3 different subextensions K_j of degree 2 over $\mathbb{Q}$. We find the algebraic relation $\zeta_K \zeta_{\mathbb{Q}}^2 = \zeta_{K_1}\zeta_{K_2}\zeta_{K_3}$.
More generally as $\zeta_K(s) = \zeta(s)\prod_{\chi\neq 1} L(s,\chi)^{\chi(1)}$ for every normal field it is clear that if $G_K := G(K/\mathbb{Q})$ has more normal subgroups than conjugacy classes, then there is a non-trivial algebraic relation between the corresponding ζ-functions.
Further algebraic relations are discussed in the article of Richard Brauer [**6**].

THEOREM 6.6. *Suppose that we have finite normal extensions $K_j/\mathbb{Q}$, $j = 1, \dots, n$ and the corresponding Dedekind Zeta-functions ζ_{K_j} do not satisfy any non-trivial algebraic relation.*
Then for every continuous function $f(x_1, \dots, x_n)$ on $\mathbb{C}^n$ the relation $f(\zeta_{K_1}, \dots, \zeta_{K_n}) \equiv 0$ implies $f \equiv 0$.

Proof: To prove this, let K be the minimal subfield of $\mathbb{C}$ containing all $K_1, \dots, K_n$. This field K is a normal extension of $\mathbb{Q}$. We may regard all the characters as characters of $G := G(K/\mathbb{Q})$ by using the restriction map $\sigma \in G(K/\mathbb{Q}) \mapsto \sigma_{|K_j} \in G(K_j/\mathbb{Q})$. The kernel of this homomorphism is a normal subgroup $N_j \lhd G$. Then for the ζ-functions we have $\zeta_{K_j}(s) = \zeta(s)\prod_{\chi\neq 1} L(s,\chi)^{\chi(1)}$, where the product is taken over all characters χ with $\chi(x) = \chi(1)$ for all $x \in N_j$. (Theorem 2.4 (2) and Corollary 2.1.)

According to Theorem 5.1 we can approximate all values $y_1 \neq 0, y_\chi \neq 0$ simultaneously by $\zeta(s)$ and $L(s,\chi)$ for $\chi \neq 1$ by taking a suitable $s \in \mathbb{C} \setminus \{1\}$.
To prove the theorem it has to be shown that the same holds for the $X_{K_j} := y_1 \prod_{\chi\neq 1, K_j} y_\chi$ (the index K_j indicates that the product is taken over the characters χ with $\chi(x) = \chi(1)$ for all $x \in N_j$): i.e., every set of non-zero values $X_{K_j}, j = 1, \dots, n$ can be simultaneously approximated.

Taking the logarithms $\log X_{K_j} = \log y_1 + \sum_{\chi\neq 1, K_j} \log y_\chi$ (each sum is taken over all χ with $\chi(x) = \chi(1)$ for all $x \in N_j$) we get n linear equations in the variables $\log y_1, \log y_\chi$ for $\chi \neq 1$. The variables X_j can be simultaneously approximated if the right sides of these equations are

linearly independent.
However if these equations were not linearly independent, then there would be a relation $0 = \sum_{j=1}^{n} m_j \Big(\log y_1 + \sum_{\chi, K_j} \log y_\chi \Big)$ with integers $m_j \neq 0$ for some j. This would result in an algebraic relation $\prod_{j=1}^{n} \zeta_{K_j}^{m_j}(s) = 1$ between the $\zeta_{K_j}(s)$. □

THEOREM 6.7. *Let K be a number field, ζ_K the corresponding Dedekind-ζ-function. Let $f(x_1, \ldots, x_m)$ be a continuous function, then the differential equation $f(\zeta_K, \zeta_K', \ldots, \zeta_K^{(m)}) \equiv 0$ implies $f \equiv 0$.*

Proof: Denote by H_K the Hilbert class field of K. Denote the principal character on $G(H_K/K)$ by 1. Then $\zeta_K(s) = L(s, 1, H_K/K)$ because of Theorem 2.4. Denote by L the normal extension of H_K over $\mathbb{Q}$. We know $L(s, 1, H_K/K) = L(s, 1, L/K) = L(s, 1^*, L/\mathbb{Q})$ as a consequence of Theorem 2.4. We have $1^* = \sum_\phi n_\phi \phi$, where the ϕ's are the irreducible characters of $G(L/\mathbb{Q})$, $n_\phi \in \mathbb{Z}^{\geq 0}$ and at least one $n_\phi \geq 1$. Denote this character by ϕ_0 and set $n_0 := n_{\phi_0}$. We get

$$\zeta_K(s) = \prod_{n_\phi \neq 0} L(s, n_\phi \phi, L/\mathbb{Q}).$$

If $f \not\equiv 0$, then there is an open set U such that $f(a) \neq 0$ for all $a \in U$. We may suppose that $a_0 \neq 0$, since the set is open, and that all points b with $|b_j - a_j| < \epsilon$ are also in U. Set $P(s) := a_0 + a_1 s + \ldots + \frac{a_m}{m!} s^m$. We may suppose that $P(s) \neq 0$ on a disc $|s| \leq r$ for some small r since $a_0 \neq 0$. Set $\sigma := 1 - \frac{1}{4[L:\mathbb{Q}]}$. According to Theorem 5.1 we may find for every $\epsilon_1 > 0$ numbers $t \in \mathbb{R}$, such that $|L(s+\sigma+it, n_\phi \phi, L/\mathbb{Q}) - 1| < \epsilon_1$ and $|L(s + \sigma + it, n_0 \phi_0, L/\mathbb{Q}) - P(s)| < \epsilon_1$. Therefore for every $\epsilon_2 > 0$ we find values $t \in \mathbb{R}$ such that $|\zeta_K(s + \sigma + it) - P(s)| < \epsilon_2$. For $\epsilon_2 > 0$ sufficiently small we thus get $|\zeta_K^{(j)}(\sigma + it) - a_j| < \epsilon$ as a consequence of Theorem 2.13 for $j = 0, \ldots, m$. Therefore $f(\zeta_K(s'), \ldots, \zeta_K^{(m)}(s')) \neq 0$ for this point $s' := \sigma + it$. □

This theorem was already proved by Reich [**26**] by different means. We may prove the analogous statement for arbitrary Hecke L-functions. The only difference in the proof is, that we replace the trivial ray character 1 by an arbitrary ray character χ of a general class group $I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ and the Hilbert class field H_K is replaced by the class field attached to this class group $I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ [**13**, I,p.9]:

THEOREM 6.8. *Let K be a number field, $I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ a general class group in the sense of class field theory* [**13**] *and χ an Abelian character on this group. Denote by $L(s,\chi)$ the corresponding Hecke L-function. If $f(x_1,\dots,x_m)$ is any continuous function, then the differential equation $f(L(s,\chi),L'(s,\chi),\dots,L^{(m)}(s,\chi))\equiv 0$ implies $f\equiv 0$.*

THEOREM 6.9. *Let $L(s,\chi)$ be a Hecke L-function of a number field K attached to ray class character χ. Suppose that the equation*

$$\sum_{k=0}^{N} s^k F_k(L(s,\chi),L'(s,\chi),\dots,L^{(m)}(s,\chi))=0$$

holds for all $s\in\mathbb{C}$ and fixed continuous functions $F_k:\mathbb{C}^{m+1}\to\mathbb{C}$. Then we get $F_k\equiv 0$ for $k=0,\dots,N$.

Proof: Denote by K_χ the class field attached to the character χ [**7**, p.219]. (This is the class field of the ideal group $H_\chi := \{\mathfrak{a}\mid\chi(\mathfrak{a})=1\}$ [**12**, p.88].) Then we have $L(s,\chi,K_\chi/K)=L(s,\chi)$. The character χ may be regarded as a character of $G(K_\chi/K)$ by using the Artin-Isomorphism. Denote the normal extension of $\mathbb{Q}$, which contains K_χ, by L. Then because of Theorem 2.4

$$L(s,\chi,K_\chi/K)=L(s,\chi,L/K)=L(s,\chi^*,L/\mathbb{Q}).$$

We have $\chi^*=\sum_\phi n_\phi\phi$, where the ϕ's are the irreducible characters of $G(L/\mathbb{Q})$, $n_\phi\in\mathbb{Z}^{\geq 0}$ and at least one $n_\phi\neq 0$. Let this be n_{ϕ_0} with the character ϕ_0. Set $n_0:=n_{\phi_0}$. Then

$$L(s,\chi)=L(s,\chi^*,L/\mathbb{Q})=\prod_{n_\phi\neq 0}L(s,n_\phi\phi,K/\mathbb{Q}).$$

Suppose that $F_N\not\equiv 0$. We get an open set U such that $|F(a)|>c$ for some positive constant $c>0$ and all $a\in U$. Since this set is open we may even suppose that the first coordinate of points in U satisfy $a_0\neq 0$. Further we may suppose that U is contained in a compact set. Set $P(s):=a_0+a_1s+\ldots+\frac{a_m}{m!}s^m$. We may suppose that $P(s)\neq 0$ for all $|s|\leq r$ for some small $r>0$ since $a_0\neq 0$. Set $\sigma:=1-\frac{1}{4[L:\mathbb{Q}]}$. According to Theorem 5.1 we find for every $\epsilon_1>0$ a set A_{ϵ_1} with $\liminf\limits_{T\longrightarrow\infty}\frac{vol(A_{\epsilon_1}\cap(0,T))}{T}>0$, such that for all $t\in A_{\epsilon_1}$ we have $|L(s+\sigma+it,n_0\phi_0,L/\mathbb{Q})-P(s)|<\epsilon_1$ and $|L(s+\sigma+it,n_\phi\phi,L/\mathbb{Q})-1|<\epsilon_1$ for all $|s|\leq r$. Then $|L(s+\sigma+it,\chi)-P(s)|<\epsilon_2$ for some set A_{ϵ_1}, if we choose ϵ_1 sufficiently small, and because of Theorem 2.13 $|L^{(j)}(\sigma+it,\chi)-a_j|<\epsilon$ for $j=0,\dots,m$.

Thus we have $|F_N(L(\sigma+it,\chi),\dots,L^{(m)}(\sigma+it,\chi))| > c$ for small $\epsilon > 0$ and for all $t \in A_{\epsilon_1}$. Then

$$\begin{aligned} c &< |F_N(L(\sigma+it,\chi),\dots,L^{(m)}(\sigma+it,\chi))| \\ &= |\sum_{k=0}^{N-1}(\sigma+it)^{k-N}F_k(L(\sigma+it,\chi),\dots,L^{(m)}(\sigma+it,\chi))|. \end{aligned}$$

Since $(L(\sigma+it,\chi),\dots,L^{(m)}(\sigma+it,\chi)) \in U$ is contained in a compact set, the values of the functions F_k are bounded on the set U. However the set A_{ϵ_1} is unbounded and we get an infinite sequence of values $t_l \in A_{\epsilon_1}$ with $t_l \to \infty$. Taking the limit, we get $0 < c \leq 0$ as a contradiction.

$\square$

Symbols

$\mathbb{Z}$	rational integers
$\mathbb{Z}^{\geq 0}$	rational integers ≥ 0
$x \in \gamma \mod \mathbb{Z}$	*page 15*
$\lvert x - x_0 \mod \mathbb{Z}\rvert < \epsilon$	*page 15*
$\mathbb{Q}$	rational numbers
$\mathbb{P}$	rational primes
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	real numbers > 0
$\mathbb{R}^{\mathbb{P}}$	functions $\theta : \mathbb{P} \to \mathbb{R}$
$\theta \in \mathbb{R}^{\mathbb{P}}$	$(\theta_p)_{p \in \mathbb{P}}$
$[\alpha]$	greatest integer such that $[\alpha] \leq \alpha$
$\{\alpha\}$	$\{\alpha\} := \alpha - [\alpha]$
$\mathbb{C}$	complex numbers
$\#M$	cardinality of a finite set M
Y^c	the complement of a set $Y \subset M$
$M \setminus Y$	$\{x \in M \mid x \notin Y\}$
$(a, b]$	$a, b \in \mathbb{R}$, interval $a < x \leq b$
k, K	algebraic number fields
$\mathcal{O}_k$	ring of integers of the number field k
H_k	Hilbert class field of k
K_χ	ray class field of the ray character χ [**7**, p.219]
$G(K/k)$	Galois group of K/k
$[K : k]$	degree of K relative to k
$Trace(\alpha)$	trace of the algebraic number α
$N(\alpha)$	norm of the algebraic number α
$N(\mathfrak{a})$	norm of the ideal $\mathfrak{a}$
$I^{(\mathfrak{f})}$	group of fractional ideals prime to $\mathfrak{f}$
$S_{\mathfrak{f}}$	ray $\mod \mathfrak{f}$
$H_{\mathfrak{f}}$	ideal group with conductor $\mathfrak{f}$
$(\mathfrak{P}, K/k)$	Frobenius Automorphism
$\sigma_{\mathfrak{p}}$	Frobenius Automorphism
χ^*	induced character of χ
$f_{\vert U}$	map f restricted to U

(ϕ, ψ)	scalar product of the class functions ϕ, ψ
$f = O(g)$	Landau symbol
$Re(z)$	real part of $z \in \mathbb{C}$
$Im(z)$	imaginary part of $z \in \mathbb{C}$
$vol(M)$	Lesbegue-measure of a set
$\mathcal{H}$	Hilbert space
$\langle x, y \rangle$	scalar product of $x, y \in \mathcal{H}$
$\|x\|$	norm of a vector
$\|x\|$	for a scalar product: $\|x\| = \sqrt{\langle x, x \rangle}$
$\|L\|$	$= \sup_{\|x\|=1} \|L(x)\|$ for continuous linear operator L
$L(s, \phi, K/k)$	Artin L-function of the non-Abelian character ϕ
$\zeta_k(s)$	Dedekind Zeta-function of the number field k
$L(s, \chi)$	Hecke L-function for ray characters χ or
$L(s, \chi)$	$= L(s, \chi, K/\mathbb{Q})$ for non-Abelian character χ of $G(K/\mathbb{Q})$
$L_M(s, \chi, \theta)$	finite Euler-product, *Def. 6, p.17*
$GL_k(\mathbb{C})$	general linear group of $k \times k$ matrices
$\det(A)$	determinant of a matrix A
E	unit matrix
$U \lhd G$	U is a normal subgroup of G
$L_2(a, b)$	space of square integrable functions with support in (a, b)
$Res_{s=s_0} f(s)$	residue of the function f at $s = s_0$
$v_p(d)$	p-valuation of d for $p \in \mathbb{P}$

Bibliography

[1] Achieser, N.I. Vorlesungen über Approximationstheorie. (German) Mathematische Lehrbücher und Monographien. I. Abt. Band II. Berlin: Akademie-Verlag. (1967).

[2] Ahlfors, Lars.V. Complex analysis. 2nd ed. Maidenhead, Berkshire: McGraw'Hill Publishing Comp. (1966).

[3] Artin, Emil. Collected papers. Ed. by Serge Lang, John T. Tate. (Unaltered repr. of the 1965 orig., publ. by Addison-Wesley Publishing Company, Inc.).

[4] Bohr, Harald; Jessen, B. On the distribution of the values of the Riemann zeta function. Am. J. Math. 58, 35-44 (1936).

[5] Brauer, Richard. On Artin's L-series with general group characters. Ann. Math., Princeton, II. Ser. 48, 502-514 (1947).

[6] Brauer, Richard. Beziehungen zwischen Klassenzahlen von Teilkörpern eines galoisschen Körpers. Math. Nachr. 4, 158-174 (1951).

[7] Cohn, Harvey. A classical invitation to algebraic numbers and class fields. New York, Springer-Verlag.(1988).

[8] Davenport, Harold; Heilbronn, H. On the zeros of certain Dirichlet series. J. London Math. Soc. 11, 181-185, 307-312 (1936).

[9] Fischer, Wolfgang; Lieb, Ingo. Ausgewählte Kapitel aus der Funktionentheorie. Vieweg, Braunschweig (1988).

[10] Garunkstis, Ramunas ; Steuding, Jörn. On the zero distributions of Lerch zeta-functions, Preprint 00-3, Vilnius University, Faculty of Mathematics and Informatics (2000).

[11] Gonek, Stephen M. Analytic properties of zeta and L-functions, Thesis, Univ. Michigan, Ann. Arbor, 1979.

[12] Hasse, Helmut. Vorlesungen über Klassenkörpertheorie. Würzburg, Physica-Verlag. (1967).

[13] Hasse, Helmut. Zahlbericht.Würzburg, Physica-Verlag (1970).

[14] Karatsuba, Anatolii A. ; Voronin, Sergei M.. The Riemann Zeta-Function. Walter de Gruyter. 1992.

[15] Lagarias, J.C.; Odlyzko, A.M. Effective versions of the Chebotarev density theorem. Algebraic Number Fields, Proc. Symp. London math. Soc., Univ. Durham 1975, 409-464 (1977).

[16] Lang, Serge. Algebraic Number Theory. Springer Verlag. 1991.

[17] Laumon, Gérard. La correspondance de Langlands sur les corps de fonctions (d'apres Laurent Lafforgue). Séminaire Bourbaki. 52ème année, 1999-2000, nr. 873, Mars 2000.

[18] Laurincikas, Antanas; Matsumoto, Kohji. Joint value-distribution theorems for the Lerch zeta-functions. Lith. Math. J. 38, No.3, 238-249 (1998).

[19] Lavrik, A.F. : Izv. Akad. Nauk SSSR, Ser. Mat. Tom. 32 (1968) No. 1. (english).

[20] Marcus, Daniel A. Number fields. New York - Heidelberg - Berlin 1977.

[21] Murty, Ram. On Artin L-Functions. In: Proceedings of the Class Field Theory Conference, Tokyo, 1998 (edited by K. Miyake).

[22] Narkiewicz, Wladyslaw. Elementary and analytic theory of algebraic numbers. 2nd ed., Berlin etc.: Springer-Verlag; PWN-Polish Scientific Publishers, Warszawa (1990).

[23] Neukirch, Jürgen. Algebraische Zahlentheorie. Springer Verlag. 1992.

[24] Paley, Ramond E.A.C.; Wiener, Norbert. Fourier transforms in the complex domain. New York (1964).

[25] Pohst, Michael E.; Zassenhaus, Hans. Algorithmic algebraic number theory. Cambridge etc.: Cambridge University Press. (1989)

[26] Reich, Axel. Zetafunktionen und Differenzen-Differentialgleichungen. Arch. Math. 38, 226-235 (1982).

[27] Serre, Jean-Pierre. Linear Representations of Finite Groups. 4th ed. New York Heidelberg Berlin (1993).

[28] Stark, Harold M. L-functions at s=1. IV: First derivatives at s=0. Adv. Math. 35, 197-235 (1980). Oxford Science Publications. Oxford: Clarendon Press

[29] Titchmarsh, E.C. The theory of functions. 2nd ed. London: Oxford University Press. (1975)

[30] Titchmarsh, E.C. The theory of the Riemann zeta-function. 2nd ed., rev. by D. R. Heath- Brown. Oxford (1986).

Ich danke Prof. M. E. Pohst für die Möglichkeit zu diesem Thema meine Dissertation zu verfassen, Prof. F. Grunewald für die Übernahme des Koreferats und Prof. E. Friedman für seine Ermutigung und sein Interesse am Thema dieser Arbeit. Bei Robert Fraatz und Maike Henningsen bedanke ich mich für ihre Mühe bei der Durchsicht meiner Arbeit.